REVIEW of Some Algebra Facts

- ***z^* is called a root of a polynomial ϕ if $\phi(z^*) = 0$.***
- ***Polynomial ϕ of order p has p roots, $z_1, \ldots, z_p$, some might be the same number.***

  Then, ϕ can be rewritten as

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = \left(1 - \frac{1}{z_1} z\right)\left(1 - \frac{1}{z_2} z\right) \ldots \left(1 - \frac{1}{z_p} z\right).$$

Check:

$\phi(z) = 0$ iff $\left(1 - \frac{1}{z_k} z\right) = 0$ for some k. This happens iff $z = z_k$ = one of the roots of ϕ.

Example:

$\phi(z) = 1 - 2.5z + 2z^2 - 0.5z^3$ has $z_1 = z_2 = 1$ and $z_3 = 2$ and can be written as $\phi(z) = (1 - \frac{1}{2} z)(1 - z)^2$.

A fact from calculus: $\sum_{n=0}^{\infty} q^n$ **is convergent iff $|q| < 1$. Then,** $\frac{1}{1-q} = \sum_{n=0}^{\infty} q^n$.

Proof:

As $N \to \infty$, $\sum_{n=0}^{N} q^n = \frac{1-q^{N+1}}{1-q} \to \frac{1}{1-q}$ iff $q^{N+1} \to 0$.

$q^{N+1} \to 0$ iff $|q| < 1$.

If $|q| > 1 \Rightarrow |q|^{N+1} \to \infty$, that is, the series does not converge.

If $q = 1$ the limit is $0/0$.

Some Algebra REVIEW and MA Representation for AR(p)

- **Inverting polynomials:**
- ***Let polynomial ϕ of order p have roots, z_1, …, z_p. Then, ϕ can be rewritten as***

$$\phi(z) = 1 - \phi_1 z - \phi_2 z^2 - \cdots - \phi_p z^p = \left(1 - \frac{1}{z_1} z\right)\left(1 - \frac{1}{z_2} z\right) \ldots \left(1 - \frac{1}{z_p} z\right).$$

Then, $\phi^{-1}(z) \equiv \frac{1}{\phi(z)} = \left\{\frac{1}{1-(1/z_1)z}\right\}\left\{\frac{1}{1-(1/z_2)z}\right\} \cdots \left\{\frac{1}{1-(1/z_p)z}\right\}$

Each of the fractions can be represented by a convergent geometric series

$$\frac{1}{1-\frac{1}{z_k}z} = \sum_{n=0}^{\infty}\left(\frac{1}{z_k}z\right)^n,\ k = 1, 2, \ldots, p.$$

if $\left|\frac{1}{z_k} z\right| < 1$. This is guaranteed for all $|z| \leq 1$, if $\left|\frac{1}{z_k}\right| < 1 \Leftrightarrow |z_k| > 1$.

Conclude: $\phi^{-1}(z)$ *has a convergent infinite series representation if its roots z_1, …, z_p satisfy conditions:* $|z_k| > 1$, k = 1, 2, …, p. The infinite series is a product of geometric series corresponding to terms $\left(1 - \frac{1}{z_k} z\right)^{-1}$.

If $\phi^{-1}(z) = \sum_{j=0}^{\infty} \psi_j z^j$, then $\phi^{-1}(B) = \sum_{j=0}^{\infty} \psi_j B^j$ and therefore

X: $\phi(B) X_t = Z_t$ has MA(∞) representation $X_t = \phi^{-1}(B) Z_t = \sum_{j=0}^{\infty} \psi_j B^j Z_t = \sum_{j=0}^{\infty} \psi_j Z_{t-j}$.
Condition: All *roots z_1, …, z_p of polynomial ϕ satisfy conditions:* $|z_k| > 1$, k = 1, 2, …, p.

Some Algebra REVIEW

For complex roots, condition $|z_k|$ > 1, k = 1, …, p, means that the roots z_1, …, z_p of the polynomial $\phi(z)$ lie outside of the unit circle:

Math facts:

z* is a root of polynomial $\phi(z)$ if it satisfies the characteristic equation: ϕ (z*) = 0.

Roots of polynomial of order p > 1 may be complex and lie on a complex plane (x,y).

For z*= x+iy, condition |z*| = $|\sqrt{\{x^2+y^2\}}|$ > 1 means that on the complex plane (x,y), z* lies is outside the unit circle x^2+y^2 = 1. Points z: |z| < 1 are inside the circle.

Example:

Roots of *ϕ (z) = 1 – 1.3z + 0.7 z^2 corresponding to* AR(2): $X_t = 1.3X_{t-1} - 0.7X_{t-2} + Z_t$.

Roots are complex and are outside unit circle:

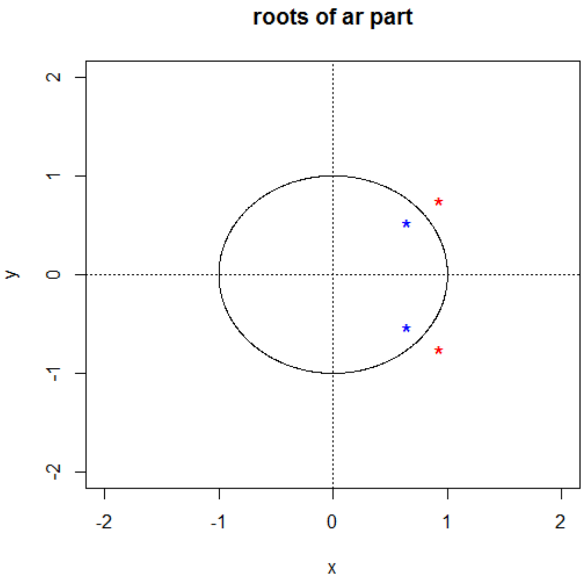
> polyroot(c(1, -1.3, 0.7))

0.9285714+0.7525467i

0.9285714-0.7525467i

On the plot, these are red stars.

Blue stars are inversed roots $\frac{1}{z_k}$.



End of Algebra Review