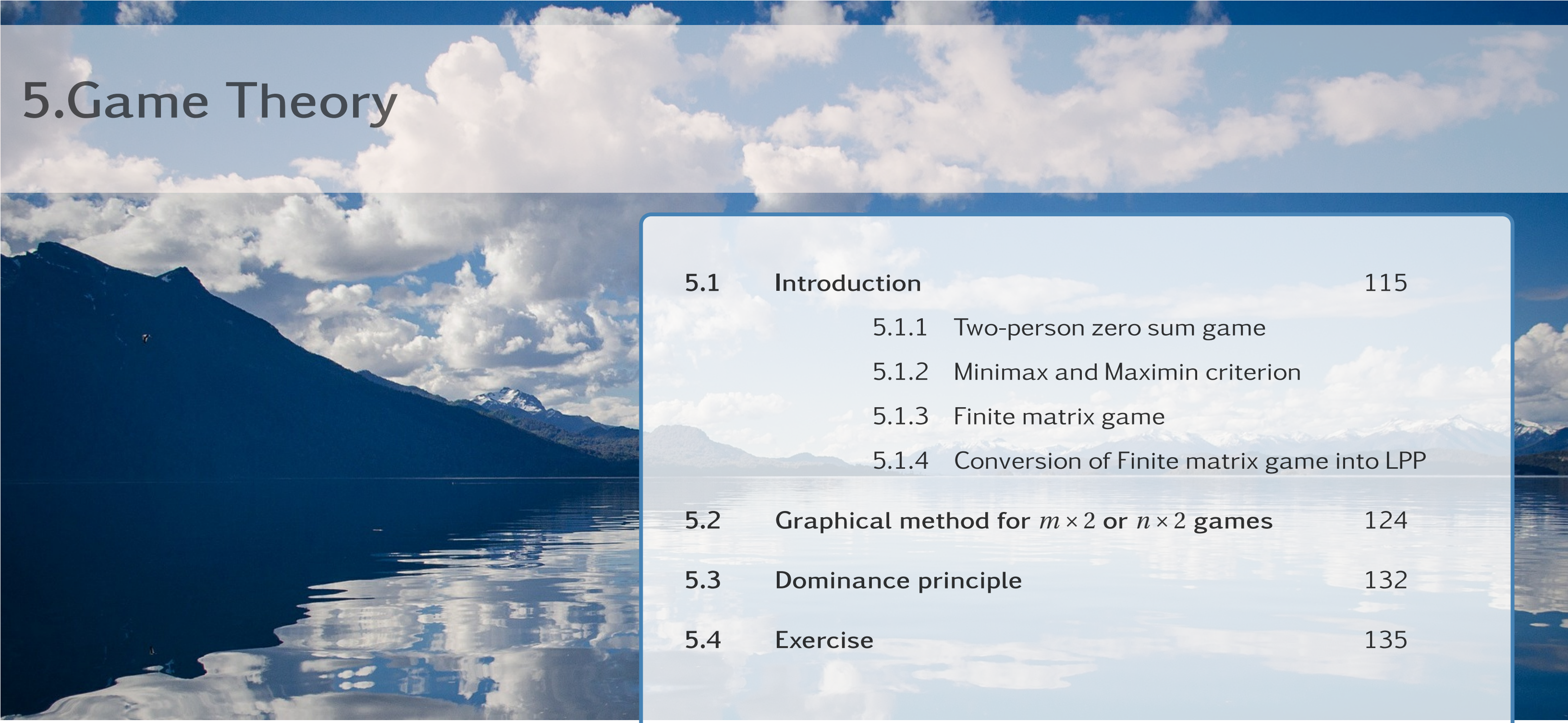

5.Game Theory

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5.1 Introduction

Definition 5.1 A complicating or competitive situation involving two or more participants is said to be a game. The participants are called the players.

Definition 5.2 A strategy for a player is a complete enumeration of all the action that they will take for every contingency that might arise.

Definition 5.3 The payoff is a connecting link between the set of strategies open to all players

Remark. Suppose, at the end of the play of a game, a player P_i $(i = 1, 2, \cdots, n)$ is expected to amount v_i called the pay-off to the player P_i. The pay-off v_i depends on the possible strategies adopted by all the n players. The total pay-off to all the n players. The total pay-off to all the n players in a plane of the game is equal the $\sum_{i=1}^{n} v_i$.

Definition 5.4 The game is called zero sum game if the total amount of the game is zero i.e.$\sum_{i=1}^{n} v_i = 0$ at end of the game.

In a zero-sum game the total amount of money lost by the losing players is equal to the total amount of money own by the winning players.

5.1.1 Two-person zero sum game

Suppose, at the end of the play of a game, a player P_i $(i = 1, 2, \cdots, n)$ is expected to amount v_i called the pay-off to the player P_i.

The pay-off v_i depends on the possible strategies adopted by all the n players. The total pay-off to all the n players. The total pay-off to all the n players in a plane of the game is equal to $\sum_{i=1}^{n} v_i$.

Definition 5.5 A two person zero sum game can be defined as the triplet $\{X, Y, K\}$, where X denotes the space of strategies for player I, Y presents the space of strategies for player II, and K is a real valued function, defined on $X \times Y$. We assume that X and Y are closed and bounded sets in $\mathbb{R}^m$ and $\mathbb{R}^n$. Also $K(x, y)$ is a continuous function of its argument.

If player I chooses a strategy $x \in X$ and player II chooses a strategy $y \in Y$, then $K(x, y)$ represents the pay-off to player I and $-K(x, y)$ is the pay-off to the player II. The pay-off function k is known as the pay-off kernel.

5.1.2 Minimax and Maximin criterion

Suppose that player I is conservative i.e. he believes that strategy $x^0 \in X$ is known to his opponent (player II), in this case the player II chooses his strategy $y^o \in Y$ such that

$$K(x^0, y^0) = \min K\ (x^0, y)$$

But for the maximum gain, the player I choose strategy $x^0 \in X$ such that

$$\min_y K(x^0, y) = \max_x \left[\min_y k(x, y)\right] = \underline{v} \tag{5.1}$$

where $\underline{v}$ is called the maximin or lower value of the game. By using (5.1), the strategy for player I can be determined. The player I can win at least value $\underline{v}$. Player I is a maximim player, since he uses a maximin criterion find his strategy.

Similarly, if player II is conservative then he will use strategy y^0 such that

$$\max_x \left(K, y^0\right) = \min_y \left[\max_x K\left(x, y\right)\right] = \overline{v} \tag{5.2}$$

where $\overline{v}$ is called the minimax or upper value of the game and it is interpreted as the most that player I can gain if player II uses the strategy y^0. Player II is a minimax player.

In this case player II is minimax player pair $\left(x^0, y^0\right) \in X \times Y$ is saddle point of the game. In general, $K\left(x, y^0\right) \leq K\left(x^0, y^0\right) \leq K\left(x^0, y\right)$ for all $x \in X$ and $y \in Y$.

Theorem 5.1 If X and Y be the closed and bounded sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, respectively. Further, suppose the pay-off Kernel $K\left(x,\ y\right)$ is a continuous function then $\overline{v} \geq \underline{v}$.

Proof. Clearly,

$$\begin{aligned} K\left(x, y\right) &\leq \max K\left(x, y\right) \\ \Rightarrow \min_y\ K\left(x, y\right) &\leq \max_x K\left(x, y\right) \\ \Rightarrow \min_y K\left(x, y\right) &\leq \min_y \max_x K\left(x, y\right) \end{aligned}$$

Thus,

$$\max_x \min_y K\left(x, y\right) \leq \min_y \max_x K\left(x, y\right).$$

Hence $\overline{v} \geq \underline{v}$.

5.1.3 Finite matrix game

Let

$$X = S_m = \left\{ x = \left(\underline{x}_1, \underline{x}_2, \ldots, x_m\right)^T, x_i \geq 0\ for\ all\ i\ and \sum_{i=1}^{m} x_i = 1 \right\},$$

$$Y = S_n = \left\{ y = \left(y_1, y_2, \ldots, y_n\right)^T, y_i \geq 0\ for\ all\ i\ and \sum_{i=1}^{n} y_i = 1 \right\},$$

$$K\left(x,\ y\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j = x^T A y,$$

where $A = \left(a_{i,j}\right)$ is given matrix of real number called the pay of matrix. The triplet $\{S_m, S_n, K\}$ is called the finite matrix game, when m and n are finite.

A strategy for player I is a probability m vector $x = \left(\underline{x}_1, \underline{x}_2, \cdots, x_m\right)^T$ and a strategy for player II is a probability n vector $y = \left(y_1, y_2, \cdots, y_n\right)^T$.

Definition 5.6 A strategy $x = \left(\underline{x}_1, \underline{x}_2, \cdots, x_m\right)^T$ for player I is called pure strategy if all $x_i = 0$ except one which is unity. Thus, there are m pure strategies i.e. $\{(1,0,\cdots,0),\ (0,1,\cdots,0),\ \cdots,\ (0,0,\cdots,1)\}$.

Definition 5.7 A strategy is called mixed strategy which is not pure strategy.

Payoff. Suppose that at any move there exist m choices for player I and n choices for player II. Let x_i be the probability that player I uses the i^{th} choice, i.e. x_i is the probability that player I uses his pure strategy α_i. Similarly, let y_j be the probability that player II uses the pure strategy β_j (j^{th} choice). Suppose $a_{i,j}$ is the pay-off to player I when players I and II use the pure strategies α_i and β_j respectively. Then, the expected (average) pay-off to player I, when player II uses his pure strategy β_j is

$$E\left(x, \beta_j\right) = \sum_{i=1}^{m} a_{i,j}\ x_i.$$

The expected pay-off to the player I when player II uses the strategy $y = \left(y_1, y_2, \cdots, y_n\right)^T$ is given by

$$E\left(x, y\right) = \sum_{j=1}^{n} y_j\ E\left(x, \beta_j\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{i,j}\ x_i y_j$$

Which is the pay-off for the kernel for the matrix game $A = \left(a_{i,j}\right)$.

5.1.4 Conversion of Finite matrix game into LPP

Consider an $m \times n$ game with the pay-off $A = \left(a_{ij}\right)$. Let $x=(x_1, \cdots, x_m)^T \in S_m$ and $y=\left(y_1, \cdots, y_n\right)^T \in S_n$ be the strategies for players I and II, respectively. Suppose that both the players play conservatively. Then, player I chooses x which maximizes

$$\min_{y \in S_n} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j \right)$$

since every strategy $y \in S_n$ is a convex combination of n pure strategies for player II, we

$$\min_{y \in S_n} \left(\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j \right) = \min_{1 \le j \le n} \left(\sum_{i=1}^{m} a_{ij} x_i \right) \quad for\ all\ x \in S_m. \tag{5.3}$$

Thus, the problem of player I is to find an $x \in S_m$ which maximizes expression (5.3) equivalent to the linear program:

(A) Maximize (L)

subject to

$$\sum_{i=1}^{m} a_{ij} x_i \geq L \quad \big(j = 1, \cdots, n\big)$$
$$\sum_{i=1}^{m} x_i = 1,$$

where L is unrestricted in sign and $x_i \geq 0,\ (i = 1, \cdots, m)$.

Similarly, player II chooses a strategy $y \in S_n$ that minimizes

$$\max_{x \in S_m} \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j = \max_{i \leq i \leq m} \left(\sum_{j=1}^{n} a_{ij} y_j \right)$$

This is equivalent to the linear program:

(B) Minimize (M)

subject to

$$\sum_{j=1}^{n} a_{ij} y_j \leq M \quad (i = 1, \cdots, m)$$
$$\sum_{j=1}^{n} y_j = 1,$$

where M is unrestricted in sign and $y_j \geq 0, (j = 0, \cdots, n)$.

Since the strategy spaces S_m and S_n are non-empty, the linear programs (A) and (B) have feasible solutions. Also, these are a pair of dual linear programs. Hence, by the duality theory, there exist optimal solutions to (A) and (B), we have, $\min(M) = \max(L)$.

Reformulation of Program (A) and (B): Consider the programs (L) and (M) and we restrict L and M of programs (A) and (B) to positive values. Define the new variables x_i' and y_i' by

$$x_i' = \frac{x_i}{L}, \quad y_i' = \frac{y_j}{M} \quad \big(i = 1, \cdots, m, j = 1, \cdots, n\big)$$

Then, the linear programming problems we obtain from (A) and (B) are:

(A') Minimize $\frac{1}{L} = \sum_{i=1}^{m} x_i'$

subject to

$$\sum_{i=1}^{n} a_j x'_i \geq 1 \quad \left(j = 1, \cdots, n\right)$$
$$x_i \geq 0 \quad (i = 1, \cdots, m).$$

(B') Maximize $\frac{1}{M} = \sum_{j=1}^{n} y'_j$

subject to

$$\sum_{j=1}^{n} a_i y'_j \leq 1 \quad (i = 1, \cdots, m)$$
$$y'_j \geq 0 \quad \left(j = 1, \cdots, n\right).$$

The pair of dual linear programs $\left(A'\right)$ and $\left(B'\right)$ is equivalent to programs (A) and (B), respectively. The optimal value of these two programs is $\frac{1}{v}$.

Example 5.1 Use linear programming formulation to solve the game with pay-off matrix

$$A = \begin{pmatrix} 1 & 0 & 4 & -1 \\ -1 & 1 & -2 & 5 \end{pmatrix}.$$

Solution. Here, min $a_{i,j} = -2$. Therefore, we add 3 to each element of A and get a modified pay-off matrix as follows

$$B = \begin{pmatrix} 4 & 3 & 7 & 2 \\ 2 & 4 & 1 & 8 \end{pmatrix}.$$

The corresponding LPP is

$$\text{Minimize } z = -\frac{1}{M} = -y'_1 - y'_2 - y'_3 - y'_4$$

subject to

$$\begin{aligned} 4y'_1 + 3y'_2 + 7y'_3 + 2y'_4 + s_1 &= 1, \\ 2y'_1 + 4y'_2 + y'_3 + 8y'_4 + s_2 &= 1, \\ y'_i,\ s_j \geq 0,\ i = 1,2,3,4,&\ j = 1,2. \end{aligned}$$

Now, we solve this LPP by simplex method.

			-1	-1	-1	-1	0	0	
CB	B. V	b	y_1'	y_2'	y_3'	y_4'	s_1	s_2	Min +ve ratio
0	$\leftarrow s_1$	1	4	3	7	2	1	0	$\frac{1}{4}$
0	s_2	1	2	4	1	8	0	1	$\frac{1}{2}$
	$\overline{z_j}=0$	$\overline{c_j}$	$-1\uparrow$	-1	-1	-1	0	0	
-1	y_1'	$\frac{1}{4}$	1	$\frac{3}{4}$	$\frac{7}{4}$	$\frac{1}{2}$	$\frac{1}{4}$	0	$\frac{1}{3}$
0	$\leftarrow s_2$	$\frac{1}{4}$	0	$\frac{5}{2}$	$-\frac{5}{2}$	7	$-\frac{1}{2}$	1	$\frac{1}{5}$
	$\overline{z_j}=-\frac{1}{4}$	$\overline{c_j}$	0	$-\frac{1}{4}$	$\frac{3}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	0	
-1	$y_1{}'$	$\frac{1}{10}$	1	0	$\frac{5}{2}$	$-\frac{8}{5}$	$\frac{2}{5}$	0	
-1	$y_2{}'$	$\frac{1}{5}$	0	1	-1	$\frac{14}{5}$	$-\frac{1}{5}$	$\frac{2}{5}$	
	$\overline{z_j}=-\frac{3}{10}$	$\overline{c_j}$	0	0	$\frac{1}{2}$	$\frac{1}{5}$	$\frac{1}{5}$	$\frac{1}{10}$	

By observing all $\overline{c_j} \geq 0$. The optimal solution is

$$y_1' = \frac{1}{10},\ y_2' = \frac{1}{5},\ y_3' = 0,\ y_4' = 0.$$

with minimum $z = -\frac{1}{M} = -\frac{3}{10}$. So, maximum $M = \frac{10}{3}$. The optimal strategy for player II, is (y_1, y_2, y_3, y_4), where $y_j' = \dfrac{y_j}{M}$. Thus $y_1 = y_1'M = \dfrac{1}{10} \times \dfrac{10}{3} = \dfrac{1}{3}$, $y_2 = y_2'M = \dfrac{1}{5} \times \dfrac{10}{3} = \dfrac{2}{3}$, $y_3 = y_4 = 0$. The value of the game is $L = M - 3 = \dfrac{10}{3} - 3 = \dfrac{1}{3}$.

The optimal strategy for player I is $x_1' = \dfrac{1}{5}$, $x_2' = \dfrac{1}{10}$. (see green values in table). Since $x_j' = \dfrac{x_i}{M}$, so $\underline{x}_1 = x_1'M = \dfrac{1}{5} \times \dfrac{10}{3} = \dfrac{2}{3}$ and $\underline{x}_2 = x_2'M = \dfrac{1}{10} \times \dfrac{10}{3} = \dfrac{1}{3}$.

Thus, the optimal strategy for player I is $\left(\frac{2}{3}, \frac{1}{3}\right)$.

Example 5.2 Solve the game,

$$A = \begin{pmatrix} 2 & 7 \\ 4 & 5 \\ 10 & 4 \end{pmatrix}.$$

Solution. We first convert this game in the equivalent LPP as follows

Minimize $z = -\frac{1}{M} = -y_1' - y_2'$

subject to

$$\begin{aligned} 2y_1' + 7y_2' &\leq 1, \\ 4y_1' + 5y_2' &\leq 1, \\ 10y_1' + 4y_2' &\leq 1, \\ y_1',\ y_2' &\geq 0. \end{aligned}$$

We solve it by simplex method. For this we convert in SLP

Minimize $z = -\frac{1}{M} = -y_1' - y_2'$

subject to

$$\begin{aligned} 2y_1' + 7y_2' + s_1 &= 1, \\ 4y_1' + 5y_2' + s_2 &= 1, \\ 10y_1' + 4y_2' + s_3 &= 1, \\ y_1',\ y_2', s_1, s_2, s_3 &\geq 0. \end{aligned}$$

		-1	-1	0	0	0		
C.B.	B. V.	y_1'	y_2'	s_1	s_2	s_3	b	Minimum positive ratio
0	s_1	2	7	1	0	0	1	$\frac{1}{2}$
0	s_2	4	5	0	1	0	1	$\frac{1}{4}$
0	$\leftarrow s_3$	10	4	0	0	1	1	$\frac{1}{10}$
	$\overline{z_j} =$	$-1\uparrow$	-1	0	0	0	$\frac{4}{5}$	
0	$\leftarrow s_1$	0	$\frac{31}{5}$	1	0	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{4}{31}$
0	s_2	0	$\frac{17}{5}$	0	1	$-\frac{2}{5}$	$\frac{1}{10}$	$\frac{3}{17}$
-1	y_1'	1	$\frac{2}{5}$	0	0	$\frac{1}{10}$	$-\frac{1}{10}$	$\frac{1}{4}$
	$\overline{z_j} =$	0	$-\frac{3}{5}\uparrow$	0	0	$\frac{1}{10}$	$\frac{1}{10}$	
-1	y_2'	0	1	$\frac{5}{31}$	0	$-\frac{1}{31}$	$\frac{4}{31}$	
0	s_2	0	0	$-\frac{17}{31}$	1	$-\frac{9}{31}$	$\frac{5}{31}$	
-1	y_1'	1	0	$-\frac{2}{31}$	0	$\frac{7}{62}$	$\frac{3}{62}$	
	$\overline{z_j} =$	0	0	$\frac{3}{31}$	0	$\frac{5}{62}$	$\frac{5}{62}$	

The optimal solution is $y_1' = \frac{3}{62}$, $y_2' = \frac{4}{31}, s_1 = 0, s_2 = \frac{5}{31}$, $s_3 = 0$ with minimum $z = -\frac{11}{62}$. So, Maximum $M = \frac{62}{11}$.

The optimal strategy for player II, is $(y_1, \ y_2)$, where $y_j' = \frac{y_j}{M}$. Thus

$$y_1 = y_1'M = \frac{3}{62} \times \frac{62}{11} = \frac{3}{11}, \gamma_2 = \gamma_2'M = \frac{4}{31} \times \frac{62}{11} = \frac{8}{11}, y_3 = y_4 = 0.$$

The value of the game is $L = M = \frac{62}{11}$. The optimal strategy for player I: $x_1' = \frac{3}{31}$, $x_2' = 0, x_3' = \frac{5}{62}$. Since $x_j' = \frac{x_i}{M}$, so $x_1 = x_1'M = \frac{3}{31} \times \frac{62}{11} = \frac{6}{11}$, $x_2 = x_2'M = 0$ and $x_3 = \frac{5}{62} \times \frac{62}{11} = \frac{5}{11}$.

Thus, the optimal strategy for player I is $\left(\frac{6}{11}, 0, \frac{5}{11}\right)$.

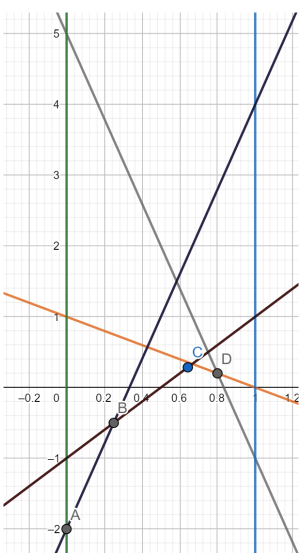
5.2 Graphical method for $m \times 2$ or $n \times 2$ games

Example 5.3 Solve graphically the 2×4 rectangular game with pay off matrix

$$A = \begin{pmatrix} 1 & 0 & 4 & -1 \\ -1 & 1 & -2 & 5 \end{pmatrix}$$

Solution. We plot the lines,

$$l_1(x) = x - (1 - x) = 2x - 1$$
$$l_2(x) = (1 - x)$$
$$l_3(x) = 4x - 2(1 - x) = 6x - 2$$
$$l_4(x) = 5 - 6x.$$



where, $0 \le x \le 1$ and $(x,\ 1-x)^T$, is the strategy for row player I. The $l_i(x)$ is the average pay off to player I, when he uses this strategy and player II uses the pure strategy β_j. Let

$$\phi(x) = \min_{1 \le x \le 4} l_i(x)$$

This function $\phi(x)$ is shown in figure, by the curve ABCD. The point C is maximum and it is intersection of l_1 and l_2 which gives the point x_0 that maximizes $\phi(x)$. Since at C, we have

$$l_1 = 2x - 1 = 1 - x = l_2$$

$2x - 1 = 1 - x \Rightarrow 3x = 2$. It gives $x = \frac{2}{3}$. Consequently $1 - x = 1 - \frac{2}{3} = \frac{1}{3}$. Hence, the optimal strategy for player I is $(x, 1-x)^T = \left(\frac{2}{3}\ ,\ \frac{1}{3}\right)^T$. The value of the game $= \max \phi(x) = 2x - 1 = 2\left(\frac{2}{3}\right) - 1 = \frac{4}{3} - 1 = \frac{1}{3}$.

The optimal strategy for player II, is

$$\begin{aligned}
w_0(l_1(x)) + (1-w_0)(l_2(x)) &= \frac{1}{3}\\
w_0(2x-1) + (1-w_0)(1-x) &= \frac{1}{3}\\
2xw_0 - w_0 + 1 - w_0 - x + w_0 x &= \frac{1}{3}\\
3w_0 x - x + 1 - 2w_0 &= \frac{1}{3}\\
(3w_0-1)x + (1-2w_0) &= \frac{1}{3}
\end{aligned}$$

On comparing, $3w_0 - 1 = 0$ or $1 - 2w_0 = \frac{1}{3}$. It gives that $w_0 = \frac{1}{3}$. So, $1 - w_0 = 1 - \frac{1}{3} = \frac{2}{3}$. Thus, the optimal strategy for player II is $\left(\frac{1}{3}, \frac{2}{3}, 0, 0\right)$.

Example 5.4 Solve, the game by graphically

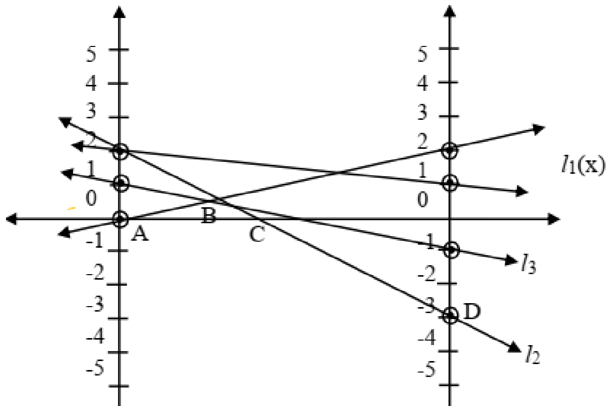
$$B = \begin{pmatrix} 2 & -3 & -1 & 1 \\ 0 & 2 & 1 & 2 \end{pmatrix}.$$

Solution. We plot the lines

$$\begin{aligned}
l_1(x) &= 2x + 0 = 2x\\
l_2(x) &= -3x + 2(1-x) = -3x = 2(1-x) = 2 - 5x\\
l_3(x) &= -x + 1 - x = 1 - 2x\\
l_4(x) &= x + 2(1-x) = 2 - x.
\end{aligned}$$

Here, $0 \le x \le 1$ and $(x,\ 1-x)^T$ is the strategy for players I. We plot the graph $\phi(x)$ is the curve ABCD and $\phi(x)$ attains its maximum value at B. The point B is the intersection of line $l_1(x)$ and $l_2(x)$ and $(\beta(x)$ attains its maximum value at B. Thus, at B, we have $2x = 1 - 2x \Leftrightarrow 4x = 1 \Leftrightarrow x = \frac{1}{4}$. Hence $1 - x = 1 - \frac{1}{4} = \frac{3}{4}$.

Thus, the optimal strategy for player I is $\left(\frac{1}{3}, \frac{1}{4}\right)$. The value of the game, $v = 2x = 2 \times \frac{1}{4} = \frac{1}{2}$.



Figure

The optimal strategy for player II is the convex combination of $l_1(x)$ and $l_3(x)$. So

$$\begin{aligned} w_0 l_1(x) + (1-w_0) l_2(x) &= v \\ \Rightarrow 2x w_0 + (1-w_0)(1-2x) &= \frac{1}{2} \\ (1-w_0) + (4w_0 - 2)x &= \frac{1}{2}. \end{aligned}$$

On comparing, we have $1 - w_0 = \frac{1}{2}$ or $4w_0 - 2 = 0$. Hence $w_0 = \frac{1}{2}$ and $1 - w_0 = \frac{1}{2}$. The optimal strategy for player II is $\left(\frac{1}{2}, 0, \frac{1}{2}, 0\right)$.

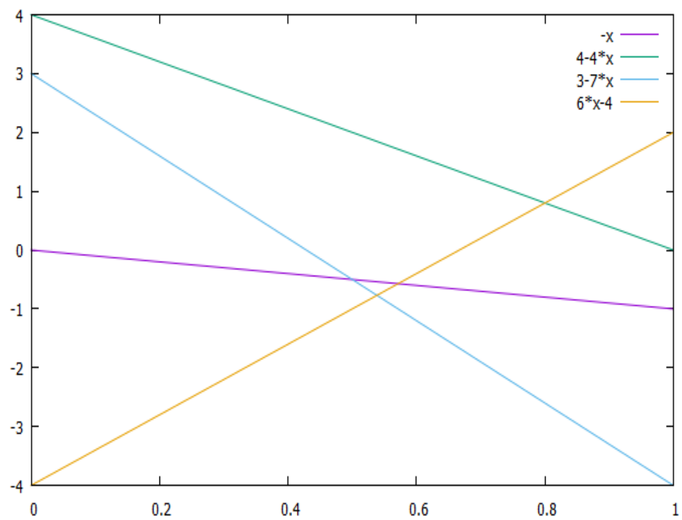
Example 5.5 Solve the game by graphically

$$\begin{pmatrix} -1 & 0 \\ 0 & 4 \\ -4 & 3 \\ 2 & -4 \end{pmatrix}$$

Solution: The lines are

$$\begin{aligned} l_1(x) &= -x \\ l_2(x) &= 4(1-x) = 4 - 4x \\ l_3(x) &= -4x + 3(1-x) = 3 - 7x \\ l_4(x) &= 2x + (1-x)(-4) = 2x - 4 + 4x = 6x - 4. \end{aligned}$$

Here $0 \le x \le 1$ and $(x,\ 1-x)^T$ is the strategy for player II, we plot the graph



The point B is the intersection of line $l_2(x)$ and $l_4(x)$, at B, the $\phi(x)$ attains its minimum value. At $B, l_2(x) = l_4(x)$ i.e. $4-4x = 6x-4 \Rightarrow 8 = 10x \Rightarrow x = \frac{4}{5}$. Hence $1-x = 1-\frac{4}{5} = \frac{1}{5}$. Thus the optimal strategy for player II is $(\frac{4}{5}, \frac{1}{5})^T$. The value of the game $v = 4-4x|_{x=\frac{4}{5}} = 4-4\times\frac{4}{5} = \frac{4}{5}$.

The optimal strategy for player I is the convex combination of line $l_2(x)$ and $l_4(x)$, hence it is convex combination of $l_2(x)$ and $l_4(x)$,

$$\begin{aligned}
w_0(4-4x)+(1-w_0)l_4(x) &= \frac{4}{5}\\
w_0(4-4x)+(1-w_0)(6x-4) &= \frac{4}{5}\\
\Rightarrow 4w_0-4xw_0+6x-4-6xw_0+4w_0 &= \frac{4}{5}\\
\Rightarrow 8w_0-10xw_0+6x-4 &= \frac{4}{5}\\
\Rightarrow (6-10w_0)x+(8w_0-4) &= \frac{4}{5}
\end{aligned}$$

On comparing, $6-10w_0 = 0$ or $8w_0-4 = 0$. It yield $w_0 = \frac{3}{5}$, so $1-x = 1-\frac{3}{5} = \frac{2}{5}$. Therefore $(\frac{3}{5}, 0, 0, \frac{2}{5})$ be the optimal strategy for player I.

Definition 5.8 A matrix game is symmetric if the number of pure strategies available to the both players is same and the pay-off matrix is skew symmetric i.e. $A^T = -A$.

In the symmetric game, the strategies S_m and S_n are equal, since $n = m$. The symmetric game is always a fair game. A game is said to fair if its value is zero.

Theorem 5.2 Every symmetric game has the value $v = 0$ and each player has the same set of optimal strategies.

Proof. Suppose A is skew-symmetric, then $A^T = -A$, and hence

$$x^T Ax = -x^T Ax = 0 \quad for\ all\ x \in S_m. \tag{5.4}$$

Let $x^0 \in S_m$ and $y^0 \in S_n$ be the optimal strategies for player I and II respectively. Then using above equation,

$$0 = y^{0^T} Ay^0 \le v \le x^{0^T} Ax^0 = 0$$

so that $v = 0$. Since x^0 is the optimal strategy for player I, therefore

$$0 = v = x^{0^T} Ax^0 \le x^{0^T} Ax \quad for\ all\ x \in S_m. \tag{5.5}$$

Taking the transpose of the terms in (5.5) and using the skew-symmetry of A, we have

$$x^T Ax^0 \le 0 \quad for\ all\ x \in S_m. \tag{5.6}$$

From (5.5) and (5.6), we get

$$x^T Ax^0 \le 0 = x^{0^T} Ax^0 \le x^{0^T} Ax \quad for\ all\ x \in S_m.$$

It shows that $(x^0,\ x^0)$ is the saddle point of $E(x,\ y) = x^T Ax$. Thus, the both players can use the same optimal strategy.

Theorem 5.3 Let E be the pay off Kernel of an $m \times n$ matrix game whose value in v. Then a necessary and sufficient condition for $x^0 \in S_m$ to be an optimal strategy for player I is

$$v \le E(x^0, y) \quad for\ all\ y \in S_n.$$

Similarly, a necessary and sufficient condition for a $y^0 \in S_n$ be an optimal strategy for player II is

$$E(x,\ y^0) \le v \quad for\ all\ x \in S_m.$$

Proof. Suppose x^0 is an optimal strategy for player I, then there exists a $y^0 \in S_n$ such that $(x^0,\ y^0)$ is a saddle point of $E(x,\ y)$. Hence

$$v = E(x^0,\ y^0) \le E(x^0, y) \quad \text{for all } \in S_n.$$

Conversely, suppose

$$v \le E(x^0, y) \quad for\ all\ y \in s_n. \tag{5.7}$$

Since every finite matrix game has a solution. Suppose the solution of the game is (x', y') . Then (x', y') is a saddle point of $E(x,\ y)$. Hence for all x and y, we have

$$E(x, y') \le E(x',\ y') \le E(x', y) \tag{5.8}$$

$$v = E(x', y') \tag{5.9}$$

Taking $y = y'$ in (5.7), we have

$$v = E(x', y') \le E(x^0, y') \tag{5.10}$$

Taking $x = x^0$ in (5.8) and then comparing it with inequality (5.10), we obtain

$$v = E(x', y') = E(x^0, y').$$

Therefore, we have

$$E(x,\ y') \le E(x^0,\ y') \le E(x^0,\ y).$$

for all $x \in S_m$ *and* $y \in S_n$. This means that $(x^0,\ y')$ is a saddle point of $E(x, y)$. Hence x^0 is an optimal strategy for player I. Similarly, we can show $y' = y^0$ optimal strategy for player II.

Theorem 5.4 Let E be pay-off kernel of an $m \times n$ matrix game whose value is v. Then necessary and sufficient condition for $x^0 \in S_m$ to be an optimal strategy for player I is

$$v \le E(x^0, \beta_j),\ j = 1, 2,\ \cdots, n.$$

Similarly, a necessary and sufficient condition for $y^0 \in S_n$ to be an optimal strategy for player II is

$$E(\alpha_i, y^0) \le v,\ i = 1, 2, \cdots,\ m.$$

Proof. First assume that x^0 is an optimal strategy for the player I. Put $y = \beta_j$ in the first part of the Theorem 5.3, we get

$$v \le E(x^0, \beta_j),\ j = 1, 2, \ , \ n. \tag{5.11}$$

Suppose (5.11) is true. Then for $y \in S_n$, we have

$$E(x^0, y) = \sum_{j=1}^{n} y_j E(x^0,\ \beta_j) \geq v \sum_{j=1}^{n} y_j = v.$$

Using Theorem 5.3, it follows that x^0 is an optimal strategy for the player I. Similarly, we can prove for the player II.

Theorem 5.5 A necessary and sufficient condition for v to be the value of the game an $m \times n$ matrix game and $x^0 \in S_m$ and $y^0 \in S_n$ to be the optimal strategy for player I and II respectively is

$$E(\alpha_j, y^0) \leq v \leq E(x^0, \beta_j)\ ,\ i = 1, \cdots,\ m,\ j = 1, \cdots,\ n. \tag{5.12}$$

Proof. Let x^0 and y^0 be the optimal strategy for player I and II respectively. By theorem 5.4, we have

$$E(\alpha_j, y^0) \leq v \leq E(x^0, \beta_j)\ ,\ i = 1, \cdots,\ m,\ j = 1, \cdots,\ n.$$

Suppose (5.12) hold, then we have

$$E(x, y^0) = \sum_{i=1}^{m} x_i E(\alpha_i, y^0) \leq v \sum_{i=1}^{m} x_i = v. \tag{5.13}$$

$$E(x^0, y) = \sum_{j=1}^{n} \gamma_j E(x^0, \beta_j) \geq v \sum_{j=1}^{n} y_j = v. \tag{5.14}$$

Put $x = x^0$ and $y = y^0$ in (5.13) and (5.14), it follows that (x^0, y^0) is a saddle point of $E(x,\ y)$. Hence x^0 and y^0 be the optimal strategy for player I and II respectively.

Theorem 5.6 In a finite matrix game, the set of all optimal strategies for each player is convex and closed.

Proof. Let v be the value of the game. Suppose that x_1^0 and x_2^0 are two optimal strategies for player I. From Theorem 5.3, we have $v \leq E(x_1^0,\ y)$ and $v \leq E(x_2^0,\ y)$ for all strategies $y \in S_n$.

Thus, for each $y \in S_n$ and $0 \leq \lambda \leq 1$, we have

$$E(\lambda x_1^0 + (1-\lambda)x_2^0, y) = \lambda E(x_1^0, y) + (1-\lambda)E(x_2^0, y) \geq v.$$

Hence, $\lambda x_1^0 + (1-\lambda)x_2^0$ is also an optimal strategy for player I and hence the set of all optimal strategies is convex. Further the set of optimal strategies is closed because $E(x,\ y)$ is continuous function x. The proof for player II is similar.

Theorem 5.7 Let E be the pay off kernel of an $m \times n$ matrix game whose value is v. Then a necessary and sufficient condition for $x^0 \in S_n$ to be an optimal strategy for player I is

$$v \le E(x^0, \beta_j), j = 1, 2, \cdots, n.$$

Proof. Suppose

$$E(x^0,\ y) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j = \sum_{j=1}^{n}\sum_{i=1}^{m} a_{ij}x_i y_j = \sum_{j=1}^{n} E(x^0,\ \beta_j)y_j \ge v\sum_{j=1}^{n} y_j = v.$$

Thus x^0 is an optimal strategy or player I.

Suppose y^0 is the optimal strategy for player II. By taking $x = \alpha_i$ in the theorem, we have

$$v \ge E(\alpha_i, y^0),\ j = 1, 2, \cdots,\ m. \tag{5.15}$$

Now, suppose (5.15) holds, then for $x \in S_m$, we have

$$E(x,\ \gamma^0) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_i y_j = \sum_{i=1}^{m} E(\alpha_i, y^0)x_i \le v\sum_{i=1}^{m} x_i = v.$$

Thus, y^0 is an optimal strategy for the player II.

Example 5.6 Solve the $m \times n$ rectangular game whose pay off matrix is $A = (a_{ij})$, where

$$a_{ij} = \begin{cases} +1\ if\ i \ne j \\ -1\ if\ i = j \end{cases}$$

Solution. Suppose $x^0 = (x_1^0,\ x_2^0, \cdots, x_m^0)^T$ and $y^0 = (y_1^0,\ y_2^0, \cdots,\ y_n^0)^T$ be optimal strategies for player I and II respectively. By Theorem 5.7, we have

$$E(\alpha_i, y^0) \le v \le E(x^0,\ \beta_j)\ ,\ i, j = 1, \cdots,\ m.$$

Now,

$$E(\alpha_i,\ y^0) = \sum_{j=1}^{m} a_{ij}y_j^0 = -y_i^0 + \sum_{j \ne i} y_j^0 = 1 - 2y_i^0$$
$$E(x^0,\ \beta_j) = \sum_{i=1}^{m} a_{ij}x_i^0 = -x_j^0 + \sum_{i \ne j} x_i^0 = 1 - 2X_j^0.$$

Hence, the necessary and sufficient conditions reduce to

$$1 - 2y_i^0 \le v \le 1 - 2x_j^0 \ (i, j = 1, \cdots, \ m).$$

Summing the first inequality over i and the second over j, we get

$$m - 2 \le mv \le m - 2, \quad i.e. \ \ v = \frac{m-2}{m}.$$

Thus, we have

$$y_i^0 \ge \frac{1}{m}, \quad x_j^0 \le \frac{1}{m} \ \ for \ \ i, j = 1, \cdots, m,$$

since x^0 and y^0 are required to be probability m vectors, we obtain

$$y_i{}^0 = \frac{1}{m}, \ x_i{}^0 = \frac{1}{m} \ (i = 1, \cdots, m).$$

This means that an optimal strategy for each player is to make each of the choices 1 to m with equal probability. The expected pay-off to player I is $v = \dfrac{m-2}{m}$.

5.3 Dominance principle

Example 5.7 Solve the game by dominance principle

$$\begin{pmatrix} 1 & 6 & 3 \\ 4 & 2 & 5 \\ 2 & 1 & 4 \end{pmatrix}.$$

Solution. There are three pure strategies available for player I, among these players I always choose pure strategy than inferior strategy. So, second row is best strategy than the third row, therefore, we delete the third row from the payoff matrix, we get the reduced pay off matrix as follows

$$\begin{pmatrix} 1 & 6 & 3 \\ 4 & 2 & 5 \end{pmatrix}.$$

By observing, the first column dominated the third column. Thus, we delete the third column. We have pay-off, matrix as follows

$$\begin{pmatrix} 1 & 6 \\ 4 & 2 \end{pmatrix}.$$

Suppose $(x,\ 1-x)^T$ be the optimal strategy for player I. Thus, we have

$$x+4(1-x)=6x+2(1-x) \Rightarrow x=\frac{2}{7}.$$

Consequently, $1-x=1-\frac{2}{7}=\frac{5}{7}$. Thus the optimal strategy for player I is $(\frac{2}{7},\ \frac{5}{7}\ ,\ 0)^T$ Suppose $(y,\ 1-y)^T$ be the optimal strategy for player II. Thus, we have

$$y+6(1-y)=4y+2(1-y) \Rightarrow y=\frac{4}{7}.$$

$1-x=1-\frac{4}{7}=\frac{3}{7}$. Thus $\left(\frac{4}{7},\frac{3}{7},0\right)$ is optimal strategy fir the player II. The value of the game is

$$v=E(x)=x-4(1-x)=4-3x=4-3\times\frac{2}{7}=\frac{22}{7}.$$

Example 5.8 Solve 4×3 game with the pay-off matrix

$$A=\begin{pmatrix} 8 & 5 & 8 \\ 8 & 6 & 5 \\ 7 & 4 & 5 \\ 6 & 5 & 6 \end{pmatrix}.$$

Solution. There are four pure strategies available. Here, row I is best strategy than the row II. Therefore, we delete the III row from pay off matrix. We get the reduced pay off matrix as follows

$$A=\begin{pmatrix} 8 & 5 & 8 \\ 8 & 6 & 5 \\ 6 & 5 & 6 \end{pmatrix}.$$

By observing , column I dominated by the column III. Thus, we remove the column I form pay-off matrix and we have

$$A=\begin{pmatrix} 5 & 8 \\ 6 & 5 \\ 5 & 6 \end{pmatrix}.$$

Also, row I is dominated by row III. Thus, we delete row III, we have reduced pay off matrix as follows

$$A=\begin{pmatrix} 5 & 8 \\ 6 & 5 \end{pmatrix}.$$

Let $(x,\ 1-x)^T$ be optimal strategy for the player I, then we have

$$5x+6(1-x)=8x+5(1-x)\Rightarrow x=\frac{1}{4}.$$

Also $1-x=1-\frac{1}{4}=\frac{3}{4}$. Hence $\left(\frac{1}{4},\frac{3}{4},0,0\right)$ be the optimal strategy for the player I.

Similarly, we assume $(y,\ 1-y)^T$ is optimal strategy for player II. We have

$$5y+8(1-y)=6y+5(1-y)\Rightarrow y=\frac{3}{4}.$$

Hence $1-y=1-\frac{3}{4}=\frac{1}{4}$. Therefore $\left(0,\ \frac{3}{4},\ \frac{1}{4}\right)$ be optimal strategy for the player II. The value of the game is $v=\frac{1}{4}\times 5+\frac{3}{4}\times 6=\frac{23}{4}$.

Example 5.9 Use dominance relations to solve the following matrix game

$$\begin{pmatrix} 2 & 1 & 3\\ 3 & -1 & 1\\ 5 & 2 & -1 \end{pmatrix}.$$

Solution. We have pay-off matrix as follows,

$$\begin{pmatrix} 2 & 1 & 3\\ 3 & -1 & 1\\ 5 & 2 & -1 \end{pmatrix}.$$

In this game, no single row is dominate remaining two rows. We look for the columns. The first column is dominated by second column. Therefore, we delete first column from pay-off matrix and we get reduced matrix as follows

$$\begin{pmatrix} 0 & 3\\ -1 & 1\\ 2 & -1 \end{pmatrix}.$$

Now, first row dominates the second row. So, we delete the second row.

$$\begin{pmatrix} 0 & 3\\ 2 & -1 \end{pmatrix}.$$

Let $(x,\ 1-x)^T$ be optimal strategy for the player I, then we have

$$2-2x=3x+(1-x)(-1)\Rightarrow x=\frac{1}{2}.$$

So, $1-x = 1-\dfrac{1}{2} = \dfrac{1}{2}$. Hence $\left(\frac{1}{2}, 0, \frac{1}{2}\right)$ *be* the optimal strategy for the player I. The value of the game is $v = E(x) = 2-2x = 2-2\times\frac{1}{2} = 1$. We assume $(y,\ 1-y)^T$ is optimal strategy for player II. We have

$$3-3y = 2y - 1(1-y) \Rightarrow \frac{2}{3}.$$

and $1-x = 1-\dfrac{2}{3} = \dfrac{1}{3}$. The optimal strategy for player II is $\left(0, \frac{2}{3}, \frac{1}{3}\right)$

5.4 Exercise

Exercise 5.1 Solve the game

$$\begin{pmatrix} 1 & 4 \\ 4 & 0 \\ 2 & 5 \end{pmatrix}.$$

Exercise 5.2 Find the strategy of both players of the following game

$$\begin{pmatrix} 2 & 2 & 1 \\ 4 & 1 & 0 \end{pmatrix}.$$

Exercise 5.3 Solve the game

$$\begin{pmatrix} 3 & 5 & 1 \\ 2 & 6 & 2 \\ 4 & 5 & 7 \end{pmatrix}.$$

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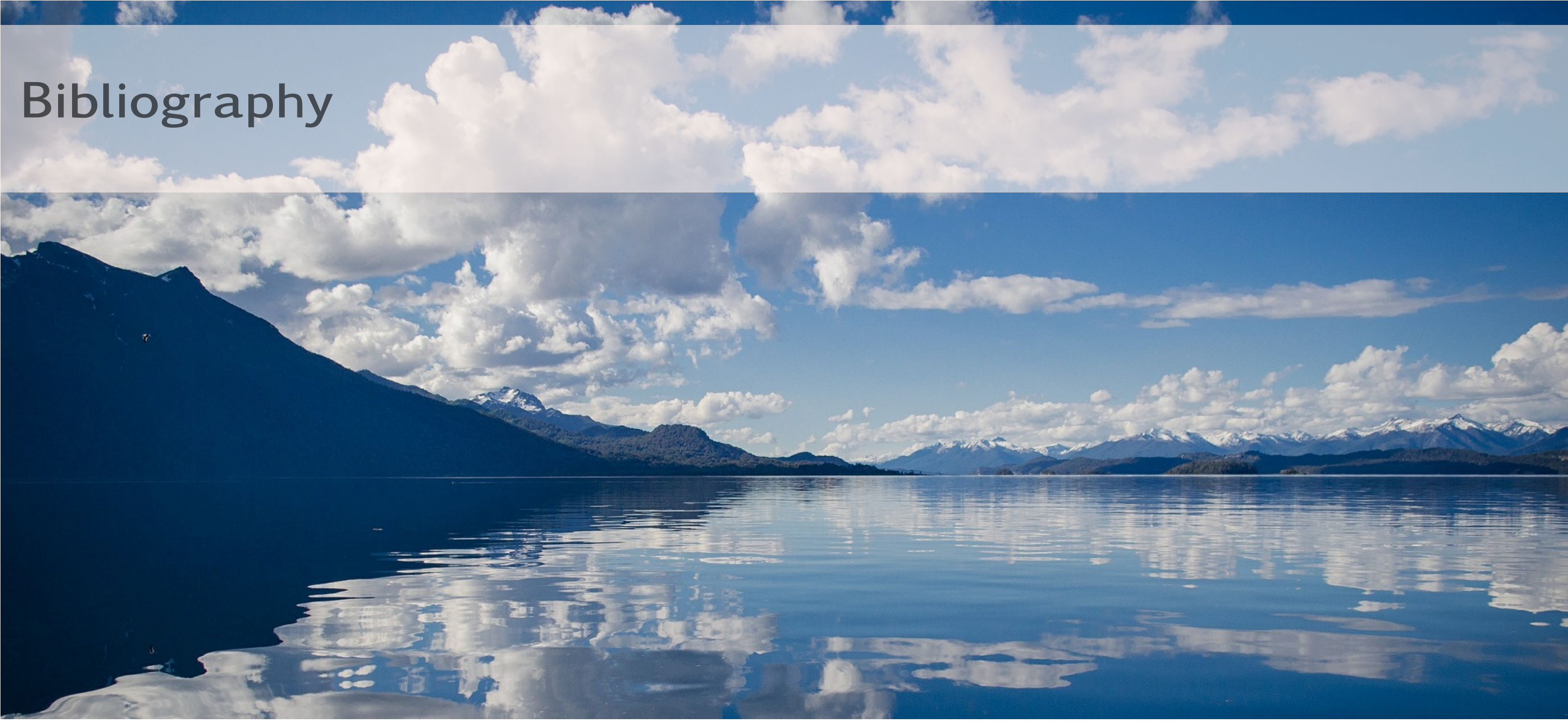

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