Week 2: MA and AR models. Shift Operator. Invertibility. Causality.

Last time we defined MA(1): $X_t = Z_t + \theta_1 Z_{t-1}$ where $Z_t \sim WN(0, \sigma_Z^2)$.

Conditions for Stationarity of X_t (weak):
$EX_t = \mu_X = const.$, $\gamma(t, t+k) = \gamma(k)$, does not depend on t.

Note that for $k = 0$, the conditions give: $\gamma_X(0) = Cov(X_t, X_t) = Var(X_t)$ is independent of t. That is, $Var(X_t) = \gamma_X(0) = \sigma_X^2$ for all t.

MA(1) always stationary, $\rho_X(k) = 0,\ k > 1$**,** $\rho_X(1) = \frac{\theta_1}{1+\theta_1^2}$.

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3.3 Shift Operator B. Invertibility (pp. 29-30 and §2.3 of [BD]).

Note on why in MA(1) model of example 3.2 we assume that $|\theta_1| < 1$:

It has to do with the property of **invertibility** which roughly says that the current observation does not depend overwhelmingly on observations in the remote past.

Definition 3.3.1. The time series $\{X_t\}$ is **invertible** if the shocks $\{Z_t\}$ can be expressed via values of X_t as *convergent* series $Z_t = X_t + \pi_1 X_{t-1} + \pi_2 X_{t-2} + \dots$.

Note that if the TS is invertible, one can write $X_t = Z_t - \pi_1 X_{t-1} - \dots$. Since the series converges, the coefficients $\pi_k \to 0$ and thus, contribution of the first terms only (the most recent values) is important.

To study invertibility of MA(1) model, we introduce the shift operator B:

Definition 3.3.2. For any TS X define the **shift operator** B: $BX_t = X_{t-1}$, $B^k X_t = X_{t-k}$.

Invertibility of MA(1): $X_t = Z_t + \theta_1 Z_{t-1} = (1 + \theta_1 B) Z_t \equiv \theta(B) Z_t$. Then,

$Z_t = \frac{1}{1-(-\theta_1 B)} X_t = (1 + (-\theta_1)B + (-\theta_1)^2 B^2 + \dots) X_t = X_t - \theta_1 X_{t-1} + \dots + (-1)^k \theta_1^k X_{t-k} + \dots$
(using geometric series expansion: $\frac{1}{1-q} = \sum_{k=0}^{\infty} q^k$ for all $q : |q| < 1$.)

The last series converges iff $|\theta_1| < 1$ (geometric series).

Note on model fitting, using an ACF $\rho_X(k)$:

Assume that we have times series data with estimated acf $\hat{\rho}(k) = 0$ when $k > 1$ and $\hat{\rho}(1) \neq 0$. We then suspect a MA(1) model $X_t = Z_t + \theta_1 Z_{t-1}$ with the preliminary estimate of coefficient θ_1 found from equation $\hat{\rho}_X(1) = \hat{\theta}_1/(1 + \hat{\theta}_1^2)$.

When $|\theta_1| < 1$, the value $\hat{\theta}_1$ can be obtained uniquely from this equation:

> the equation $\hat{\theta}_1^2 \hat{\rho}_X(1) - \hat{\theta}_1 + \hat{\rho}_X(1) = 0$ has two roots. Because their product is equal to 1 (Vieta's Theorem applied to quadratic equation $\hat{\theta}_1^2 - \Big(1/\hat{\rho}_X(1)\Big)\hat{\theta}_1 + 1 = 0$), only one satisfies the condition $|\hat{\theta}_1| < 1$.

3.4 Moving average of order q, MA(q)

Definition of MA(q): $X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}$ where $Z_t \sim WN(0, \sigma_Z^2)$.

Stationarity: MA models are always stationary with ACF:

$\rho_X(k) = \frac{\theta_k + \theta_1\theta_{k+1} + \ldots + \theta_{q-k}\theta_q}{1+\theta_1^2+\ldots+\theta_q^2}$, $k = 1, 2, \ldots, q$, $\rho_X(k) = 0$, $k > q$.

Note: the ACF is cut off beyond the order q of the model.

Derivation of the ACVF –shows that autocovariance function depends on time-lag only, i.e. stationarity of the model: Denote $\theta_0 = 1$. With $EX_t = 0$, we have:

$$\gamma_X(t, t+k) = EX_tX_{t+k} = E\Big[\Big(\textstyle\sum_{i=0}^q \theta_i Z_{t-i}\Big)\Big(\sum_{j=0}^q \theta_j Z_{t+k-j}\Big)\Big] \quad \textit{(denote } k-j = -m, j = k+m\textit{)}$$

$$= E\Big[\Big(\textstyle\sum_{i=0}^q \theta_i Z_{t-i}\Big)\Big(\sum_{m=-k}^{q-k} \theta_{k+m} Z_{t-m}\Big)\Big] \quad \textit{(recall: } EZ_{t-i}Z_{t-m} = 0, i \neq m\textit{).}$$

$$= \textstyle\sum_{i=0}^{q-k} \theta_i\theta_{k+i} E(Z_{t-i}^2) + 0 = \sigma_Z^2 \sum_{i=0}^{q-k} \theta_i\theta_{k+i} \quad \textit{(recall: } \theta_0 = 1,\ EZ_t^2 = \sigma_Z^2\textit{)}$$

$$= \sigma_Z^2(\theta_k + \theta_1\theta_{k+1} + \ldots + \theta_{q-k}\theta_q) = \gamma_X(k).$$

If $k = 0$: $\gamma_X(0) = \sigma_Z^2(1+\theta_1^2+\ldots+\theta_q^2) \equiv \sigma_X^2$ *(Used: $\theta_0 = 1$.)*

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Invertibility. To study invertibility, rewrite MA(q) equation using polynomial $\theta(z)$:

(i) use shift operator to rewrite $Z_{t-k} = B^k Z_t$ and

(ii) introduce polynomial $\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q$.

Then, $\theta(B) = 1 + \theta_1 B + \ldots + \theta_q B^q$ and we can write

$X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q} = (1 + \theta_1 B + \ldots + \theta_q B^q) Z_t \equiv \theta(B) Z_t$.

Invertibility of MA(q):

For MA(q) X_t to be invertible, we should be able to express shocks $\{Z_t\}$ via values of X_t as a *convergent* series $Z_t = X_t + \pi_1 X_{t-1} + \pi_2 X_{t-2} + \ldots$. Roughly, the property says that the current observation does not depend overwhelmingly on observations in the remote past.

Invertibility – Restrictions on coefficients:

For MA(q) process $X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}$ with $Z_t \sim WN(0, \sigma_Z^2)$, define polynomial $\theta(z) = 1 + \theta_1 z + \ldots + \theta_q z^q$.

Then, $X_t = \theta(B) Z_t$.

MA(q) are invertible if the roots of the polynomial $\boldsymbol{\theta(z)}$ lie outside of the unit circle, that is, $\boldsymbol{\theta(z) \neq 0}$ for $\boldsymbol{|z| \leq 1}$.

In this case, $\theta(z)^{-1} = 1 + \pi_1 z + \pi_2 z^2 + \ldots$ has a representation as a convergent series and from $X_t = \theta(B) Z_t$ we have:

$$Z_t = \frac{1}{\theta(B)} X_t = (1 + \pi_1 B + \pi_2 B^2 + \ldots) X_t \equiv X_t + \pi_1 X_{t-1} + \pi_2 X_{t-2} + \ldots.$$

3.4.1 Example: MA(2)– Lab 2: $X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2}$, $Z_t \sim WN(0, \sigma_Z^2)$.

Derive restrictions on coefficients θ_1, θ_2 to assure invertibility:

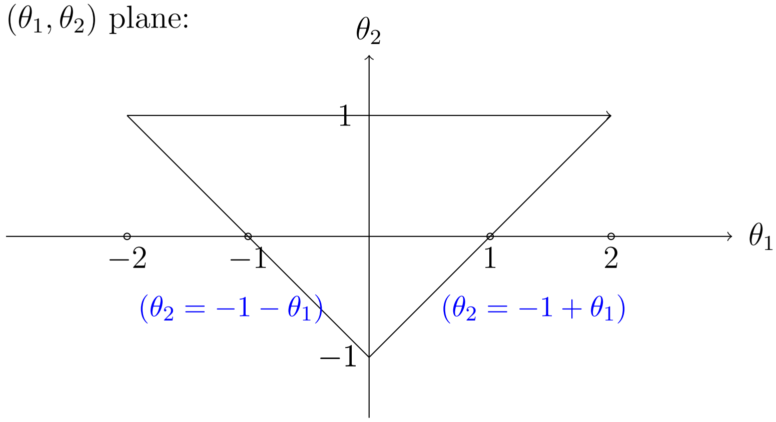
Write $\theta(z) = 1 + \theta_1 z + \theta_2 z^2 = (1 - \alpha_1 z)(1 - \alpha_2 z)$, that is, $\theta(z) = 0$ when $z_i = 1/\alpha_i, i = 1, 2$. Using geometric series,

$$\frac{1}{\theta(z)} = \frac{1}{1+\theta_1 z+\theta_2 z^2} = \left(\frac{1}{1-\alpha_1 z}\right)\left(\frac{1}{1-\alpha_2 z}\right) = (1+\alpha_1 z+\alpha_1^2 z^2+\ldots+\alpha_1^k z^k+\ldots)(1+\alpha_2 z+\ldots+\alpha_2^k z^k+\ldots).$$

These series converge for $|z| \leq 1$ iff $|\alpha_i| < 1$ so that the roots of $\theta(z)$ satisfy $|1/\alpha_i| > 1$.

Note that $\theta_1 = -(\alpha_1 + \alpha_2)$; $\theta_2 = \alpha_1 \alpha_2$.

Conditions $|\alpha_1| < 1, |\alpha_2| < 1$ are satisfied when θ_1, θ_2 take values in the following triangle on (θ_1, θ_2) plane:



$\theta_2 + \theta_1 > -1$
$\theta_2 - \theta_1 > -1$
$-1 < \theta_2 < 1.$

Note that θ_1 is allowed to be larger than 1.

Stationarity: MA(2) is stationary with ACF (special case of formula in 3.4):

$$\rho_X(k) = \begin{cases} \frac{\theta_1+\theta_1\theta_2}{1+\theta_1^2+\theta_2^2} & \text{if } k = 1, -1, \\ \frac{\theta_2}{1+\theta_1^2+\theta_2^2} & \text{if } k = 2, -2, \\ 0 & \text{if } |k| > 2. \end{cases}$$

As in MA(1) case, we suspect MA(2) model if ACF $= 0$ after lag 2. We can obtain preliminary estimates of the parameters θ_1, θ_2 from the estimates of ACF obtained from data.

Question: How does graph of $\rho_X(k)$ look like?

4. Autoregressive Models, AR(p)

4.1 Autoregressive Model of order 1, AR(1) (Based on §§1.4 and 2.2 of [BD])

$$X_t = \phi_1 X_{t-1} + Z_t,\ |\phi_1| < 1$$

The current value depends on the previous one plus a shock.

AR(1) is always invertible: $Z_t = X_t - \phi_1 X_{t-1} + 0$.

AR(1) has a MA(∞) representation as long as $|\phi_1| < 1$:
$X_t = Z_t + \theta_1 Z_{t-1} + \theta_2 Z_{t-2} - \dots$ with coefficients $\theta_n = \phi_1^n$.

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Derivation:
Use shift operator $BX_t = X_{t-1}$ to rewrite the equation for X, $X_t - \phi X_{t-1} = Z_t$ as $(1-\phi_1 B)X_t = Z_t$.

Then $X_t = \frac{1}{1-\phi_1 B} Z_t = (1 + \phi_1 B + \phi_1^2 B^2 + \dots)Z_t = Z_t + \phi_1 Z_{t-1} + \phi_1^2 Z_{t-2} + \dots$ as long as $|\phi_1| < 1$.

MA(∞) representation says that the current deviation of the TS depends on the shocks entering the system in all time periods.

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AR(1) is stationary as long as $|\phi_1| < 1$ because it then has a MA(∞) representation and MA models are always stationary (see 3.4 of these lecture notes).

ACF of AR(1): $\rho_X(k) = \phi_1^k$ for all $k : |k| \geq 1$ and $\sigma_X^2 = \gamma_X(0) = \frac{\sigma_Z^2}{1-\phi_1^2}$.
Note: for AR(1) ACF decays exponentially.

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Derivation: *For $k \geq 0$*

$$\gamma_X(k) = EX_t X_{t-k} = E\{(\phi_1 X_{t-1} + Z_t)X_{t-k}\} = \begin{cases} \phi_1 \gamma_X(k-1), & k \geq 1, \\ \phi_1 \gamma_X(1) + EZ_t X_t, & k = 0. \end{cases}$$

Substitute into the upper line, sequentially, $k = 1, 2, 3, \dots$, to get:

$\gamma_X(1) = \phi_1 \gamma_X(0)$, $\gamma_X(2) = \phi_1 \gamma_X(1) = \phi_1^2 \gamma_X(0)$, $\gamma_X(3) = \phi_1 \gamma_X(2) = \phi_1^3 \gamma_X(0)$, *etc.*

Divide these values by $\gamma_X(0)$, (we will see in a moment that it is finite) to obtain acf $\rho_X(k)$:
$\rho_X(1) = \phi_1, \rho_X(2) = \phi_1^2, \dots, \rho_X(k) = \phi_1^k$.

Remains to complete calculation of $\gamma_X(0)$. Use MA(∞) representation of X_t:

$EZ_t X_t = EZ_t \sum_{j=0}^{\infty} \phi_1^j Z_{t-j} = \phi_1^0 EZ_t^2 = \sigma_Z^2$. *We thus have:*

$\gamma_X(0) = \phi_1 \gamma_X(1) + EZ_t X_t = \phi_1^2 \gamma_X(0) + \sigma_Z^2$ *or* $\gamma_X(0) = \sigma_Z^2/(1-\phi_1^2) \equiv \sigma_X^2$

$\square$.

Smoothness as function of ϕ_1: If ϕ_1 is near 1, then there is a lot inertia in the process: the current value depends heavily on the previous value. The TS tends to be smooth. Also, the ACF tends to die out slowly (i.e. it is an ACF for the TS with heavy dependence). If ϕ_1 is close to 0, then there is little dependence on the previous value, the ACF decays to zero fast, and the TS is choppy.

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4.2 Causality (p. 47 of [BD])

Definition. $\{X_t\}$ is **causal** or a causal function of $\{Z_t\}$ if its value can be expressed in terms of current or past values of Z: $X_t = \text{function}(Z_s, s \le t)$.

MA(q) is always causal: $X_t = Z_t + \theta_1 Z_{t-1} + \ldots + \theta_q Z_{t-q}$.

AR(1) is causal for $|\phi_1| < 1$ (when it has MA representation):

$X_t = \frac{1}{1-\phi_1 B} Z_t = (1 + \phi_1 B + \ldots) Z_t = Z_t + \phi_1 Z_{t-1} + \phi_1^2 Z_{t-2} + \ldots$.

AR(1) is NOT causal for $|\phi_1| > 1$: Since $|(\phi_1)^{-1}| < 1$, we can write

$X_t = \frac{1}{1-\phi_1 B} Z_t = -\frac{(\phi_1 B)^{-1}}{1-(\phi_1 B)^{-1}} Z_t = -\sum_{j=1}^{\infty} \phi_1^{-j} B^{-j} Z_t = -\sum_{j=1}^{\infty} \phi_1^{-j} Z_{t+j}$ (Z_{t+j}=future shocks).

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4.3 AR(p) models: $\boldsymbol{X_t = \phi_1 X_{t-1} + \phi_2 X_{t-2} + \ldots + \phi_p X_{t-p} + Z_t,\ \{Z_t\} \sim WN(0, \sigma_Z^2)}$.

AR(p) is stationarity when $\phi(z) \equiv (1 - \phi_1 z - \ldots - \phi_p z^p) \ne 0$ **for all** $|z| \le 1$**.**

AR(p) is always invertible.

ACF: We show that ACF satisfies the **Yule-Walker equations**:

(4.3.1) $\rho_X(k) - \phi_1 \rho_X(k-1) - \phi_2 \rho_X(k-2) - \ldots - \phi_p \rho_X(k-p) = 0.$

which has a solution of the form

(4.3.2) $\rho_X(k) = A_1 \alpha_1^k + \ldots + A_p \alpha_p^k$, where $\alpha_1, \ldots, \alpha_k$ are the roots of $x^p - \phi_1 x^{p-1} - \ldots - \phi_p = 0$ with $|\alpha_i| < 1$.

We will continue discussing AR(p) models in Week 3 lectures. We will start with deriving Yule-Walker equations (4.3.1).