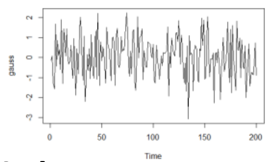
PSTAT 174/274, Week 2, Lecture 3: Moving Average Models

Creating Time Series Models from White Noise

(Lego blocks)

Unobserved building blocks

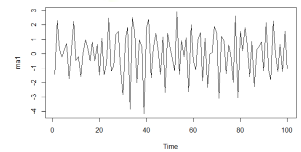


(White noise, shocks)

White Noise Z_t, t=1,2, ...

(Lego construction)

Observed



(Time series)

$$X_t = Z_t + \theta_1 Z_{t-1}, |\theta_1| < 1.$$

Lecture 3 Outline:

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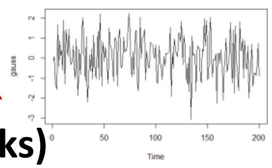
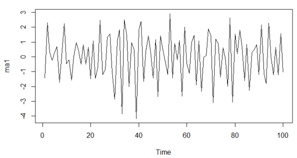




(White noise, shocks)

(Time series)

White Noise Z_t, t=1,2, …

$$X_t = Z_t + \theta_1 Z_{t-1}, |\theta_1| < 1.$$

LECTURE 3 PART 1: Review of MA(1)

Review of Moving Average of Order 1 Model

3.1: MA(0) (Moving Average of Order Zero) process Z_t is the White Noise process $\{Z_t\} \sim WN(0, \sigma_Z^2)$. It is characterized by properties

• $E(Z_t) = 0$, for all t; • $Var(Z_t) = E(Z_t^2) = \sigma_Z^2$ for all t; • $Cov(Z_t, Z_s) = E(Z_t Z_s) = 0$, $t \neq s$.

3.2: MA(1) (Moving Average of Order One) process X_t satisfies equation

$$X_t = Z_t + \theta_1 Z_{t-1} \text{ where } Z_t \sim WN(0, \sigma_Z^2) \text{ and } |\theta_1| < 1$$

It's properties are:

Recall: $\gamma_X(k) = Cov(X_t, X_{t+k})$ is covariance at lag k

- $EX_t = 0$ for all t ;
- ACVF (autocovariance function) is

  $\gamma_X(0) = Var(X_t) = \sigma_Z^2 (1 + \theta_1^2)$; $\gamma_X(1) = \gamma_X(-1) = \sigma_Z^2 \theta_1$; $\gamma_X(k) = 0$, $|k| > 1$.

Recall: $\rho_X(k) = \gamma_X(k) / \gamma_X(0)$

- ACF (autocorrelation function)

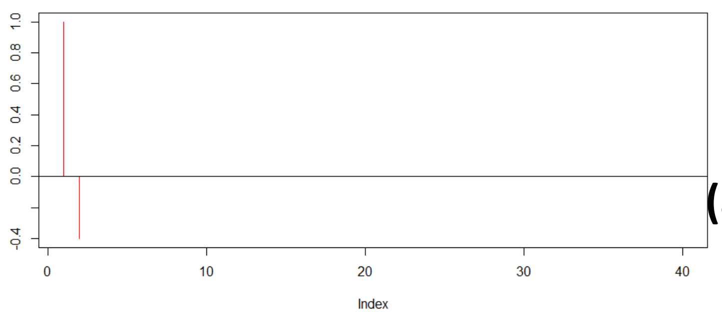
  $\rho_X(0) = 1$; $\rho_X(1) = \theta_1/(1 + \theta_1^2)$; $\rho_X(k) = 0$ for $|k| > 1$.
- Because ACVF of X_t depends only on lag k and mean is constant (zero), MA(1) process is always (weakly) stationary.

ACVF and ACF of Moving Average Models: MA(1)

$X_t = Z_t + \theta_1 Z_{t-1}$ where $Z_t \sim$ WN $(0, \sigma_Z^2)$ and $|\theta_1| < 1$.

X is stationary with $\mu_x = E(X_t) = 0$, and ACF
$\rho_X(0) = 1$; $\rho_X(k) = 0$ for $|k| > 1$, and $\rho_X(1) = \theta_1/(1+\theta_1^2)$

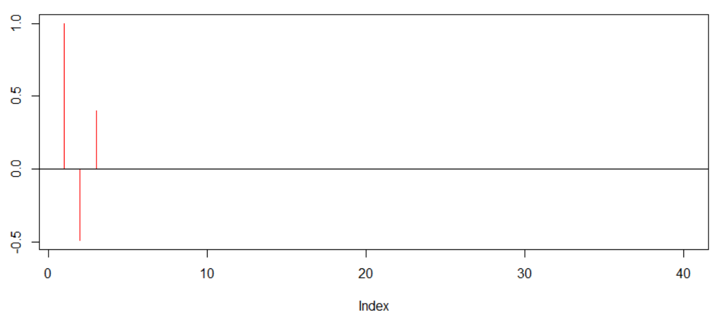
Important: For MA(1), autocorrelations for lags k ≠ 0, 1 are zero:
$\rho_X(k) = 0$ for $|k| > 1$. Sign $(\rho_X(1))$ = Sign (θ_1)



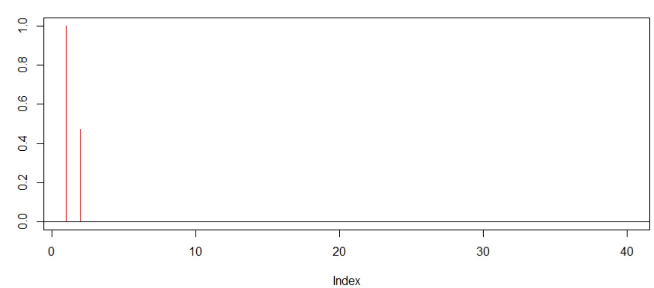
(a)

Check your understanding:
Figures (a), (b), & (c) plot acfs for different TS

- Which graphs correspond to MA(1) ?
- For graphs corresponding to MA(1) what is the sign of its coefficient θ_1?



(b)



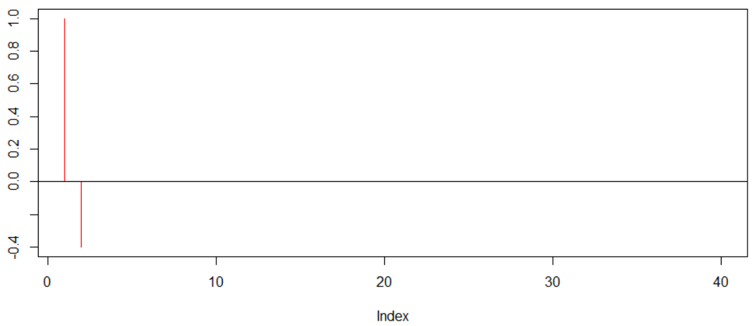
(c)

ACVF and ACF of Moving Average Models: MA(1)

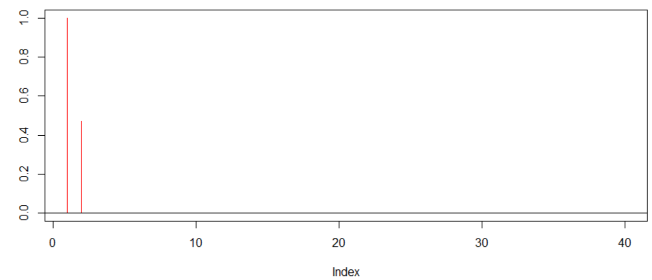
$X_t = Z_t + \theta_1 Z_{t-1}$ where $Z_t \sim$ WN (0, σ_Z^2) and $|\theta_1| < 1$.

Important: For MA(1), autocorrelations for lags k ≠ 0, 1 are zero:

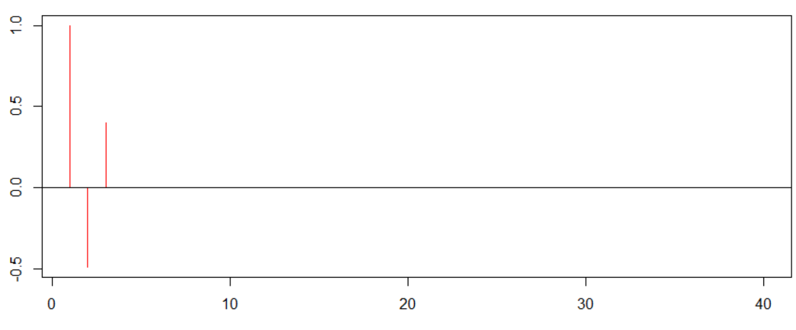
ρ_X (k) = 0 for |k| >1. Sign ($\rho_X(1)$) = Sign (θ_1)



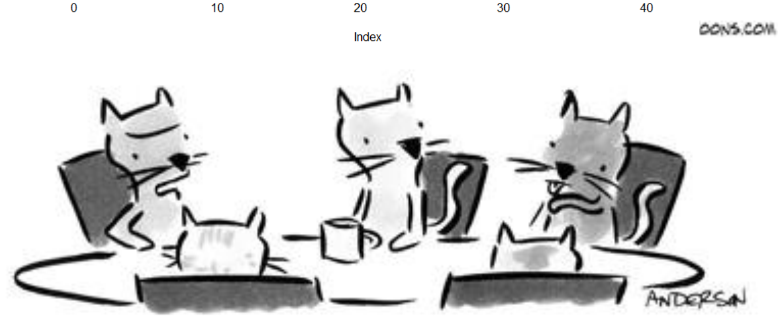
(a)



(c)



(b)



"For crying out loud, Jensen! Stop licking yourself and pay attention!"

Remote learning is hard. It requires focus and patience…

- **Which graphs correspond to MA(1) ? Graphs (a) & (c). Because (b) has ρ_X (2) ≠ 0, it is not MA(1).**
- **Because sign ($\rho_X(1)$) = Sign (θ_1), for (a), $\theta_1 < 0$, and for (c), $\theta_1 > 0$.**

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Creating Time Series Models from White Noise

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(Lego construction)

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Building blocks

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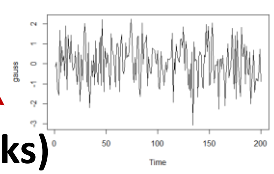
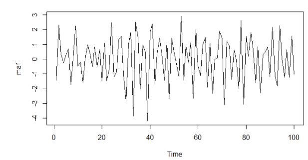




(Time series)

(White noise, shocks)

White Noise Z_t, t=1,2, …

$$X_t = Z_t + \theta_1 Z_{t-1}, |\theta_1| < 1.$$

LECTURE 3 PART II:

Backshift Operator B: pp. 7 - 8;
Definition of Invertibility: p. 9
Invertibility of MA(1): pp. 10 - 11;
Estimating θ_1 from data: pp. 12 - 13

Start New Material: (Backward) Shift Operator B

The shift operator B acts on a time series X as follows:

$$B\,X_t = X_{t-1}; \qquad B^k\,X_t = X_{t-k}\,.$$

B also is called Backshift operator.

What is operator?
Do you have an example?
If not, check the next slide.

Using shift operator B, MA(1) process can be written as

$$X_t = Z_t + \theta_1 Z_{t-1} \quad \text{or} \quad X_t = 1\cdot Z_t + \theta_1 B\,Z_t \equiv (1+\theta_1 B)Z_t$$

Polynomials in B are manipulated in the same way as polynomial functions of real variables.

Define MA(1) operator $\theta\,(B) = 1 + \theta_1 B$. **Treating B as a (complex) number z,**

This defines a polynomial θ **(z) =** $1 + \theta_1 z$ **(sometimes called characteristic)**

Shift Operator is a useful technical tool!

We use it to investigate *INVERTIBILITY* of MA(1)

New notion!

Let's practice: Name

$B^2\,X_t$ = ?

$B^5\,Y_{t-1}$ = ?

You may check your answers on the next slide Or skip to slide 9.

More about Operators

- In calculus, function f(x) maps a real value x into another real value y=f(x).

For example, y= x^2 will map x=1 into y =1; x=2 into y =4, etc.

- In TS, we consider operators that map sequences into different sequences or numbers.

For example, $B\, X_t = X_{t-1}$ will map sequence $\{X_5, X_6, X_7, \ldots\}$ into $\{X_4, X_5, X_6, \ldots\}$

$B^3\, X_t = X_{t-3}$ will map TS $\{X_5, X_6, X_7, \ldots\}$ into TS $\{X_2, X_3, X_4, \ldots\}$

- You have seen example of operators in your classes before:

Calculus: derivative $\frac{\partial y}{\partial x}$ maps a real function y= x^2 into another real function : $\frac{\partial}{\partial x}(x^2) = 2x$.

integral ∫ maps real function into another function: $\int x^2\, dx = x^3/3$.

summation ∑ maps a sequence $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots)$ into a number: $\sum_{i=0}^{\infty} \frac{1}{2^i} = 2$.

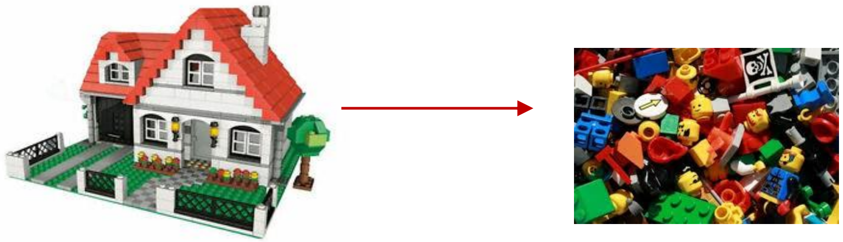
Probability: Expectation takes a r.v. X and maps it into a real number: $E(X) = \mu$.

Let's practice:

$B^1 X_t = X_{t-1}$; $B^2 X_t = B(B X_t) = B X_{t-1} = X_{t-2}$; $B^3 X_t = X_{t-3}$; …

$B^1 Y_{t-1} = Y_{t-2}$ (one step back) ; $B^5 Y_{t-1} = Y_{t-6}$ (five steps back)

Back to Time Series: Property of Invertibility of X



Time Series $\{X_t\}$ is *Invertible* if the shocks $\{Z_t\}$ can be expressed via values of X_t as a *convergent* series:

$$Z_t = X_t + \pi_1 X_{t\text{-}1} + \pi_2 X_{t\text{-}2} + \pi_3 X_{t\text{-}3} + \ldots$$

Why important? Because then

$$X_t = Z_t - \pi_1 X_{t\text{-}1} - \pi_2 X_{t\text{-}2} - \pi_3 X_{t\text{-}3} - \ldots$$

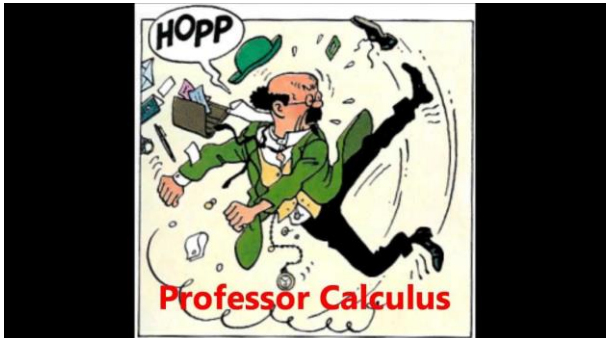
Since the series is convergent, $|\pi_n| \to 0$, so that the contribution of the most recent terms is most important.

Invertibility roughly says that remote past has less influence on current values.

A fact from calculus:

If an infinite series $\sum_{n=0}^{\infty} a_n$ converges then its common term $|a_n| \to 0$ as $n \to \infty$.

Forgot your calculus? Meet



Better : review your Math class ...

Invertibility of MA(1)

Definition: Time Series $\{X_t\}$ is *Invertible* if the shocks (that is, values of the White Noise) $\{Z_t\}$ can be expressed via values of X_t as a *convergent* series:

$$Z_t = X_t + \pi_1 X_{t\text{-}1} + \pi_2 X_{t\text{-}2} + \pi_3 X_{t\text{-}3} + \ldots$$

To check invertibility of MA(1), rewrite the model using backshift operator B:

$$X_t = Z_t + \theta_1 Z_{t-1} = 1 \cdot Z_t + \theta_1 B \cdot Z_t \equiv (1 + \theta_1 B) Z_t$$

By property of geometric series $\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n$ (iff |q| < 1), with q = $-\theta_1 B$:

$$Z_t = \frac{1}{1+\theta_1 B} X_t = \frac{1}{1-(-\theta_1 B)} X_t = (1 + (-\theta_1)\mathrm{B} + (-\theta_1)^2 \mathrm{B}^2 + \ldots + (-\theta_1)^k \mathrm{B}^k + \ldots)\, X_t$$

$$= X_t - \theta_1 X_{t-1} + \ldots + (-\theta_1)^k X_{t-k} - \ldots$$

Polynomials in B are manipulated in the same way as polynomial functions of real variables.

This series converges only for $|\theta_1| < 1$ (geometric series).

A fact from calculus:

An infinite series $\sum_{n=0}^{\infty} q^n$ is convergent iff |q| < 1. Then, $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$.

More on Invertibility of MA(1)

Another way to check invertibility of MA(1): $X_t = Z_t + \theta_1 Z_{t\text{-}1}$

Write:

$$
\begin{aligned}
Z_t &= -\theta_1 Z_{t-1} + X_t \\
&= -\theta_1(-\theta_1 Z_{t-2} + X_{t-1}) + X_t = (-\theta_1)^2 Z_{t-2} + (-\theta_1) X_{t-1} + X_t \\
&\dots \\
&= (-\theta_1)^n Z_{t-n} + \sum_{k=0}^{n-1} (-\theta_1)^k X_{t-k}
\end{aligned}
$$

This method suggests that, by continuing to iterate backward, and provided that $|\theta_1| < 1$, one can represent shocks (White Noise) $\{Z_t\}$ as a convergent infinite series of observations $\{X_t\}$:

$$Z_t = \sum_{k=0}^{\infty} (-\theta_1)^k X_{t-k}$$

Justification (convergence in m.s.):

$$\lim_{n\to\infty} E[Z_t - \sum_{k=0}^{n-1} (-\theta_1)^k X_{t-k}]^2 = \lim_{n\to\infty} (\theta_1)^{2n} E[Z_{t-n}]^2 = 0.$$

Note that we used $|\theta_1| < 1$ in the last expression: **E$[Z_{t-n}]^2 = \sigma_Z^2$ = constant;**

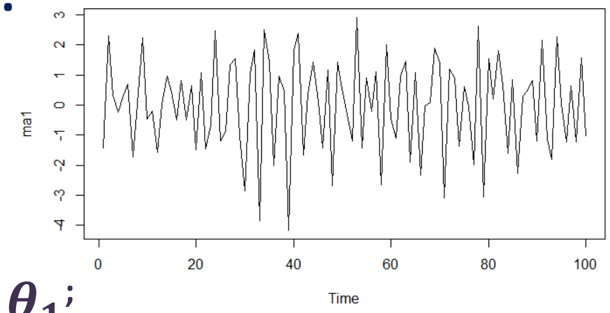
$|(-\theta_1)^{2n}| \rightarrow 0$ iff $|\theta_1| < 1$.

Still MA(1)

Some food for thought …

We have equation: $X_t = Z_t + \theta_1 Z_{t-1}$; θ_1 unknown.

We observe a time series data:



Question: How to estimate θ_1 from data?

From theory: ACF $\rho_X(1)$= $\theta_1/(1+\theta_1{}^2)$ depends on θ_1;

From data: find estimate of the ACF, called sample ACF, $\hat{\rho}_X(1)$;

(formulas for sample mean and sample covariance are given (i) on slide 38 of Week 1; (ii) in § 1.2 of Lecture Notes , Week 1); and (iii) in Problem 3 of Homework 1.)

By comparing theoretical and sample ACFs, try to find an estimate of θ_1.

(check your solution on the next slide)

MA(1) Still …

Some food for thought …

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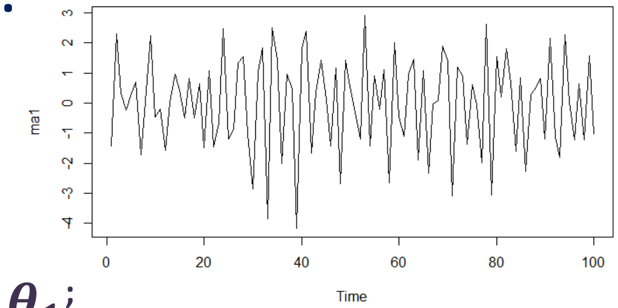

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Vieta's Formulas:
Roots x_1 and x_2 of $x^2 + px + q = 0$ satisfy: $x_1 x_2 = q$; $x_1 + x_2 = -p$

By comparing theoretical and sample ACFs, find an estimate of θ_1:

From $\hat{\rho}_X(1) = \frac{\hat{\theta}_1}{1+\hat{\theta}_1^2}$ obtain the quadratic equation

$$\hat{\theta}_1^2 \hat{\rho}_X(1) - \hat{\theta}_1 + \hat{\rho}_X(1) = 0 \text{ or } \hat{\theta}_1^2 - (1/\hat{\rho}_X(1))\hat{\theta}_1 + 1 = 0.$$

This equation has two roots.
Because their product is equal 1, only one root satisfies $|\hat{\theta}_1| < 1$.

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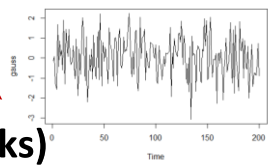
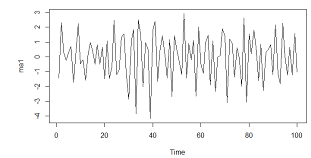


(White noise, shocks)



(Time series)

White Noise Z_t, t=1,2, …

$$X_t = Z_t + \theta_1 Z_{t-1}, |\theta_1| < 1.$$

Outline of Lecture 3, Part III:

Moving to … Moving Average Models, MA(q)

$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$ where $Z_t \sim WN(0, \sigma_Z^2)$

For example,

MA(2): $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$

MA(3): $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \theta_3 Z_{t-3}$

$X_t = Z_t + \theta_1 Z_{t-1} + \theta_3 Z_{t-3}$

$\theta_0 = 1$. All missing terms correspond to a zero coefficient.

What order is $X_t = Z_t - 0.2 Z_{t-1} - 0.7 Z_{t-2}$?

What order is $X_t = Z_t - 0.6 Z_{t-1} + 0.08 Z_{t-2} + 0.3\, Z_{t-5}$?

Mean and ACVF for Moving Average Models: MA(q)

$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$ where $Z_t \sim$ WN $(0, \sigma_Z^2)$

Moment Calculation: $E(X_t) = 0$ (linearity; $E(Z_t) = 0$ by def'n of WN)

Denote $\theta_0 = 1$. Then,

$$X_t = 1 \cdot Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q} = \sum_{i=0}^{q} \theta_i Z_{t-i}$$

$$\gamma_X(k) = E(X_t X_{t+k}) = E\{(\sum_{i=0}^{q} \theta_i Z_{t-i})(\sum_{j=0}^{q} \theta_j Z_{t+k-j})\}$$ (k-j = -m or m=j-k)

$$= E\{(\sum_{i=0}^{q} \theta_i Z_{t-i})(\sum_{m=-k}^{q-k} \theta_{k+m} Z_{t-m})\}$$ (t-i=t-m)

For $\{Z_t\} \sim WN(0, \sigma^2)$,
$E(Z_t) = 0$ for all t
$E(Z_t Z_s) = 0$, $t \neq s$.
$E(Z_t^2) = \sigma_Z^2$ for all t

$$= \sum_{i=0}^{q-k} \theta_i \, \theta_{k+i} E(Z_{t-i} Z_{t-i})$$

$E(Z_t Z_s) = 0$, $t \neq s$.
$E(Z_t^2) = \sigma_Z^2$ for all t

$$= \sigma_Z^2 \sum_{i=0}^{q-k} \theta_i \, \theta_{k+i}$$

$$= \sigma_Z^2 (\theta_k + \theta_1 \theta_{k+1} + \theta_2 \theta_{k+2} + \ldots + \theta_{q-k} \theta_q)$$

$\theta_0 = 1$

$$Note: \gamma_X(0) \equiv Var(X_t) = \sigma_Z^2 (1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2) = \sigma_X^2 .$$

Moving Average Models: MA(q)

$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$ where $Z_t \sim WN(0, \sigma_Z^2)$

Conclude: MA(q) is always stationary with

Mean $\mu_X = 0$, $Var(X_t) = \sigma_X^2 = \sigma_Z^2 (1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2)$, ← **constants**

Autocovariance for lags k = 1, 2, … q,

$\gamma_X(k) = Cov(X_t, X_{t+k}) = \sigma_Z^2 (\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \ldots + \theta_{q-k}\theta_q)$

Autocovariance for lags k > q are 0: $\gamma_X(k) = 0$ for $|k| > q$.

depends on lag k but not on t

Autocorrelation $\rho_X(k) = \dfrac{\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \ldots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2}$, k=1, …, q

$\rho_X(k) = 0$, k > q.

Moving Average Models: MA(q)

$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$ **where Z_t ~ WN $(0, \sigma_Z^2)$**

ACF $\rho_X(k) = \dfrac{\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \ldots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2} = \dfrac{\sum_{i=0}^{q-k} \theta_i\, \theta_{k+i}}{1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2}$, k = 1, …, q

$\rho_X(k) = 0$, k > q.

Write expression for ACF for MA(2) $X_t = Z_t - 0.2Z_{t-1} - 0.7Z_{t-2}$

Hint: identify θ_1 = - 0.2. What is the value of θ_2 ?

Moving Average Models: MA(q)

$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$ **where $Z_t \sim$ WN $(0, \sigma_Z^2)$**

ACF $\rho_X(k) = \dfrac{\theta_k + \theta_1\theta_{k+1} + \theta_2\theta_{k+2} + \ldots + \theta_{q-k}\theta_q}{1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2} = \dfrac{\sum_{i=0}^{q-k} \theta_i\, \theta_{k+i}}{1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2}$, $k = 1, \ldots, q$

$\rho_X(k) = 0,\ k > q.$

Write expression for ACF for MA(2) $X_t = Z_t - 0.2Z_{t-1} - 0.7Z_{t-2}$

Answer: Substitute $q = 2$, $\theta_1 = -0.2$, and $\theta_2 = -0.7$: $\rho_X(k) = \dfrac{\theta_k + \theta_{2-k}\,\theta_2}{1 + \theta_1^2 + \theta_2^2}$, $k = 1, 2$

$\rho_X(1) = \dfrac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2} = \dfrac{(-0.2) + (-0.2)(-0.7)}{1 + (-0.2)^2 + (-0.7)^2}$; $\rho_X(2) = \dfrac{\theta_2}{1 + \theta_1^2 + \theta_2^2} = \dfrac{(-0.7)}{1 + (-0.2)^2 + (-0.7)^2}$

$\rho_X(0) = 1$ (always!); $\rho_X(k) = 0,\ k > 2.$

Check your understanding : ACF for MA(q)

The MA(6) model $X_t = Z_t + \theta_1 Z_{t-1} + \theta_3 Z_{t-3} + \theta_6 Z_{t-6}$ is being analyzed, with θ_1, θ_3 & θ_6 all non-zero.
How many of the following autocorrelations must be 0:
I. $\rho_X(2)$ II. $\rho_X(3)$ III. $\rho_X(4)$ IV. $\rho_X(5)$

Check your answer of the next slide

$E(Z_t Z_s) = 0,\ t \neq s.$
$E(Z_t^2) = \sigma_Z^2$ for all t

One way to think about this problem:

If an MA model does not have any pair of Z terms that are k time lags apart, then $\rho_X(k)$ must be 0:

$$E(X_t X_{t+k}) = E\{(Z_t + \theta_1 Z_{t-1} + \theta_3 Z_{t-3} + \theta_6 Z_{t-6})(Z_{t+k} + \theta_1 Z_{t+k-1} + \theta_3 Z_{t+k-3} + \theta_6 Z_{t+k-6})\} =$$

$$= E(Z_t Z_{t+k}) + \theta_1 E(Z_t Z_{t+k-1}) + \theta_3 E(Z_t Z_{t+k-3}) + \theta_6 E(Z_t Z_{t+k-6})$$

$$+ \theta_1 \{E(Z_{t-1} Z_{t+k}) + \theta_1 E(Z_{t-1} Z_{t+k-1}) + \theta_3 E(Z_{t-1} Z_{t+k-3}) + \theta_6 E(Z_{t-1} Z_{t+k-6})\}$$

$$+ \theta_3 \{E(Z_{t-3} Z_{t+k}) + \theta_1 E(Z_{t-3} Z_{t+k-1}) + \theta_3 E(Z_{t-3} Z_{t+k-3}) + \theta_6 E(Z_{t-3} Z_{t+k-6})\}$$

$$+ \theta_6 \{E(Z_{t-6} Z_{t+k}) + \theta_1 E(Z_{t-6} Z_{t+k-1}) + \theta_3 E(Z_{t-6} Z_{t+k-3}) + \theta_6 E(Z_{t-6} Z_{t+k-6})\}$$

Indices are the same (sign does not matter) if
k=0, 1, 3, 6
k= -1 (or 1), 0, 2, 5
k = -3 (or 3), -2(or 2), 0, 3
k= -6, -5, -3, 0 (or 6, 5, 3)

Pairs of Z terms that are k time lags apart: 0, 1 (Z_t & Z_{t-1}), 2 (Z_{t-1} & Z_{t-3}), 3 (Z_{t-3} & Z_{t-6}), 5 (Z_{t-1} & Z_{t-6}), 6 (Z_t & Z_{t-6}).

Index k = 4 is missing

Check your understanding : ACF for MA(q)

The MA(6) model $X_t = Z_t + \theta_1 Z_{t-1} + \theta_3 Z_{t-3} + \theta_6 Z_{t-6}$ is being analyzed, with θ_1, θ_3 & θ_6 all non-zero. How many of the following autocorrelations must be 0:
I. $\rho_X(2)$ II. $\rho_X(3)$ III. $\rho_X(4)$ IV. $\rho_X(5)$

$E(Z_t Z_s) = 0$, $t \neq s$.
$E(Z_t^2) = \sigma_Z^2$ for all t

This MA (6) does not have any pair of Z terms that are 4 time lags apart, so that $\rho_X(4)$ must be 0.

Z_{t-1} & Z_{t-3} are 2 lags apart and for k=2 term $E(Z_{t-1} Z_{t+k-3}) = E(Z_{t-1} Z_{t+2-3}) = \sigma_Z^2$, *so that* $\rho_X(2) \neq 0$.

Z_t & Z_{t-3} are 3 lags apart and for k=3 term $E(Z_t Z_{t+k-3}) = E(Z_t Z_{t+3-3}) = \sigma_Z^2$, *so that* $\rho_X(3) \neq 0$.

Z_{t-1} & Z_{t-6} are 5 lags apart and for k=5 term $E(Z_{t-1} Z_{t+k-6}) = E(Z_{t-1} Z_{t+5-6}) = \sigma_Z^2$, *so that* $\rho_X(5) \neq 0$.

Answer: Only one acf $\rho_X(4)$ must be zero.

$$E(X_t X_{t+k}) = E\{(Z_t + \theta_1 Z_{t-1} + \theta_3 Z_{t-3} + \theta_6 Z_{t-6})(Z_{t+k} + \theta_1 Z_{t+k-1} + \theta_3 Z_{t+k-3} + \theta_6 Z_{t+k-6})\} =$$

$$= E(Z_t Z_{t+k}) + \theta_1 E(Z_t Z_{t+k-1}) + \theta_3 E(Z_t Z_{t+k-3}) + \theta_6 E(Z_t Z_{t+k-6})$$

$$+ \theta_1 \{E(Z_{t-1} Z_{t+k}) + \theta_1 E(Z_{t-1} Z_{t+k-1}) + \theta_3 E(Z_{t-1} Z_{t+k-3}) + \theta_6 E(Z_{t-1} Z_{t+k-6})\}$$

$$+ \theta_3 \{E(Z_{t-3} Z_{t+k}) + \theta_1 E(Z_{t-3} Z_{t+k-1}) + \theta_3 E(Z_{t-3} Z_{t+k-3}) + \theta_6 E(Z_{t-3} Z_{t+k-6})\}$$

$$+ \theta_6 \{E(Z_{t-6} Z_{t+k}) + \theta_1 E(Z_{t-6} Z_{t+k-1}) + \theta_3 E(Z_{t-6} Z_{t+k-3}) + \theta_6 E(Z_{t-6} Z_{t+k-6})\}$$

Indices are the same (sign does not matter) if

k= 0, 1, 3, 6

k= -1 (or 1), 0, 2, 5

k = -3 (or 3), -2(or 2), 0, 3

k= -6, -5, -3, 0 (or 6, 5, 3)

PSTAT 174/274, Week 2, Lecture 3: Moving Average Models

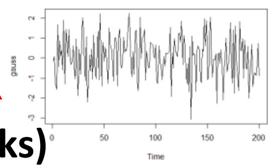
Creating Time Series Models from White Noise

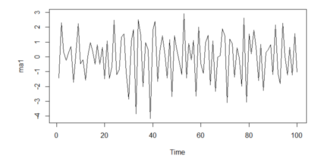
(Lego blocks)

(Lego construction)

Unobserved
Building blocks

Observed





(White noise, shocks)

(Time series)

White Noise Z_t, t=1,2, …

$X_t = Z_t + \theta_1 Z_{t-1}, |\theta_1| < 1.$

Lecture 3 part IV Outline:

Invertibility of MA(q)

Time Series $\{X_t\}$ is *Invertible* if the shocks $\{Z_t\}$ can be expressed via values of X_t as a *convergent* series: $Z_t = X_t + \pi_1 X_{t\text{-}1} + \pi_2 X_{t\text{-}2} + \pi_3 X_{t\text{-}3} + \ldots$

To investigate invertibility of MA(q) rewrite the model using shift operator:

MA(q): $$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$$

(i) Use shift operator to rewrite $Z_{t-k} = B^k Z_t$. Then,

$$X_t = 1 \cdot Z_t + \theta_1 B Z_t + \theta_2 B^2 Z_t + \cdots + \theta_q B^q Z_t$$

(ii) Introduce notation for polynomial θ of order q: (see slides 7 - 11 for MA(1))

$$\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q.$$

Then, $\theta(B) = 1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q$ **and** $X_t = \theta(B)\, Z_t$

…continued on the next slide

Invertibility of MA(q)

Time Series $\{X_t\}$ is *Invertible* if the shocks Z_t can be expressed via values of X_t as a *convergent* series:

$$Z_t = X_t + \pi_1 X_{t\text{-}1} + \pi_2 X_{t\text{-}2} + \pi_3 X_{t\text{-}3} + \ldots$$

MA(q) $X_t = \theta(B)\, Z_t = (1 + \theta_1 B + \theta_2 B^2 + \cdots + \theta_q B^q)\, Z_t$

with $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$

$$\boxed{Z_t = \frac{1}{\theta(B)} X_t}$$

is invertible, that is, $Z_t = \theta^{-1}(B) X_t$, if $\theta(z) \neq 0$ for $|z| \leq 1$, that is,

the roots of the polynomial $\theta(z)$ lie outside of the unit circle.

Math: z* is a root of $\theta(z)$ if θ (z*) = 0. Condition for invertibility: |z*| > 1.

In this case, $\theta^{-1}(z) = 1 + \pi_1 z + \pi_2 z^2 + \cdots$ has a representation as a convergent series so that

$$Z_t = \theta^{-1}(B) X_t = X_t + \pi_1 X_{t-1} + \pi_2 X_{t-2} + \cdots$$

...continued on the next slide

Math facts:

z* is a root of polynomial $\theta(z)$ if it satisfies the characteristic equation: θ (z*) = 0.

Roots of polynomial of order q > 1 may be complex and lie on a complex plane (x,y).

For z*= x+iy, condition |z*| = $|\sqrt{\{x^2 + y^2\}}|$ > 1 means that on the complex plane (x,y), z* lies is outside the unit circle $x^2 + y^2$ = 1. Points z: |z| < 1 are inside the circle.

Invertibility of MA(q): Check for MA(1) Model

MA(q) $X_t = \theta(B)\, Z_t$ **with** $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ **is**
Invertible if $\theta(z) \neq 0$ **for** $|z| \leq 1$**, that is,**
the roots z* of the polynomial $\theta(z)$ **lie outside of the unit circle: |z*| > 1.**

Check for MA(1): $X_t = Z_t + \theta_1 Z_{t-1} \equiv (1 + \theta_1 B)\ \mathrm{Z_t} \equiv \ \theta(\mathrm{B})\, \mathrm{Z_t}$ **with** $\theta(z) = 1 + \theta_1 z$.

Check that it is invertible if the roots z* of the polynomial $\theta(z)$ **lie outside of the unit circle:**

$\theta(z*) = 1 + \theta_1 z^* = 0$ **if** $z^* = -1/\theta_1$. $|z^*| = |-1/\theta_1| > 1$ **iff** $|\theta_1| < 1$.

(same as on slides 10-11)

$$\theta^{-1}(z) = \frac{1}{1+\theta_1 z} = \frac{1}{1-(-\theta_1 z)} = 1 + (-\theta_1 z) + (-\theta_1 z)^2 + \cdots$$

$= 1 - \theta_1 z + \theta_1{}^2 z^2 + \cdots$ **, convergent for** $|\theta_1| < 1$ **and** $|z| \leq 1$.

Substitute z = B to get

$$Z_t = \theta^{-1}(B) X_t = \frac{1}{1+\theta_1 B} \mathrm{X_t} = (1 - \theta_1 B + \theta_1^2 B^2 - \ldots) X_t$$

$$= X_t - \theta_1 X_{t-1} + \theta_1^2 X_{t-2} - \cdots$$

A fact from calculus: If |q| < 1, then $\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n$ **.**

Results of MA(2) Example – Lab 2

$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$ **with Z_t ~ WN (0, σ_Z^2)**

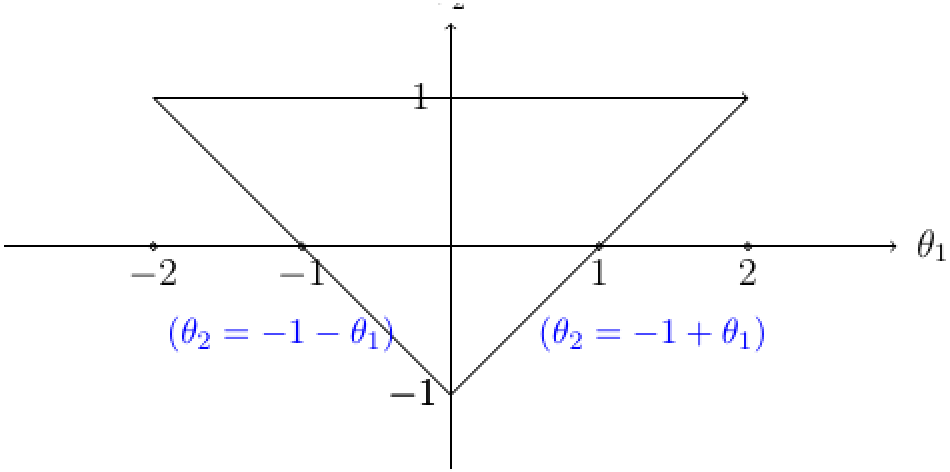
Corresponding polynomial $\theta(z) = 1 + \theta_1 z + \theta_2 z^2$;

Stationary with mean zero and ACF:

$\rho_X(1) = \dfrac{\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2}$, $\rho_X(2) = \dfrac{\theta_2}{1+\theta_1^2+\theta_2^2}$, $\rho_X(k) = 0$, $k > 2$. **(see slide #19)**

Invertible if

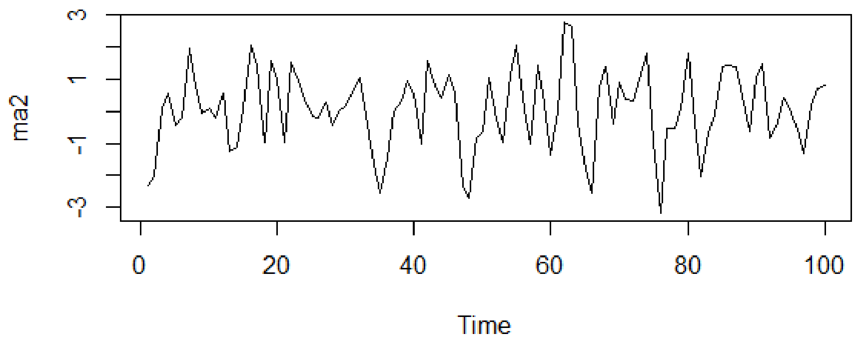
$\theta_2 + \theta_1 > -1 \quad \theta_2 - \theta_1 > -1, \quad -1 < \theta_2 < 1$



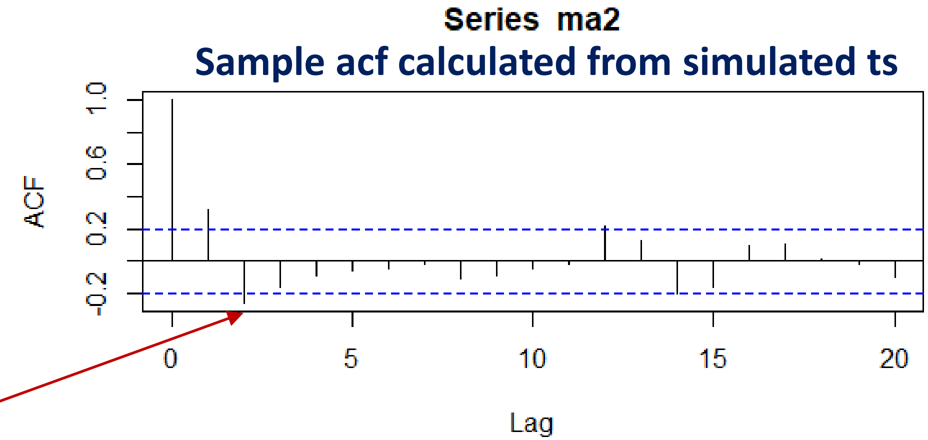
$$\begin{aligned} &\theta_2 + \theta_1 > -1 \\ &\theta_2 - \theta_1 > -1 \\ &-1 < \theta_2 < 1. \end{aligned}$$

Note: for invertibility, $|\theta_2| < 1$.
This property extends to MA(q): $|\theta_q| < 1$.

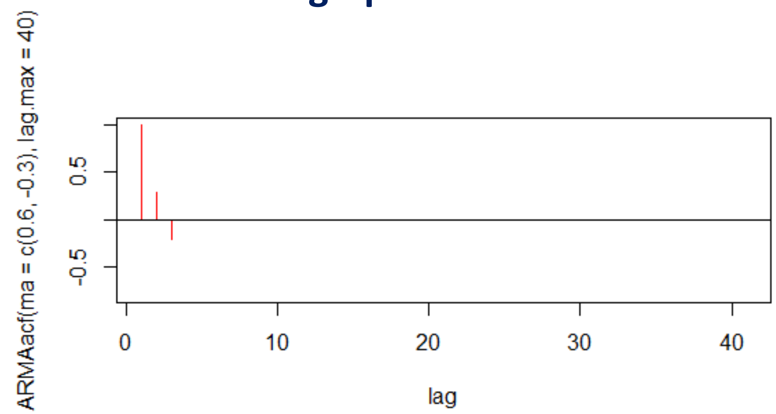
ACF for 100 simulated values of MA(2) $X_t = Z_t + 0.6\, Z_{t-1} - 0.3\, Z_{t-2}$



Sample acf calculated from simulated ts



Theoretical acf graphed from formulas below

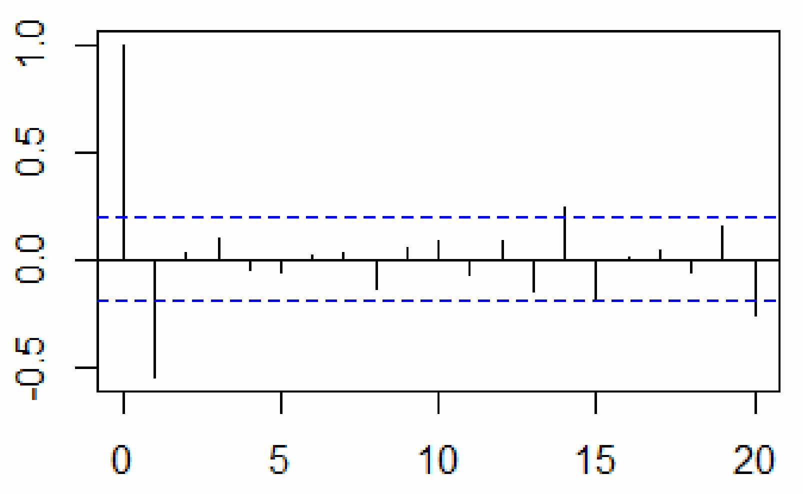


ACF=0, k > 2. Negative correlation at lag 2 corresponds to negative θ_2.

$$\rho_X(1) = \frac{\theta_1 + \theta_1\theta_2}{1 + \theta_1^2 + \theta_2^2}, \quad \rho_X(2) = \frac{\theta_2}{1 + \theta_1^2 + \theta_2^2}, \quad \rho_X(k) = 0,\ k > 2.$$

Check your Understanding:
MA(q): Model Identification from Sample ACF

A



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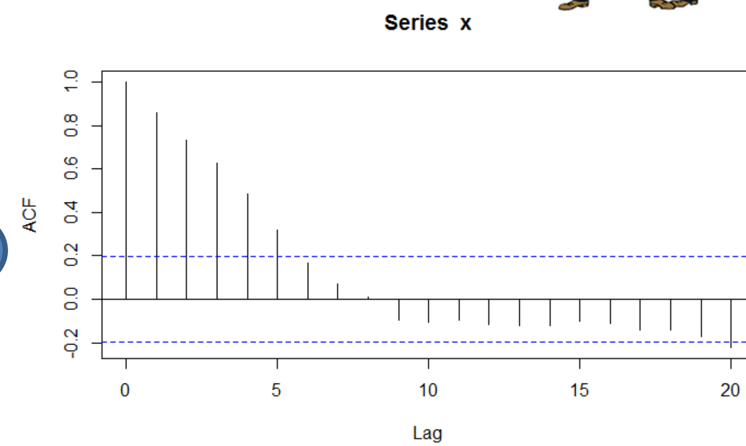

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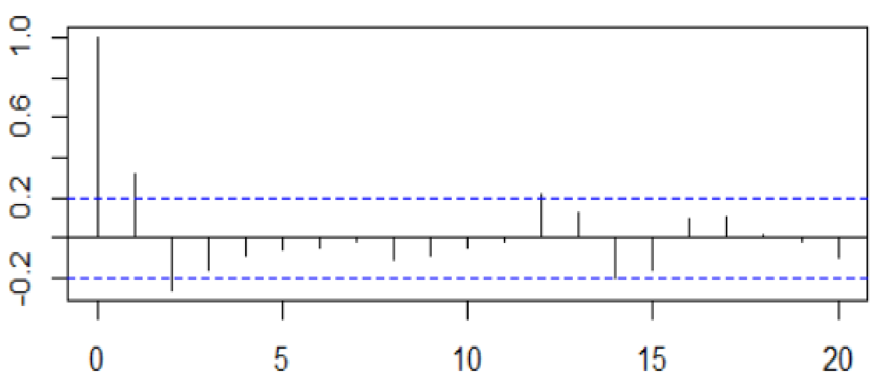

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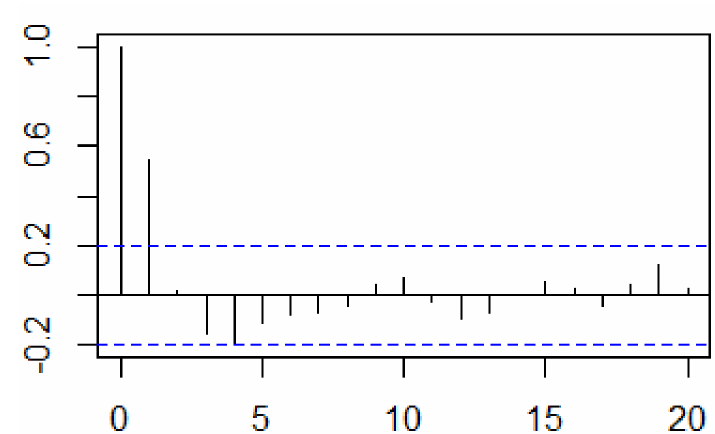
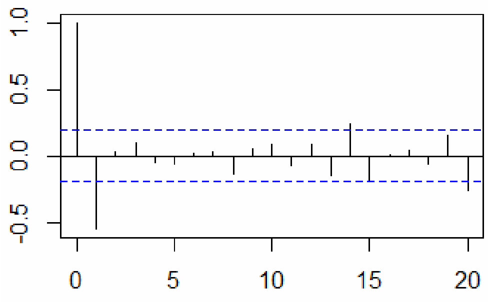


Identify graphs of sample ACFs corresponding to MA(q) processes. Determine q.

… check your answers on the next slide

MA(q): Model Identification from Sample ACF

A



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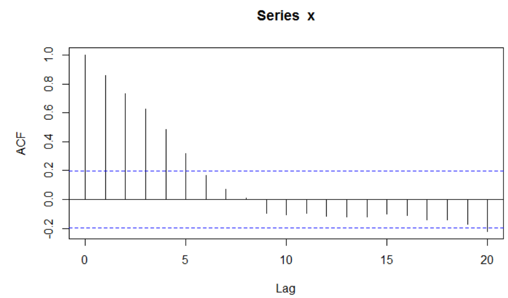

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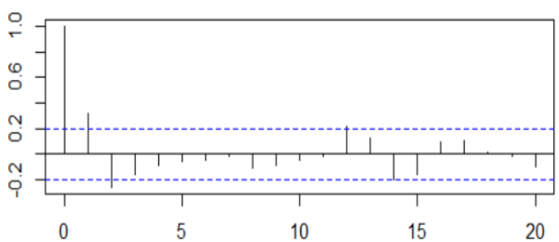
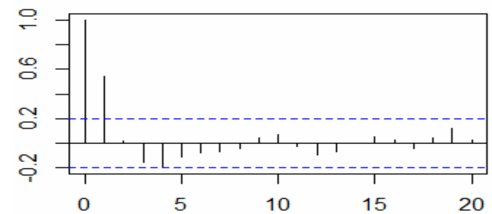


D



Identify graphs of sample ACFs corresponding to MA(q) processes. Determine q.

Discussion:

(A) Sample ACFs at lag 1 is outside the confidence interval, but is inside the intervals for lags k >1. Thus, the model is MA(1). (ρ_X(14) and ρ_X(20) are almost in the interval.)

(B) Sample ACFs at lag 2 is outside the confidence interval, but is inside the intervals for lags k >2. Thus, the model is MA(2).

(C) Sample ACFs are large and do not follow pattern of MA(q).

(D) Sample ACFs at lag 1 is outside the confidence interval, but is inside the intervals for lags k >1. Thus, the model is MA(1).

MA(q): Check of Understanding

You are given the following statements about stationarity:

I. Linear models for time series are stationary when they include functions of time.

II. All moving averages processes are stationary.

III. All random walk processes are nonstationary.

Determine which of the above statements are true:

A. None are true
B. I and II only
C. I and III only
D. II and III only
E. The answer is not given by (A), (B), (C) or (D).

… check your answers on the next slide

MA(q): Check of Understanding -- Discussion

You are given the following statements about stationarity:

I. Linear models for time series are stationary when they include functions of time.

II. All moving averages processes are stationary.

III. All random walk processes are nonstationary.

Determine which of the above statements are true:

A. None are true

B. I and II only

C. I and III only

D. II and III only

E. The answer is not given by (A), (B), (C) or (D).

Quick answer: D

I is false: Random walk (discussed in § 2.1.5, slide 55 of week 1) is linear but not stationary.

II is correct: All MA(q) models are stationary, (see slide 17 of this lecture)

III is correct: Random walk is not stationary, its variance depends on time t (§ 2.1.5 of lecture notes; slide 55 of week 1)

Main Points to Take from Lecture 3

- **Backshift or Shift Operator B, $B\,X_t = X_{t-1};\ B^k\,X_t = X_{t-k}$.**
- **Notions of invertibility and stationarity**
- **MA(1) and MA(q) processes:**
  - **Model equation ($\text{e.g.}, X_t = Z_t + \theta_1 Z_{t-1}$ for MA(1))**
  - **Restrictions on model coefficients for invertibility**

    (e.g., $|\theta_q|$ <1);
  - **Fact that MA models are always stationary;**
  - **ACVF and ACF formulas for MA(1), MA(2) and MA(q) models or simply remember where to find them;**

  -- Fact that for MA(q) model,

  ACF at lag q is NOT zero, and ACFs at lags k > q are all zeros.
  - **Sample ACF, its' use for model identification**

Welcome to Lecture 4: Autoregressive Models

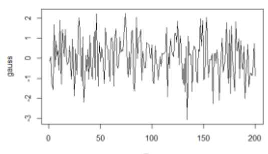
Creating Time Series Models from White Noise

(Lego blocks)

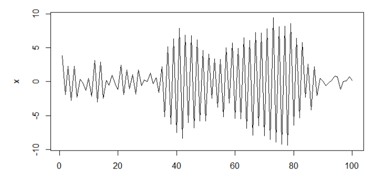
(Lego construction)17

Unobserved
Building blocks

Observed



(White noise)



(Time series)

White Noise Z_t, t=1,2, …

$X_t = \phi_1 X_{t-1} + Z_t, |\phi_1| < 1.$

Welcome to Week 2 Lecture 4: Autoregressive Models
Part 1 : AR(1) Models

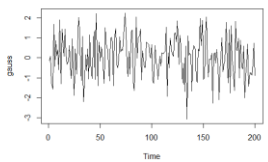
Creating Time Series Models from White Noise

(Lego blocks)

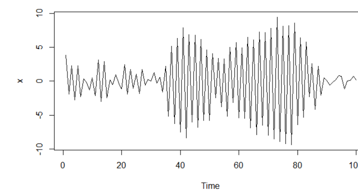
(Lego construction)

Unobserved
Building blocks

Observed



(White noise)



(Time series)

White Noise Z_t, t=1,2, …

$$X_t = \phi_1 X_{t-1} + Z_t, |\phi_1| < 1.$$

Lecture 4 Part 1:

Short Review

Lectures 2 and 3 discussed MA(q) processes:

- **Model equation for MA(q):** $X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}$
- **Stationarity: MA(q) is always stationary, i.e., E(**X_t**)** $= 0$; **Cov(**X_t, X_{t+k} **)= γ**$_X$ **(k)** **that is, ACVF does not depend on time t, but only on lag k.**
- **Formulas for ACVF and ACF for MA(q) are known, see slide 17.**
- **Important characteristic of MA(q): ACF** $\rho_X(q) \neq 0$; $\rho_X(k) = 0$ **for all k > q.**
- **Invertibility means that the shocks** $\{Z_t\}$ **can be expressed via values of** X_t **as a** ***convergent*** **series:** $Z_t = X_t + \pi_1 X_{t\text{-}1} + \pi_2 X_{t\text{-}2} + \pi_3 X_{t\text{-}3} + \ldots$
- **MA(q) are invertible under the condition that the roots of polynomial**

  $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 + \cdots + \theta_q z^q$ **(i.e., z* such that θ (z*) = 0)**

  are outside unit circle: |z*| > 1 (for q > 1, the roots can be complex numbers)

A math fact:

For a complex number z = a + ib, |z| = $\sqrt{a^2 + b^2}$ (https://en.wikipedia.org/wiki/Complex_number)

4.1 Autoregressive models: AR(1)

4.1: AR(1) Autoregressive of Order One

$X_t = \phi_1 X_{t-1} + Z_t$ **where Z_t ~ WN $(0, \sigma_Z^2)$ and $|\phi_1| < 1$.**

In AR(1) model, current observation X_t depends on today's shock Z_t and yesterday's observation X_{t-1}!

AR(1) processes are created from past observations and WN!

Examples of AR(1):

$X_t = 0.8\, X_{t-1} + Z_t$; $\phi_1 = 0.8$

$X_t = -0.6 X_{t-1} + Z_t$; $\phi_1 = -0.6$

Can you give an intuitive reason why ϕ_1 should be in the (-1,1) interval ?

4.1 Autoregressive models: AR(1)

AR(1) Autoregressive of Order One

$X_t = \phi_1 X_{t-1} + Z_t$ **where** $Z_t \sim WN(0, \sigma_Z^2)$ **and** $|\phi_1| < 1$.

Invertibility: Unlike in MA models, AR(1) is always invertible:

$$Z_t = X_t - \phi_1 X_{t-1} + 0$$

Stationarity: We will see that AR(1) is not always stationary!

Condition: $|\phi_1| < 1$.

Under this condition, AR(1) has MA(∞) representation

Follow derivation on the next slide…

Recall: Invertibility means that the shocks $\{Z_t\}$ can be expressed via values of X_t as a *convergent* series: $Z_t = X_t + \pi_1 X_{t\text{-}1} + \pi_2 X_{t\text{-}2} + \pi_3 X_{t\text{-}3} + \ldots$

Recall: Stationarity means $E(X_t) = \mu_X$; $Cov(X_t, X_{t+k}) = \gamma_X(k)$, does not depend on t

4.1 MA representation for AR(1)

$AR(1)$: $X_t = \phi_1 X_{t-1} + Z_t$ **where $Z_t \sim$ WN $(0, \sigma_Z^2)$ and $|\phi_1| < 1$.**

Plan to determine stationarity of AR(1) and find its moments:

(i) Show that for $|\phi_1| < 1$, AR(1) model has a MA(∞) representation

$X_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j} = \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}$ **that is,** $\psi_j = \phi_1^j$.

(ii) Because MA processes are always stationary, (i) will imply that AR(1) process

$X_t = \phi_1 X_{t-1} + Z_t$ **is stationary as long as** $|\phi_1| < 1$.

(iii) MA(∞) representation can be used to find ACVF of X_t similar to the method used to find ACVF of MA(q).

Derivation: **Use shift operator B $X_t = X_{t-1}$ to write $X_t - \phi_1 X_{t-1} = (1 - \phi_1 B) X_t = Z_t$**

Then,

$$X_t = \frac{1}{1-\phi_1 B} Z_t = \left(1 + \phi_1 B + \phi_1^2 B^2 + \cdots\right) Z_t$$

$$= Z_t + \phi_1 Z_{t-1} + \phi_1^2 Z_{t-2} + \cdots$$

The series is convergent if $|\phi_1| < 1$. (geometric series)

A fact from calculus:

An infinite series $\sum_{n=0}^{\infty} q^n$ is convergent iff $|q| < 1$. Then, $\sum_{n=0}^{\infty} q^n = \frac{1}{1-q}$.

More on MA(∞) Representation of AR(1)

Another way to see MA(∞) representation of AR(1) $X_t = \phi_1 X_{t-1} + Z_t$:

$$X_t = \phi_1 X_{t-1} + Z_t$$
$$= \phi_1(\phi_1 X_{t-2} + Z_{t-1}) + Z_t = \phi_1{}^2 X_{t-2} + \phi_1 Z_{t-1} + Z_t$$
$$= \phi_1{}^2 (\phi_1 X_{t-3} + Z_{t-2}) + \phi_1 Z_{t-1} + Z_t = \phi_1{}^3 X_{t-3} + \phi_1{}^2 Z_{t-2} + \phi_1 Z_{t-1} + Z_t$$
$$\ldots$$
$$= \phi_1{}^k X_{t-k} + \sum_{j=0}^{k-1} \phi_1^j Z_{t-j}$$

This method suggests that, by continuing to iterate backward, and provided that $|\phi_1| < 1$, and that $\{X_t\}$ is stationary, $\{X_t\}$ has a representation as a convergent infinite series of $\{Z_t\}$, that is, MA(∞) :

$$X_t = \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}$$

Justification (convergence in m.s.):

$$\lim_{k\to\infty} E[X_t - \sum_{j=0}^{k-1} (\phi_1)^j Z_{t-j}]^2 = \lim_{k\to\infty} (\phi_1)^{2k} E[X_{t-k}]^2 = 0.$$

Main Points to Take from Part 1 Lecture 4

AR(1) model

- **Model equation: $X_t = \phi_1 X_{t-1} + Z_t$ where Z_t ~ WN (0, σ_Z^2) and $|\phi_1| < 1$.**
- **AR(1) is always invertible;**
- **AR(1) process has a MA(∞) *representation* $X_t = \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}$ *when* $|\phi_1| < 1$.**

Moving to Part 2 of Lecture 4

Welcome to Week 2 Lecture 4: Autoregressive Models
Part 2: AR(1) Models

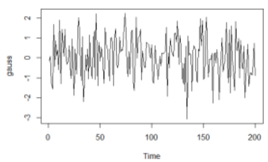
Creating Time Series Models from White Noise

(Lego blocks)

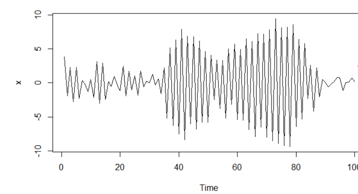
(Lego construction)

Unobserved Building blocks

Observed



(White noise)



(Time series)

White Noise Z_t, t=1,2, …

$$X_t = \phi_1 X_{t-1} + Z_t, |\phi_1| < 1.$$

Outline of Lecture 4 Part 2:

4.1 Calculation of ACVF and ACF of AR(1)

$AR(1)$: $X_t = \phi_1 X_{t-1} + Z_t$ where $Z_t \sim WN(0, \sigma_Z^2)$ and $|\phi_1| < 1$.
MA(∞) representation: $X_t = Z_t + \phi_1 Z_{t-1} + \phi_1^2 Z_{t-2} + \cdots$

For $\{Z_t\} \sim WN(0, \sigma^2)$,
$E(Z_t) = 0$ *for all t*
$E(Z_t Z_s) = 0,\ t \neq s.$
$E(Z_t^2) = \sigma_Z^2$ *for all* t

Calculate first and second moments:

- $E(X_t) = 0$ (b/c MA process has mean zero.)
- ACVF $\gamma_X(k) = E(X_t X_{t-k}) = E\{(\phi_1 X_{t-1} + Z_t)\, X_{t-k}\}$;
- For $k \geq 1$, $E(Z_t X_{t-k}) = 0$

  b/c X_{t-k} depends on Z_{t-k} , Z_{t-k-1}, etc., uncorrelated with Z_t

➢ For $k \geq 1$, we obtained recursive formula for ACVF:

$$\gamma_X(k) = E\{(\phi_1 X_{t-1} + Z_t)\, X_{t-k}\} = \phi_1 E[X_{t-1} X_{t-k}] = \phi_1 \gamma_X((t-1) - (t-k))$$
$$= \phi_1\, \gamma_X(k-1)$$

$\gamma_X(1) = \phi_1 \gamma_X(0)$, $\gamma_X(2) = \phi_1 \gamma_X(1) = \phi_1{}^2 \gamma_X(0)$, $\gamma_X(3) = \phi_1 \gamma_X(2) = \phi_1{}^3 \gamma_X(0)$, …

➢ For $k \geq 1$, ACF $\rho_X(k) = \gamma_X(k)/\gamma_X(0)$, that is,

$$\rho_X(1) = \phi_1,\ \rho_X(2) = \phi_1^2, \ldots, \rho_X(k) = \phi_1^k.$$

Remains to find $\gamma_X(0)$ …

Calculation of ACVF of AR(1)

$AR(1)$: $X_t = \phi_1 X_{t-1} + Z_t$ where $Z_t \sim WN(0, \sigma_Z^2)$ and $|\phi_1| < 1$.

MA(∞) representation: $X_t = Z_t + \phi_1 Z_{t-1} + \phi_1^2 Z_{t-2} + \cdots$

ACVF $\gamma_X(k) = E(X_t X_{t-k}) = E\{(\phi_1 X_{t-1} + Z_t) X_{t-k}\}$;

Calculation of $\gamma_X(0) = Var(X_t)$:

For k = 0, $E(Z_t X_t) = E\{Z_t \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}\}$

$= \sum_{j=0}^{\infty} \phi_1^j E\{Z_t Z_{t-j}\}$

$= \phi_1^0 E\{Z_t^2\} = \sigma_Z^2$

$E(Z_t Z_s) = 0$, $t \neq s$.
$E(Z_t^2) = \sigma_Z^2$ for all t

Thus, $\gamma_X(0) = E\{(\phi_1 X_{t-1} + Z_t) X_t\} = \phi_1 \gamma_X(1) + \sigma_Z^2 = \phi_1^2 \gamma_X(0) + \sigma_Z^2$

$E(X_{t-1} X_t) = \gamma_X(1)$

So that

$$\boxed{\gamma_X(0) = \frac{\sigma_Z^2}{1-\phi_1^2}, \quad \gamma_X(k) = \phi_1^k \gamma_X(0). \quad \rho_X(1) = \phi_1,\ \rho_X(2) = \phi_1^2, \ldots, \rho_X(k) = \phi_1^k.}$$

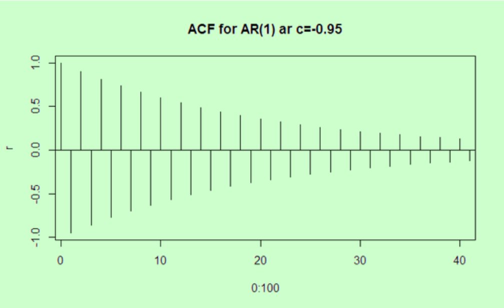
Summary of Properties of AR(1)

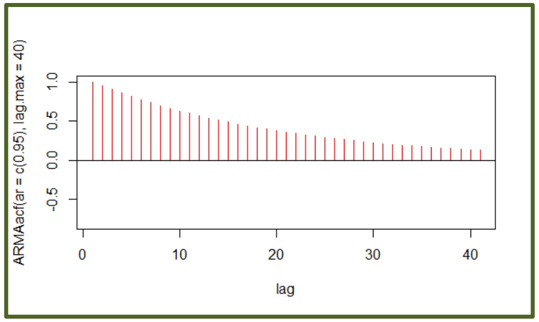
$AR(1)$: $X_t = \phi_1 X_{t-1} + Z_t$ **where Z_t ~ WN $(0, \sigma_Z^2)$ and $|\phi_1| < 1$.**

Summary:

- **Always invertible!**
- **MA(∞) representation:** $X_t = Z_t + \phi_1 Z_{t-1} + \phi_1^2 Z_{t-2} + \cdots$ *when* $|\phi_1| < 1$
- **Stationary when $|\phi_1| < 1$.**
- **Variance: Var (X) = $\gamma_X(0) = \frac{\sigma_Z^2}{1-\phi_1^2}$;**
- **ACF: $\rho_X(1) = \phi_1,\ \rho_X(2) = \phi_1^2, \ldots, \rho_X(k) = \phi_1^k$ ← converges to 0**

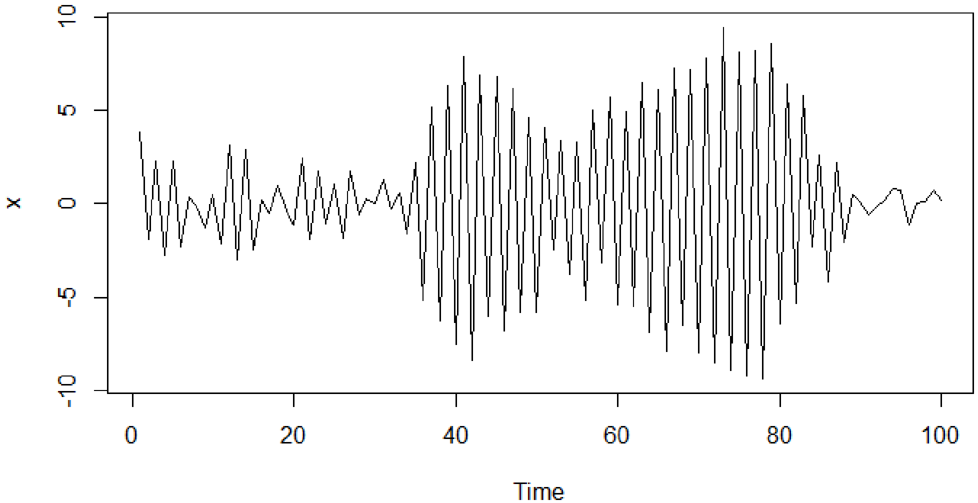
Notice: ACFs of AR(1) decay exponentially, but are never zero!





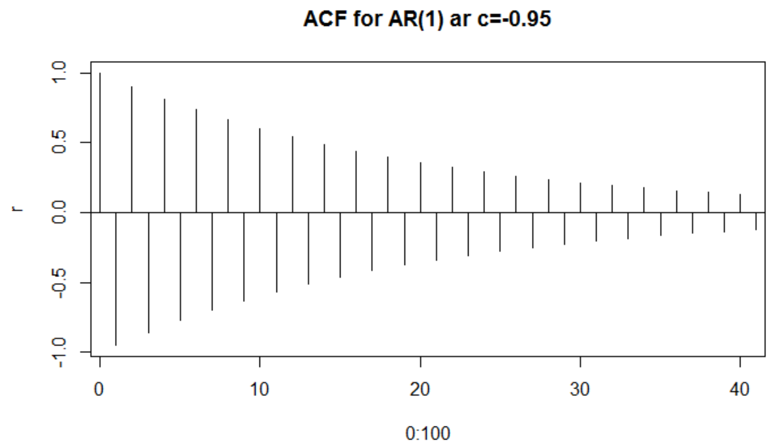
For both graphs $|\phi_1|$ = 0.95.
Name the graph with ϕ_1 <0.

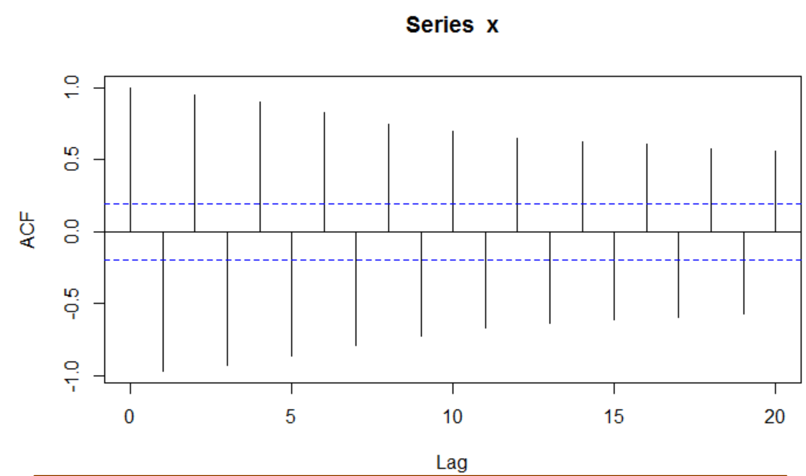
100 simulated values of AR(1) $X_t = -0.95\, X_{t-1} + Z_t$ and its ACF



Correlations die out slowly because $\varphi = -0.95$ is near -1.
Signs alternate, exponential decay.

Graph choppy because of strong negative correlation.



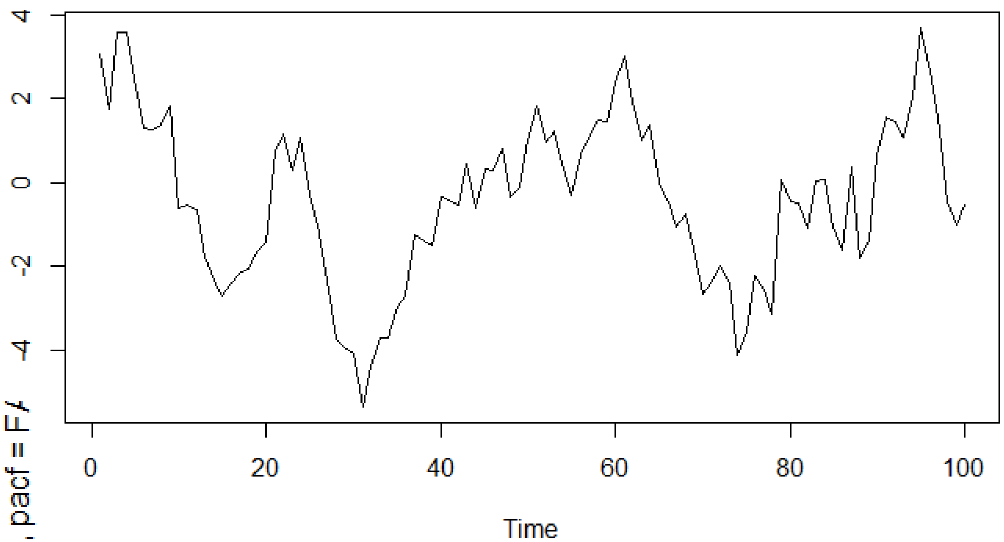


(the same acf with confidence intervals)

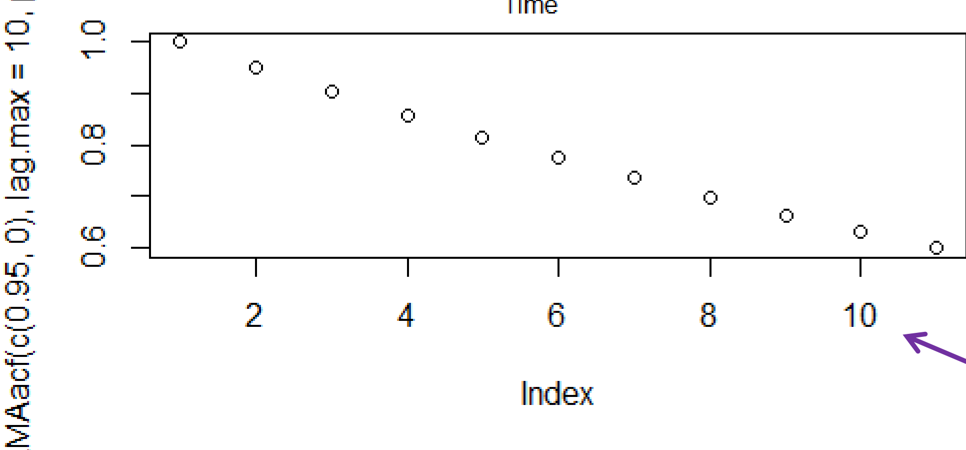
Commands used in R:

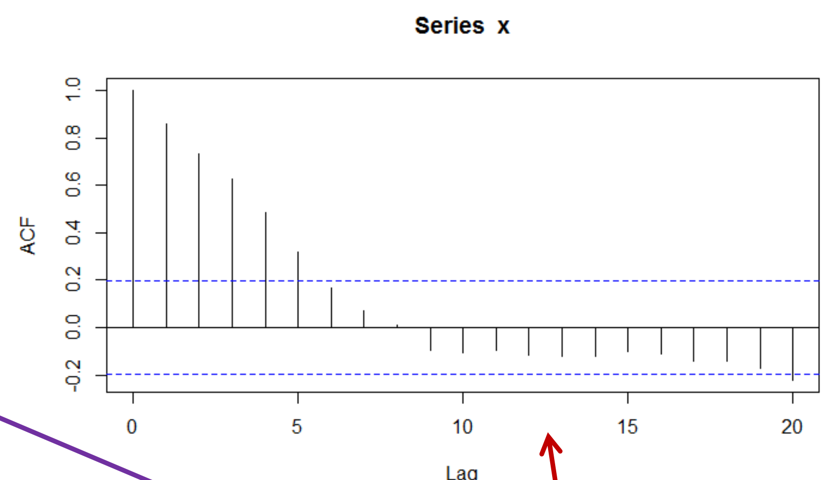
```
> ar1 <- arima.sim(model=list(ar=c(-0.95)), n=100, sd=1)
> plot(ar1)
> plot(ARMAacf(ar=c(-0.95), lag.max=40), ylab="r",type="h", main="ACF for AR(1)  c=-0.95");  abline(h=0)
```

100 simulated values of $X_t = 0.95\, X_{t-1} + Z_t$ (AR(1)) and its ACF



- **Correlations die out slowly because $\varphi = 0.95$ is close to 1.**
- **Sample ACF: exponential decay is slow.**
- **Graph of X is smooth because of strong positive correlation.**



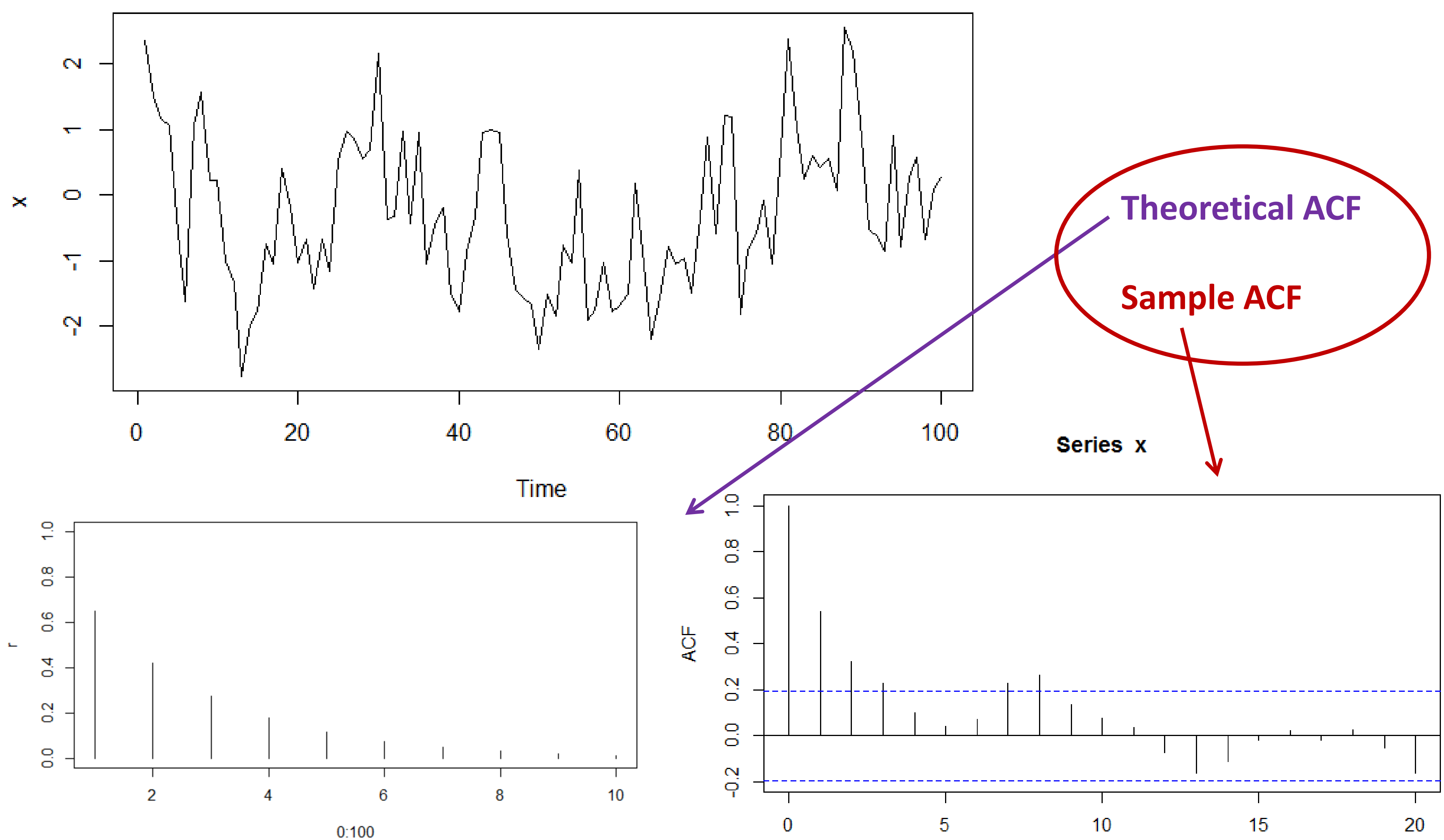


Theoretical ACF **Sample ACF**

Commands used in R:

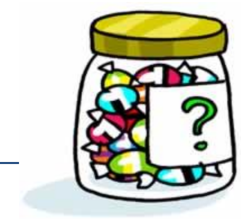
```
> ar1 <- arima.sim(model=list(ar=c(0.95)), n=100, sd=1)
> plot(ar1)
> acf(ar1)
> plot(ARMAacf(c(0.95,0), lag.max = 10, pacf = FALSE))
```

100 simulated values of $X_t = 0.65\ X_{t-1} + Z_t$ (AR(1)) and its ACF



- **ACF dies out faster than on the previous slide because φ=0.65 is not close to 1;**
- **Graph of X less smooth (less correlation, less linear dependence) .**

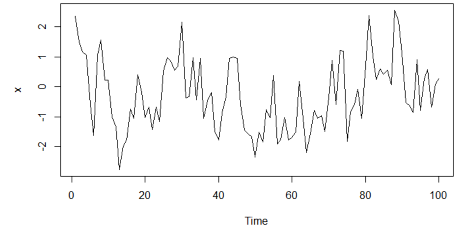
AR(1) still: Estimating ϕ_1 from Data



Some food for thought …

We have equation: $X_t = \phi_1 X_{t-1} + Z_t$; ϕ_1 unknown

We observe a time series data:



Question: How to estimate ϕ_1 from data?

From theory: ACF $\rho_X(1) = \phi_1$, $\rho_X(2) = \phi_1^2, \dots, \rho_X(k) = \phi_1^k$

From data: find estimate of the ACF, called sample ACF, $\hat{\rho}_X(1)$;

(formulas for sample mean and covariance are given (i) on slide 38 of Week 1; (ii) in § 1.2 of Lecture Notes , Week 1); and (iii) in Problem 3 of Homework 1.)

By comparing theoretical and sample ACFs, find an estimate of ϕ_1:

$$\hat{\phi}_1 = \hat{\rho}_X(1)$$

4.2 Property of Causality of X

Time Series $\{X_t\}$ is *causal or a causal function of* shocks $\{Z_t\}$ if its value can be expressed in terms of current or past values of Z_t :

$$X_t = function\,(Z_s, s \le t)$$

Why causality important? → **FUTURE-INDEPENDENCE**

Invertibility: remote past has less influence on current values.

Causality: observations are future independent.

MA(q) is always causal: $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} + \cdots + \theta_q Z_{t-q}$

AR(1) is causal for $|\phi_1| < 1$, when it has MA(∞) representation.

What happens when $|\phi_1|$ *> 1 ? see next slide …*

4.2 Property of Causality of X: Example of non-causal time series

Time Series $\{X_t\}$ is *causal or a causal function of* shocks $\{Z_t\}$ if its value can be expressed in terms of current or past values of Z_t:

$$X_t = function\,(Z_s, s \le t)$$

Example of non-causal time series: AR (1) with ϕ_1 >1:

- Write $X_t - \phi_1 X_{t-1}$ = (1 - ϕ_1B) X_t = Z_t
- $X_t = \frac{1}{1-\phi_1 B} Z_t = -\frac{(\phi_1 B)^{-1}}{1-(\phi_1 B)^{-1}} Z_t$ (multiply numerator & denominator by $(\phi_1 B)^{-1}$)

  = - $(\phi_1 B)^{-1}\left(1 + (\phi_1 B)^{-1} + (\phi_1 B)^{-2} + \cdots + (\phi_1 B)^{-n} + \cdots\right) Z_t$

  (b/c ϕ_1 > 1, its inverse 0 $<$ $(\phi_1)^{-1}$ < 1, so that $\frac{1}{1-q}$ = $\sum_{n=0}^{\infty} q^n$ holds for 0 < q = $(\phi_1)^{-1}$ < 1)

- Recall: $B^{-k} Z_t$ = $Z_{t-(-k)}$ = $\mathrm{Z}_{\mathrm{t+k}}$. Thus, final expression for X is:
- $X_t = -(\phi_1)^{-1} Z_{t+1} - (\phi_1)^{-2} Z_{t+2} - \cdots - (\phi_1)^{-\mathrm{k}} \mathrm{Z}_{\mathrm{t+k}}$ – ...

Is this series causal? Can it be used for forecast ?

Main Points to Take from Parts 1 — 2 of Lecture 4

AR(1) model

- **Model equation:** $X_t = \phi_1 X_{t-1} + Z_t$ **where** $Z_t \sim WN(0, \sigma_Z^2)$ **and** $|\phi_1| < 1$.
- **AR(1) is always invertible;**
- **AR(1) process has a MA(∞)** ***representation*** $X_t = \sum_{j=0}^{\infty} \phi_1^j Z_{t-j}$ ***when*** $|\phi_1| < 1$.
- **ACVF:** $\gamma_X(0) = \frac{\sigma_Z^2}{1-\phi_1^2}$, $\gamma_X(k) = \phi_1^k \gamma_X(0)$.
- **ACF:** $\rho_X(k) = \phi_1^k$.
- **Estimate of** ϕ_1 **from data:** $\hat{\phi}_1 = \hat{\rho}_X(1)$
- **Causality: observations are future independent**

Moving to Part 3 of Lecture 4

Welcome to Week 2 Lecture 4: Autoregressive Models
Part 3: AR(p) Models

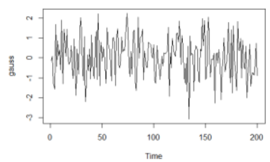
Creating Time Series Models from White Noise

(Lego blocks)

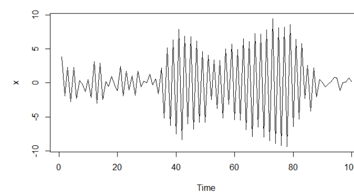
(Lego construction)

Unobserved
Building blocks

Observed



(White noise)



(Time series)

White Noise Z_t, t=1,2, …

$X_t = \phi_1 X_{t-1} + Z_t, |\phi_1| < 1.$

Outline of Part 3 of Lecture 4:

4.3 Autoregressive of Order p Models: AR(p)

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + Z_t$ where $Z_t \sim$ WN $(0, \sigma_Z^2)$

For example,

AR(2): $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + Z_t$

AR(3): $X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \phi_3 X_{t-3} + Z_t$

$X_t = \phi_2 X_{t-2} + \phi_3 X_{t-3} + Z_t$

What order is $X_t = -0.2 X_{t-1} - 0.7 X_{t-2}$ $+ Z_t$?

What order is $X_t = 0.6 X_{t-1} + 0.08 X_{t-2} + 0.3 X_{t-5}$ $+ Z_t$?

$\phi_0 = 1$. **All missing terms correspond to a zero coefficient.**

4.3 AR(p) and its Autoregressive Polynomial ϕ *(B)*

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + Z_t$ where $Z_t \sim$ WN $(0, \sigma_Z^2)$

Writing AR(p) model concisely via shift operator:

- **Use shift operator to rewrite $X_{t-k} = B^k X_t$.**

  Move all terms with X to left-hand side:

  $$1 \cdot X_t - \phi_1 B X_t - \phi_2 B^2 X_t - \cdots - \phi_p B^p X_t = Z_t$$

  or $$(1 - \phi_1 B - \phi_2 B^2 \ldots - \phi_p B^p)\, X_t = Z_t$$

- **Introduce notation for polynomial ϕ of order p:**

  $$\phi\,(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p.$$

  Notice signs of ϕ's

- **Substitute z = B:** $\phi\,(B) = 1 - \phi_1 B - \phi_2 B^2 - \cdots - \phi_p B^p$
- **Write** $\phi\,(B)\, X_t = Z_t$, $Z_t \sim$ WN $(0, \sigma_Z^2)$

4.3 Autoregressive of Order p Models: AR(p)

$X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + Z_t$ **where** $Z_t \sim$ **WN** $(0, \sigma_Z^2)$

or $\phi(B) X_t = Z_t$, $Z_t \sim$ **WN** $(0, \sigma_Z^2)$

Properties of AR(p):

- ***AR(p) always invertible by its construction:*** $Z_t = \phi(B) X_t$.

- ***AR(p) has MA(∞) representation when***

  $\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p \neq 0$ **for** $|z| \leq 1$,

  that is, when the roots of the polynomial $\phi(z)$ **lie outside of the unit circle**

  (z* is a root of $\phi(z)$ **if** $\phi(z^*) = 0$. **Condition:** $|z^*| > 1$**)**

- ***AR(p) is stationary when it has MA(∞) representation,***

  that is, if $\phi(z) \neq 0$ **for** $|z| \leq 1$ **OR when** $|z^*| > 1$ **for z* such that** $\phi(z^*) = 0$.

- ***NEW: ACF is found by solving Yule-Walker equations.***

Compare graphs of AR(2) models for different combinations of signs of coefficients.

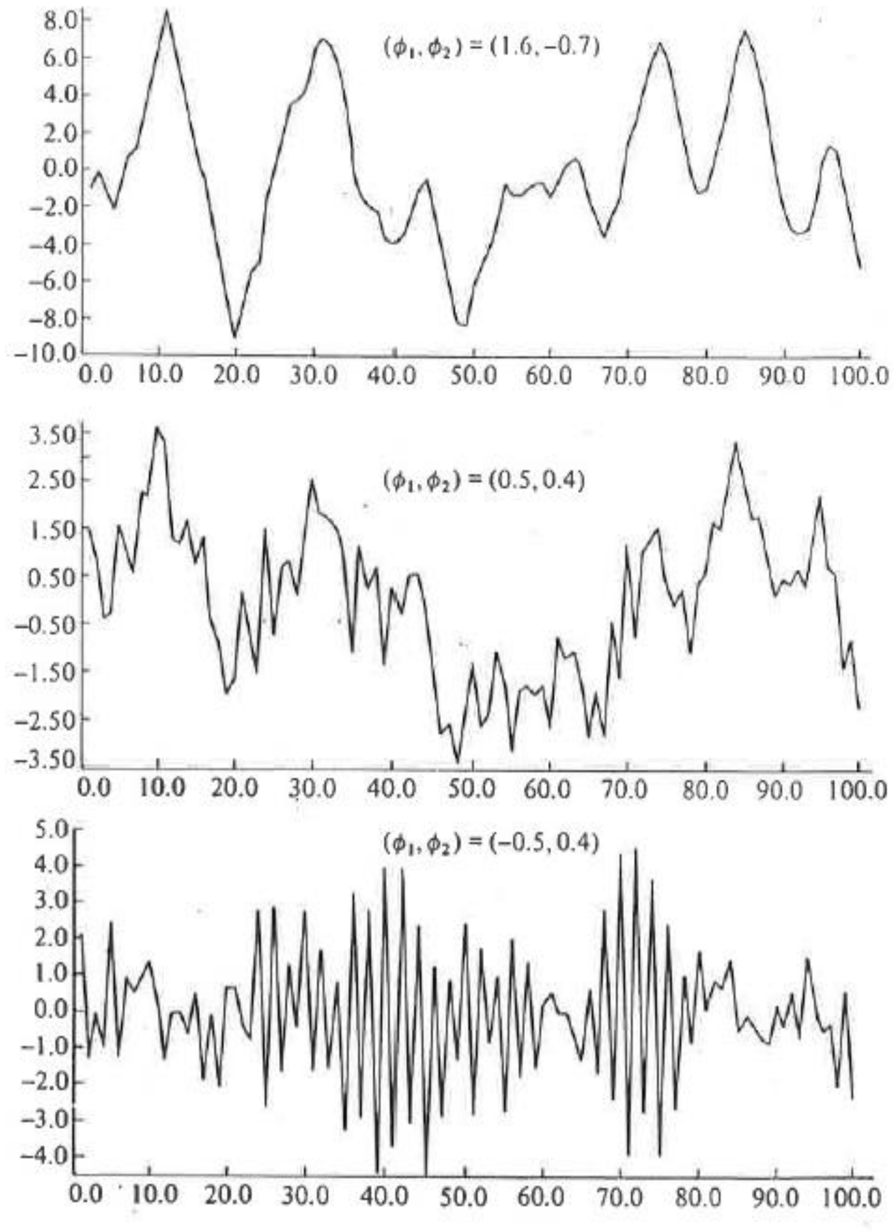
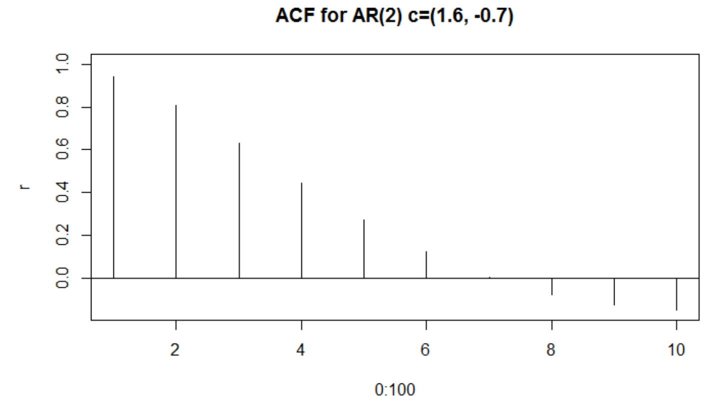
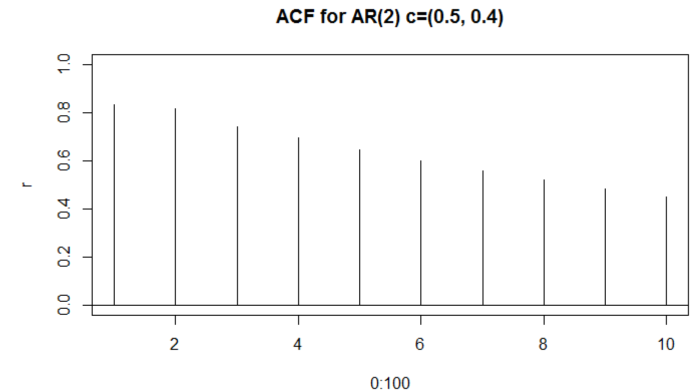
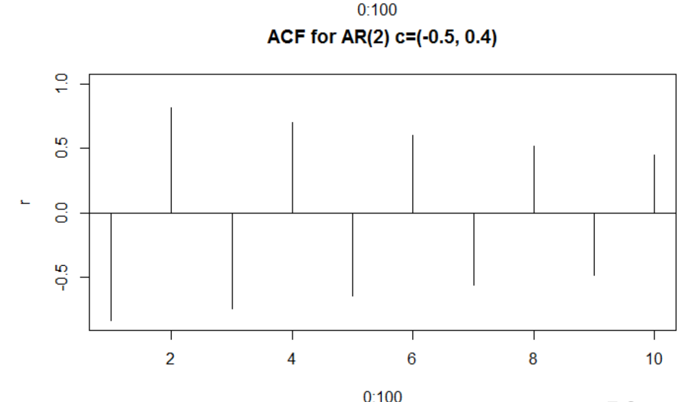
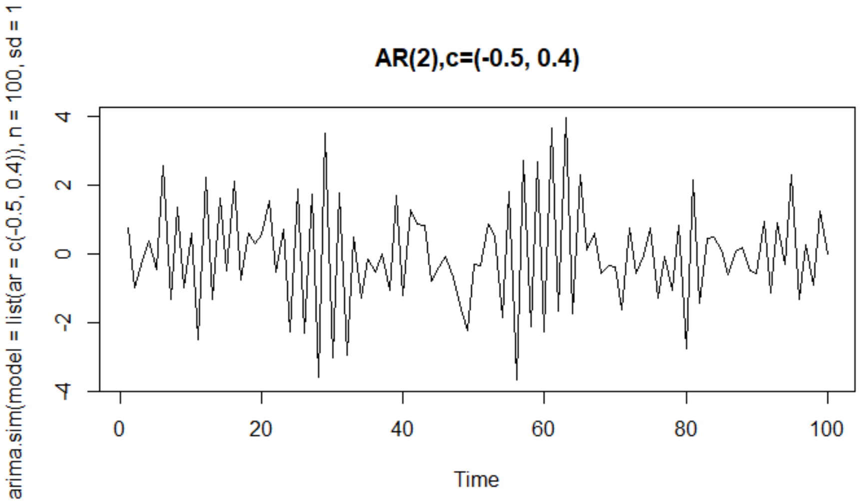


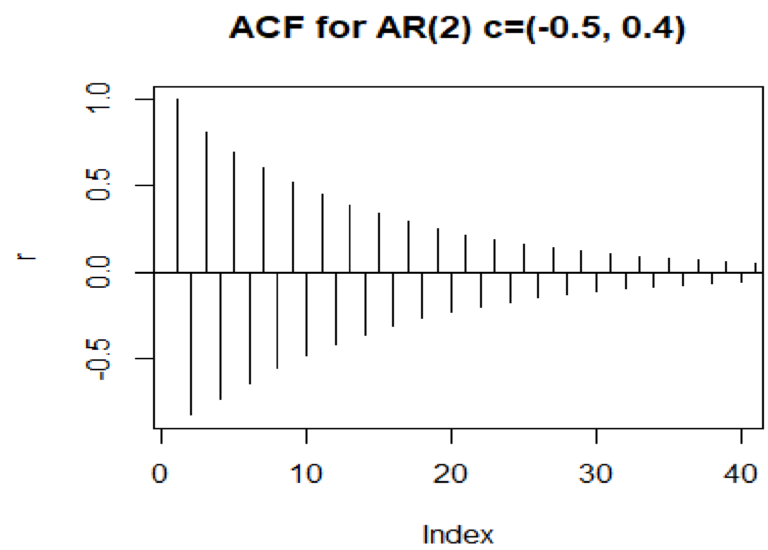
Figure 10.5 Realizations of the process: $Y_t = \phi_1 Y_{t-1} + \phi_2 Y_{t-2} + e_t$

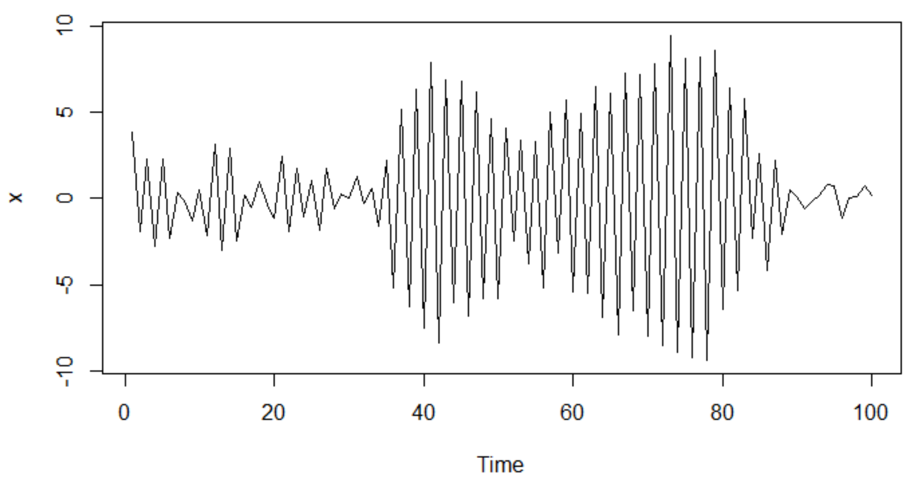


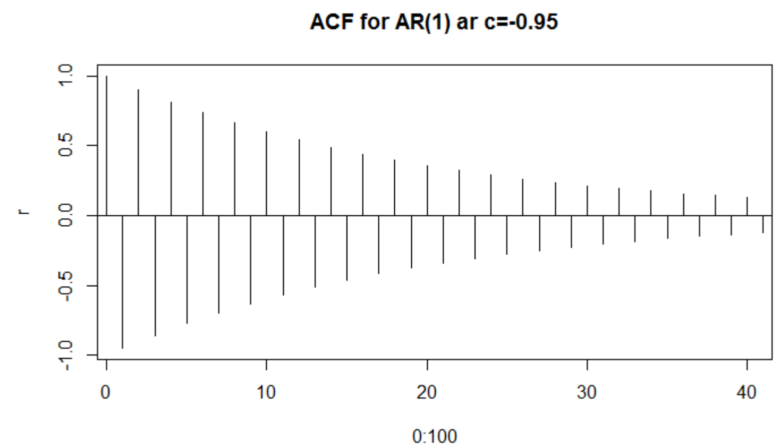








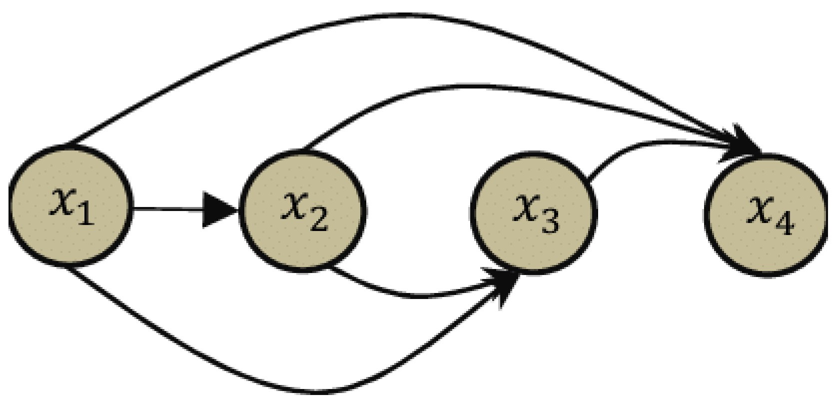




Compare plots and graphs ACFs for AR(1) and AR(2) models. Can we guess the order from these plots?

Main Points to Take from Lecture 4

- **Notion of causality**
- **AR(1) and AR(p):**
  - **Model equation;**
  - ***AR models are stationary when they have MA(∞) representation,***
    Restriction on coefficients: $|z^*| > 1$ for z^* such that $\phi(z^*) = 0$.
  - **Fact that AR models are always invertible;**
  - **ACVF and ACF formulas for AR(1) model**
  - **Sample ACF, its' use for model identification for AR(1)**



Some simple R commands used to create previous slides

- To simulate 100 values of AR(1): $X_t = -0.95X_{t-1} + Z_t$

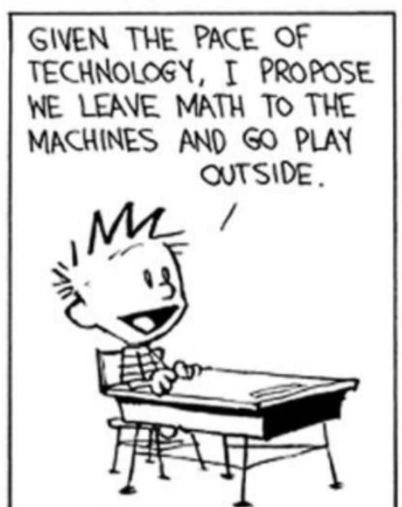
```
ar1 <- arima.sim(model=list(ar=c(-0.95)), n=100, sd=1)
plot(ar1)
```

- To plot ACF for simulated values stored in file ar1:

`acf (ar1, type="correlation", plot=T)` or simply `acf(ar1)`

- To plot theoretical ACF for this AR(1)

```
plot(ARMAacf(ar=c(-0.95), lag.max=40), col="red", type="h", xlab="lag", ylim=c(-.8,1));
abline(h=0)
```



Have a nice weekend!
Take advice from Calvin:
Go for a walk!
Spend time with friends and family!