APPRENTICESHIP WEEK - FITNESS EXERCISE

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In order to understand the incidence correspondence of the 27 lines on a smooth cubic surface in $\mathbb{P}^3$ over an algebraically closed field we will utilize the following result, which is stated in Remark 4.7.1 of [Har77].

Theorem 1.1. *Any smooth cubic surface X in $\mathbb{P}^3$ over an algebraically closed field is isomorphic to the blow-up of $\mathbb{P}^2$ at six points $p_1,\ldots,p_6$ where no three of the p_i are colinear and all six are not on a cubic.*

Using this fact we see two things:

(i) the incidence graph of lines on a smooth cubic surface in $\mathbb{P}^3$ (over an algebraically closed field) is well-defined i.e. does not depend on our choice of X;

(ii) this graph can be constructed by considering the lines on X the blow-up of $\mathbb{P}^2$ at six sufficently generic chosen points $p_1,\ldots,p_6$.

Restricting our attention to the cubic surface X described above we can visualize the lines as shown below:

In particular, from this picture above that our 27 lines come in three flavors:

(i) one line, the exceptional divisor E_i, over the point p_i

(ii) one line $\ell_{i,j}$ coming from the strict transform of the line between p_i and p_j

(iii) one line C_i coming from strict transform of the unique plane conic going through $p_1,\ldots,\hat{p}_i,\ldots,p_6$.

(Note we will often abuse notation and use the same symbol for a curve and its strict transform.) Shift towards trying to compute the incidence graph G of these lines recall the strict transforms of two curves $C_1, C_2 \subset \mathbb{P}^2$ will intersect in the blow-up X if and only if C_1 and C_2 have a point of intersection outside of $\{p_1,\ldots,p_6\}$. ♠♠♠ DJ: [NEDED]

For example, in the plane two lines $\ell_{i,j}$ and $\ell_{s,t}$ intersect in exactly one point. Moreover, the intersection $\{i,j\}\cap\{s,t\}$ is non-empty if and only if this point of intersection is one of the p_i. Therefore, we see that:

$$\ell_{i,j}\cap\ell_{s,t} =$$

Similarly,

if two curves in $\mathbb{P}^2$ intersect in a point other than $\{p_1,\ldots,p_6\}$.Coding these relations into Macaulay2 we can construct the adjacency matrix of our incidence graph G via the following code:

```
-- The incidence matrix is a block matrix where the blocks are given by the subspaces:
-- The E = {e1,e2,e3,..,e6} with e_i corresponding to the exceptional divisors at point p_i
-- The C = {c1,c2,c3,..,c6} with c_i corresponding to the conic not containing the poitn p_i
-- The L = {{1,2},{1,3},...,{5,6}} where {i,j} with i>j is the line from p_i to p_j.

L = {{1,2},{1,3},{1,4},{1,5},{1,6},{2,3},{2,4},{2,5},{2,6},{3,4},{3,5},{3,6},{4,5},{4,6},{5,6}}

-- Constructs the needed block matrices.
EE = matrix apply(6,i->apply(6, j->0))
```

```
EC = matrix apply(6,i->apply(6, j->(if i==j then 0 else 1)))
EL = matrix apply(6,i->apply(L, j->(if isSubset({i+1},j)==true then 1 else 0)))
CL = matrix apply(6,i->apply(L, j->(if isSubset({i+1},j)==true then 1 else 0)))
CCC = matrix apply(6,i->apply(6, j->0))
LL = matrix apply(L, i->apply(L, j->(if set i * set j===set {} then 1 else 0)))

-- Combines the blocks into the block rows.
R1 = EE | EC | EL
R2 = EC | CCC | CL
R3 = (transpose EL) | (transpose CL) | LL

-- The adjacency matrix for the graph.
A = transpose ( (transpose R1)|(transpose R2)|(transpose R3))
```

From this information it is easy to verify – for example by summing the rows of the adjacency matrix – that the graph is 10-regular i.e. every vertex has degree 10. Additionally, we can try to visualize it as seen below: ♠♠♠ DJ: [NEDEDED]

Turning our attention to finding independent sets Macaulay2 shows the maximal ones are of size six. For example, the the six exceptional divisors $E_1,\ldots,E_6$ do not intersect, and so the corresponding vertices are an independent set. In fact, all the maximal independent sets can be classified as follows:

Type	**Example**	**Number**
$\{E_1,\ldots,E_6\}$	$\{E_1,\ldots,E_6\}$	1
$\{C_1,\ldots,C_6\}$	$\{C_1,\ldots,C_6\}$	1
$\left\{\{E_i,E_j,E_k,\ell_{a,b},\ell_{c,d},\ell_{e,f}\} \;\middle\vert\; i,j,k \notin \{a,b,c,d,e,f\}\right\}$	$\{E_1,E_2,E_3,\ell_{4,5},\ell_{4,6},\ell_{5,6}\}$	20
$\left\{\{C_i,C_j,C_k,\ell_{a,b},\ell_{c,d},\ell_{e,f}\} \;\middle\vert\; i,j,k \notin \{a,b,c,d,e,f\}\right\}$	$\{C_1,C_2,C_3,\ell_{4,5},\ell_{4,6},\ell_{5,6}\}$	20
$\left\{\{E_i,C_i,\ell_{a,b},\ell_{c,d},\ell_{e,f},\ell_{g,h}\} \;\middle\vert\; \right\}$	$\{E_1,C_1,\ell_{2,3},\ell_{2,4},\ell_{2,5},\ell_{2,6}\}$	30
	Total	**72**

Similarly, we can also classify the independent sets of size five, i.e. one less than maximal, as follows:

♠♠♠ DJ: [NEDEDEDED]

Thus, we see that the complex of independent sets of G, which we denote $\mathcal{C}_{\mathrm{Ind}}(G)$, has exactly 72 faces of dimension 5 and 648 face of dimension 4. ♠♠♠ DJ: [Can we say something more from this....] In order to get our "hands" on the full complex $\mathcal{C}_{\mathrm{Ind}}(G)$ we use the fact that the independence complex is the clique complex of complement graph. (Recall the complement of a graph is the graph with the same vertices where two vertices are adjacent if they are not adjacent in the initial graph.) Hence the starting with the adjaceny matrix A for G the following Macaulay2 code, using the Graphs ♠♠♠ DJ: [NEDEDED] package, computes $\mathcal{C}_{\mathrm{Ind}}(G)$.

```
loadPackage "Graphs"
G = graph A
G' = complementGraph G
S = cliqueComplex G'
```

Doing this we can confirm that $\mathcal{C}_{\mathrm{Ind}}(G)$ is five-dimensional, and moreover, its f-vector is:

$$\mathrm{fVec}(\mathcal{C}_{\mathrm{Ind}}(G)) = (27,216,720,1080,648,72).$$

References

[Eis95] David Eisenbud, *Commutative algebra*, Graduate Texts in Mathematics, vol. 150, Springer-Verlag, New York, 1995. With a view toward algebraic geometry. ↑

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[M2] Daniel R. Grayson and Michael E. Stillman, *Macaulay 2, a software system for research in algebraic geometry*. Available at `http://www.math.uiuc.edu/Macaulay2/`. ↑

[Mus11] Mircea Mustaţă, *Zeta functions in algebraic geometry* (2011). Available at `http://www.math.lsa.umich.edu/~mmustata/zeta_book.pdf`. ↑

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