# Solutions to the Fitness Exercise

*Based on problems written by: Bernd Sturmfels*
*Notes by: the Apprentices*

Early-career mathematicians - those who are close to finishing their doctorates or have recently finished - are invited to a fortnight of apprenticeship in combinatorial algebraic geometry. For an intense two weeks (21 August 2016 - 3 September 2016), we will get our hands dirty exploring new problems and keep our computers spinning with many calculations. Led by Bernd Sturmfels, participants will learn new skills of the trade, networks with peers, and practice their craft.

at the Fields Institute, Summer 2016

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# 1. Monday, August 22, 2016 - Curves

## 1.1 Exercise #2 - Madeline Brandt

In this chapter, we study the tropicaliztion of genus 2 curves. Given a genus 2 curve $C$ over a valued field $K$, there is a genus 2 tropical curve we can associate to $C$ uniquely. This is the dual metric graph of the special fiber of a semistable model for $C$, with appropriate edge lengths. This is also the Berkovich skeleton of the Berkovich analytification of $C$. It is a nontrivial task compute this tropical curve, and the problem has been studied in the papers [Hel] and [RSS14], each using a different method. In what follows, we will discuss both methods and provide examples which demonstrate that the two methods yield the same results.

### 1.1.1 Moduli Space of Tropical Curves

For our purposes, a *tropical curve* will be a triple $(\Gamma, w, l)$ where $\Gamma = (V, E)$ is a connected graph, and $w$ is a function from $V \to \mathbb{Z}_{\geq 0}$ assigning weights to the vertices, and $l$ is a function from $E \to \mathbb{R}_{\geq 0}$ assigning a length to each edge. The genus of a tropical curve is the sum over all weights of the vertices, plus the classical genus of the graph $\Gamma$. We will say that two tropical curves are isomorphic if one can be obtained from the other via the following operations:

(1) Graph automorphisms.

(2) Removing a leaf of weight 0, together with the edge connected to it.

(3) Removing a vertex of degree 2 and weight 0, and replacing the corresponding edges by one edge whose length is the sum of the lengths of the old edges.

(4) Removing an edge of length 0 and adding the weights of the corresponding vertices.

In this way, every tropical curve has a *minimal skeleton*. This is a tropical curve with no vertices of weight 0 and degree less than or equal to two, or edges of length zero.

Given a fixed pair $(\Gamma, w)$, or *combinatorial type*, the moduli space of tropical curves of this type is given by $\mathbb{R}_{\geq 0}^{|E|}/\mathrm{Aut}(\Gamma)$. The coordinates in $\mathbb{R}_{\geq 0}^{|E|}$ give the edge lengths in the graph. The boundary of these cones corresponds to curves with at least one edge of length 0. Then, we glue the cones along the boundaries to form $M_g^{tr}$. The moduli space $M_g^{tr}$ is a *stacky fan*, and has been well studied. This is a fan together with some identifications, as described above. The stacky fan $M_2^{tr}$ is displayed below.

### 1.1.2 The dual graph of a curve

We have that there exist coarse moduli spaces $\overline{M}_g$ and $\overline{M}_{g,n}$ of stable curves and $n$-pointed stable curves, and each is a projective variety. These are called the *stable compactifications* of $M_g$ and $M_{g,n}$. The space $\overline{M}_g$ is compact and separated. The evaluative criterion for properness tells us that any regular map from the complement of a point on a smooth curve to $\overline{\mathcal{M}}_g$ admits an extension to a regular map on the whole curve. The separability can be thought of as showing that this extension is unique.

In what follows we will describe how to map curves over a valued field to tropical curves. Let $R$ be a complete discrete valuation ring with maximal ideal $m$. Let $K$ be the field of fractions of $R$, and let $k = R/m$ be the residue field. Let $t \in R$ be a uniformizing parameter. Fix a genus $g$ curve $C$ over $K$. The curve $C$ gives a morphism from $\mathrm{spec}(K) \to M_g$. Since $\overline{M}_g$ is proper, a stable curve over $K$ uniquely extends to a stable curve $C/\mathrm{spec}(R')$, where $R'$ is the valuation ring of a finite extension $K'$ over $K$. Then, we have a morphism $\mathrm{spec}(R') \to \overline{M_g}$. This is called a *stable model* of $C$. Reducing modulo $m'$ gives a map $\mathrm{spec}(k) \to \overline{M_g}$. Then this is a stable curve $C_S$ over $k$. The stable model may not be unique, but the stable curve is unique. This curve has at worst nodal singularities.

We will define the dual tropical curve as follows. Given a stable model for $C$, the vertex set of the graph will be the collection of irreducible components, where each vertex has a weight equal to the genus of the components it came from. For each node, make an edge between the corresponding vertices. Given a

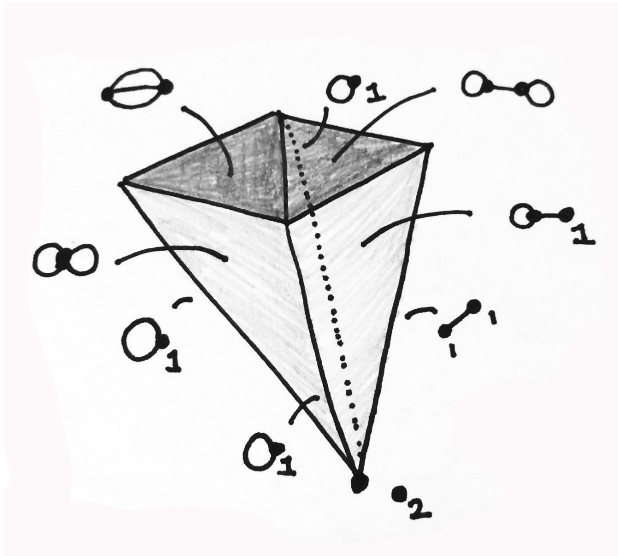


Figure 1. The moduli space $M_2^{tr}$ of genus 2 tropical curves.

vertex-weighted graph $\Gamma$, the genus of $\Gamma$ is the sum over all weights of the vertices, plus the classical genus of the graph $\Gamma$. Then, if $\Gamma$ is the dual graph of a curve $C$, it has the same genus as $C$. We will discuss how to obtain the length function (to make this a metric graph). Given an edge $e$ in $\Gamma(C_S)$, we have a node $p \in C_S$. Near $p$, the curve $C_S$ admits a local equation of the form $xy = f$, for $f \in R$. Let $l(e) = val(f)/d$, where $d$ is the degree of the field extension $K \subset K'$. We note that this is independent of the local equation chosen. Now, we have a tropical curve $(\Gamma(C_S), l)$.

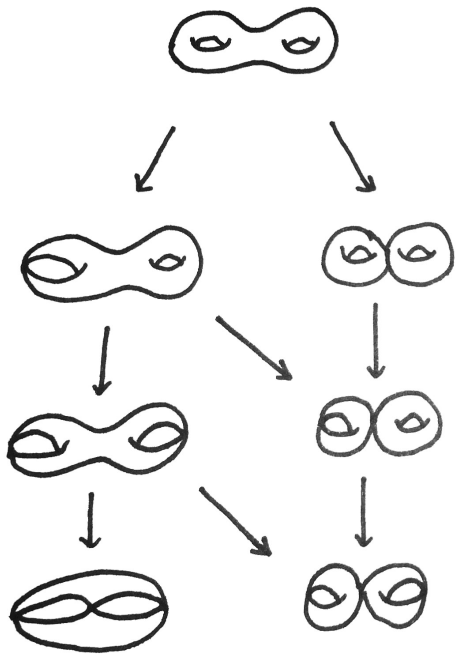
Let $M_\Gamma$ be the space parametrizing curves with weighted graph $\Gamma$. Then its codimension in $\overline{M}_g$ is equal to the number of edges in $\Gamma$, and we have that

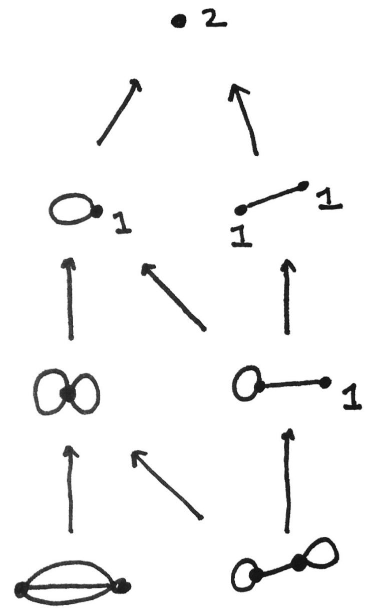
$$\overline{M}_g = \bigsqcup_{\Gamma \text{ st } g(\Gamma)=g} M_\Gamma.$$

Then, we have a stratification of $\overline{M}_g$ in the sense that the closure of $M_\Gamma$ is the disjoint union of pieces of the same kind, with the closure containing surfaces in which more loops were pulled. Dually, this corresponds to contracting edges in the graph. Then, we have the following containment:

$$M_{\Gamma'} \subset \overline{M_\Gamma} \leftrightarrow \text{there exists a contraction } \Gamma' \to \Gamma.$$

Combinatorially, we obtain the following pictures. Taking the dual graph of a curve gives an arrow-reversing bijection between the seven combinatorial types of stable curves of genus two and the seven types of vertex-weighted graphs of genus two.






We know that the combinatorial nature of these two moduli spaces is similar, so the natural question to ask is: what is the right way to tie these two spaces together?

### 1.1.3 The two methods

#### 1.1.3.1 Method One

Our goal is to understand the map

$$M_2 \to M_2^{tr}.$$

We now discuss the methods presented in [RSS14] for carrying this out.

**Theorem 1.1.1** ([RSS14])**.** *There is a commutative diagram*

$$\begin{array}{ccc} M_{0,6} & \longrightarrow & M_{0,6}^{tr} \\ \downarrow & & \downarrow \\ M_2 & \longrightarrow & M_2^{tr} \end{array}$$

Now, we may study the bottom map by instead studying the top horizontal map and the right vertical map, which we understand well.

The map from

$$M_{0,6} \to M_2$$

takes the genus 2 curve coming from the hyperelliptic cover of $\mathbb{P}^1$ with the 6 marked ramification points. All genus 2 curves are hyperelliptic, meaning any one can be defined by giving 6 points in $\mathbb{P}^1$. The curve is then the double cover of $\mathbb{P}^1$ branched at the 6 points. If the curve is given in the form

$$y^2 = f(x).$$

For a polynomial $f$ of degree 5 or 6, then the points are precisely the roots of $f$, plus possibly the point at infinity depending upon if $f$ is degree 5.

The space $M_{0,6}^{tr}$ is one we know well: this is the space of trees with 6 taxa. Using the distances $d_{ij}$ between the 6 marked points, we get a fan in $\mathbb{TP}^{14}$. Combinatorially, it agrees with the tropical Grassmannian $trop(Gr(2,6))$, as described in [MS15]. We know that $M_{0,6}^{tr}$ has a tropical basis given by the Plücker relations for $Gr(2,6)$. It has one 0 dimensional cone, 25 rays, 105 two dimensional cones, and 105 three dimensional cones. The dimension corresponds to the number of interior edges in the tree. Then the map

$$M_{0,6} \to M_{0,6}^{tr}$$

can be described as follows. Denote the 6 points in $\mathbb{P}^1$ by $(a_i : b_i)$. Then, valuations of all $2\times 2$ minors of the matrix

$$\begin{bmatrix} a_1 & a_2 & \cdots & a_6 \\ b_1 & b_2 & \cdots & b_6 \end{bmatrix}.$$

This describes a point in $\mathbb{P}^{14}$:

$$\Delta = (p_{12} : p_{13} : \cdots : p_{56}),$$

and this point gives a tree metric on a tree with 6 taxa by taking $d(i,j) = -2p_{ij} + \mathbf{1}$ for a suitable constant $n$.

The map

$$M_{0,6}^{tr} \to M_2^{tr}$$

is a morphism of generalized cone complexes, and can be described as follows. Given a point in $M_{0,6}^{tr}$, find the tree associated to it, with interior edge lengths. This tree maps to the corresponding tropical curve, as in the figures displayed below. For example, the caterpillar tree maps to the dumbell graph. Then, we must define the lengths of the edges in the corresponding tropical curve. If an interior edge in the tree has length $l$ and maps to an edge in the tropical curve which contributes to the genus, then the edge in the tropical curve receives length $2l$. Otherwise, the edge in the tropical curve receives length $l/2$. So, in the case of the dumbell, if all interior edges of the tree have length $l$, then the two loops of the dumbell obtain length $2l$, and the edge joining them obtains length $l/2$.

There is also a map $\mathbb{P}^{14} \to \mathbb{P}^{14}$ coming from the Segre cubic which sends

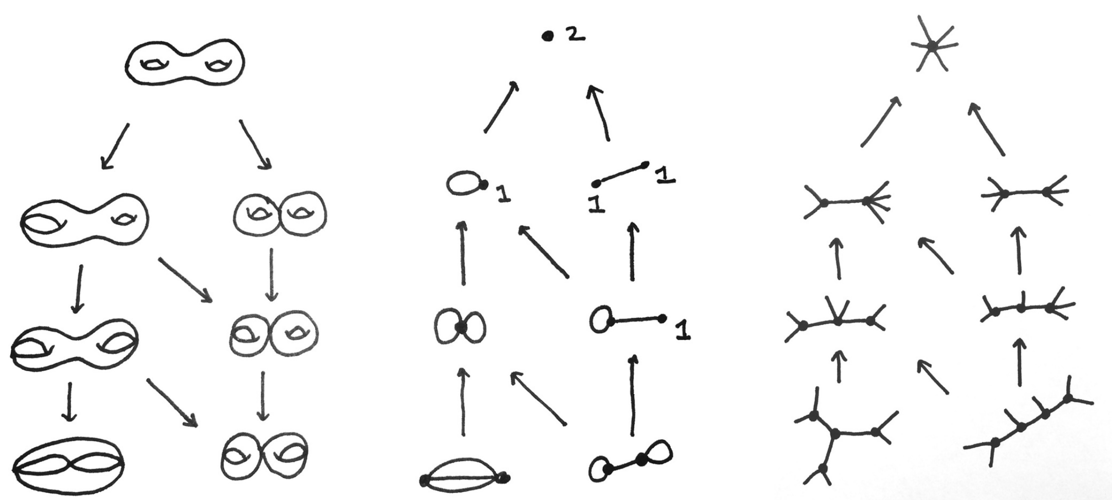
$$\begin{aligned}(p_{12} : p_{13} : \cdots : p_{56}) \mapsto (&p_{12}p_{34}p_{56} : p_{12}p_{35}p_{46} : p_{12}p_{36}p_{45} : p_{13}p_{24}p_{56} : p_{13}p_{25}p_{46} \\ &: p_{13}p_{26}p_{45} : p_{14}p_{23}p_{56} : p_{14}p_{25}p_{36} : p_{14}p_{26}p_{35} : p_{15}p_{23}p_{46} \\ &: p_{15}p_{24}p_{36} : p_{15}p_{26}p_{34} : p_{16}p_{23}p_{45} : p_{16}p_{24}p_{35} : p_{16}p_{25}p_{34}).\end{aligned}$$

Call these coordinates $m_0, \ldots, m_{14}$. Then, we can calculate the lengths in the tree using the $m_i$. For instance, in the snowflake tree, we have that the interior edge lengths will be

$$v(m_2) - v(m_{13}), v(m_6) - v(m_{13}), v(m_{14}) - v(m_{13}).$$

Suppose that the polynomial $f$ has roots $r_1, \ldots, r_6$. Additionally, suppose that $r_1, r_2$, and $r_3, r_4$, and $r_5, r_6$ come together in the special fiber, and let valuations of the pairwise differences be called $v_{12}, v_{34}, v_{56}$. Then (after an elementary calculation) we find that the interior edge lengths in the snowflake are $v_{12}, v_{34}, v_{56}$. The example in Section 1.1.5.1 is a special case of this.

This method for finding the tropical curve takes advantage of the fact that all genus 2 curves are hyperelliptic. This allows us to look at the space of trees on 6 taxa by mapping valuations of the differences of the roots into the tropical Grassmannian. Then, the difficult task is to find the correct scaling factors for the edge lengths in the tropical curve. For this reason, a promising direction for future work would be to study tropicalizations of hyperelliptic curves.




### 1.1.4 Method Two

In the case of genus 1 curves, the valuation of the $j$-invariant determines the semistable reduction type of the curve, so it is natural to wonder whether or not there are other invariants which play a similar role for higher genus. In [Hel], Helminck completely determines the semistable reduction type of a genus two curve using the *Igusa invariants*. These were first defined by Igusa in [Igu60]. Helminck gives the following theorem.

**Theorem 1.1.2** ([Hel]). *Let $C$ be a semistable curve of genus 2 over $K$. Then the cycle lengths and reduction type of a faithful tropicalization can be completely described in terms of the tropical Igusa invariants.*

One can introduce the Igusa invariants $\{J_2, J_4, J_6, J_8, J_{10}\}$ and $\{I_2, I_4, I_6, I_8, I_{12}\}$ of the curve $C$ as follows. Since $C$ is hyperelliptic we have that $C$ is given by an equation of the form $y^2 = f(x)$ where $f$ has degree 5 or 6. The exact values of the invariants can then be written down in terms of the roots of the polynomial $f$ (See the *Mathematica* notebook). For instance, if the roots are $x_1, \dots, x_6$, then

$$J_2 = \frac{1}{8} \sum_{\text{fifteen}} (x_1 - x_2)^2 (x_3 - x_4)^2 (x_5 - x_6)^2,$$

where the sum is over all 15 ways of grouping 6 objects into pairs.

**Definition 1.1.1.** The *tropical Igusa invariants* are the valuations of $J_i$ and $I_i$.

Using the tropical Igusa invariants, Helminck's theorem can determine which of the seven possible reduction types $\mathcal{C}_S$. The full theorem statement may be found in [Hel], and we implemented this in *Mathematica* (see the appendix). Helminck also determines the *thickness* of the singular points on $\mathcal{C}_S$ in terms of the tropical Igusa invariants, which gives us the lengths of the edges in the tropical curve.

### 1.1.5 Examples

#### 1.1.5.1 Two roots coincide

Suppose exactly two roots coincide in the residue field. Call these two points $a_5, a_6$. Then the tropical Igusa invariants are

$$(0, 0, 0, 0, 2a, 0, 0, 0, 0, 0),$$

where $a = v(a_5 - a_6) > 0$. This tells us that the curve is irreducible with one singular point of thickness $2a$. This corresponds to a single loop of length $2a$ and a vertex of weight 1.

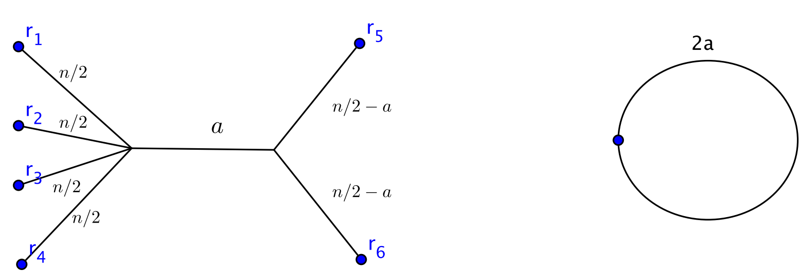
On the other hand, in $M_{0,6}^{tr}$ we have a point of the form

$$(0, \dots, 0, a) = (p_{1,2}, p_{1,3}, \dots, p_{1,5}).$$

This gives us a tree metric

$$(n, \dots, n, n - 2a) = (d_{1,2}, d_{1,3}, \dots, d_{1,5}).$$

So the tree has the desired type, with an interior edge length of $a$. Then, we double this length to find the length of the corresponding loop in the tropical curve.

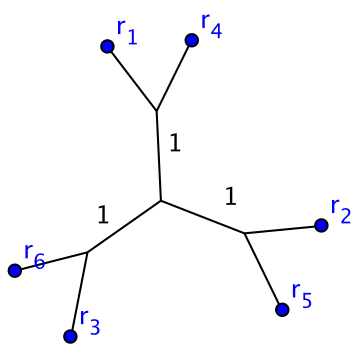


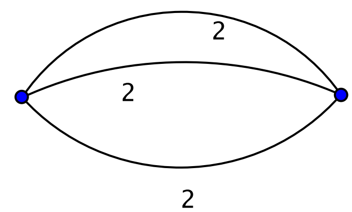

We see that in both cases, we obtained the same result using the two methods.

#### 1.1.5.2 A Snowflake

Consider the polynomial
$$y^2 = (x-1)(x-2)(x-3)(x-6)(x-7)(x-8),$$
with the 5-adic valuation. Its tropical Igusa invariants are
$$(0, 2, 2, 2, 6, 0, 0, 2, 2, 2).$$
This tells us that the reduction type is two projective lines intersecting in three points, with thicknesses $(2, 2, 2)$, so the graph is the theta graph with each edge having length 2.

On the other hand, in $M_{0,6}^{tr}$ we have the point
$$(0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0) = (p_{1,2}, p_{1,3}, \dots, p_{1,5}).$$
So that a possible tree metric (up to all 1's vector) is
$$(4, 4, 2, 4, 4, 4, 4, 2, 4, 4, 4, 2, 4, 4, 4) = (d_{1,2}, d_{1,3}, \dots, d_{1,5}).$$
This means that this is a metric for the snowflake with edge length 1 on each interior edge, so this corresponds to a theta graph with edges of length 2.




## 1.2 Exercise #3 - David J. Bruce

In order to understand, and eventually find equations for, when two plane conics $C_1$ and $C_2$ are tangent let us set up the following incidence correspondence:
$$\mathcal{C} := \left\{ (C_1, C_2, p) \quad \middle| \quad \begin{array}{c} C_1 \text{ and } C_2 \text{ plane conics} \\ p \in C_1 \cap C_2 \text{ and } T_pC_1 = T_pC_2 \end{array} \right\} \subset \mathbb{P}^5 \times \mathbb{P}^5 \times \mathbb{P}^2,$$
where we think of our coordinates as $([s_0 : s_1 : s_2 : s_3 : s_4 : s_5], [t_0 : t_1 : t_2 : t_3 : t_4 : t_5], [x : y : z])$. Notice we are thinking of the space of plane conics as the $\mathbb{P}^5$ given by its coefficients. To find equations for this incidence correspondence $\mathcal{C}$ notice that we essentially have three conditions:

(i) the point $p = [x : y : z]$ lies on $C_1$ meaning that $s_0x^2 + s_1y^2 + s_2z^2 + s_3xy + s_4xz + s_5yz = 0$,

(ii) the point $p = [x : y : z]$ lies on $C_2$ meaning that $t_0x^2 + t_1y^2 + t_2z^2 + t_3xy + t_4xz + t_5yz = 0$,

(iii) the curves $C_1$ and $C_2$ are tangent at the point $p$.

This last condition is the most complicated, but since we know if we let $J_p(C_i)$ be the Jacobian matrix of $C_i$ evaluated at $p$ then
$$T_pC_1 = \mathbb{P}\Big(\ker J_p(C_1)\Big) = \mathbb{P}\Big(\ker\begin{pmatrix}2s_0x + s_3y + s_4z & 2s_1y + s_3x + s_5z & 2s_2z + s_4x + s_5y\end{pmatrix}\Big)$$
$$T_pC_2 = \mathbb{P}\Big(\ker J_p(C_1)\Big) = \mathbb{P}\Big(\ker\begin{pmatrix}2t_0x + t_3y + t_4z & 2t_1y + t_3x + t_5z & 2t_2z + t_4x + t_5y\end{pmatrix}\Big)$$

it reduces to linear algebra. In particular, we have reduced to the question when do $J_p(C_1)$ and $J_p(C_1)$ define the same kernel, which is equivalent to saying the above gradient vectors are linearly dependent. Hence the equations corresponding to condition (iii) are given by:

$$\text{Minors}_2\left(\begin{pmatrix} 2s_0x+s_3y+s_4z & 2s_1y+s_3x+s_5z & 2s_2z+s_4x+s_5y \\ 2t_0x+t_3y+t_4z & 2t_1y+t_3x+t_5z & 2t_2z+t_4x+t_5y \end{pmatrix}\right).$$

Combining these equations with the equations of type (i) and (ii) it would seem that the incidence correspondence $\mathcal{C}$ is given by the following ideal:

$$I = \left\langle \begin{matrix} s_0x^2+s_1y^2+s_2z^2+s_3xy+s_4xz+s_5yz, \\ 2t_0x+t_3y+t_4z+2t_1y+t_3x+t_5z+2t_2z \end{matrix} \right\rangle + \text{Minors}_2\left(\begin{pmatrix} 2s_0x+s_3y+s_4z & 2s_1y+s_3x+s_5z & 2s_2z+s_4x+s_5y \\ 2t_0x+t_3y+t_4z & 2t_1y+t_3x+t_5z & 2t_2z+t_4x+t_5y \end{pmatrix}\right),$$

however, this is not quite true. In particular, we failed to ensure that the point $p$ and our conics $C_1$ and $C_2$ are actually well-defined subsets of projective space. That is to say we never included restrictions to eliminate the points where $x=y=z=0$ or $s_0=s_1=s_2=s_3=s_4=s_5=0$ or $t_0=t_1=t_2=t_3=t_4=t_5=0$ from our correspondence despite these points not corresponding to actual points or conics in $\mathbb{P}^2$. The remedy for this is to saturate our ideal with respects to the ideals $\langle x,y,z\rangle$, $\langle s_0,s_1,s_2,s_3,s_4,s_5\rangle$ and $\langle t_0,t_1,t_2,t_3,t_4,t_5\rangle$. (Note we must saturate by each ideal one at a time.) The resulting ideal, call it $J$, now defines the correspondence $\mathcal{C}$.

Now the incidence correspondence $\mathcal{C}$ carries a natural projection:

$$\pi : \mathcal{C} \longrightarrow \mathbb{P}^5 \times \mathbb{P}^5$$

given by sending a tuple $(C_1, C_2, p)$ to the pair of conics $(C_1, C_2)$. Moreover, by construction the image of $\pi$ is precisely the loci of conics that are tangent. So in order to find the Tact invariant we are left to compute defining equations for the image $\pi(\mathcal{C}) \subset \mathbb{P}^5 \times \mathbb{P}^5$.

Given the ideal $J$ this can an easily be accomplished via Macaulay2 via the `eliminate` command. In particular, using the command:

$A =$ `eliminate({x,y,z}, J)`

will produce the ideal $A$ generated by all the elements of $J$ not using the variables $x,y,z$. This resulting ideal clearly defines $\pi(\mathcal{C})$, and so should hopefully be generated by the Tact invariant. Implementing this construction of $J$ and $A$ in Macaulay2 we find that $A$ is generated by a bi-degree $(6,6)$ polynomial with 3210 terms, the first few being:

$$s_0^2s_3^2s_4^2t_t^6 - 4s_0^3s_1s_4^2t_5^6 - 4s_0^3s_2s_3^2t_5^6 - 2s_0^2s_3s_4^2s_5t_3t_5^5 + 8s_0^3s_1s_3s_4t_2t_5^2 + \cdots$$

### 1.2.1 Complete Code - (Macaulay2)

```
restart

S = QQ[x,y,z,s0,s1,s2,s3,s4,s5,t0,t1,t2,t3,t4,t5]

f = s0*x^2+s1*y^2+s2*z^2+s3*x*y+s4*x*z+s5*y*z;
g = t0*x^2+t1*y^2+t2*z^2+t3*x*y+t4*x*z+t5*y*z;

M = matrix{
{2*s0*x+s3*y+s4*z, 2*s1*y+s3*x+s5*z, 2*s2*z+s4*x+s5*y},
 {2*t0*x+t3*y+t4*z, 2*t1*y+t3*x+t5*z, 2*t2*z+t4*x+t5*y}};

J = ideal(f,g)+minors(2,M);
I = saturate(J, ideal(x,y,z));

A = eliminate({x,y,z},I);
```

# 2. Wednesday, August 24, 2016 - Surfaces

## 2.1 Exercise #9 - Barbara Bolognese, Lars Kastner, & Julie Rana

### 2.1.1 Problem

State the *Hodge Index Theorem*. Verify this theorem for cubic surfaces in $\mathbb{P}^3$, by explicitly computing the matrix for the intersection pairing.

### 2.1.2 Answer

#### 2.1.2.1 Statement

**Theorem 2.1.1.** *The signature of the intersection pairing on $H^2(X,\mathbb{R})$ equals $\sum_{p,q}(-1)^p h^{p,q}(X)$.*

Explanation: If the the pairing is given by a symmetric matrix $A$, then the signature equals the number of positive eigenvalues minus the number of negative eigenvalues.

#### 2.1.2.2 Hodge diamond

To verify this theorem, we will compute the Hodge diamond of a cubic surface $X$.

$$\begin{array}{ccccc} & & h^{00} & & \\ & h^{10} & & h^{01} & \\ h^{20} & & h^{11} & & h^{02} \\ & h^{21} & & h^{12} & \\ & & h^{22} & & \end{array}$$

We have the following symmetries:

(1) $h^{pq} = h^{n-q,n-p}$ because of Serre duality.

(2) $h^{pq} = h^{qp}$ because $H^{pq}(X)$ is the complex conjugate $H^{qp}(X)$.

So we just need to compute the upper left triangle of the Hodge diamond.

$h^{00}$: We have $h^{00} = h^0(X,\mathcal{O}_X) = 1$, since $X$ is compact connected.

$h^{01}$: $h^{01}$ is zero, because the Lefschetz hyperplane theorem implies that a hypersurface in $\mathbb{P}^n$ is simply connected, hence its first homology/cohomology group is also zero. Therefore $0 = b_1(X) = 2h^{01}$.

$h^{02}$: Since $\bigwedge^2 \Omega_X = \omega_X$, we want to know $h^0(X,\omega_X)$ (use symmetry). Using the adjunction formula we obtain

$$\omega_X \cong (\omega_{\mathbb{P}^3} \otimes \mathcal{O}(X))|_X \cong \mathcal{O}_X(-1),$$

since $\omega_{\mathbb{P}^3} \cong \mathcal{O}(-4)$ and $X$ has degree 3. But $\mathcal{O}_X(-1)$ does not have any global sections, so $h^{02} = 0$.

$h^{11}$: We have $\chi_{top}(X) = \sum_{i=0}^4 (-1)^{i+1} b_i$. The $b_i$ are sums of the Hodge numbers, namely the rows in the Hodge diamond. Inserting the known numbers, we obtain

$$\chi_{top}(X) \;=\; h^{11} + 2.$$

There are two ways to compute the holomorphic Euler characteristic $\chi(\mathcal{O}_X)$, the second way involves $\chi_{top}(X)$.

(1)
$$1 = \chi(\mathcal{O}_X) = h^0(\mathcal{O}_X) - h^1(\mathcal{O}_X) + h^2(\mathcal{O}_X) = h^{00} - h^{01} + h^{02} = h^2(X,\mathcal{O}_X).$$

(2)

$$\begin{aligned}\chi(\mathcal{O}_X) &= \int_X \mathrm{ch}(\mathcal{O}_X)\,\mathrm{td}(\mathcal{O}_X)\\ = \int_X (1,0,0)(1, \frac{c_1(X)}{2}, \frac{c_1^2(X)+c_2(X)}{12}) &= \int_X \frac{c_1^2(X)+c_2(X)}{12}\\ &= \frac{1}{12}(K_X^2 + \chi_{top}(X)).\end{aligned}$$

We have $K_X = -H$ for $H$ a hyperplane section and hence $K_X^2 = 3$.

This yields $\chi_{top}(X) = 9$ and hence $h^{11} = 7$.

This gives the Hodge diamond

$$\begin{matrix} & & 1 & & \\ & 0 & & 0 & \\ 0 & & 7 & & 0 \\ & 0 & & 0 & \\ & & 1 & & \end{matrix}$$

and for the signature we get $-5$. Denote by $a$ the number of positive eigenvalues of $A$ and by $b$ the number of negative eigenvalues of $A$. We now have the following two equations

$$a - b = -5 \quad \text{and} \quad a + b = 7.$$

Thus $a = 1$ and $b = 6$.

### Intersection pairing

Every cubic surface can be given as the blowup of $\mathbb{P}^2$ in six points. Hartshorne Prop 4.8 explicitly gives a basis for $\mathrm{Pic}(X)$. Using this basis, the intersection form is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}$$

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