MATH 632: ALGEBRAIC GEOMETRY II

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This is a supplement to Hartshorne's book, designed to fill in the details regarding the question:

What is f^* of a sheaf, how is it different from f^{-1}, and how is it related to f_*?

The ensuing explanation consists of "follow your nose" technicalities, which Hartshorne apparently deemed better left as exercises. *It is very important to think this through yourself at least once,* or you will find yourself confused whenever f^* comes up, as it often does. Because the definitions rest on a towering foundation of abstraction, understanding f^* and its relationship to f^{-1} and f_* can be confusing at first.

Notation. Throughout, X and Y denote topological spaces, and $X \xrightarrow{f} Y$ is a continuous map. For example, X and Y could be schemes, and f a morphism of schemes. We are given two sheaves of abelian groups: a sheaf $\mathcal{F}$ on X and a sheaf $\mathcal{G}$ on Y. The word "sheaf" below will always be taken to mean "sheaf of abelian groups."

Direct image. Given a sheaf $\mathcal{F}$ on X, it is easy to define an "image sheaf" $f_*\mathcal{F}$ on Y: for any open set $V \subset Y$, we simply define $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$. Since $f^{-1}(V)$ is an open set of X, this makes sense. One readily verifies this defines a sheaf. The sheaf $f_*\mathcal{F}$ is called the direct image of $\mathcal{F}$. The direct image f_* is a functor from the category of sheaves on X to the category of sheaves on Y. This means that whenever $\mathcal{E} \longrightarrow \mathcal{F}$ is a map of sheaves on X, there is an induced map of sheaves $f_*\mathcal{E} \longrightarrow f_*\mathcal{F}$ on Y (and this association takes any commutative diagram of sheaves on X to a commutative diagram of sheaves on Y). The functor f_* is *covariant,* meaning "arrowing preserving" as opposed to a *contravariant* functor, which would reverse the arrows. If the sheaf $\mathcal{F}$ also happens to have the structure of a sheaf of rings on X, then $f_*\mathcal{F}$ has the structure of a sheaf of rings on Y, and f_* is a functor from the category of sheaves of rings on X to sheaves of rings on Y.

Inverse image. Given a sheaf $\mathcal{G}$ on Y, it is also possible to define an "inverse image" sheaf $f^{-1}\mathcal{G}$ on X. We'd like it to be defined as follows: for an open set $U \subset X$, set $f^{-1}\mathcal{G}(U) = \mathcal{G}(f(U))$. But because the set $f(U)$ need not be open in Y, this doesn't work. Of course, if $X \xrightarrow{f} Y$ is an open map, there is no problem: we can define $f^{-1}\mathcal{G}$ as the sheaf determined by the presheaf sending $U \subset X$ to $\mathcal{G}(f(U))$. If $X \xrightarrow{f} Y$ is not open, we have to approximate $f(U)$ by open sets using a limit.

[1] Thanks to Manuel Blickle, Dennis Keeler, and Brian Conrad for their comments on an earlier version.

In general, the sheaf $f^{-1}\mathcal{G}$ is defined as follows. It is the sheaf associated to the presheaf which sends an open set $U \subset X$ to

$$f^{-1}\mathcal{G}(U) = \varinjlim_{V \supset f(U)} \mathcal{G}(V).$$

Here, the maps in the direct limit system are the restriction maps $\mathcal{G}(V') \to \mathcal{G}(V)$ induced whenever $V \subset V'$ are two open subsets of Y containing $f(U)$. The limit is taken over *all* open sets containing $f(U)$, but as usual, we are free to consider a smaller limit system (provided it is cofinal with the given system). For instance, we can consider only those V that are in some basis of the topology on Y, or only those "sufficiently small" in a sense that arises in context. In particular, if Y is a scheme, the V's can be assumed affine.

If $\mathcal{G}$ is a sheaf of abelian groups, so is $f^{-1}\mathcal{G}$. If $\mathcal{G}$ is a sheaf of rings, so is $f^{-1}\mathcal{G}$. This is because direct limits of abelian groups are abelian groups and direct limits of rings are rings—in fancier language, direct limits exist in the category of abelian groups and in the category of rings. The operation f^{-1} is a covariant functor from sheaves of abelian groups (or rings) on Y to sheaves of abelian groups (or rings) on X.

Exercise 1. Let P be a point in X and let $Q = f(P) \in Y$. Show that there is a natural isomorphism of stalks

$$(f^{-1}\mathcal{G})_P = \mathcal{G}_Q.$$

Adjointness. How are these operations, f_* and f^{-1}, related to each other? In spite of the name, they are not usually inverse to each other: it is *not true* that $f_*f^{-1}\mathcal{G} \cong \mathcal{G}$ or that $f^{-1}f_*\mathcal{F} \cong \mathcal{F}$. However, there are natural maps

$$\mathcal{G} \to f_*f^{-1}\mathcal{G} \quad \text{of sheaves on } Y$$

$$f^{-1}f_*\mathcal{F} \to \mathcal{F} \quad \text{of sheaves on } X,$$

and these maps induce a relationship between f^{-1} and f_* called *adjointness.* These maps are little more than glorified restriction maps, and defining them is just a matter of unraveling the definitions. Try figuring out what they are before reading my explanation.

The map $\mathcal{G} \to f_*f^{-1}\mathcal{G}$ is defined as follows. First, remember that a map of presheaves uniquely determines a map of the associated sheaves, so we are free to worry only about defining a map of presheaves. For each open set U of Y, we would like a map of groups

$$\mathcal{G}(U) \to f_*\{f^{-1}\mathcal{G}\}(U) = \{f^{-1}\mathcal{G}\}(f^{-1}(U)) = \varinjlim_{V \supset f(f^{-1}U)} \mathcal{G}(V).$$

Such a map is naturally defined as a limit of restriction maps. Indeed, since $U \supset f(f^{-1}(U))$, in computing the direct limit we can assume that each V is sufficiently small so as to be contained in U. Then, for each such V, there is a restriction map

$$\mathcal{G}(U) \to \mathcal{G}(V)$$

and whenever $V \subset V'$ are both contained in U, we have restriction maps

$$\mathcal{G}(U) \to \mathcal{G}(V') \to \mathcal{G}(V).$$

These maps naturally induce a map to the direct limit:[2]

$$\mathcal{G}(U) \to \varinjlim_{V} \mathcal{G}(V).$$

This defines the needed natural map of groups (or rings...)

$$\mathcal{G}(U) \to f_* f^{-1}\mathcal{G}(U)$$

for each open set $U \subset Y$. It is easy to verify that when $U \subset U'$ are two open sets of Y, these maps of groups commute with the restriction maps, which gives the map of sheaves on Y:

$$\mathcal{G} \to f_* f^{-1}\mathcal{G}.$$

The notation is a nightmare, which is why we usually abuse it, but the idea is incredibly simple. It is important to study this until you agree.

The map $f^{-1}f_*\mathcal{F} \to \mathcal{F}$ is equally straightforward (?) to define. For each $U \subset X$, we seek a map

$$f^{-1}(f_*\mathcal{F})(U) \to \mathcal{F}(U).$$

Giving a map from a direct limit is the same as giving a map from each object in the direct limit in such a way that these maps commute with the maps in the limit system[3]. In our case, this means the needed map

$$f^{-1}f_*\mathcal{F}(U) = \varinjlim_{V \supset f(U)} f_*\mathcal{F}(V) \to \mathcal{F}(U)$$

will be described by a map

$$f_*\mathcal{F}(V) \to \mathcal{F}(U),$$

for each V in the limit system of open sets containing $f(U)$, subject to the additional condition that whenever $V' \subset V$, the following diagram commutes

$$\begin{tikzcd}
f_*\mathcal{F}(V) \arrow[r] \arrow[d, "restrict"'] & \mathcal{F}(U) \\
f_*\mathcal{F}(V') &
\end{tikzcd}$$

[2] If this is not obvious to you, you need to review the definition of a direct limit.

[3] If you are not sure what I mean, review the definition of a direct limit.

Now it is obvious what to do. Since the limit is taken over open sets V containing $f(U)$, obviously $f^{-1}(V) \supset f^{-1}(f(U)) \supset U$. Thus the usual restriction maps $\mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U)$ define the needed map $f_*\mathcal{F}(V) \to \mathcal{F}(U)$ in such a way that the diagram obviously commutes.[4]

So what is adjointness? Let $\mathcal{F}$ be a sheaf on X and let $\mathcal{G}$ be a sheaf on Y. We can not discuss maps between the sheaves $\mathcal{F}$ and $\mathcal{G}$ because they are sheaves on two different spaces. But we could, for example, consider maps from $f^{-1}\mathcal{G}$ to $\mathcal{F}$ (these would be maps of sheaves on X), or from $\mathcal{G}$ to $f_*\mathcal{F}$ (these would be maps of sheaves on Y). It is more or less trivial to see that a map

$$f^{-1}\mathcal{G} \to \mathcal{F}$$

is essentially "the same" as a map

$$\mathcal{G} \to f_*\mathcal{F}.$$

Indeed, to define either map, one winds up needing to find group maps $\mathcal{G}(V) \to \mathcal{F}(U)$ where $V \subset Y$ contains $f(U)$ for an open set $U \subset X$. This "sameness" of maps is expressed more precisely by saying there is a natural bijection of sets

$$\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) = \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

Here $\mathrm{Hom}_X(\mathcal{A}, \mathcal{B})$ denotes the set of sheaf maps from $\mathcal{A}$ to $\mathcal{B}$, with the subscript 'X' alerting us that these are sheaves on X. This natural bijection of Hom-sets called *the adjointness of* f^{-1} *and* f_*.

Exercise 2. Prove the adjointness of f^{-1} and f_*.

The bijection is *natural* in the following sense. Given a map $\mathcal{G} \to \mathcal{H}$ of sheaves on Y, there is, as usual, an induced map of Hom-sets for each sheaf $\mathcal{F}$ on X

$$\mathrm{Hom}_Y(\mathcal{H}, f_*\mathcal{F}) \to \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

On the other hand, pulling back to X we have a map $f^{-1}\mathcal{G} \to f^{-1}\mathcal{H}$, and there is an induced map of Hom-sets

$$\mathrm{Hom}_X(f^{-1}\mathcal{H}, \mathcal{F}) \to \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}).$$

Naturality means these maps are compatible with the bijections of Hom-sets; that is, the following diagram commutes:

$$\begin{array}{ccc}
\mathrm{Hom}_X(f^{-1}\mathcal{H}, \mathcal{F}) & \xrightarrow{nat.\,bij.} & \mathrm{Hom}_Y(\mathcal{H}, f_*\mathcal{F}) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) & \xrightarrow{nat.\,bij.} & \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).
\end{array}$$

[4]If this is not obvious to you, review the definition of a sheaf.

Likewise, we can consider what happens when we have a map of sheaves $\mathcal{E} \to \mathcal{F}$ on X. Naturality means that the induced diagram of Hom-sets commutes for each sheaf $\mathcal{G}$ on Y:

$$\begin{array}{ccc} \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{E}) & \xrightarrow{nat.\,bij.} & \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{E}) \\ \downarrow & & \downarrow \\ \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}) & \xrightarrow{nat.\,bij} & \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}). \end{array}$$

You should convince yourself that both commutative diagrams are obvious. When this is clear in your head, you understand the adjointness of f_* and f^{-1}.

Furthermore, the set of sheaf homomorphisms from one sheaf of abelian groups to another naturally has the structure of an abelian group and the natural bijections above respect this group structure.[5]

Adjointness can be defined in the general setting of two functors between two categories: it always amounts to a natural bijection of Hom-Sets. In our case, we have the functor f_* from the category of sheaves on X to the category of sheaves on Y, which is the (right) adjoint of the functor f^{-1} of sheaves of Y to sheaves on X. Likewise, f^{-1} is the (left) adjoint of f_*. The fancy language of adjoint functors is just an extremely concise way of expressing something that is usually obvious in any particular setting once one has unraveled the definitions, although unraveling definitions can often be a daunting chore. As usual, abstraction can be a tremendously powerful intellectual tool, as it often encodes *a lot* of non-trivial ideas in a sentence or two. If you understand adjunction between f_* and f^{-1}, you should have no trouble writing down the abstraction to arbitrary categories. Consult any book on category theory or homological algebra.[6]

So what about f^*?

There is another operation f^* that allows us to pull back sheaves from Y to X. This operation is *not the same* as f^{-1}. The operation f^* is not defined for an arbitrary map of topological spaces; instead, it is a functor from *sheaves of* $\mathcal{O}_Y$*-modules to sheaves of* $\mathcal{O}_X$*-modules,* and can be defined only when $X \xrightarrow{f} Y$ is a morphism of *ringed spaces.* Since this operation is supposed to be very natural, we expect that it should transform the structure sheaf of rings $\mathcal{O}_Y$ on Y into the structure sheaf of rings $\mathcal{O}_X$ on X; that is, we expect $f^*\mathcal{O}_Y$ will be $\mathcal{O}_X$.

To define f^*, recall that a map of ringed spaces $X \xrightarrow{f} Y$ includes the data of map of sheaves of rings $\mathcal{O}_Y \xrightarrow{f^\sharp} f_*\mathcal{O}_X$ on Y. This naturally induces a map of sheaves of rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ on X. Indeed, for an open set $U \subset X$, we define a ring map $f^{-1}\mathcal{O}_Y(U) \to \mathcal{O}_X(U)$ as follows. Consider the system of all open sets $V \subset Y$

[5] Stop and check why!

[6] Such as MacLane's *Categories for the working mathematician* or Hilton and Stammbach's *A course in homological algebra,* both in the Springer Graduate Texts series.

containing $f(U)$; since $V \supset f(U)$, it follows that $f^{-1}(V) \supset f^{-1}(f(U)) \supset U$. Thus, for each such V, there is a natural ring map obtained as the composition

$$\mathcal{O}_Y(V) \xrightarrow{f^\sharp(V)} \mathcal{O}_X(f^{-1}(V)) \overset{restrict}{\longrightarrow} \mathcal{O}_X(U).$$

Because the maps of rings $\mathcal{O}_Y(V) \to \mathcal{O}_X(U)$ commute with the restriction maps whenever $V' \supset V$, they induce a unique map of the limit $f^{-1}\mathcal{O}_Y(U) \to \mathcal{O}_X(U)$ for each U.[7] Because the maps of rings $f^{-1}\mathcal{O}_Y(U) \to \mathcal{O}_X(U)$ commute with restriction whenever $U \supset U'$, these maps naturally define a map of sheaves of rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Of course, the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on X constructed above corresponds to the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y under the natural bijection given by adjointness. Adjointness means that the abuse of notation $\mathcal{O}_Y \to \mathcal{O}_X$ can be interpreted, more or less equivalently, either as the map of sheaves of rings $\mathcal{O}_Y \to f_*\mathcal{O}_X$ on Y or as the map of sheaves of rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ on X, depending on what seems more convenient in the context. Some caution is required: being maps of sheaves of rings on different spaces, the induced maps of stalks can be quite different. For a point $P \in X$ mapping to the point Q in Y, the map of stalks $(f^{-1}\mathcal{O}_Y)_P \to \mathcal{O}_{X,P}$ is the the the induced map $\mathcal{O}_{Y,Q} \to \mathcal{O}_{X,P}$. But the map of stalks $\mathcal{O}_{Y,Q} \to (f_*\mathcal{O}_X)_Q$ can be different. Indeed, the stalk $(f_*\mathcal{O}_X)_Q$ need not equal $\mathcal{O}_{X,P}$: otherwise, what would happen when there are two points P and P' in the fiber over Q? In this sense, the map $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ of sheaves of rings on X carries more information than the map $\mathcal{O}_Y \to f_*\mathcal{O}_X$.

Now given a sheaf $\mathcal{M}$ of $\mathcal{O}_Y$-modules, one immediately checks that $f^{-1}\mathcal{M}$ is a $f^{-1}\mathcal{O}_Y$-module. Although $f^{-1}\mathcal{M}$ is a sheaf on the topological space X which happens to have the structure of a ringed space $(X, \mathcal{O}_X)$, and it is a sheaf of modules over the sheaf of rings $f^{-1}\mathcal{O}_Y$ on X, *it is not usually a sheaf of $\mathcal{O}_X$-modules.* The purpose of f^* is to make it an $\mathcal{O}_X$-module.

The pull back $f^*\mathcal{M}$ is defined as follows. For each open set $U \subset X$, there is an obvious way to create a $\mathcal{O}_X(U)$-module out of $f^{-1}\mathcal{M}(U)$:

$$f^*\mathcal{M}(U) := \mathcal{O}_X(U) \otimes_{f^{-1}\mathcal{O}_Y(U)} f^{-1}\mathcal{M}(U).$$

Now $f^*\mathcal{M}$ is defined as the sheaf of $\mathcal{O}_X$-modules associated to this obvious presheaf of $\mathcal{O}_X$-modules. If $\mathcal{M} \to \mathcal{N}$ is map of $\mathcal{O}_Y$-modules, then $f^*\mathcal{M} \to f^*\mathcal{N}$ is a map of $\mathcal{O}_X$-modules. For any map $X \to Y$ of ringed spaces, the operation f^* is a covariant functor from the category of sheaves of $\mathcal{O}_Y$-modules to the category of sheaves of $\mathcal{O}_X$-modules.

Exercise 3. Show that if $X \xrightarrow{f} Y$ is a continuous map of topological spaces and Y is a ringed space with structure sheaf $\mathcal{O}_Y$, then X can be endowed with a ringed space structure with structure sheaf $f^{-1}\mathcal{O}_Y$. If X is itself a ringed space with

[7] Stop and make sure this is obvious to you.

structure sheaf $\mathcal{O}_X$, then $f^{-1}\mathcal{O}_Y$ usually defines a *different* ringed space structure on X. Find examples where these two ringed space structures on X are non-isomorphic. When are they isomorphic?

Exercise 4. Check that there is a natural isomorphism $f^*\mathcal{O}_Y = \mathcal{O}_X$ of sheaves of rings on X.

The main case for us is where $X \xrightarrow{f} Y$ is a morphism of schemes and $\mathcal{G}$ is a quasicoherent $\mathcal{O}_Y$-module. In this case, $f^*\mathcal{G}$ is also quasicoherent (think it through!). If both schemes are Noetherian, then if $\mathcal{G}$ is coherent, so is $f^*\mathcal{G}$ (think it through!). By contrast, the direct image $f_*\mathcal{F}$ of a coherent $\mathcal{O}_X$-module need not be a coherent $\mathcal{O}_Y$-module, although it will be quasicoherent (when X is Noetherian).

A tempting abuse of notation is to write $f^*\mathcal{M}$ as $\mathcal{O}_X \otimes_{\mathcal{O}_Y} \mathcal{M}$, which is a reasonable extension of the abuse of notation $\mathcal{O}_Y \to \mathcal{O}_X$. Of course, when $X = \text{Spec } S$ and $Y = \text{Spec } R$ are affine schemes and $\mathcal{M}$ is the quasi-coherent $\mathcal{O}_Y$-module corresponding to the R-module M, then $f^*\mathcal{M}$ is the quasicoherent $\mathcal{O}_X$-module corresponding to the S-module $S \otimes_R M$. (Think it through!) More generally, thinking of $\mathcal{O}_Y$ as a generalization of a ring, and the map $\mathcal{O}_Y \to \mathcal{O}_X$ as a generalization of a ring map, then the functor f^* is the generalization of the idea of transforming a module over $\mathcal{O}_Y$ into a module over $\mathcal{O}_X$ by using tensor product to enact a "base change." By contrast, when $\mathcal{N}$ is the quasi-coherent $\mathcal{O}_X$-module corresponding to the S-module N, then $f_*\mathcal{N}$ is the quasicoherent $\mathcal{O}_Y$-module associated to the R-module M (using the map $R \to S$ to define an R-module structure on the S-module M). So f_* is the generalization of "restriction of scalars."

Exercise 5. Let $X \xrightarrow{f} Y$ be a map of ringed spaces and let $\mathcal{M}$ be a $\mathcal{O}_Y$-module. If $f(P) = Q$, show there is a natural isomorphism of stalks $(f^*\mathcal{M})_P = \mathcal{O}_{X,P} \otimes_{\mathcal{O}_{Y,Q}} \mathcal{M}_Q$.

Exercise 6. Prove that f_* and f^* are adjoint functors. That is, if $X \xrightarrow{f} Y$ is morphism of ringed spaces, $\mathcal{F}$ is an $\mathcal{O}_X$-module, and $\mathcal{G}$ is a $\mathcal{O}_Y$-module, then there is a natural bijection

$$\text{Hom}_{\mathcal{O}_X}(f^*\mathcal{G}, \mathcal{F}) = \text{Hom}_{\mathcal{O}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

It is important to realize that the Hom-sets here are different from the Hom-sets expressing the adjointness of f_* and f^{-1}. For sheaves of $\mathcal{O}_X$-modules $\mathcal{A}$ and $\mathcal{B}$, $\text{Hom}_{\mathcal{O}_X}(\mathcal{A}, \mathcal{B})$ denotes the set of maps of sheaves of $\mathcal{O}_X$-modules from $\mathcal{A}$ to $\mathcal{B}$. This is the subset of all the sheaf maps from $\mathcal{A}$ to $\mathcal{B}$ consisting of those maps *which preserve the $\mathcal{O}_X$-module structure.*

Exercise 7. Check that if $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, then $f^*\mathcal{E}$ is a locally free $\mathcal{O}_X$-module of the same rank.

Exercise 9. Prove the projection formula: If $\mathcal{E}$ is a locally free $\mathcal{O}_Y$-module of finite rank, and $\mathcal{F}$ is an arbitrary $\mathcal{O}_X$-module, then there is a natural isomorphism

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) = f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E}.$$

<u>*Derived Functors of f_*.*</u> Neither the functor f_* nor the functor f^* is exact in general. However, something can be said:

Exercise 10. Show that the functor f_* is left exact. Show that the functor f^* is right exact.

Because the category of sheaves of abelian groups on any topological space has enough injectives, we can define the right derived functors of the left exact covariant functor f_* for any continuous map of topological spaces $X \xrightarrow{f} Y$. For each sheaf $\mathcal{F}$ on X, these will be a collection of sheaves $R^p f_* \mathcal{F}$ on Y, one for each non-negative number p. The functors $R^p f_*$ are called the *higher direct image functors*, and arise frequently in algebraic geometry. They are discussed in Hartshorne, III, Section 8.

The sheaf $R^p f_* \mathcal{F}$ is defined the same way that right derived functors are always defined:

(1) Construct a resolution of $\mathcal{F}$ by injective sheaves on X,
(2) Apply the functor f_* to get a complex of sheaves on Y,
(3) Compute the sheaf $R^p f_* \mathcal{F}$ as the "kernel mod image" of this complex at the p^{th} spot. This is independent of the choice of resolution.

Instead of working with sheaves of abelian groups on an arbitrary topological space, all of this can be done in the category of sheaves of modules on a ringed space. The category of $\mathcal{O}_X$-modules on a ringed space has enough injectives, so we can define $R^p f_* \mathcal{F}$ in this category.[8] Thus if $\mathcal{F}$ is an $\mathcal{O}_X$-module and $X \xrightarrow{f} Y$ is a morphism of ringed spaces, then the higher direct image sheaves $R^p f_* \mathcal{F}$ are sheaves of $\mathcal{O}_Y$-modules. Likewise, because the category of quasicoherent sheaves on a Noetherian scheme has enough injectives, one sees that if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_X$-module and $X \xrightarrow{f} Y$ is a morphism of Noetherian schemes, then each $R^p f_* \mathcal{F}$ is a quasi-coherent $\mathcal{O}_Y$-modules. It is not completely obvious, but it does follow from the general principles of homological algebra, that all the resulting objects $R^p f_* \mathcal{F}$ are the same when computed in any of the categories where the computation makes sense.

Exercise 11. Let $X \xrightarrow{f} Y = \text{ Spec } R$ be a separated morphism of Noetherian schemes, and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. Show that $f_* \mathcal{F}$ is the quasi-coherent $\mathcal{O}_Y$-module corresponding to the R-module of global sections of $\mathcal{F}$. Furthermore, show that $R^p f_* \mathcal{F}$ is naturally isomorphic to the quasi-coherent $\mathcal{O}_Y$-module associated to the R-module $H^p(X, \mathcal{F})$.

More generally, whenever $X \xrightarrow{f} Y$ is a continuous map of topological spaces, the sheaf $R^p f_* \mathcal{F}$ on X can be identified with the sheaf associated to the presheaf on Y defined by

$$U \mapsto H^p(f^{-1}(U), \mathcal{F}|_{f^{-1}(U)}).$$

[8]This means that in step (1) above, the resolution consists of injective $\mathcal{O}_X$-modules and the maps are $\mathcal{O}_X$-linear.

So we can think of the sheaves $R^pf_*\mathcal{F}$ as being some sort of "relative cohomology" for the map $X \xrightarrow{f} Y$. In fact, there is a natural spectral sequence[9]

$$H^q(Y, R^pf_*\mathcal{F}) \Rightarrow H^{p+q}(X, \mathcal{F})$$

relating cohomology of $\mathcal{F}$ on X to cohomology of $f_*\mathcal{F}$ on Y. When $X \xrightarrow{f} Y$ is an affine morphism of Noetherian schemes, the higher direct images $R^pf_*\mathcal{F}$ are all zero, so the sequence degenerates. This shows that $H^p(X, \mathcal{F}) \cong H^p(Y, f_*\mathcal{F})$ for an affine morphism $X \to Y$, which we have already seen using more down-to-earth machinery. The spectral sequence be viewed as an analog of the Kunneth formula relating the cohomology of a fiber space to the cohomology of the base and the fibers; the classical Kunneth formula from topology can be deduced from this spectral sequence by taking $X = Y \times Z$ and $\mathcal{F}$ to be a constant sheaf.

You can read more about the higher direct image functors $R^pf_*\mathcal{F}$ in Hartshorne, III, Section 8.

Derived Functors of f^*. By contrast, we do not usually speak of the (left) derived functors of the right exact covariant functor f^* because the category of sheaves of modules on a ringed space does not, in general, have enough projectives. When such derived functors can be defined, we would expect them to be some sort of sheafified analog of Tor. This does come up, for instance on affine schemes, but does not seem to occupy a central position in algebraic geometry like the derived functors of f_*.

However, the situation where f^* is actually an *exact functor* is extremely important. Let $X \xrightarrow{f} Y$ be a morphism of schemes. We say that X is *flat over Y* that the induced maps of stalks $\mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,Q}$ are flat for all $P \in X$. This is the same as saying that $\mathcal{O}_X$ is flat when considered as a $f^{-1}\mathcal{O}_Y$ module.

Exercise 12. Show that the functor f^* is exact on quasi-coherent $\mathcal{O}_Y$-modules whenever $X \xrightarrow{f} Y$ is a flat map. In fact, Deligne proves in SGA that this condition is equivalent to flatness.

Surjective flat maps of schemes are sometimes called *flat families:* we think of a map $X \to Y$ as a family of schemes parameterized by the points of Y whose members are the (scheme theoretic) fibers over the points of Y. Flat maps are important because, loosely speaking, the members of a flat family all "look roughly the same." This is made precise with cohomology: numerical invariants such as ranks of cohomology groups are constant in flat families. For example, let $X \to Y$ be a morphism of projective varieties over an algebraically closed field k. The fibers over any closed point of Y are projective varieties over k. More generally, the fiber over any point $P \in Y$ is a projective variety over the residue field $\kappa(P)$ of P. This is because the fiber over P is, by definition, the fiber product $X \times_Y$ Spec $\kappa(P)$ and

[9]We won't make use of this or any spectral sequence in 632, so don't worry if you're unfamiliar with this bit of machinery from homological algebra.

we have seen before that projective maps are preserved under base change (Think this through!). Let X_P denote the fiber over P, with structure sheaf $\mathcal{O}_{X_P}$. If the map $X \to Y$ is flat, then under a mild extra condition, the dimension of the $\kappa(P)$-vector space $H^i(X_P, \mathcal{O}_{X_P})$ (which is finite by Serre's theorem) will be the same for all members X_P in the family. This "mild extra condition" is that the higher direct images $R^i f_* \mathcal{O}_X$ are all locally free, which is always satisfied in characteristic zero by the so-called Lefshetz principle. A theorem of Grauert says that that local freeness of the $R^i f_* \mathcal{O}_X$ is equivalent to the cohomologies of the fibers having the same dimension.

For example, if the fibers of the flat family $X \to Y$ are all curves, then they all have the same genus. Likewise, the Hilbert polynomial will be the same for all the members in the flat family.

You can read more about flat families in Hartshorne, III, Section 9.