Solutions to the Fitness Exercise

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Notes by: the Apprentices

Early-career mathematicians - those who are close to finishing their doctorates or have recently finished - are invited to a fortnight of apprenticeship in combinatorial algebraic geometry. For an intense two weeks (21 August 2016 - 3 September 2016), we will get our hands dirty exploring new problems and keep our computers spinning with many calculations. Led by Bernd Sturmfels, participants will learn new skills of the trade, networks with peers, and practice their craft.

at the Fields Institute, Summer 2016

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1. Monday, August 22, 2016 - Curves

1.1 Exercise #3 - David J. Bruce

In [Har77] order to understand, and eventually find equations for, when two plane conics C_1 and C_2 are tangent let us set up the following incidence correspondence:

$$\mathcal{C} := \left\{ (C_1, C_2, p) \quad \middle| \quad \begin{matrix} C_1 \text{ and } C_2 \text{ plane conics} \\ p \in C_1 \cap C_2 \text{ and } T_pC_1 = T_pC_2 \end{matrix} \right\} \subset \mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^2,$$

where we think of our coordinates as $([s_0 : s_1 : s_2 : s_3 : s_4 : s_5], [t_0 : t_1 : t_2 : t_3 : t_4 : t_5], [x : y : z])$. Notice we are thinking of the space of plane conics as the $\mathbb{P}^5$ given by its coefficients. To find equations for this incidence correspondence $\mathcal{C}$ notice that we essentially have three conditions:

(i) the point $p = [x : y : z]$ lies on C_1 meaning that $s_0x^2 + s_1y^2 + s_2z^2 + s_3xy + s_4xz + s_5yz = 0$,

(ii) the point $p = [x : y : z]$ lies on C_2 meaning that $t_0x^2 + t_1y^2 + t_2z^2 + t_3xy + t_4xz + t_5yz = 0$,

(iii) the curves C_1 and C_2 are tangent at the point p.

This last condition is the most complicated, but since we know if we let $J_p(C_i)$ be the Jacobian matrix of C_i evaluated at p then

$$T_pC_1 = \mathbb{P}\Big(\ker J_p(C_1)\Big) = \mathbb{P}\Big(\ker \begin{pmatrix} 2s_0x + s_3y + s_4z & 2s_1y + s_3x + s_5z & 2s_2z + s_4x + s_5y \end{pmatrix}\Big)$$
$$T_pC_2 = \mathbb{P}\Big(\ker J_p(C_1)\Big) = \mathbb{P}\Big(\ker \begin{pmatrix} 2t_0x + t_3y + t_4z & 2t_1y + t_3x + t_5z & 2t_2z + t_4x + t_5y \end{pmatrix}\Big)$$

it reduces to linear algebra. In particular, we have reduced to the question when do $J_p(C_1)$ and $J_p(C_1)$ define the same kernel, which is equivalent to saying the above gradient vectors are linearly dependent. Hence the equations corresponding to condition (iii) are given by:

$$\mathrm{Minors}_2\left(\begin{pmatrix} 2s_0x + s_3y + s_4z & 2s_1y + s_3x + s_5z & 2s_2z + s_4x + s_5y \\ 2t_0x + t_3y + t_4z & 2t_1y + t_3x + t_5z & 2t_2z + t_4x + t_5y \end{pmatrix}\right).$$

Combining these equations with the equations of type (i) and (ii) it would seem that the incidence correspondence $\mathcal{C}$ is given by the following ideal:

$$I = \left\langle \begin{matrix} s_0x^2 + s_1y^2 + s_2z^2 + s_3xy + s_4xz + s_5yz, \\ 2t_0x + t_3y + t_4z + 2t_1y + t_3x + t_5z + 2t_2z \end{matrix} \right\rangle + \mathrm{Minors}_2\left(\begin{pmatrix} 2s_0x + s_3y + s_4z & 2s_1y + s_3x + s_5z & 2s_2z + s_4x + s_5y \\ 2t_0x + t_3y + t_4z & 2t_1y + t_3x + t_5z & 2t_2z + t_4x + t_5y \end{pmatrix}\right),$$

however, this is not quite true. In particular, we failed to ensure that the point p and our conics C_1 and C_2 are actually well-defined subsets of projective space. That is to say we never included restrictions to eliminate the points where $x = y = z = 0$ or $s_0 = s_1 = s_2 = s_3 = s_4 = s_5 = 0$ or $t_0 = t_1 = t_2 = t_3 = t_4 = t_5 = 0$ from our correspondence despite these points not corresponding to actual points or conics in $\mathbb{P}^2$. The remedy for this is to saturate our ideal with respects to the ideals $\langle x, y, z\rangle$, $\langle s_0, s_1, s_2, s_3, s_4, s_5\rangle$ and $\langle t_0, t_1, t_2, t_3, t_4, t_5\rangle$. (Note we must saturate by each ideal one at a time.) The resulting ideal, call it J, now defines the correspondence $\mathcal{C}$.

Now the incidence correspondence $\mathcal{C}$ carries a natural projection:

$$\pi : \mathcal{C} \longrightarrow \mathbb{P}^5 \times \mathbb{P}^5$$

given by sending a tuple (C_1, C_2, p) to the pair of conics (C_1, C_2). Moreover, by construction the image of π is precisely the loci of conics that are tangent. So in order to find the Tact invariant we are left to compute defining equations for the image $\pi(\mathcal{C}) \subset \mathbb{P}^5 \times \mathbb{P}^5$.

Given the ideal J this can an easily be accomplished via Macaulay2 via the `eliminate` command. In particular, using the command:

$A =$ `eliminate({x,y,z}, J)`

will produce the ideal A generated by all the elements of J not using the variables x, y, z. This resulting ideal clearly defines $\pi(\mathcal{C})$, and so should hopefully be generated by the Tact invariant. Implementing this construction of J and A in Macaulay2 we find that A is generated by a bi-degree $(6,6)$ polynomial with 3210 terms, the first few being:

$$s_0^2 s_3^2 s_4^2 t_t^6 - 4s_0^3 s_1 s_4^2 t_5^6 - 4s_0^3 s_2 s_3^2 t_5^6 - 2s_0^2 s_3 s_4^2 s_5 t_3 t_5^5 + 8s_0^3 s_1 s_3 s_4 t_2 t_5^2 + \cdots$$

1.1 Complete Code - (Macaulay2)

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restart

S = QQ[x,y,z,s0,s1,s2,s3,s4,s5,t0,t1,t2,t3,t4,t5]

f = s0*x^2+s1*y^2+s2*z^2+s3*x*y+s4*x*z+s5*y*z;
g = t0*x^2+t1*y^2+t2*z^2+t3*x*y+t4*x*z+t5*y*z;

M = matrix{
{2*s0*x+s3*y+s4*z, 2*s1*y+s3*x+s5*z, 2*s2*z+s4*x+s5*y},
 {2*t0*x+t3*y+t4*z, 2*t1*y+t3*x+t5*z, 2*t2*z+t4*x+t5*y}};

J = ideal(f,g)+minors(2,M);
I = saturate(J, ideal(x,y,z));

A = eliminate({x,y,z},I);
```

2. Wednesday, August 24, 2016 - Surfaces

References

[Har77] Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977 (English). ↑