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Stochastic transient of a noisy van der Pol oscillator

H.K. Leung [1]

Physics Department, National Central University, Chung-li, Taiwan 32054, ROC

Abstract

Transient characteristics of a van der Pol oscillator perturbed by additive and multiplicative noises of different non-linearities are studied and compared systematically. We numerically determine the noise dependence of the period and of the life-time of a noisy oscillator. Attraction basins and stability are also investigated for the trivial fixed point and the limit cycle.

1. Introduction

Non-equilibrium systems are characterized by their transient behaviors and are usually modeled by a set of rate equations which involves small number of variables and parameters. Most of these systems are open ones which are subject to external fluctuations of various types and strengths. Stochastic analysis of the transient properties could provide a more realistic understanding of the non-equilibrium processes.

Non-linearity of the model systems may allow for fixed points and oscillation patterns. A transition among them resembles a phase transition which occurs in equilibrium systems [2]. Complex phenomena associated with their appearances and transitions have attracted considerable attention in recent years [3]. The interplay of non-linearity, periodicity and randomness could lead to many interesting phenomena including stochastic resonance [5], harmonic mixing [6] and noise-induced resonance [7], etc.

In this report, we choose a well-known model of van der Pol oscillation [4], which possesses a trivial fixed point and a limit cycle attractor. We consider all possible types of white noise which might influence the model system, and compare their effects

[1] E-mail: leung@joule.phy.ncu.edu.tw.

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on the transient properties. Approximation schemes are employed to derive the phase trajectories so that we are able to probe how the self-sustaining mechanism of the van der Pol oscillator is perturbed by external noises, and to determine the period and lifetime of oscillation. Furthermore, detailed analysis along this line enables us to estimate the attraction basins for both fixed point and limit cycle.

2. The model system

The classic van der Pol oscillator can be modeled by the following deterministic rate equation:

$$\ddot{x} = -\alpha x - \beta \dot{x}(x^2 - 1) , \tag{1}$$

which is so simple that the restoring and damping forces in mechanical system can be readily identified. The self-sustaining mechanism which is responsible for the perpetual oscillation rests on the last term of Eq. (1). Energy exchange with the external agent depends on the magnitude of displacement $|x|$ and on the sign of velocity $\dot{x}$. During a complete cycle of oscillation, the energy is dissipated-out and feded-in two times each. Among the four shiftings of these opposite operations, the two with $|x| = 1$ and maximum $|\dot{x}|$ play a significant role in sustaining the oscillation since they occur drastically within a short moment and involve large quantities.

The second order Eq. (1) can be rewritten as two first order ones,

$$\dot{x} = f_1(x, y) = y , \tag{2}$$

$$\dot{y} = f_2(x, y) = -\alpha x - \beta y(x^2 - 1) . \tag{3}$$

These equations allow only one steady state solution of $(x^s, y^s) = (0, 0)$, which is found to be unstable with respect to perturbation and can be reached only if the system initiates there. A system starts with $(x_0, y_0) = (0, 0)$ will stay at this trivial fixed point forever. All other initial configurations $(x_0, y_0) \neq (0, 0)$ belongs to the attraction basin of the limit cycle.

3. Stochastic treatment

By considering the influences of external randomness, the rate equations (2) and (3) are now stochastic [5],

$$\dot{x} = f_1(x, y) , \tag{4}$$

$$\dot{y} = f_2(x, y) + Dg(x, y)\, \xi_t , \tag{5}$$

where D represents noise amplitude and ξ_t is a random process which satisfies

$$\langle \xi_t \rangle = 0 , \qquad \langle \xi_t\, \xi_{t'} \rangle = \delta(t - t') . \tag{6}$$

The multiplicative function $g(x,y)$ specifies the types of noise involved. For the case of additive noise which represents a random driving force added onto the rate equation (1), $g(x,y) = 1$. Multiplicative noises due to fluctuations in rate parameters α and β are respectively represented by $g(x,y) = -x$ and $g(x,y) = (1-x^2)y$.

The stochastic system is now described by a probability function $P(x,y;t)$ which is found to satisfy a Fokker-Planck equation [8],

$$\partial_t P = -\partial_x [f_1 P] - \partial_y \left[\left(f_2 + \frac{\nu - 1}{2} D^2 g \partial_y g \right) P \right] + \frac{1}{2} D^2 \partial_{yy} [g^2 P] . \tag{7}$$

In the above, $\nu = 1$ and $\nu = 2$ stands respectively for the results derived from Ito's and Stratonovich's interpretations of stochastic calculi.

Should we be able to solve $P(x,y;t)$, statistical moments

$$\overline{x^m y^n} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} P(x,y;t) x^m y^n \, dx dy \tag{8}$$

can calculated to provide pertinent informations, e.g. the means $\bar{x}_i$ and the variances $\sigma_{ij} = \overline{x_i x_j} - \bar{x}_i \bar{x}_j$. where $i = 1$ and $i = 2$ represent respectively the x and y parts. Since an explicit solution of P is unlikely, we turn to the moment equations which can be derived from Eqs. (7) and (8) as

$$\dot{\overline{x_i}} = \bar{f}_i + \frac{\nu - 1}{2} D^2 \overline{g \partial_y g} \, \delta_{i2} , \tag{9}$$

$$\begin{aligned} \dot{\sigma}_{ij} &= \overline{x_i f_j} - \bar{x}_i \bar{f}_j + \overline{x_j f_i} - \bar{x}_j \bar{f}_i \\ &+ D^2 \left[\overline{g^2} \delta_{ij} + \frac{\nu - 1}{2} \left(\overline{x_i g \partial_{x_j} g} - \bar{x}_i \, \overline{g \partial_{x_j} g} \right) \right] \delta_{j2} . \end{aligned} \tag{10}$$

To obtain a set of closed form equations for these five moments, approximation schemes [9] is required. The final results can be expressed as

$$\dot{\bar{x}} = \bar{y} , \tag{11}$$

$$\dot{\bar{y}} = -\alpha \bar{x} - \beta [\bar{y}(\bar{x}^2 - 1 + \sigma_{xx}) + 2\bar{x}\sigma_{xy}] + \frac{m(m-1)(\nu-1)}{4} H_1 , \tag{12}$$

$$\dot{\sigma}_{xx} = 2\sigma_{xy} , \tag{13}$$

$$\begin{aligned} \dot{\sigma}_{yy} &= -2[(\alpha + 2\beta\bar{x}\bar{y})\sigma_{xy} + \beta(\bar{x}^2 - 1)\sigma_{yy}] \\ &+ (m-2) \left[\frac{m-1}{2} - m(\sigma_{xx} + \bar{x}^2) \right] D^2 \\ &+ m(m-1)[(H_1 + H_2)\bar{y} + (\nu - 1) H_2] , \end{aligned} \tag{14}$$

$$\dot{\sigma}_{xy} = -(\alpha + 2\beta\bar{x}\bar{y})\sigma_{xx} - \beta(\bar{x}^2 - 1)\sigma_{xy} + \sigma_{yy} + m(m-1) H_3 , \tag{15}$$

where $m = 0, 1$ and 2 stand respectively for the results of additive, linear and non-linear multiplicative noises. In the above, we defined

$$H_1 = D^2 [\bar{y}(\bar{x}^2 - 1)^2 + 4\bar{x}(\bar{x}^2 - 1)\sigma_{xy} + 2\bar{y}(3\bar{x}^2 - 1)\sigma_{xx}] , \tag{16}$$

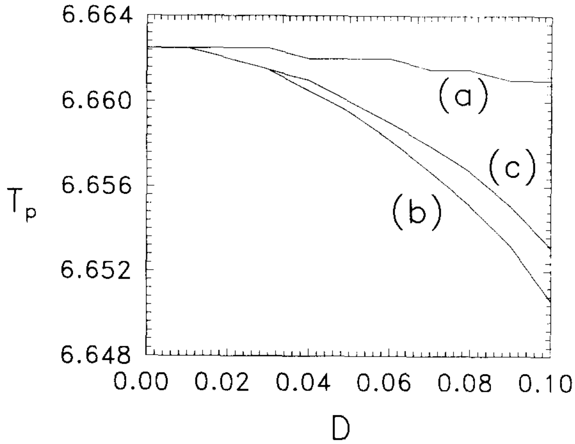

Fig. 1. The oscillation period is plotted against the noise amplitude. Curves (a), (b) and (c) stand respectively for the results of additive, linear and non-linear multiplicative noises. In curve (c) only the results due to Stratonovich's interpretation are plotted; the curve from Ito's interpretation is slightly below curve (c). $\alpha = \beta = 1$ is assumed.

$$H_2 = D^2(\bar{x}^2 - 1)[(\bar{x}^2 - 1)\sigma_{yy} + 4\bar{x}\sigma_{xy}], \tag{17}$$

$$H_3 = \frac{\nu - 1}{4} D^2(\bar{x}^2 - 1)[(\bar{x}^2 - 1)\sigma_{xy} + 4\bar{x}\bar{y}\sigma_{xx}]. \tag{18}$$

The stochastic transient is now approximated by the five moments which are solved numerically from above with a given initial condition.

4. Noise effects on limit cycle

At small and moderate noises, stochastic trajectories described by $(\bar{x}, \bar{y})$ remain practically deterministic. This is true for both pre-cycle relaxation and the subsequent oscillation. The period of oscillation is found to be a decreasing function of noise intensity. The dependence is very weak, as is shown in Fig. 1. It is reasonable to find that fluctuating α causes a most significant drop of the period, since this parameter is responsible for determining the period of a simple harmonic oscillator. This phenomenon of noise-induced speeding-up is in contrast to the finding that noise could slow down relaxation processes [9].

The stochastic aspect of the noise affected transient rests on the behaviors of variances $\sigma_{ij}(t)$, which are found to oscillate with increasing amplitude. This result agrees qualitatively with those derived from other oscillating models [10–12]. The oscillating σ_{ij} reach their peak values near the regions with $|\bar{x}| \approx 1$ and $|\bar{y}| = |\dot{\bar{x}}| \approx$ max, at which the system undergoes a drastic shift between the energy-dissipating and energy-feeding operations. Since the self-sustaining mechanism is perturbed by noise-induced fluctuations, the stability of the oscillator is weakened. After prolonged perturbation of noise, deformation of the limit cycle trajectory occurs around these regions. As is shown in

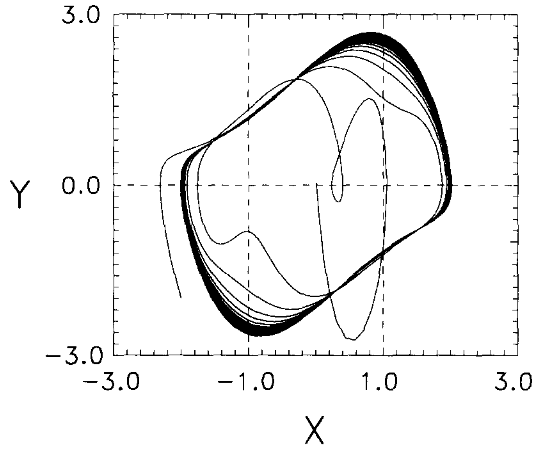

Fig. 2. Deformation of the limit cycle trajectory is plotted for a system perturbed by linear multiplicative noise. $D = 0.02$ and $\alpha = \beta = 1$ is assumed. Similar behaviors are found for other noises.

Fig. 2, the limit cycle is gradualy compressed towards the trivial steady state $(0,0)$ which would otherwise repel any phase paths not initiating there. Transition of the limit cycle to a fixed point was predicted in numerical studies of the steady state probability [13].

The deformed trajectory is found to spiral toward the vicinity of $(0,0)$ before turning to be erratic. Our approximation fails to describe the detail of this final process. Non-equilibrium transition is stochastic in nature, a quantitative description of the transition process is neither necessary nor appropriate.

The life-time T of the noisy oscillator is also a decreasing function of D. This dependence follows a simple power law relationship,

$$T \propto D^{-\Gamma}. \tag{19}$$

For all noises considered, the exponent is numerically found to be $\Gamma = 1$. Eq. (19) remains valid at the deterministic limit of $D = 0$, at which a perpetual motion implies that $T \to \infty$.

Deviation from the universal behavior of Eq. (19) occurs for large D, with which the system oscillates only a few cycles with large deformation. For even larger values of $D \approx O(1)$, the phase path spirals directly toward the trivial fixed point $(0,0)$ without being attracted to the limit cycle.

5. Stability of the trivial fixed point

It is a stochastic phenomenon that the deterministic fixed point turns to be attractive for phase paths at the later phase of the transient process. However, this trivial point is not a stable steady state in both deterministic and stochastic analyses. It is not a real attractor, instead it is a special case of transient modality [14] at which the probability function $P(x,y;t)$ will dwell for some portion of the transient time.

With the special initial condition that $(x_0, y_0) = (0,0)$ and $\sigma_{ij}(0) = 0$, this trivial state does behave uniquely in stochastical analysis. The averaged phase path $(\bar{x}(t), \bar{y}(t))$ remains stationary there, since the moment equations (11)–(15) allow that $\dot{\bar{x}} = \dot{\bar{y}} = 0$ and that

$$\dot{\sigma}_{xx} = 2\sigma_{xy}\,, \tag{20}$$

$$\dot{\sigma}_{yy} = -2\alpha\sigma_{xy} + 2\beta\sigma_{yy} + \frac{m(m-1)}{2}\nu D^2\sigma_{yy} + (m-2)\left[\frac{m-1}{2} - m\sigma_{xx}\right] D^2\,, \tag{21}$$

$$\dot{\sigma}_{xy} = -\alpha\sigma_{xx} + \sigma_{yy} + \left[\beta + m(m-1)\frac{\nu-1}{4}D^2\right]\sigma_{xy}\,. \tag{22}$$

In the above, $m = 0,1,2$ stand respectively for the additive, linear and non-linear multiplicative noises.

For the case of additive noise, the trivial state $(0,0)$ is found to be metastable since Eqs. (20)–(22) predict that $\bar{x}(t) = \bar{y}(t) = 0$ and that σ_{ij} oscillate with increasing amplitude. In other words, $P(x,y;t)$ remains peaked at the initial location $(0,0)$ and expands its width in an oscillating manner. This resembles the stochastic metastability found in other models [10,15].

On the other hand, multiplicative noise preserves this fixed point with time independent $\bar{x}^s = \bar{y}^s = 0$ and $\sigma_{ij}^s = 0$. Linear stability analysis [1] shows that this steady state is unstable with respect to small perturbation. The stability eigenvalues, for the case of $\alpha = \beta = 1$ and at the weak noise limit, are found to satisfy

$$\text{Max}\,[\text{Real}\,\{\lambda\}] = \frac{1+\sqrt{5}}{2} > 0\,. \tag{23}$$

This value is larger than the deterministic value of $\frac{1}{2}$. Therefore, the stability of trivial point attractor $(0,0)$ is weakened by multiplicative noise.

6. Noise effects on attraction basins

The deterministic fixed point $(0,0)$ remains to be a stochastic one when multiplicative noises are considered. Its attraction basin, which consists of nothing but itself, is preserved.

Though the self-sustaining mechanism is perturbed by noise, the limit cycle attractor is quite noise-robust. Our studies show that noises affect not only the stability and shape of the limit cycle, but also the pre-cycle relaxation. As in deterministic analysis, it takes a very long time to approach the limit cycle if (x_0, y_0) is located close to $(0,0)$ or very far away from the limit cycle. Since σ_{ij} is increasing with time, the fluctuations

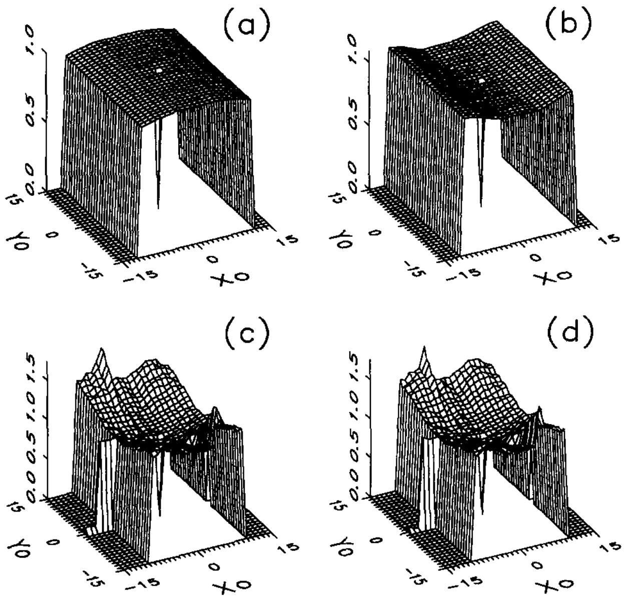

Fig. 3. Exponent Γ' defined in Eq. (24) is plotted over a range of initial configuration for (a) additive, (b) linear, and non-linear multiplicative noise in (c) Ito's and (d) Stratonovich's interpretations. $\alpha = \beta = 1$ is assumed.

might cause significant deviation from the deterministic path. The limit cycle could lose its attraction for these paths if noise is large.

The transient time T' is defined as the total time interval starting from an initial state to the final stage of destruction of the limit cycle. Since T' consists of the pre-cycle relaxation time $T_{\rm rel}$ plus the life-time of the limit cycle oscillation, the T'–D relationship cannot be expressed by a simple power-law with universal exponent. By adapting the form of Eq. (19), this relationship can be expressed as

$$T' = T_{\rm rel} + T \propto D^{-\Gamma'} , \qquad (24)$$

with exponent Γ' depending on (x_0, y_0). In Fig. 3, we plot the numerical value of Γ' over a wide range of initial state configurations. The result that $\Gamma' \approx \Gamma \approx 1$ is valid for any (x_0, y_0) which is reasonably close to the limit cycle. Deviation from this *universal* value of unity reflects the noise effect on the relaxation process. The region with $\Gamma' = 0$ contains the initial states from which phase paths relax toward an erratic oscillation without being attracted to the limit cycle. In other words, the attraction basin of the limit cycle is depleted by noise.

7. Conclusions and discussions

In the present case study, we probe how and to what extend a noise could exhibit its destructive effects on a non-linear system which would otherwise behave coherently. All noises we considered are found to disturb the self-sustain mechanism of an oscillator. Stability and attraction basins are also affected by noise.

The fact that the limit cycle is compressed toward the center and that the deformed trajectory approaches the vicinity of $(0,0)$ in a spiralling manner suggests that noise plays a role in tilting the competition between the two coexisting attractors. This implication echoes the results that noise could postpone the bifurcation between fixed point and limit cycle [16] and that noise could induce a transition from a stable fixed point to a stable limit cycle, and vice versa [17].

Noise could slow down or speed up the transient process of a non-linear system. Stochastic resonance, which exhibits a maximum amplification of a periodic signal at an optimum noise intensity [5], is a result of compromise between its constructive and destructive roles. Noises are found to be able to retrieve intrinsic periodicity which was damped out by the dissipation process [18]. Since the response of a dynamic system to noise is non-linear, noisy systems could reveal more complex and interesting phenomena than deterministic systems could do. More investigation of noise effects on non-linear systems is needed.

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