Pergamon

0005-1098(94)00084-0

Automatica, Vol. 31, No. 2, pp. 275–279, 1995
Elsevier Science Ltd
Printed in Great Britain
0005-1098/95 $9.50 + 0.00

Brief Paper

Numerical Methods to Design the Reaching Phase of Output Feedback Variable Structure Control*

BONNIE S. HECK,† SUBBARAO V. YALLAPRAGADA‡ and MICHAEL K. H. FAN†

Key Words—Variable structure control; optimization; linear matrix inequalities: numerical methods.

Abstract—Two numerical methods are developed to design a variable structure control that satisfies the reaching condition using static output feedback. The design is formulated as a nonsmooth convex optimization problem for which existing algorithms are available. It is shown how the resulting control law can be modified to be robust in the presence of parameter uncertainty or a disturbance. Numerical examples successfully demonstrate the developed techniques.

1. *Introduction*

Variable structure control (VSC) is a robust nonlinear control strategy employing feedback of a discontinuous signal. Most of the previous work in VSC employed either full state or estimated state feedback which may be impractical or overly complicated to implement. Two alternatives which are easier to implement are static output feedback and dynamic output feedback (Heck and Ferri, 1989; Diong and Medanic, 1992, El-Khazali and DeCarlo, 1992, 1993; Zak and Hui, 1993). Dynamic feedback is harder to design but generally has better performance than static feedback. This paper addresses the design of a static gain matrix for use in the reaching phase of a VSC, but the method is applicable for both static and dynamic output feedback. In particular, once the compensator poles are chosen, the compensator states can be augmented to the plant and the procedure developed in this paper applied to the combined system.

1.1. *Problem formulation.* The system to be controlled is assumed to be linear and time invariant:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx, \end{aligned} \tag{1}$$

where $x \in R^n$, $u \in R^m$, $y \in R^p$, and CB has full rank. The VSC control is based on output feedback, so the i^{th} component can be written as:

$$u_i(y) = \begin{cases} u_i^+(y) & \text{if } s_i(y) \geq 0 \\ u_i^-(y) & \text{if } s_i(y) < 0. \end{cases} \tag{2}$$

The switching function has the form $s = Gy$, where $s = 0$ denotes the sliding surface. Setting $\dot{s} = 0$ and solving for the equivalent control yields the equations of motion on the sliding surface:

$$\dot{x} = (A - B(GCB)^{-1}GCA)x, \tag{3}$$

where it was assumed that $(GCB)^{-1}$ exists. The two step design process is to choose the matrix G to give good behavior on the sliding surface and to choose the control components to insure that the sliding mode exists and is reachable. References (El-Khazali and DeCarlo, 1992; Zak and Hui, 1993) can be consulted to obtain methods for selecting the matrix G.

The focus of this paper is to develop design techniques for choosing the control to satisfy the reaching condition, i.e. ensuring that trajectories are directed toward the switching surface from any point in the state space. This corresponds to the global stability of the switching surface $s = 0$. Therefore, if s^Ts is the Lyapunov function, a suitable control $u = f(y)$ must be chosen to guarantee that $s^T\dot{s} < 0$. The next section presents two numerical algorithms used to design output feedback that satisfies the reaching condition. The algorithms are based on efficient numerical optimization procedures that are used for convex nonsmooth problems, such as those found in linear matrix inequalities. Section 3 contains a discussion of the effect of parameter uncertainty or an unknown disturbance on the reaching condition. Section 4 contains illustrative examples and Section 5 contains concluding remarks.

2. *Design methods*

Consider a control law of the form:

$$u = -(GCB)^{-1}Ny - \alpha(GCB)^{-1}SGN(s), \tag{4}$$

where $s = Gy$ and $SGN(s)$ is a vector with components $sgn(s_i) = 1$ if $s_i \geq 0$ and $sgn(s_i) = -1$ if $s_i < 0$. The gain matrix N is chosen to satisfy the reaching condition given by:

$$s^T\dot{s} = x^TC^TG^T(GCA - NC)x - \alpha \|s\|_1 < 0, \tag{5}$$

where $\|s\|_1 = \sum_{i=1}^{m} |s_i|$. Let $L(N)$ represent the symmetric part of $C^TG^T(GCA - NC)$:

$$L(N) \triangleq \frac{C^TG^T(GCA - NC) + (GCA - NC))^TGC}{2}. \tag{6}$$

Note that if $\alpha > 0$, then the reaching condition can be restated as follows:

$$\lambda_{\max}\{L(N)\} \leq 0. \tag{7}$$

The matrix N can be found using a numerical optimization method which minimizes the maximum eigenvalue of $L(N)$. Note that in general, there will be multiple eigenvalues at 0. This is due to the fact that the maximum rank of GC is m, so the maximum rank of $L(N)$ is $2m$ which is usually less than its dimension of n. As a result, solving for N requires an algorithm that can handle nonsmooth optimization, such as

* Received 24 February 1993; revised 10 January 1994; received in final form 28 March 1994. This paper was not presented at any IFAC meeting. This paper was recommended for publication in revised form by Associate Editor V. Utkin under the direction of Editor Huibert Kwakernaak. Corresponding author Professor Bonnie S. Heck. Tel. +1 404 894 3145; Fax +1 404 853 9171; E-mail bonnie.heck@ee.gatech.edu.

† School of Electrical and Computer Engineering, Georgia Institute of Technology, Atlanta, GA 30332-0250, U.S.A.

‡ Qualcomm Inc., 10555 Sorrento Valley Rd, San Diego, CA 92121, U.S.A.

the Cutting Plane Method (Kelly, 1960) or Interior Point Methods (Boyd and El Ghaoui, 1993; Fan and Nekooie, 1994). These methods are commonly used in the solution of linear matrix inequalities (LMI), which have received a great deal of attention in the control community.

Since N is a gain matrix, a straightforward minimization of the eigenvalue might yield too large a gain. To avoid this problem, we minimize the weighted average of the norm of N and the maximum eigenvalue of $L(N)$. Let $\phi(N)$ be defined as the quantity to be minimized:

$$\phi(N) = \lambda_{\max}\{L(N)\} + \rho\|N\|^r, \tag{8}$$

where $\rho \geq 0$ is a constant and $r \in \{1, 2\}$ depending on the numerical method used.

We now show that $\phi(N)$ is convex in N. Since $\rho \geq 0$ and the sum of convex functions is convex, it suffices to show that $\lambda_{\max}\{L(N)\}$ and $\|N\|^r$ are both convex in N. The convexity of $\lambda_{\max}\{L(N)\}$ follows from the variational characterization of largest eigenvalues for symmetric matrices and the fact that the maximum of convex functions is convex. The convexity of $\|N\|^r$ is a direct consequence of the triangular and Cauchy–Schwarz inequalities. Since $\phi(N)$ is convex, any local minima is also global.

Two numerical methods will be examined and the problem will be reformulated for each of the methods. The Cutting Plane Method is most suited for using the Frobenius norm with $r = 1$ in (8), while the Interior Point Method is most suited for the induced l_2 norm with $r = 2$ in (8). For application of the numerical methods, the matrix defined in (6) is rewritten as:

$$L(N) = L_0 + \sum_{k=1}^{mp} n_k L_k, \tag{9}$$

where the L_k's are $n \times n$ real symmetric matrices and the n_k's are the elements of N. Specifically, the value of n_k in (9) is defined as the k^{th} element of the vector formed by stacking the columns of the N matrix: $[n_{11}, n_{21}, \ldots, n_{m1}; n_{12}, n_{22}, \ldots, n_{m2}; \ldots; n_{1p}, n_{2p}, \ldots, n_{mp}]^{\mathrm{T}}$. Let L_0 be defined by:

$$L_0 = \frac{C^{\mathrm{T}}G^{\mathrm{T}}GCA + (GCA)^{\mathrm{T}}GC}{2} \tag{10}$$

and let L_k be defined by:

$$L_k = -\frac{C^{\mathrm{T}}G^{\mathrm{T}}N_kC + C^{\mathrm{T}}N_k^{\mathrm{T}}GC}{2}, \tag{11}$$

where N_k is an $m \times p$ matrix with zeros everywhere except a '1' in one position. For $k = 1$, the 1, 1 element of N_1 is 1. As k is increased, the 1 moves down the first column, then the second column and so on. For example, if $m = 2$ and $p = 2$, then the N_k's are defined below:

$$N_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad N_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad N_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad N_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

2.1. *Cutting Plane Method.* To apply the Cutting Plane Method of Kelly (1960), let $\|N\|$ in equation (8) refer to the Frobenius norm and let $r = 2$.

Cutting Plane Algorithm:

Step 0. Initialize $N = 0$ and $\phi_{lb} = -1 \times 10^6$.

Step 1. Compute the maximum eigenvalue of $L(N)$.

Step 2. Form the matrices f, H and b, where:

$$f^{\mathrm{T}} = [0_{mp\times 1} \quad 1],$$

$$H = [v^{\mathrm{T}}L_1v + 2\rho n_1, v^{\mathrm{T}}L_2v + 2\rho n_2, \ldots, v^{\mathrm{T}}L_{mp}v + 2\rho n_{mp}, -1],$$

$$b = \left[-v^{\mathrm{T}}L_0v + \rho\sum_{k=1}^{mp} n_k^2\right],$$

where v is the eigenvector associated with the largest eigenvalue of $L(N)$. Note that H is $1 \times (mp + 1)$, b is 1×1, and f is $(mp + 1) \times 1$.

Step 3. Solve the linear programming problem:

$$\min_x f^{\mathrm{T}}x \quad \text{subject to } Hx \leq b,$$

where

$$x^{\mathrm{T}} = [n_1\, n_2 \cdots n_{mp} \quad \phi_{lb}(N)].$$

Step 4. Compute $\phi(N)$ from equation (8). Stop if $\phi_{lb}(N) \geq \phi(N)$ or if $\phi_{lb}(N)$ converges.

Step 5. Go to Step 2. In the next iteration of Step 2, augment the new H and b row vectors to the last H and b. Therefore, the row dimensions of H and b increase with each iteration. (To increase computational efficiency, only the last mp to $2mp$ rows for which the Lagrange multiplier in the linear program was nonzero need to be included in H and b.)

Note, that some of the terms in the matrices defined in Step 2 are actually the subgradients of ϕ, thus providing the descent direction.

2.2. *Interior Point Method.* Let $\|N\|$ in equation (8) refer to the l_2 induced norm and let $r = 1$. To use an Interior Point Method, the problem first has to be reformulated as a generalized eigenvalue minimization problem with LMI constraints. Let $\gamma \in R$ and let $\Gamma(\gamma, N)$ be defined by:

$$\Gamma(\gamma, N) = \begin{bmatrix} \gamma I & N^{\mathrm{T}} \\ N & \gamma I \end{bmatrix}. \tag{12}$$

Using the well-known fact that $\|N\| \leq \gamma$ if and only if $\Gamma(\gamma, N) \geq 0$, the minimization of (8) can be restated as:

$$\min_{N,\alpha,\beta,\lambda} \{\lambda : \alpha + \beta \leq \lambda,\ L(N) \leq \alpha I,\ \Gamma(\beta\rho^{-1}, N) \geq 0\}, \tag{13}$$

where α, β, $\lambda \in R$ and the optimum λ equals the optimum ϕ, i.e. $\lambda^* = \phi^*$. Note that α and β are introduced to provide upper bounds on the first and second terms in (8), respectively.

Let $z = [\alpha, \beta, n_{11} \cdots n_{m1}, n_{12} \cdots n_{m2}, \ldots, n_{1p} \cdots n_{mp}]$ be the vector used for the minimization and let F be the feasible set consisting of elements (z, λ), such that the inequalities in (13) are satisfied. Define a barrier function by:

$$\begin{aligned}\pi(z, \lambda) = &-\log(\lambda - \alpha - \beta) - \log\det(\alpha I - L(N)) \\ &- \log\det(\Gamma(\beta\rho^{-1}, N)),\end{aligned} \tag{14}$$

if $(N, \alpha, \beta, \lambda)$ is in the interior of F, and $\pi(z, \lambda) = \infty$ otherwise. For $\lambda > \lambda^*$, the function $\pi(z, \lambda)$ can be easily shown to be analytic and, under a mild assumption, it is strictly convex in z. Consequently, for $\lambda > \lambda^*$, the optimization problem:

$$z_c(\lambda) := \arg\min_z \pi(z, \lambda) \tag{15}$$

has a unique minimizer, which is called the 'analytic center'. The analytic center $z_c(\lambda)$ can be efficiently computed using Newton's method with an appropriate step size rule.

The method proposed in Fan and Nekooie (1994) in the context of solving (13) is summarized as follows:

Interior Point Algorithm:

Step 0. Set $k = 0$, $\theta = 0.001$. Initialize $z^{(0)}$ as $\alpha = 2 + \lambda_{\max}(L_0)$, $\beta = 1$ and $N = 0$.

Step 1. Find $z_c(\lambda^{(k)})$ from (15).

Step 2. Set:

$$\hat{z} = z_c(\lambda^{(k)}) - \mu\frac{\mathrm{d}z_c(\lambda^{(k)})}{\mathrm{d}\lambda},$$

where μ is the unique positive number such that $(\hat{z}, \lambda^{(k)} - \mu)$ lies on the boundary of the set F.

Step 3. Write $(\hat{\alpha}, \hat{\beta}, \hat{N}) = \hat{z}$. Set $\lambda^{(k+1)} = (1 - \theta)(\hat{\alpha} + \hat{\beta}) + \theta\lambda^{(k)}$ and $z^{(k+1)} = (1 - \theta)\hat{z} + \theta z_c(\lambda^{(k)})$.

Step 4. Stop if $\lambda^{(k)} - \lambda^{(k+1)} < 1e - 6$. Otherwise, set $k = k + 1$ and go to Step 1.

The derivative in Step 2 can be obtained as follows: in view of the definition of $z_c(\lambda)$, the gradient of $\pi(z, \lambda)$ with respect to z at $(z_c(\lambda), \lambda)$ is zero, i.e.:

$$\nabla_z \pi(z_c(\lambda), \lambda) = 0 \tag{16}$$

for all $\lambda > \lambda^*$. Differentiating (16) and solving for $\mathrm{d}z_c(\lambda)/\mathrm{d}\lambda$ yields:

$$\frac{\mathrm{d}z_c(\lambda)}{\mathrm{d}\lambda} = (\nabla_z^2 \pi(z_c(\lambda), \lambda))^{-1} \frac{\partial}{\partial\lambda} \nabla_z \pi(z_c(\lambda), \lambda), \tag{17}$$

where the Hessian matrix $\nabla_z^2 \pi(z_c(\lambda), \lambda)$ is positive definite with components given by:

$$\frac{\partial^2 \pi}{\partial \alpha^2} = \frac{1}{(\lambda - \alpha - \beta)^2} + \mathrm{tr}\,((\alpha I - L(N))^{-2}),$$

$$\frac{\partial^2 \pi}{\partial \beta^2} = \frac{1}{(\lambda - \alpha - \beta)^2} + \mathrm{tr}\,(\Gamma(\beta\rho^{-1}, N)^{-2}\rho^{-2}),$$

$$\frac{\partial^2 \pi}{\partial n_i^2} = \mathrm{tr}\,((\alpha I - L(N))^{-1} L_i (\alpha I - L(N))^{-1} L_i),$$

$$\frac{\partial^2 \pi}{\partial \alpha \partial \beta} = \frac{\partial^2 \pi}{\partial \beta \partial \alpha} = \frac{1}{(\lambda - \alpha - \beta)^2},$$

$$\frac{\partial^2 \pi}{\partial \alpha \partial n_i} = \frac{\partial^2 \pi}{\partial n_i \partial \alpha} = -\mathrm{tr}\,((\alpha I - L(N))^{-2} L_i),$$

$$\frac{\partial^2 \pi}{\partial \beta \partial n_i} = \frac{\partial^2 \pi}{\partial n_i \partial \beta} = \mathrm{tr}\left(\Gamma(\beta\rho^{-1}, N)^{-2} \begin{bmatrix} 0 & N_i^{\mathrm{T}} \\ N_i & 0 \end{bmatrix} \rho^{-1}\right).$$

The components of $\partial(\nabla_z \pi)/\partial\lambda$ are given by:

$$\frac{\partial^2 \pi}{\partial \lambda \partial \beta} = \frac{\partial^2 \pi}{\partial \lambda \partial \alpha} = \frac{-1}{(\lambda - \alpha - \beta)^2}, \quad \frac{\partial^2 \pi}{\partial \lambda \partial n_i} = 0.$$

The value of μ in Step 2 can be found numerically by the following bisection procedure: choose $\mu_a > 0$, such that

$$y_a := \left(z_c(\lambda^{(k)}) - \mu_a \frac{\mathrm{d}z_c(\lambda^{(k)})}{\mathrm{d}\lambda}, \quad \lambda^{(k)} - \mu_a\right)$$

does not belong to the set F (μ_a can be chosen successively larger until this condition is met, e.g. $\mu_a = 1$, 10, 100, 1000, etc.). Define $y_b = (z_c(\lambda^{(k)}), \lambda^{(k)})$. Let $y_c = (y_a + y_b)/2$. If y_c is in F, set $y_b = y_c$. Otherwise, set $y_a = y_c$. The process repeats until $\|y_a - y_b\|$ is sufficiently small.

2.3. *Comments.* Both algorithms can be programmed using MATLAB. The Cutting Plane Algorithm requires the Optimization Toolbox (needed for the linear programming solution), while the Interior Point Method does not require any Toolbox. Both algorithms can be used to minimize the function (8) reliably. The Cutting Plane Method is well-proven and is fairly easy to program if a linear program solver is available. Interior Point Methods have received a great deal of attention recently due to their fast convergence properties; however, both methods had comparable speed in this application due to the added weighting on the norm of N. The choice of norm for each method was based on the computation efficiency of the resulting algorithm. This difference in the selection of the norm allows for freedom in the design. Moreover, the Frobenius norm used in the Cutting Plane Method can be modified so that there is a different ρ for each n_k. This gives the designer the choice in weighting certain elements of the gain matrix more than others.

If an N does not exist that results in $L(N)$ being negative semi-definite, then the gain α in (4) can be chosen appropriately to satisfy the reaching condition in a region of the state space. The smaller the maximum eigenvalue of $L(N)$, the larger the region. Consider, for example, the control law in (4) with the term $s/\|s\|^2$ in place of $SGN(s)$. The following expression is obtained:

$$s^{\mathrm{T}}\dot{s} = x^{\mathrm{T}}L(N)x - \alpha.$$

An upper bound for this expression is:

$$s^{\mathrm{T}}\dot{s} \le \lambda_{\max}\{L(N)\}\, \|x\|^2 - \alpha. \tag{18}$$

To satisfy the reaching condition, $s^{\mathrm{T}}\dot{s} < 0$, it is necessary for $\|x\| < \sqrt{\alpha/\lambda_{\max}\{L(N)\}}$. Thus, the smaller the value of $\lambda_{\max}\{L(N)\}$, the larger $\|x\|$ may be while still satisfying the reaching condition.

A similar result occurs if the term $SGN(s)$ is retained. In particular:

$$s^{\mathrm{T}}\dot{s} \le \lambda_{\max}\{L(N)\}\, \|x\|_2^2 - \alpha\|s\|_1.$$

If x is in the region defined by $\{x : \|x\|_2 \le \Omega\}$, then the results on ultimate boundedness (Khalil, 1992) can be used to show that s is ultimately bounded by:

$$\|s\|_2 \le \frac{\lambda_{\max}\{L(N)\}\Omega^2}{\alpha}. \tag{19}$$

Since $\|s\|_1 \ge \|s\|_2$, the same bound is achieved for a control of the common form $s/\|s\|_2$. The bound is made small by an obvious trade-off between $\lambda_{\max}\{L(N)\}$, α and Ω. The fact that the system trajectory is only guaranteed to be in a small band surrounding $s = 0$ (i.e. not asymptotic to $s = 0$) is acceptable for VSC applications since nonideal relays or chatter reduction boundary layers also result in a similar band. It should be noted that existence of the sliding mode is guaranteed in the region $\{x : \|x\| \le \Omega\}$ for the discontinuous controls $s/\|s\|$ and $SGN(s)$, if α is chosen such that $\alpha > \|GCA - NC\|$ as seen from (5).

3. *Robustness properties*

The control law given in (4) can be modified easily for application to a system in the following form:

$$\dot{x} = Ax + Bu + h(x), \tag{20}$$

where $h(x)$ represents a bounded uncertainty, unknown disturbance and/or nonlinearities. It can be shown easily that the sliding mode equation (3) is invariant to $h(x)$ with the control in (4), if the standard matching condition is assumed, $h(x) \in \mathbb{R}(B)$. The control must be modified to guarantee that the reaching condition is satisfied in the presence of $h(x)$. Suppose a matrix N that satisfies the condition in (7) is found numerically, then the following theorem can be used to modify the control law given in (4).

Theorem 1. Suppose $h(x) = Be$, where e is bounded by $\|e\|_2 \le \kappa(t, y)$. If the control is chosen as:

$$u = -(GCB)^{-1}Ny - (\alpha + \gamma\kappa)(GCB)^{-1}SGN(s), \tag{21}$$

where $\alpha > 0$ and $\gamma \ge \|GCB\|_2$, and if $\lambda_{\max}\{L(N)\} \le 0$, then the reaching condition, $s^{\mathrm{T}}\dot{s} < 0$, is satisfied for the system in (20).

Proof. The following expression is derived for $s^{\mathrm{T}}\dot{s}$:

$$s^{\mathrm{T}}\dot{s} = x^{\mathrm{T}}L(N)x - (\alpha + \gamma\kappa)\|s\|_1 + s^{\mathrm{T}}GCBe.$$

Since $\lambda_{\max}\{L(N)\} \le 0$, $\|e\|_2 \le \kappa$ and $\|s\|_2 \le \|s\|_1$, the following is obtained:

$$s^{\mathrm{T}}\dot{s} \le -(\alpha + \gamma\kappa)\|s\|_2 + \|s\|_2\|GCB\|_2\kappa,$$

where $\|GCB\|_2$ represents the induced l_2 norm. Therefore, if $\gamma \ge \|GCB\|_2$, then the reaching condition, $s^{\mathrm{T}}\dot{s} < 0$, is satisfied.

The above theorem requires that $h(x) \in \mathbb{R}(B)$. This condition is removed in the following theorem:

Theorem 2. Suppose $\|h(x)\|_2 \le \Delta$. If the control is chosen as:

$$u = -(GCB)^{-1}Ny - \alpha(GCB)^{-1}SGN(s), \tag{22}$$

where $\alpha > \|GC\|_2\Delta$ and if $\lambda_{\max}\{L(N)\} \le 0$, then the reaching condition $s^{\mathrm{T}}\dot{s} < 0$ is satisfied.

Proof. The following expression is derived for $s^{\mathrm{T}}\dot{s}$:

$$s^{\mathrm{T}}\dot{s} = x^{\mathrm{T}}L(N)x - \alpha\|s\|_1 + s^{\mathrm{T}}GCh.$$

An upper bound is found to be:

$$s^{\mathrm{T}}\dot{s} \le x^{\mathrm{T}}L(N)x - \alpha\|s\|_1 + \|s\|_2\|GC\|_2\|h\|_2.$$

With the conditions in the theorem and the fact that $\|s\|_1 \ge \|s\|_2$, it is obvious that $s^{\mathrm{T}}\dot{s} < 0$.

4. *Numerical examples*

An L-1011 aircraft model modified from Andry *et al.* (1983) will be used to demonstrate the design methods.

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -0.154 & -0.0042 & 1.54 & 0 & -0.744 & -0.032 \\ 0 & 0.249 & -1 & -5.2 & 0 & 0.337 & -1.12 \\ 0.0386 & -0.996 & -0.0003 & -2.117 & 0 & 0.02 & 0 \\ 0 & 0.5 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -20 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -25 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 20 & 0 \\ 0 & 25 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & -0.154 & -0.0042 & 1.54 & 0 & -0.744 & -0.032 \\ 0 & 0.249 & -1 & -5.2 & 0 & 0.337 & -1.12 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix}.$$

The inputs are the aileron and the rudder commands. The outputs chosen for this example are yaw and roll accelerations, bank angle and the wash out filter state, which ensures that CB is full rank. Note also that the actual compensator is dynamic due to a wash-out filter; however, the wash-out filter is augmented to the plant resulting in a static output feedback formulation. The following G matrix yields good sliding mode dynamics:

$$G = \begin{bmatrix} -0.0067 & 0.0167 & 0.0033 & 0 \\ 0.0167 & -0.0333 & 0 & 0.0333 \end{bmatrix}.$$

Both algorithms were implemented to obtain the gain N used in the control law given in (4). For a weighting of $\rho = 1 \times 10^{-5}$, the Cutting Plane Algorithm yielded:

$$N = \begin{bmatrix} -0.6923 & 2.189 & 0.1724 & -1.475 \\ 3.122 & -6.298 & -0.3223 & 4.681 \end{bmatrix}.$$

The resulting $\lambda_{max}\{L(N)\}$ was 1.56×10^{-3}. The Interior Point Method was run with $\rho = 2 \times 10^{-4}$ to yield:

$$N = \begin{bmatrix} -1.447 & 4.121 & 1.251 & 2.311 \\ 2.657 & -5.182 & 0.0546 & 5.423 \end{bmatrix},$$

with a corresponding $\lambda_{max}\{L(N)\} = 1.68 \times 10^{-3}$. Note that in each case, the maximum eigenvalue is very small in magnitude, but it is positive. As ρ is decreased, the eigenvalue decreases asymptotically to zero; however, some of the gains in N go to infinity. With the discontinuous part of the control choosen to be $\alpha(GCB)^{-1}s/\|s\|^2$, where $\alpha = 0.5$, the region in which the reaching condition holds is found from (18) to be approximately $\|x\| \leq 17$ for both cases. This region can be enlarged by decreasing ρ, thereby decreasing $\lambda_{max}\{L(N)\}$, or by increasing α.

A smoothing technique was used to reduce chattering, i.e. the discontinuous part of the control was replaced by $\alpha(GCB)^{-1}s/(\varepsilon + \|s\|_2)$ where $\varepsilon = 0.0005$. (Other common smoothing techniques can also be used.) Numerical simulations were performed to demonstrate the resulting system behavior. The response of the side slip angle (x_4) to an initial condition of $[0\,0\,0\,1\,0\,0\,0]^T$ is shown in Fig. 1 and the command to the aileron (u_2) is shown in Fig. 2. The gains obtained from each method yield similar plots for these graphs but do vary somewhat for other states. The value of s^Ts is plotted in Fig. 3 showing the quick decay of the reaching phase for the gain obtained from the Interior Point Method. (The gain found from the Cutting Plane Method produced virtually identical results.)

Interestingly, it was found in this example that as ρ is decreased, the direction of the eigenvector associated with the largest eigenvalue of $L(N)$ asymptotically approaches the surface $s = 0$. As a result, the components of x that are not on the surface are directed towards the surface. Moreover, as ρ increases, then N seems to approach a high gain term of the form G/ε, where ε is small and decreases as ρ increases. This correlates with the expression in (5), i.e. substitution of G/ε for N in (5) yields:

$$s^T\dot{s} = x^TC^TG^TGCAx - \frac{1}{\varepsilon}x^TC^TG^TGCx - \alpha\|s\|_1.$$

Thus, if there is a component of x that is not in the null space of GC, then the second term dominates the expression driving $s^T\dot{s}$ negative. If x is entirely in the null space of GC, then it is on the sliding surface.

5. *Conclusions*

Output feedback is much simpler to implement than either full state feedback or estimated state feedback; however, satisfying the reaching conditions had previously been a major obstacle in the use of output feedback in VSC. Two numerical methods are developed to design an output feedback gain matrix N that satisfies the reaching condition. Specifically, the reaching condition is met if a matrix $L(N)$ is negative semi-definite. Numerical optimization procedures are used to minimize the weighted average of the maximum eigenvalue of $L(N)$ and $\|N\|^r$, where $r = 1$ or 2. In this manner, the reaching condition can be satisfied without requiring an unduly large gain N. The two numerical methods differ in the selection of the norm, thereby providing some freedom in the design process. It should be noted that special numerical optimization algorithms must be used on this problem since the function to be minimized is not smooth. This problem was formulated such that the same optimization techniques used in the solution of linear matrix inequalities can be used. In the cases where $L(N)$ cannot be

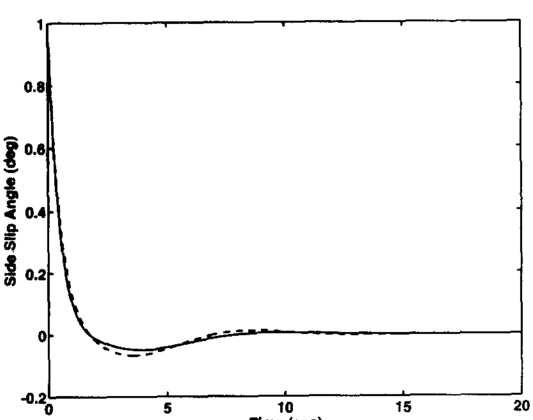

Fig. 1. Side slip angle (x_4) for the Interior Point result (solid line) and the Cutting Plane result (dashed line).

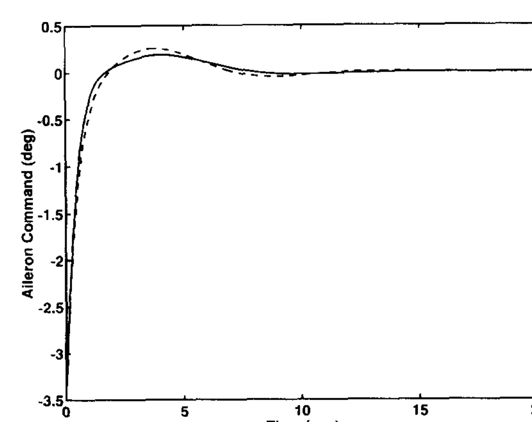

Fig. 2. Aileron command (u_2) for the Interior Point result (solid line) and the Cutting Plane result (dashed line).

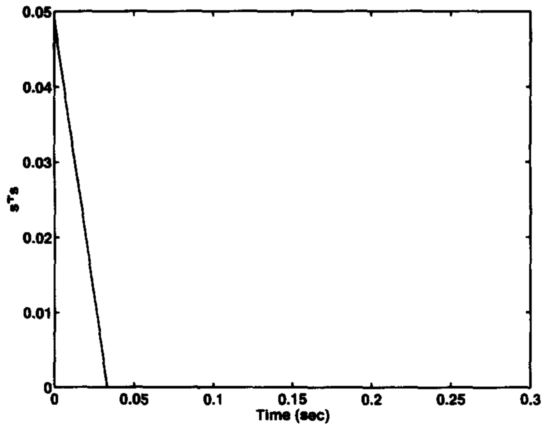

Fig. 3. Lyapunov function for the Interior Point result.

made negative semi-definite, the region in which the reaching condition holds can be made arbitrarily large by a proper selection of the weighting parameter ρ and by the additional gain parameter α.

Both of the developed control laws can be modified to be robust in the presence of bounded uncertainties and disturbances or nonlinearities. Specifically, it is shown that the modified control laws still satisfy the reaching conditions when the uncertainty, disturbance or nonlinearity is bounded in magnitude.

Acknowledgements—This work was supported by the National Science Foundation, contract ECS-9058140.

References

Andry Jr., A. N., E. Y. Shapiro and J. C. Chung (1983). Eigenstructure assignment for linear systems. *IEEE Trans. on Aerospace and Electronic Systems*, **AES-19**, 711–729.

Boyd, S. P. and L. El Ghaoui (1993). Method of centers for minimizing generalized eigenvalues. *Linear Algebra and its Applic.*, **188–189**, 63–112.

Diong, B. M. and J. V. Medanic (1992). Dynamic output feedback variable structure control for system stabilization. *Int. J. Control*, **56**, 607–630.

El-Khazali, R. and R. A. DeCarlo (1993). Output feedback variable structure control design using dynamic compensator for linear systems. In *Proc. of Am. Control Conf.*, San Francisco, CA, pp. 954–958.

El-Khazali, R. and R. A. DeCarlo (1992). Variable structure output feedback control. In *Proc. Am. Control Conf.*, Chicago, IL, pp. 871–875.

Fan, M. K. H. and B. Nekooie (1994). An algorithm on minimizing generalized eigenvalues with linear matrix inequality constraints. In *Proc. Am. Control Conf.*, pp. 831–835, Baltimore, MD.

Heck, B. S. and A. A. Ferri (1989). Application of output feedback for variable structure systems. *AIAA J. of Guidance, Control and Dynamics*, **12**, 932–935.

Kelly, J. E. (1960). The cutting plane method for solving convex programs. *J. Soc. Indust. Appl. Math.*, **8**, 703–712.

Khalil, H. (1992). *Nonlinear Systems*, pp. 202. Macmillan, New York.

Zak, S. H. and S. Hui (1993). On variable structure output feedback controllers for uncertain dynamic systems. *IEEE Trans. Autom. Control.*, **AC-38**, 1509–1512.