**Example 2.8** Solve the LPP by simplex method

$$\text{Minimize } z = -5x_1 - 2x_2$$

Subject to

$$x_1 + 4x_2 \leq 4$$
$$5x_1 + 2x_2 \leq 10$$
$$x_1,\ x_2 \geq 0.$$

**Solution.** The standard form is

$$\text{Minimize } z = -5x_1 - 2x_2$$

Subject to

$$x_1 + 4x_2 + x_3 = 4$$
$$5x_1 + 2x_2 + x_4 = 10$$
$$x_1,\ x_2,\ x_3,\ x_4 \geq 0.$$

The simplex table is

| | | $-5$ | $-2$ | $0$ | $0$ | | |
|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $b$ | Min +ve ratio |
| 0 | $x_3$ | 1 | 4 | 1 | 0 | 4 | 4 |
| 0 | $\leftarrow x_4$ | 5 | 2 | 0 | 1 | 10 | 2 |
| | $\overline{z_j}$ | $-5\uparrow$ | $-2$ | 0 | 0 | 0 | |
| 0 | $x_3$ | 0 | $\frac{18}{5}$ | 1 | $-\frac{1}{5}$ | 2 | |
| $-5$ | $x_1$ | 1 | $\frac{2}{5}$ | 0 | $\frac{1}{5}$ | 2 | |
| | $\overline{z_j}$ | 0 | 0 | 0 | 1 | $-10$ | |

By observing the simplex table, we have all $\overline{z_j} \geq 0$. Thus, the optimal solution is $x_1 = 2$, $x_2 = 0$, $x_3 = 2$ and $x_4 = 0$ with minimum value $z = -10$.

## 2.5 Two-Phase Method

**Example 2.9** Use the simplex method to solve the problem

$$\text{Minimize z} = -2x_1 - x_2$$

Subject to

$$x_1 + x_2 \geq 2$$
$$x_1 + x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

**Solution.** The standard form is

$$\text{Minimize z} = -2x_1 - x_2$$

Subject to

$$x_1 + x_2 - x_3 = 2$$
$$x_1 + x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

**Phase I:** In order to obtain an initial feasible solution, we need to add the artificial variable $w_1$ in the first constraint. Let us consider the following problem

$$\text{Minimize w} = w_1$$

Subject to

$$x_1 + x_2 - x_3 + w_1 = 2$$
$$x_1 + x_2 + x_4 = 4$$
$$x_1, x_2, x_3, x_4,\ w_1 \geq 0.$$

| | | 0 | 0 | 0 | 0 | 1 | | |
|---|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $w_1$ | $b$ | Min + ve ratio |
| 1 | $\leftarrow w_1$ | 1 | 1 | $-1$ | 0 | 1 | 2 | 2 |
| 0 | $x_4$ | 1 | 1 | 0 | 1 | 0 | 4 | 4 |
| | $\overline{w_j}$ | $-1\ \uparrow$ | $-1$ | 1 | 0 | 0 | 2 | |
| 0 | $x_1$ | 1 | 1 | $-1$ | 0 | 1 | 2 | |
| 0 | $x_4$ | 0 | 0 | 1 | 1 | $-1$ | 2 | |
| | $\overline{w_j}$ | 0 | 0 | 0 | 0 | 1 | | |

By observing each $\overline{w_j} \geq 0$. The artificial variable $w_1$ is removed from the phase I, so, we proceed to phase II.
**Phase II:** For the phase II, consider the objective function $z = -2x_1 - x_2$.

| | | $-2$ | $-1$ | $0$ | $0$ | | |
|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | b | Min +ve ratio |
| $-2$ | $x_1$ | 1 | 1 | $-1$ | 0 | 2 | $-$ |
| 0 | $\leftarrow x_4$ | 0 | 0 | 1 | 1 | 2 | 2 |
| | $\overline{z_j}$ | 0 | 1 | $-2\uparrow$ | 0 | -4 | |
| $-2$ | $x_1$ | 1 | 1 | 0 | 1 | 4 | |
| 0 | $x_3$ | 0 | 0 | 1 | 1 | 2 | |
| | $\overline{z_j}$ | 0 | 1 | 0 | 2 | $-8$ | |

Since, each $\overline{z_j} \geq 0$. The optimal solution is $x_1 = 4,\ \ x_2 = 0,\ \ x_3 = 2,\ x_4 = 0$ with $z = -8$.

**Example 2.10** Use the simplex method to solve the problem

$$\text{Minimize } z = x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 2 \\ 3x_1 + 5x_2 &\geq 15 \\ x_1,\ x_2 &\geq 0. \end{aligned}$$

**Solution.** The standard form of the LPP is

$$\text{Minimize } z = x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + 5x_2 - x_4 &= 15 \\ x_1,\ x_2,\ x_3,\ x_4 &\geq 0. \end{aligned}$$

**Phase I:** To obtain an initial feasible solution, we need to add an artificial variable $w_1$ in the second constraint equation and consider the following LPP for Phase I

$$\text{Minimize } w = w_1$$

Subject to

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 3x_1 + 5x_2 - x_4 + w_1 &= 15 \\ x_1,\ x_2,\ x_3,\ x_4,\ w_1 &\geq 0. \end{aligned}$$

The simplex table is

| | | 0 | 0 | 0 | 0 | 1 | | |
|---|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $w_1$ | $b$ | Min +ve ratio |
| 0 | $\leftarrow x_3$ | 1 | 2 | 1 | 0 | 0 | 2 | 1 |
| 1 | $w_1$ | 3 | 5 | 0 | $-1$ | 1 | 15 | 3 |
| | $\overline{w_j}$ | $-3$ | $-5\uparrow$ | 0 | 1 | 0 | 15 | |
| 0 | $\leftarrow x_2$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | 0 | 1 | 2 |
| 1 | $w_2$ | $\frac{1}{2}$ | 0 | $-\frac{5}{2}$ | $-1$ | 1 | 10 | 20 |
| | $\overline{w_j}$ | $-\frac{1}{2}\uparrow$ | 0 | $\frac{5}{2}$ | 1 | 0 | 10 | |
| 0 | $\leftarrow x_1$ | 1 | 2 | 1 | 0 | 0 | 2 | |
| 1 | $w_1$ | 0 | $-1$ | $-3$ | $-1$ | 1 | 9 | |
| | $\overline{w_j}$ | 0 | 1 | 3 | 1 | 0 | 9 | |

Clearly $\overline{w_j} \geq 0$ and the artificial variables $w_1$ present in the basic solution. Hence, the given constraints are inconsistent. Thus, the problem does not have solution.

**Example 2.11** Use the two-phase method to show that the LPP

$$\text{Minimize } z = -x_1 + 2x_2$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\geq 1 \\ -x_1 + x_2 &\leq 1 \\ x_1, x_2 &\geq 0. \end{aligned}$$

has an unbounded solution.

**Solution:** The standard form of given problem is

$$\text{Minimize } z = -x_1 + 2x_2 +$$

subject to

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ -x_1 + x_2 + x_4 &= 1 \\ x_1, x_2,\ x_3,\ x_4 &\geq 0. \end{aligned}$$

**Phase I**: We add an artificial variable $w_1$ and consider the problem

$$\text{Minimize } w = w_1$$

subject to

$$\begin{aligned} x_1 + 2x_2 - x_3 + w_1 &= 1 \\ -x_1 + x_2 + x_4 &= 1 \\ x_1, x_2,\ x_3,\ x_4,\ w_1 &\geq 0. \end{aligned}$$

| | | 0 | 0 | 0 | 0 | 1 | | |
|---|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $w_1$ | $b$ | Min + ve ratio |
| 1 | $\leftarrow w_1$ | 1 | 2 | $-1$ | 0 | 1 | 1 | $\frac{1}{2}$ |
| 0 | $x_4$ | $-1$ | 1 | 0 | 1 | 0 | 1 | 1 |
| | $\overline{w_j}$ | 1 | $2\uparrow$ | 1 | 0 | 1 | 1 | |
| 0 | $x_2$ | $\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | $\frac{1}{2}$ | |
| 0 | $x_4$ | $-\frac{3}{2}$ | 0 | $\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | |
| | $\overline{w_j}$ | 0 | 0 | 0 | 0 | 1 | 0 | |

By observing each $\overline{w_j} \geq 0$, and $w_1$ is removed from the basic variables so we proceed to phase II.
**Phase II:** The object function is minimize $z = -x_1 + 2x_2$

| | | $-1$ | 2 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $b$ | Min +ve ratio |
| 2 | $\leftarrow x_2$ | $\frac{1}{2}$ | 1 | $-\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 |
| 0 | $x_4$ | $-\frac{3}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | – |
| | $\overline{z_j}$ | $-2\uparrow$ | 0 | 1 | 0 | 1 | |
| $-1$ | $x_1$ | 1 | 2 | $-1$ | 0 | 1 | – |
| 0 | $x_4$ | 0 | 3 | $-1$ | 1 | 2 | -- |
| | $\overline{z_j}$ | 0 | 4 | $-1\uparrow$ | 0 | $-1$ | |

Here, we cannot find the new outgoing vector. so, it concludes that the given LPP has an unbounded solution

**Example 2.12** Find the solution of the LPP

$$\text{Minimize } z = 2 - x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &= 4 \\ -x_2 - x_3 &= 0 \\ x_1,\ x_2, x_3 &\geq 0. \end{aligned}$$

**Solution:** The given problem is in the standard form. We treat $x_1$ as a slack variable and $x_3$ as a surplus variable. We add an artificial variable $w_1$ in view of initial basic feasible solution.

**Phase I:** consider the problem

$$\text{Minimize } w = w_1$$

subject to

$$\begin{aligned} x_1 - x_2 &= 4 \\ -x_2 - x_3 + w_1 &= 0 \\ x_1,\ x_2, x_3, w_1 &\geq 0. \end{aligned}$$

| | | 0 | 0 | 0 | 1 | |
|---|---|---|---|---|---|---|
| $C_B$ | BV | $x_1$ | $x_2$ | $x_3$ | $w_1$ | $b$ |
| 0 | $x_1$ | 1 | −1 | 0 | 0 | 4 |
| 1 | $w_1$ | 0 | −1 | −1 | 1 | 0 |
| | $\overline{w_j}$ | 0 | 1 | 1 | 0 | 0 |
| 0 | $x_1$ | 1 | −1 | 0 | 0 | 4 |
| 1 | $\leftarrow w_1$ | 0 | 1 | 1 | −1 | 0 |
| | $\overline{w_j}$ | 0 | $-1 \uparrow$ | −1 | 2 | |
| 0 | $x_1$ | 1 | 0 | 0 | −1 | |
| 0 | $x_2$ | 0 | 1 | 1 | −1 | |
| | $\overline{w_j}$ | 0 | 0 | 0 | 1 | |

All $\overline{z_j} \geq 0$ and $w_1$ is removed from the basic variable. So, we proceed for the phase II.

**Phase II:** The objective function is $z = 2 - x_2$

| | | 0 | -1 | 0 | |
|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $b$ |
| 0 | $x_1$ | 1 | 0 | 1 | 4 |
| $-1$ | $x_2$ | 0 | 1 | 1 | 0 |
| | $\overline{z_j}$ | 0 | 0 | 1 | 2 |

All $\overline{z_j} \geq 0$. Thus, the optimal solution is $x_1 = 4,\ x_2 = 0,\ x_3 = 0$ and minimum $z\ =\ 2$.

**Example 2.13** Solve the LPP by two-phase method

$$\text{Maximize } z = -3x_1 + 7x_2$$

Subject to

$$\begin{aligned} x_1 + 4x_2 &\geq 4 \\ 5x_2 + 2x_2 &\geq 10 \\ 4x_1 + 5x_2 &\leq 20 \\ x_1, x_2 &\geq 0. \end{aligned}$$

**Solution:** The given problem, we can write in the standard form as follows.

$$\text{Maximize } z = -3x_1 + 7x_2$$

Subject to

$$\begin{aligned} x_1 + 4x_2 - x_3 &= 4 \\ 5x_2 + 2x_2 - x_4 &= 10 \\ 4x_1 + 5x_2 + x_5 &= 20 \\ x_1, x_2,\ x_3,\ x_4,\ x_5 &\geq 0. \end{aligned}$$

**Phase I:** We need to add two artificial variables in first and second constraints.

$$\text{Maximize } w = w_1 + w_2$$

Subject to

$$\begin{aligned} x_1 + 4x_2 - x_3 + w_1 &= 4 \\ 5x_2 + 2x_2 - x_4 + w_2 &= 10 \\ 4x_1 + 5x_2 + x_5 &= 20 \\ x_1, x_2,\ x_3,\ x_4,\ x_5,\ w_1,\ w_2 &\geq 0. \end{aligned}$$

| | | 0 | 0 | 0 | 0 | 0 | 1 | 1 | | |
|---|---|---|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $w_1$ | $w_2$ | $b$ | Min + ratio |
| 1 | $\leftarrow w_1$ | 1 | 4 | $-1$ | 0 | 0 | 1 | 0 | 4 | 1 |
| 1 | $w_2$ | 5 | 2 | 0 | $-1$ | 0 | 0 | 1 | 10 | 5 |
| 0 | $x_5$ | 4 | 5 | 0 | 0 | 1 | 0 | 0 | 20 | 4 |
| | $\overline{w_j}$ | $-6$ | $-6\uparrow$ | 1 | 1 | 0 | 0 | 0 | 14 | |
| 0 | $x_2$ | $\frac{1}{4}$ | 1 | $-\frac{1}{4}$ | 0 | 0 | $\frac{1}{4}$ | 1 | 1 | 4 |
| 1 | $\leftarrow w_2$ | $\frac{9}{2}$ | 0 | $\frac{1}{2}$ | $-1$ | 1 | $-\frac{1}{2}$ | 1 | 8 | $\frac{16}{9}$ |
| 0 | $x_5$ | $\frac{1}{4}$ | 0 | $\frac{5}{4}$ | 0 | 0 | $-\frac{5}{4}$ | 0 | 15 | $\frac{60}{11}$ |
| | $\overline{w_j}$ | $-\frac{9}{2}\uparrow$ | 0 | $-\frac{1}{2}$ | 1 | -1 | $\frac{3}{2}$ | 0 | 8 | |
| 0 | $x_2$ | 0 | 1 | $-\frac{5}{8}$ | $\frac{1}{18}$ | 0 | $\frac{5}{18}$ | $-\frac{1}{18}$ | $\frac{5}{9}$ | |
| 0 | $x_1$ | 1 | 0 | $\frac{1}{9}$ | $-\frac{2}{9}$ | 0 | $-\frac{1}{9}$ | $\frac{2}{9}$ | $\frac{16}{9}$ | |
| 0 | $x_5$ | 0 | 0 | $\frac{17}{18}$ | $\frac{11}{18}$ | 1 | $-\frac{18}{18}$ | $-\frac{1}{18}$ | $\frac{11}{9}$ | |
| | $\overline{w_j}$ | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | |

All $\overline{w_j} \geq 0$ and $w_1$, $w_2$ are removed from the basic variables. We have we go to the phase II.
**Phase II:** The objective function for this phase is minimum $z = -3x_1 + 7x_2$

| | | 3 | $-7$ | 0 | 0 | 0 | | |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | b | Min +ve ratio |
| $-7$ | $x_2$ | 0 | 1 | $-\frac{5}{18}$ | $\frac{1}{18}$ | 0 | $\frac{5}{9}$ | $-2$ |
| 3 | $x_1$ | 1 | 0 | $\frac{1}{9}$ | $-\frac{2}{9}$ | 0 | $\frac{16}{9}$ | 16 |
| 0 | $\leftarrow x_5$ | 0 | 0 | $\frac{17}{18}$ | $\frac{11}{18}$ | 1 | $\frac{91}{9}$ | $\frac{182}{17}$ |
| | $\overline{z_j}$ | 0 | 0 | $-\frac{41}{18}\uparrow$ | $\frac{19}{18}$ | 0 | $\frac{13}{9}$ | |
| $-7$ | $x_2$ | 0 | 1 | 0 | $\frac{4}{17}$ | $\frac{5}{17}$ | $\frac{60}{17}$ | |
| 3 | $x_1$ | 1 | 0 | 0 | $-\frac{5}{17}$ | $-\frac{2}{17}$ | $\frac{10}{17}$ | |
| 0 | $x_3$ | 0 | 0 | 1 | $\frac{18}{17}$ | $\frac{18}{17}$ | $\frac{18}{17}$ | |
| | $\overline{z_j}$ | 0 | 0 | 0 | $\frac{28}{17}$ | $\frac{35}{17}$ | $-\frac{390}{17}$ | |

Here all $\overline{z_j} \geq 0$. Thus, the optimal solution is

$$x_1 = \frac{10}{17},\ x_2 = \frac{60}{17},\ x_3 = \frac{182}{17},\ x_4 = 0,\ x_5 = 0$$

The minimum of $z$ is

$$z = -3 \times \frac{10}{17} + 7 \times \frac{60}{17} = -\frac{30}{17} + \frac{420}{17} = \frac{390}{17}.$$

**Example 2.14** Solve the LPP

$$\text{Maximize } z = 4x_1 + 3x_2$$

such that

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 2 \\ x_1 - 2x_2 + x_4 &= 4 \\ x_1 + 3x_2 &= 9 \\ x_1, x_2, x_3, x_4 &\geq 4. \end{aligned}$$

**Solution:** The given problem we can write in the standard form as follows.

$$\text{Maximize } z = 4x_1 + 3x_2$$

subject to

$$\begin{aligned} 2x_1 + x_2 - x_3 + w_1 &= 2 \\ x_1 - 2x_2 + x_4 &= 4 \\ x_1 + 3x_2 + x_5 &= 9 \\ x_1, x_2, x_3, x_4, x_5,\ w_1 &\geq 4. \end{aligned}$$

**Phase I:** We use objective function as $w = w_1$

| | | 0 | 0 | 0 | 0 | 0 | 1 | | |
|---|---|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $w_1$ | $b$ | min +ve ratio |
| 1 | $\leftarrow w_1$ | 2 | 1 | $-1$ | 0 | 0 | 1 | 2 | 1 |
| 0 | $x_4$ | 1 | $-2$ | 0 | 1 | 0 | 0 | 4 | 4 |
| 0 | $x_5$ | 1 | 3 | 0 | 0 | 1 | 0 | 9 | 9 |
| | $\overline{z_j}$ | $-2 \uparrow$ | $-1$ | 1 | 0 | 0 | 0 | 2 | |
| 0 | $x_1$ | 1 | $\frac{1}{2}$ | $-\frac{1}{2}$ | 0 | 0 | $\frac{1}{2}$ | 1 | |
| 0 | $x_4$ | 0 | $-\frac{5}{2}$ | $\frac{1}{2}$ | 1 | 0 | $-\frac{1}{2}$ | 3 | |
| 0 | $x_5$ | 0 | $\frac{5}{2}$ | $\frac{1}{2}$ | 0 | 1 | $-\frac{1}{2}$ | 8 | |
| | $\overline{z_j}$ | 0 | 0 | 0 | 0 | 0 | 1 | | |

All $\overline{w_j} \geq 0$ and $w_1$ is removed form the basic variables, so we proceed to phase II.
**Phase II.** The objective function of this phase is minimum $z = -4x_1 - 3x_2$.

| | | $-4$ | $-3$ | $0$ | $0$ | $0$ | | |
|---|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $b$ | Min +ve ratio |
| $-4$ | $x_1$ | $1$ | $\frac{1}{2}$ | $-\frac{1}{2}$ | $0$ | $0$ | $1$ | $-$ |
| $0$ | $\leftarrow x_4$ | $0$ | $-\frac{5}{2}$ | $\frac{1}{2}$ | $1$ | $0$ | $3$ | $6$ |
| $0$ | $x_5$ | $0$ | $\frac{5}{2}$ | $\frac{1}{2}$ | $0$ | $1$ | $8$ | $16$ |
| | $\overline{z_j}$ | $0$ | $-1$ | $-2\uparrow$ | $0$ | $0$ | $-4$ | $-$ |
| $-4$ | $x_1$ | $1$ | $-2$ | $0$ | $1$ | $0$ | $4$ | $-$ |
| $0$ | $x_3$ | $0$ | $-5$ | $1$ | $2$ | $0$ | $6$ | $1$ |
| $0$ | $x_5$ | $0$ | $5$ | $0$ | $-1$ | $1$ | $5$ | |
| | $\overline{z_j}$ | $0$ | $-11\uparrow$ | $0$ | $4$ | $0$ | $-16$ | |
| $-4$ | $x_1$ | $1$ | $0$ | $0$ | $\frac{3}{5}$ | $\frac{2}{5}$ | $6$ | $\frac{10}{3}$ |
| $0$ | $x_3$ | $0$ | $0$ | $1$ | $1$ | $1$ | $11$ | $11$ |
| $-3$ | $x_2$ | $0$ | $1$ | $0$ | $-\frac{1}{5}$ | $\frac{11}{5}$ | $1$ | $-$ |
| | $\overline{z_j}$ | $0$ | $0$ | $0$ | $\frac{9}{5}$ | $\frac{11}{5}$ | $-4$ | |

All $\overline{z_j} \geq 0$. Thus, the optimal solution is $x_1 = 6,\ x_2 = 1,\ x_3 = 11,\ x_4 = 0,\ x_5 = 0$.

$$z = 4x_1 + 3x_2 = 24 + 3 = 27.$$

**Example 2.15** Find the solution of LPP

$$\text{Minimize } z = -2x_1 - x_2 - 4x_3$$

Subject to

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ 2x_1 - x_2 + x_3 &= 1 \\ 3x_1 + x_2 + 4x_3 &= 2 \\ x_1,\ x_2, x_3 &\geq 0. \end{aligned}$$

**Solution.** The standard form is

$$\text{Minimize } z = -2x_1 - x_2 - 4x_3$$

Subject to

$$x_1 + 2x_2 + 3x_3 + w_1 = 1$$
$$2x_1 - x_2 + x_3 + w_2 = 1$$
$$3x_1 + x_2 + 4x_3 + w_3 = 2$$
$$x_1,\ x_2, x_3,\ w_1,\ w_2, w_3 \geq 0.$$

**Phase I:** The objective function for this phase we use as $w = w_1 + w_2 + w_3$

| | | 0 | 0 | 0 | 1 | 1 | 1 | | |
|---|---|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $w_1$ | $w_2$ | $w_3$ | $b$ | Min + ve ratio |
| 1 | $\leftarrow w_1$ | 1 | 2 | 3 | 1 | 0 | 0 | 1 | 1/3 |
| 1 | $w_2$ | 2 | $-1$ | 1 | 0 | 1 | 0 | 1 | 1 |
| 1 | $w_3$ | 3 | 1 | 4 | 0 | 0 | 1 | 2 | 1/2 |
| | $\overline{w_j}$ | $-6$ | $-2$ | $-8\uparrow$ | 0 | 0 | 0 | 4 | |
| 0 | $x_3$ | $\frac{1}{3}$ | $\frac{2}{3}$ | 1 | $\frac{1}{3}$ | 0 | 0 | $\frac{1}{3}$ | 1 |
| 1 | $\leftarrow w_2$ | $\frac{5}{3}$ | $-\frac{5}{3}$ | 0 | $-\frac{1}{3}$ | 1 | 0 | $\frac{2}{3}$ | $\frac{2}{5}$ |
| 1 | $w_3$ | $\frac{5}{3}$ | $\frac{5}{3}$ | 0 | $-\frac{4}{3}$ | 0 | 1 | $\frac{2}{3}$ | $\frac{2}{5}$ |
| | $\overline{w_j}$ | $-\frac{10}{3}\uparrow$ | $\frac{10}{3}$ | 0 | $\frac{8}{3}$ | 0 | 0 | $\frac{4}{3}$ | |
| 0 | $x_3$ | 0 | 1 | 1 | $\frac{2}{5}$ | $-\frac{1}{5}$ | 0 | $\frac{1}{5}$ | |
| 0 | $x_1$ | 1 | $-1$ | 0 | $-\frac{1}{5}$ | $\frac{3}{5}$ | 0 | $\frac{2}{5}$ | |
| 1 | $w_3$ | 0 | 0 | 0 | $-1$ | $-1$ | 1 | 0 | |
| | $\overline{w_j}$ | 0 | 0 | 0 | 2 | 2 | 0 | 0 | |

All $\overline{w_j} \geq 0$ and observing this $w_3$ is redundant variable. So, delete the row 3 from phase I.
**Phase II:** The objective function is $z = -2x_1 - x_2 - 4x_3$

| | | $-2$ | $-1$ | $-4$ | |
|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | b |
| $-4$ | $x_3$ | 0 | 1 | 1 | $\frac{1}{5}$ |
| $-2$ | $x_1$ | 1 | $-1$ | 0 | $\frac{2}{5}$ |
| | $\overline{z_j}$ | 0 | 1 | 0 | $-\frac{3}{5}$ |

All $\overline{z_j} \geq 0$. The optimal solution is $x_1 = \frac{1}{5},\ x_2 = 0,\ x_3 = \frac{1}{5}$ with $z = -\frac{4}{3}$.

## 2.6 Big method/Charne's M-Technique the method:

**Example 2.16** Solve the LPP

$$\text{Minimize } z = 4x_1 + 8x_2 + 3x_3$$

subject to

$$x_1 + x_2 \geq 2$$
$$2x_2 + x_3 \geq 5$$
$$x_1, x_2,\ x_3 \geq 0.$$

**Solution:** We use Charne's-M technique method to solve the LPP

$$\text{Minimize } z = 4x_1 + 8x_2 + 3x_3 + Mw_1 + Mw_2$$

subject to

$$x_1 + x_2 - x_4 + w_1 = 2$$
$$2x_2 + x_3 - x_5 + w_2 = 5$$
$$x_1, x_2,\ x_3,\ x_4,\ x_5,\ w_1, w_2 \geq 0.$$

| | | 4 | 8 | 3 | 0 | 0 | $M$ | $M$ | | Min |
|---|---|---|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $w_1$ | $w_2$ | $b$ | +ve |
| $M$ | $\leftarrow w_1$ | 1 | 1 | 0 | $-1$ | 0 | 1 | 0 | 2 | 2 |
| $M$ | $w_2$ | 0 | 2 | 1 | 0 | $-1$ | 0 | 1 | 5 | $\frac{5}{2}$ |
| | $\overline{z_j}$ | 4-M | $8-3M\uparrow$ | $13-M$ | $M$ | $M$ | 0 | 0 | $7M$ | |
| $B$ | $x_2$ | 1 | 1 | 0 | $-1$ | 0 | | 0 | 2 | $-$ |
| $-M$ | $\leftarrow w_2$ | $-2$ | 0 | 1 | 2 | $-1$ | | 1 | 1 | 1 |
| | $\overline{z_j}$ | $2M-4$ | 0 | $3-M$ | $8-2M\uparrow$ | $M$ | | 0 | $16+M$ | |
| 8 | $x_1$ | 1 | 1 | $\frac{1}{2}$ | 0 | $-\frac{1}{2}$ | | | $\frac{5}{2}$ | 5 |
| 0 | $x_4$ | $-1$ | 0 | $\frac{1}{2}$ | 1 | $-\frac{1}{3}$ | | | $\frac{1}{2}$ | 1 |
| | $\overline{z_j}$ | 4 | 0 | $-1\uparrow$ | 0 | 4 | | | 20 | |
| 8 | $x_2$ | 1 | 1 | 0 | $-1$ | 0 | | | 2 | |
| 3 | $x_3$ | $-2$ | 0 | 1 | 2 | $-1$ | | | 1 | |
| | $\overline{z_j}$ | 2 | 0 | 0 | 2 | 3 | | | 19 | |

Clearly, all $\overline{z_j} \geq 0$. The optimal solution of this problem is $x_1 = 0,\ x_2 = 2,\ x_3 = 1$ and minimum value of $z$ is $z = 19$.

**Example 2.17** Solve LPP using Big-M method.

$$\text{Minimize } z = x_1 + x_2$$

subject to

$$x_1 + x_2 \leq 2$$
$$3x_1 + 5x_2 \geq 15$$
$$x_1, x_2 \geq 0$$

**Solution:** Consider the problem as follows

$$\text{Minimize } z = x_1 + x_2 + Mw_1$$

subject to

$$x_1 + x_2 + x_3 = 2$$
$$3x_1 + 5x_2 - x_3 + w_1 = 15$$
$$x_1, x_2,\ x_3,\ x_4,\ w_1 \geq 0.$$

Consider the simplex table

| | | 1 | 1 | 0 | 0 | $M$ | | Min +ve ratio |
|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $w_1$ | $b$ | |
| 0 | $\leftarrow x_3$ | 1 | 2 | 1 | 0 | 0 | 2 | 1 |
| $M$ | $w_1$ | 3 | 5 | 0 | $-1$ | 1 | 15 | 3 |
| | $\overline{z_j}$ | $1 - 3M$ | $1 - 5M \uparrow$ | 0 | $M$ | 0 | $15M$ | |
| 1 | $\leftarrow x_2$ | $\frac{1}{2}$ | 1 | $\frac{1}{2}$ | 0 | 0 | 1 | 2 |
| $M$ | $w_1$ | $\frac{1}{2}$ | 0 | $-\frac{5}{2}$ | $-1$ | 1 | 10 | 20 |
| | $\overline{z_j}$ | $\frac{1}{2} - \frac{M}{2} \uparrow$ | 0 | $M - \frac{1}{2}$ | $M$ | 0 | $1 + 10M$ | |
| 1 | $x_1$ | 1 | 2 | 1 | 0 | 0 | 2 | |
| $M$ | $w_1$ | 0 | $-1$ | $-3$ | $-1$ | 1 | $+9$ | |
| | $\overline{z_j}$ | 0 | $M - 1$ | $3M - 1$ | $M$ | 0 | $2 + 9M$ | |

All $\overline{z_j} \geq 0$ and $w_j$ is artificial variable present in the basic solution. Thus, the given system of equation is inconsistent.

**Example 2.18** Solve the LPP

$$\text{Minimize } z = -x_1 + 2x_2$$

subject to

$$x_1 + 2x_2 - x_3 = 1$$
$$-x_1 + x_2 + x_4 = 1$$
$$x_1, x_2,\ x_3,\ x_4 \geq 0.$$

**Solution:** Consider the problem

$$\text{Minimize } z = -x_1 + 2x_2 + Mw_1$$

subject to

$$x_1 + 2x_2 - x_3 + w_1 = 1$$
$$-x_1 + x_2 + x_4 = 1$$
$$x_1, x_2,\ x_3,\ x_4,\ w_1 \geq 0.$$

| | | $-1$ | $2$ | $0$ | $0$ | $M$ | | Min +ve ratio |
|---|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $w_1$ | $b$ | |
| $M$ | $w_1$ | $1$ | $2$ | $-1$ | $0$ | $1$ | $1$ | $\frac{1}{2}$ |
| $0$ | $x_4$ | $-1$ | $1$ | $0$ | $1$ | $0$ | $1$ | $1$ |
| | $\overline{z_j}$ | $-1-M$ | $2-2M\uparrow$ | $M$ | $0$ | $0$ | $M$ | |
| $2$ | $x_2$ | $\frac{1}{2}$ | $1$ | $-\frac{1}{2}$ | $0$ | $\frac{1}{2}$ | $\frac{1}{2}$ | |
| $0$ | $x_4$ | $-\frac{3}{2}$ | $0$ | $\frac{1}{2}$ | $1$ | $-\frac{1}{2}$ | $\frac{1}{2}$ | |
| | $\overline{z_j}$ | $0$ | $0$ | $+1$ | $0$ | $M\ +1$ | $+1$ | |

All $\overline{z_j} \geq 0$ and artificial variable still present in the table. It concludes that the given system is inconsistent.

**Example 2.19** Solve LPP

$$\text{Minimize } z = 4x_1 + 8x_2 + 3x_3$$

subject to

$$x_1 + x_2 \geq 2$$
$$x_2 + x_2 \geq 5$$
$$x_1, x_2, x_3 \geq 0.$$

**Solution.** We consider the problem

$$\text{Minimize } z = 4x_1 + 8x_2 + 3x_3 + Mw_1 + Mw_2$$

subject to

$$x_1 + x_2 - x_4 + w_1 = 2$$
$$x_2 + x_2 - x_5 + w_2 = 5$$
$$x_1, x_2, x_3, x_4, x_5, w_1, w_2 \geq 0.$$

We have a table

| | | 4 | 8 | 3 | 0 | 0 | M | M | | Min + ratio |
|---|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $w_1$ | $w_2$ | b | |
| M | $\leftarrow w_1$ | 1 | 1 | 0 | -1 | 0 | 1 | 0 | 2 | 2 |
| M | $w_2$ | 0 | 1 | 1 | 0 | -1 | 0 | 1 | 5 | 5 |
| | $\overline{z_j}$ | 4 - M | $8-2M\uparrow$ | 2 - M | M | M | 0 | 0 | 7M | |
| 8 | $x_2$ | 1 | 1 | 0 | -1 | 0 | | 0 | 2 | - |
| M | $\leftarrow w_2$ | -1 | 0 | 1 | 1 | -1 | | 1 | 3 | 3 |
| | $\overline{z_j}$ | -4 + M | 0 | $3-M\uparrow$ | 8 -M | M | | 0 | 1 - 3M | |
| 8 | $\leftarrow x_2$ | 1 | 1 | 0 | -1 | 0 | | | 2 | 2 |
| 3 | $x_3$ | -1 | 0 | 1 | 1 | -1 | | | 3 | - |
| | $\overline{z_j}$ | $-1\uparrow$ | 0 | 0 | 5 | 3 | | | 25 | |
| 4 | $x_1$ | 1 | 1 | 0 | -1 | 0 | | | 2 | |
| 3 | $x_3$ | 0 | 1 | 1 | 0 | -1 | | | 5 | |
| | $\overline{z_j}$ | 0 | 1 | 0 | 4 | 3 | | | 23 | |

All $\overline{z_j} \geq 0$. Therefore, the optimal solution is $x_1 = 2$, $x_2 = 0$, $x_3 = 5$, $x_4 = x_5 = 0$ and the minimum $z = 23$.

**Example 2.20** Solve the following LPP

$$\text{Minimize } z = -x_1 + x_2$$

subject to

$$x_1 - 2x_2 - x_3 = 1$$
$$-x_1 + 2x_2 - x_4 = 1$$
$$x_1,\ x_2,\ x_3,\ x_4 \geq 0.$$

**Solution.** We have

$$\text{Minimize } z = -x_1 + x_2 + Mw_1$$

subject to

$$x_1 - 2x_2 - x_3 + w_1 = 1$$
$$-x_1 + 2x_2 - x_4 + w_2 = 1$$
$$x_1,\ x_2,\ x_3,\ x_4,\ w_1,\ w_2 \geq 0.$$

| | | -1 | 1 | 0 | 0 | M | M | | Min +ve ratio |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $w_1$ | $w_2$ | b | |
| $M$ | $\leftarrow w_1$ | 1 | $-2$ | $-1$ | 0 | 1 | 0 | 1 | 1 |
| $M$ | $w_2$ | $-1$ | 2 | 0 | $-1$ | 0 | 1 | 1 | |
| | $\overline{z_j}$ | $-1\uparrow$ | 1 | $M$ | $M$ | 0 | 0 | $2M$ | |
| $-1$ | $x_1$ | 1 | $-2$ | $-1$ | 0 | | 0 | 1 | – |
| $M$ | $w_2$ | 0 | 0 | $-1$ | $-1$ | | 1 | 2 | – |
| | $\overline{z_j}$ | 0 | $-1\uparrow$ | $-1+M$ | M | | 0 | $2M-1$ | |

Here, $x_2$ is incoming vector but we can not decide the outgoing vector. So, the given LPP has an unbounded solution.

**Example 2.21** Solve the following LPP

$$\text{Minimize } z = -x_1 - x_2$$

subject to

$$x_1 - x_2 - x_3 = 1$$
$$-x_1 + x_2 + 3x_3 - x_4 = 0$$
$$x_1,\ x_2,\ x_3, x_4 \geq 0.$$

**Solution.** we have Minimize $z = -x_1 - x_2 + Mw_1 + Mw_2$

$$x_1 - x_2 - x_3 + w_1 = 1$$
$$-x_1 + x_2 + 3x_3 - x_4 + w_2 = 0$$
$$x_1,\ x_2,\ x_3, x_4,\ w_1,\ w_2 \geq 0.$$

| | | $-1$ | $-1$ | 0 | 0 | $M$ | $M$ | | |
|---|---|---|---|---|---|---|---|---|---|
| CB | BV | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $w_1$ | $w_2$ | $b$ | Min +ve ratio |
| $M$ | $\leftarrow w_1$ | 1 | $-1$ | $-1$ | 0 | 1 | 0 | 1 | 1 |
| $M$ | $w_2$ | $-1$ | 1 | 3 | $-1$ | 0 | 1 | 0 | - |
| | $\overline{z_j}$ | $-1\uparrow$ | $-1$ | $-2M$ | $M$ | 0 | 0 | $M$ | |
| $-1$ | $x_1$ | 1 | $-1$ | $-1$ | 0 | 1 | 0 | 1 | |
| $M$ | $w_2$ | 0 | 0 | 2 | $-1$ | 1 | 1 | 1 | |
| | $\overline{z_j}$ | 0 | $-2$ | $-2M-1\uparrow$ | $M$ | 1 | 0 | $M-1$ | |
| $-1$ | $x_1$ | 1 | $-1$ | 0 | $-\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ | |
| 0 | $x_3$ | 0 | 0 | 1 | $-\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | |
| | $\overline{z_j}$ | 0 | $-2\uparrow$ | 0 | $-\frac{1}{2}$ | $M+\frac{3}{2}$ | $M+\frac{1}{2}$ | $-\frac{3}{2}$ | |

Since $x_2$ is an incoming vector in the simplex table, but we could not find the minimum positive ratio and so could not decide the out going vector. It indicates that the this LPP has an unbounded solution.

## 2.7 Duality in Linear programming

Consider the primal linear program

$$\text{(SLP)} \quad \text{Minimize } \; Z = c^T x \tag{2.24}$$

Subject to

$$Ax \geq b \tag{2.25}$$
$$x \geq 0, \tag{2.26}$$

where $A = (a_{i,j})$ is a $m \times n$ coefficient matrix and $x = (x_1,\ x_2, \cdots, x_n)$ is a primal variable.

The dual of the (SLP) is given by

$$\text{(DSLP) Maximize } v = b^T y$$

Subject to

$$\begin{aligned} A^T y &\leq c \\ y &\geq 0. \end{aligned}$$

where $y = (y_2, y_2, \cdots,\ y_m)$ is the dual variable. The pair of problems (SLP)and (DSLP) is called the symmetric (canonical) forms of the dual programs.

**Example 2.22** Find the dual of the linear program

$$\text{Minimize } z = c^T x$$

subject to

$$\begin{aligned} Ax &= b \\ x &\geq 0. \end{aligned}$$

**Solution.** Given LPP, we can write in a standard primal form as follows

$$\text{Minimize } z = c^T x$$

subject to

$$\begin{aligned} Ax &\geq b \\ (-A)x &\geq (-b) \\ x &\geq 0. \end{aligned}$$

It can be written as

$$\text{Minimize } z = c^T x$$

$$\begin{pmatrix} A \\ -A \end{pmatrix} x \geq \begin{pmatrix} b \\ -b \end{pmatrix}$$
$$x \geq 0.$$

The dual is given by

$$\text{Maximize } v = (b, -b)^T (y_1, y_2)$$

$$\begin{pmatrix} A \\ -A \end{pmatrix}^T (y_1, y_2) \leq c,$$
$$y_1, y_2 \geq 0.$$

It can be written as

$$\text{Maximize } v = b^T y_1 - b^T y_2$$

$$\begin{pmatrix} A^T \\ -A^T \end{pmatrix} (y)1, y_2) \leq c,$$
$$y_1, y_2 \geq 0.$$

Substitute $y = y_1 - y_2$, then we get

$$\text{Maximize } v = b^T y$$

subject to

$$A^T y \leq c,$$

$y$ is unrestricted in sign.

**Theorem 2.9** The dual of the dual is the primal.

**Proof.** Without loss of generality, we consider the (SLP)

$$\text{Minimize} \quad z = c^T x$$

subject to

$$Ax \geq b$$
$$x \geq 0.$$

Then the corresponding dual is

$$\text{Maximize } v = b^T y$$

subject to

$$A^T y \leq c$$
$$y \geq 0.$$

Now, this dual we write in the standard primal form as follow

$$\text{Minimize } -v = (-b^T) y$$

subject to

$$\begin{aligned}(-A^T y) &\geq (-c)\\ y &\geq 0.\end{aligned}$$

The dual of the this program is

$$\text{Maximize } -z = (-c)^T x$$

subject to

$$\begin{aligned}(-A^T)^T &\leq -b\\ x &\geq 0\end{aligned}$$

That is

$$\text{Minimize } z = c^T x$$

Subject to

$$\begin{aligned}Ax &\geq b\\ x &\geq 0.\end{aligned}$$

**Theorem 2.10** If the $k^{\text{th}}$ constraint in a primal is an equality then the corresponding dual variable $y_k$ is unrestricted in sign.

**Proof.** Let the $k^{\text{th}}$ constrain of a primal be an equality. Then the LPP is of the following form

$$\text{Minimize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1\\ &\vdots\\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &= b_k\\ &\vdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m\\ x_i &\geq 0, \forall i = 1, 2, \cdots, n.\end{aligned}$$

It can be written in (SLP) form

$$\text{Minimize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

subject to

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\geq b_1\\ &\vdots\\ a_{k1}x_1 + a_{k2}x_2 + \cdots + a_{kn}x_n &\geq b_k\\ -a_{k1}x_1 - a_{k2}x_2 - \cdots - a_{kn}x_n &\geq -b_k\\ &\vdots\\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\geq b_m\\ x_j &\geq 0, \forall i = 1, 2, \cdots, n.\end{aligned}$$

The dual of above (SLP) is

$$\text{Maximize } v = b_1y_1 + b_2y_2 + \cdots + b_ky_k^+ - b_1y_k^- + \cdots + b_my_m$$

subject to

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \cdots + a_{k1}y_k^+ - a_{k1}y_k^- + \cdots + b_{m1}y_m &\leq c_1 \\ &\vdots \\ a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{kn}y_k^+ - a_{kn}y_k^- + \cdots + b_{mn}y_m &\leq c_n \\ y_1, y_2,\ y_k^+, y_k^-, \cdots, y_m &\geq 0. \end{aligned}$$

Put $y_k = y_k^+ - y_k^-$, we have

$$\text{Maximize } v = b_1y_1 + b_2y_2 + \cdots + b_ky_k^+ - b_1y_k^- + \cdots + b_my_m$$

subject to

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + \cdots + a_{k1}y_k + \cdots + b_{m1}y_m &\leq c_1 \\ &\vdots \\ a_{n1}y_1 + a_{n2}y_2 + \cdots + a_{kn}y_k + \cdots + b_{mn}y_m &\leq c_n \\ y_1, y_2, \cdots, y_{k-1}, y_{k+1}, \cdots, y_m &\geq 0, \end{aligned}$$

and $y_k$ is in restricted in sign.

**Example 2.23** Write the dual of

$$\text{Minimize } z = x_1 + x_2 + 2x_3$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\geq 3 \\ x_2 + 7x_3 &\leq 6 \\ x_1 - 3x_2 + 3x_3 &= 5 \end{aligned}$$

where $x_1 \geq 0$, $x_2 \geq 0$ and $x_3$ is unrestricted in sign.

**Solution.** Put $x_3 = x_4 - x_5$. The standard primal form is

$$\text{Minimum } z = x_1 + x_2 + 2x_4 - 2x_5$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\geq 3 \\ -x_2 - 7x_4 + 7x_5 &\geq -6 \\ x_1 - 3x_2 + 3x_4 - 3x_5 &\geq 5 \\ -x_1 + 3x_2 - 3x_4 + 3x_5 &\geq -5. \end{aligned}$$

The dual is

$$\text{Maximize } v = 3y_1 - 6y_2 + 5y_3 - 5y_4$$

subject to

$$\begin{aligned} y_1 + y_3 - y_4 &\leq 1 \\ 2y_1 - y_2 - 3y_3 + 3y_4 &\leq 1 \\ -7y_2 + 3y_3 - 3y_4 &\leq 2 \\ 7y_2 - 3y_3 + 3y_4 &\leq -2 \\ y_1, y_2, y_3, y_4 &\geq 0. \end{aligned}$$

Put $y_3 - y_4 = y_5$

$$\text{Maximize } v = 3y_1 - 6y_2 + 5y_5$$

subject to

$$\begin{aligned} y_1 + y_5 &\leq 1 \\ 2y_1 - y_2 - 3y_5 &\leq 2 \\ -7y_2 + 5y_5 &\geq 2, \\ 7y_2 - 5y_5 &\leq -2 \end{aligned}$$

i.e. last two can be written as

$$-7y_2 + 5y_5 = 2,$$

where $y_1 \geq 0, y_2 \geq 0, y_5$ are unrestricted in sign.

**Example 2.24** Minimize $z = 3x_1 - 6x_2$

subject to

$$\begin{aligned} 4x_1 + 2x_2 &= 4 \\ x_1 - x_2 &\geq -2 \end{aligned}$$

$x_1 \geq 0$, $x_2$ is unrestricted in sign.

**Solution.** Put $x_2 = x_3 - x_4$. Then

$$\text{Minimize } z = 3x_1 - 6x_3 + 6x_4$$

subject to

$$\begin{aligned} 4x_1 + 2x_3 - 2x_4 &\geq 4 \\ -4x_1 - 2x_3 + 2x_4 &\geq -4 \\ x_1 - x_3 + x_4 &\geq -2 \\ x_i &\geq 0, i = 1, 2, 3, 4. \end{aligned}$$

The dual is

$$\text{Maximize } v = 4y_1 - 4y_2 - 2y_3$$

subject to

$$
\begin{aligned}
4y_1 - 4y_2 + y_3 &\le 3 \\
2y_1 - 2y_2 - y_3 &\le -6 \\
-2y_1 + 2y_2 + y_3 &\le 6 \\
y_1, y_2, y_3 &\ge 0.
\end{aligned}
$$

we now put $y_1 - y_2 = y_4$. Then, we get

$$\text{Maximize } v = 4y_4 - 2y_3$$

subject to

$$
\begin{aligned}
4y_4 + y_3 &\le 3 \\
2y_4 - y_3 &\le -6 \\
-2y_4 + y_3 &\le 6
\end{aligned}
$$

where $y_3, y_4 \ge 0$. It also be written as

$$\text{Maximize } v = 4y_4 - 2y_3$$

Subject to

$$
\begin{aligned}
4y_4 + y_3 &\le 3 \\
2\,y_4 - y_3 &= 6
\end{aligned}
$$

$y_3 \ge 0$ and $y_4$ is unrestricted in sign.

**Theorem 2.11** If the variable $x$ of a primal is unrestricted in sign, then the corresponding $p^{th}$ constraint of the dual is an equality in sign.

**Proof.** Consider the linear program

$$\text{Minimize } z = c_1x_1 + c_2x_2 + \cdots + c_px_p + \cdots + c_nx_n$$

subject to

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p + \cdots + a_{1n}x_n &\ge b_1 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mp}x_p + \cdots + a_{mn}x_n &\ge b_m
\end{aligned}
$$

where $x_1, x_2, \cdots, x_{p-1}, x_{p+1}, \cdots, x_n \ge 0$ and $x_p$ is in unrestricted in sign. Putting $x_p = x_p^+ - x_p$, we get

$$\text{Minimize } z = c_1x_1 + c_2x_2 + \cdots + c_px_p^+ - c_px_p^- + \cdots + c_nx_n$$

Subject to

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1p}x_p^+ - a_{1p}x_p^- + \cdots + a_{1n}x_n &\ge b_1 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mp}x_p^+ - a_{mp}x_p^- + \cdots + a_{mn}x_n &\ge b_m
\end{aligned}
$$

where $x_1, x_2, \cdots, x_{p-1}, x_p^+, x_p^-, x_{p+1}, \cdots, x_n \ge 0$. The dual of the primal is

$$\text{Maximize } z = b_1y_1 + b_2y_2 + \cdots + b_ny_n$$

subject to

$$\begin{aligned}
a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\leq c_1 \\
a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m &\leq c_2 \\
&\vdots \\
a_{1p}y_1 + a_{2p}y_2 + \cdots + a_{mp}y_m &\leq c_p \\
-a_{1p}y_1 - a_{2p}y_2 - \cdots - a_{mp}y_m &\leq -c_p \\
&\vdots \\
a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_n &\leq c_n
\end{aligned}$$

where $y_1, y_2, \cdots, y_n \geq 0$. It can be written as

$$\text{Maximize } z = b_1y_1 + b_2y_2 + +b_ny_n$$

subject to

$$\begin{aligned}
a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m &\leq c_1 \\
a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m &\leq c_2 \\
&\vdots \\
a_{1p}y_1 + a_{2p}y_2 + \cdots + a_{mp}y_m &= c_p \\
&\vdots \\
a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_n &\leq c_n \\
y_1, y_2, \cdots, y_n &\geq 0.
\end{aligned}$$

Thus $p^{th}$ constraint of this dual is an equality in sign.

**Theorem 2.12** Any feasible solution to primal (SLP) has value $z$ greater than or at least equal to the value $v$ for any feasible solution to dual (DSLP).

**Proof.** Suppose $x$ and $y$ be solutions to primal (SLP) and its dual (DSLP). The (SLP) is given by

$$\text{Minimize} \quad z = c^T x$$

subject to

$$\begin{aligned}
Ax &\geq b \\
x &\geq 0.
\end{aligned}$$

The dual is

$$\text{Maximize} \quad v = b^T y$$

subject to

$$A^T y \leq c$$
$$y \geq 0.$$

Thus

$$z = c^T x \geq (A^T y)^T x = y^T (A^T)^T x = y^T (Ax) \geq y^T b = b^T y = v.$$
$$\Rightarrow Z \ \geq \ v.$$

**Theorem 2.13** (Weak duality theorem) If the optimal solution to both the primal and dual exists then $\min z \geq \max v$.

**Proof.** Let $z_0 = \min z$ and $v_0 = \max v$. Then

$$c^T X \ \geq \ (A^T y)^T x = y^T (Ax) \geq \ y^T b = b^T y. \quad \text{Thus}$$

$$z \ \geq \ v \Rightarrow\Rightarrow \min z \geq \max z \Rightarrow \ z_0 \ \geq \ v_0.$$

**Lemma 2.3** If $x^0$ and $y^0$ are optimal solution to (SLP) and (DSLP) respectively such that $c^T x^0 = b^T y^0$, then $x^0$ and $y^0$ are optimal solutions to (SLP) and (DSLP).

**Proof.** By Theorem 2.12, we know that, for any feasible solution $x$ to (SLP), we have $c^T x \geq b^T y$ for any feasible solution to (DSLP). Thus

$$c^T x \geq b^T y^0 = c^T x^0.$$

This shows that $x^0$ is an optimal solution of (SLP) Now let $y$ be any solution to (SLP), then

$$b^T y \leq c^T x^0 = b^T y^0.$$

This shows that $y^0$ is optimal solution to (DSLP).

**Theorem 2.14** If standard linear program has an unbounded objective function, then its dual has no feasible solution and vice versa.

**Proof.** Suppose the objective function of (SLP) is unbounded. That is, for each $M > 0$, there exist a feasible solution $x_M$ such that

$$c^T x_M < -M.$$

Let $y$ be the solution of its (DSLP), by Theorem, for the feasible solution $y$ and arbitrary large $M > 0$, we have a feasible solution $x_M$ to (SLP) such that $b^T y \leq C^T X_M < -M$. This shows that $b^T y \to -\infty$ for each feasible solution $y$. This is not true for maximization problem, because for maximization problem, the value of objective function either is finite or $v \to \infty$. Hence existence of feasible solution to dual linear program is wrong. Thus, the dual has no feasible solution.

**Theorem 2.15** (Complementary Slackness Theorem) Let $x^0$ and $y^0$ be feasible solutions to the dual program (SLP) and (DSLP) respectively. Then necessary and sufficient for $x^0$ and $y^0$ to be optimal solutions to (SLP) and (DSLP) respectively is

$$(y^0)^T(Ax^0 - b) = 0, \quad (x^0)^T(c - A^T y^0) = 0.$$

**Proof.** Let $x^0$ and $y^0$ are feasible solutions to primal and its dual respectively. Suppose

$$\alpha = (y^0)^{\mathrm{T}}(Ax^0 - b) \geq 0, \beta = (x^0)^T(c - A^T y^0) \geq 0.$$

Thus $\alpha + \beta = (x^0)^T c - (y^0)^T b \geq 0$. If $x^0$ and $y^0$ are optimal solutions, then we must have $c^T x^0 = b^T y^0$. This shows that $\alpha + \beta = 0$. Since $\alpha \geq 0$ and $\beta \geq 0$. This implies that $\alpha = 0$ and $\beta = 0$. Conversely, let $\alpha = 0$ and $\beta = 0$. Then $0 = \alpha + \beta = c^T x^0 - b^T y^0$. i.e. $c^T x^0 = b^T y^0$, thus $x^0$ and $y^0$ are optimal solutions to (SLP) and (DSLP) respectively.

## 2.8 Dual simplex method

**Example 2.25** Use the dual simplex method to solve the program

$$\text{Minimize } z = 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 - x_2 &\geq 2 \\ x_1 + x_2 &\leq 4 \\ x_1, x_2 &\geq 0 \end{aligned}$$

**Solution.**

| | | 2 | 1 | 0 | 0 | |
|---|---|---|---|---|---|---|
| CB | B. V. | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $b$ |
| 0 | $\leftarrow x_3$ | $-1$ | $-1$ | 1 | 0 | $-2$ |
| 0 | $x_4$ | 1 | 1 | 0 | 1 | 4 |
| | $\overline{c_j}$ | 2 | $1 \uparrow$ | 0 | 0 | 0 |
| 1 | $x_2$ | 1 | 1 | $-1$ | 0 | 2 |
| 0 | $x_4$ | 0 | 0 | 1 | 1 | 2 |
| | $\overline{c_j}$ | 1 | 0 | 1 | 0 | 2 |

We have

$$\text{Minimize } z = 2x_1 + x_2$$

subject to

$$\begin{aligned} x_1 + x_2 - x_3 &= 2 \\ x_1 + x_2 + x_4 &= 4 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Since $b = -2 < 0$. Thus $x_3$ is out going vector. Also

$$\left|\frac{\overline{c_s}}{\bar{a}_{rs}}\right| = \min\left\{\left|\frac{2}{-1}\right|, \left|\frac{1}{-1}\right|\right\} = 1$$

it is corresponding to the vector $x_2$. Thus $x_2$ is an incoming vector. The simplex table is

By observing the table, $b > 0$ and $\overline{c_j} \geq 0$ for all $j$. Thus, the optimal solution is $x_1 = 0, x_2 = 2, x_3 = 0, x_4 = 2$ and $\min z = 2$.

**Example 2.26** Use the dual simplex method to solve the program

$$\text{Minimize } z = -2x_1 - x_2 - x_3$$

subject to

$$\begin{aligned} 4x_1 + 6x_2 + 3x_3 &\leq 8 \\ x_1 - 9x_2 + x_3 &\leq -3 \\ -2x_1 - 3x_2 + 5x_3 &\leq -4 \\ x_1 \geq 0, i &= 1, 2, 3 \end{aligned}$$

**Solution.** We write the given problem in the standard form

$$\text{Minimize } z = -2x_1 - x_2 - x_3$$

subject to

$$\begin{aligned} 4x_1 + 6x_2 + 3x_3 + x_4 &= 8 \\ x_1 - 9x_2 + x_3 + x_5 &= -3 \\ -2x_1 - 3x_2 + 5x_3 + x_6 &= -4 \\ x_i \geq 0, \quad i &= 1, 2, 3, 4, 5, 6. \end{aligned}$$

The initial solution is $x_4 = 8, x_5 = -3, x_6 = -4, x_1 = x_2 = x_3 = 0$. Clearly $\overline{c_1} = -2, \overline{c_2} = -1, \overline{c_3} = -1$, so the corresponding solution is not feasible.$\overline{c_j} = \min\{\overline{c_1}, \overline{c_2}, \overline{c_3}\} = \min\{-2, -1, -1\} = -2$. So, we add a artificial constraint as

$$x_1 = M - x_0 - x_2 - x_3.$$

Then, we have a program

$$\text{Minimum } z = -2(M - x_0 - x_2 - x_3) - x_2 - x_3 = -2M + 2x_0 + x_2 + x_3$$

subject to

$$\begin{aligned} 4(M - x_0 - x_2 - x_3) + 6x_2 + 3x_3 + x_4 &= 8 \\ 4x_0 - 2x_2 - x_5 + x_4 &= 8 - 4M \\ -x_0 - 10x_2 + x_3 &= -3 - M \\ 2x_0 - x_2 + 7x_3 + x_6 &= -4 + 2M \\ x_0 + x_1 + x_2 + x_3 &= M \\ x_0\,,\ x_2\,,\ x_3\,,\ x_4\,,\ x_5\,,\ x_6 &\geq 0. \end{aligned}$$

| | | | 2 | 0 | 1 | 1 | 0 | 0 | 0 |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | B. V. | $b$ | $x_0$ | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $x_5$ | $x_6$ |
| 0 | $x_4$ | $8-4M$ | $-4$ | 0 | 2 | $-1$ | 1 | 0 | 0 |
| 0 | $x_5$ | $-3-M$ | $-1$ | 0 | $-10$ | 0 | 0 | 1 | 0 |
| 0 | $x_6$ | $-4+2M$ | 2 | 0 | $-1$ | 7 | 0 | 0 | 1 |
| 0 | $x_1$ | $M$ | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| $\overline{c_j}=0$ | | | $2\uparrow$ | 0 | 1 | 1 | 0 | 0 | |
| 2 | $x_0$ | $M-2$ | 1 | 0 | $-\frac{1}{2}$ | $\frac{1}{4}$ | $-\frac{1}{4}$ | 0 | 0 |
| 0 | $x_5$ | $-5$ | 0 | 0 | $-\frac{21}{2}$ | $\frac{1}{4}$ | $-\frac{1}{4}$ | 1 | 0 |
| 0 | $x_6$ | 0 | 0 | 0 | 0 | $\frac{13}{2}$ | $\frac{1}{2}$ | 0 | 1 |
| 0 | $x_1$ | 2 | 0 | 1 | $\frac{3}{2}$ | $\frac{3}{4}$ | $\frac{1}{4}$ | 0 | 0 |
| $\overline{c_j}=2M-4$ | | | 0 | 0 | $2\uparrow$ | $\frac{1}{2}$ | $\frac{1}{2}$ | 0 | 0 |
| 2 | $x_0$ | $M-\frac{37}{21}$ | 1 | 0 | 0 | $\frac{5}{21}$ | $-\frac{5}{21}$ | $-\frac{1}{21}$ | 0 |
| 1 | $x_2$ | $\frac{10}{21}$ | 0 | 0 | 1 | $-\frac{1}{42}$ | $\frac{1}{42}$ | $-\frac{2}{21}$ | 0 |
| 0 | $x_6$ | 0 | 0 | 0 | 0 | $\frac{13}{2}$ | $\frac{1}{2}$ | 0 | 1 |
| 0 | $x_1$ | $\frac{9}{7}$ | 0 | 1 | 0 | $\frac{11}{4}$ | $\frac{3}{14}$ | $\frac{1}{7}$ | 0 |
| $\overline{c_j}=2M-4$ | | | 0 | 0 | 0 | | | | |

Thus $x_1 = \frac{9}{7}$ , $x_2 = \frac{10}{21}$ , $x_3 = 0$, is optimal solution.

**Example 2.27** Minimize $z = x_1 + 2x_2$

subject to

$$
\begin{aligned}
4x_1 + 2x_2 + x_3 &= 4\\
3x_1 - 3x_2 + x_4 &= -2\\
x_i \geq 0, i &= 1, 2,3, 4.
\end{aligned}
$$

**Solution.** We have

$$\text{Minimize} \quad z = x_1 + 2x_2$$

subject to

$$
\begin{aligned}
4x_1 + 2x_2 + x_3 &= 4\\
3x_1 - 3x_2 + x_4 &= -2\\
x_i \geq 0, i &= 1, 2,3, 4.
\end{aligned}
$$

| | | 1 | 2 | 0 | 0 | |
|---|---|---|---|---|---|---|
| CB | B.V | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $b$ |
| 0 | $x_3$ | 4 | 2 | 1 | 0 | 4 |
| 0 | $x_4$ | 3 | −3 | 0 | 1 | −2 |
| | $\overline{c_j}$ | 1 | 2↑ | 0 | 0 | 0 |
| 0 | $x_3$ | 6 | 0 | 1 | $\frac{2}{3}$ | $\frac{8}{3}$ |
| 2 | $x_2$ | −1 | 1 | 0 | $-\frac{1}{3}$ | $\frac{2}{3}$ |
| | $\overline{c_j}$ | 3 | 0 | 0 | $\frac{2}{3}$ | $\frac{4}{3}$ |

Here all $\bar{c_j} \geq 0$, and all $b_j \geq 0$. Thus, optimal solution is

$$x_1 = 0, x_2 = 2/3, x_3 = 8/3, x_4 = 0 \text{ and Min } z = x_4 + 2x_2 = 0 + 2\frac{2}{3} = \frac{4}{3}.$$

**Example 2.28** Solve

$$\text{Minimize} \quad z = 3x_2 + 5x_4$$

subject to

$$x_1 - 3x_2 - x4 = -4$$
$$x_2 + x_3 + x_4 = 3$$
$$x_1.x_2, x_3, x_4 \geq 0.$$

| | | 3 | 5 | 0 | 0 | |
|---|---|---|---|---|---|---|
| CB | B.V. | $x_1$ | $x_2$ | $x_3$ | $x_4$ | $b$ |
| 0 | $x_1$ | 1 | −3 | 0 | −1 | −4 |
| 0 | $x_3$ | 0 | 1 | 1 | 1 | 3 |
| $\overline{z_j} = 0$ | $c_j$ | 0 | 3 ↑ | 0 | 5 | |
| 3 | $x_2$ | $-\frac{1}{3}$ | 1 | 0 | $\frac{1}{3}$ | $\frac{4}{3}$ |
| 0 | $x_3$ | $\frac{1}{3}$ | 0 | 1 | $\frac{2}{3}$ | $\frac{5}{3}$ |
| $\overline{z_j} = 4$ | $c_j$ | 1 | 0 | 0 | 4 | 4 |

All $\bar{c_j} \geq 0$ and all $b_j \geq 0$. The optimal solution is $x_1 = 0, x_2 = \frac{4}{3}, x_3 = \frac{5}{3}, x_4 = 0$ with minimum $z = 4$.

## 2.9 Exercise

**Exercise 2.1** Find the optimal solution of the linear program

$$\text{Minimize } z = -x - y - z$$

subject to

$$\begin{aligned} x + y - z &\leq 1 \\ 2x - 4y + z &\geq 7 \\ x, y, z &\geq 0. \end{aligned}$$

**Exercise 2.2** Find the optimal solution of the linear program

$$\text{Minimize } z = 4x - y$$

subject to

$$\begin{aligned} 2x + y + z &= 1 \\ x - y + w &= 3 \\ x, y, z, w &\geq 0. \end{aligned}$$

**Exercise 2.3** Use two Phase method to find the solution of linear program

$$\text{Minimize } z = -x - 6y$$

subject to

$$\begin{aligned} x + 5y &\geq 3 \\ x - y &= 5 \\ x, y &\geq 0. \end{aligned}$$

**Exercise 2.4** Use two Phase method to find the solution of linear program

$$\text{Minimize } z = -x - y - z$$

subject to

$$\begin{aligned} x + y - z &\geq 1 \\ x - y + z &\geq 4 \\ x + 2y + 3z &\leq 2 \\ x, y, z &\geq 0. \end{aligned}$$

**Exercise 2.5** Use Big M-method to find the solution of linear program

$$\text{Minimize } z = -x - y + 4z$$

subject to

$$\begin{aligned} x - 2y - 2z &\geq 3 \\ 5x - y &\leq 2 \\ x &\leq 4 \\ x, y, z &\geq 0. \end{aligned}$$

**Exercise 2.6** Use Big M-method to find the solution of linear program

$$\text{Maximize } z = x + 2y + 2z$$

subject to

$$\begin{aligned} x - y - z &\leq 3 \\ z - y &\leq 2 \\ x - y &\leq 4 \\ x, y, z &\geq 0. \end{aligned}$$