2. Linear Programming

Linear programming problem is one of rich developed optimization technique. Linear programming models arise in a several of decision problems in government, engineering, computer science, economics etc. These models is effective for taking a decisions in a critical positions. This technique also help for getting maximum benefit or reduce the time problems.

2.1 Linear Programming Model

The general problem of linear programming is to optimize a linear function subject to linear equality or inequality constraint. i.e. to determine the values of $x_1, x_2, \cdots, x_n$ that solve the problem (LP)

$$\text{Minimize } z = \sum_{i=1}^{n} c_i x_i$$

Subject to

$$\sum_{j=1}^{n} a_{i,j} x_j \ \{\leq,\ =,\ \geq\}\ b_i,\ i = 1, 2, \cdots, m. \tag{2.1}$$

where one and only one of the sign $\leq, =, \geq$ holds for each constraint in (2.1) and this sign may varies from one constraint to another. Here c_i, b_i and $a_{i,j}$ are known real numbers.

Definition 2.1 A vector $(x_1,\ x_2,\ \cdots, x_n) \in \mathbb{R}^n$ is called a feasible solution, to problem (LP) if constraint (2.1) is satisfied by the vector.

Definition 2.2 A feasible solution is said to be an optimal solution to program (LP) if it gives minimum value of the objective function provided minimum value exists.

It is clear that feasible solution/optimal solution may or may not be exists for a linear programming problem. The set of feasible solution to (LP) is given by

$$T = \left\{(x_1, x_2, \cdots, x_n) : (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \ and \ (2.1) \ hold \ at \ (x_1, x_2, \cdots, x_n)^T\right\}. \tag{2.2}$$

The set T is also called the constraint set, feasible set or feasible region of (LP).

Lemma 2.1 Every constraint set is convex.

Proof. We know that the constraint set T is a closed convex polyhedron. Hence it is convex.

Lemma 2.2 The set of optimal solution to the linear program (LP) is convex.

Proof. Let $x^0 = (x_1^0,\ x_2^0, \cdots,\ x_n^0)$ and $y^0 = (y_1^0,\ y_2^0, \cdots, y_n^0)$ be two optimal solution to program (LP) then $c^T x_0 = c^T y_0 = \min(z)$, where $c^T = (c_1,\ c_2, \cdots, c_n)^T$ By assumption, x^0 and y^0 are feasible optimal solution for the linear program (LP). The feasible set T is convex, for $0 \leq \lambda \leq 1, \lambda x^0 + (1-\lambda) y^0 \in T$. Consider

$$\begin{aligned} c^T(\lambda x^0 + (1-\lambda) y^0) &= \lambda c^T x^0 + (1-\lambda) c^T y^0 \\ &= \lambda c^T x^0 + c^T y^0 - \lambda c^T y^0 \\ &= c^T y^0 \\ &= \min z \\ \Rightarrow c^T(\lambda x^0 + (1-\lambda) y^0) &= \min z. \end{aligned}$$

Hence, the set of optimal solution is a convex set.

Theorem 2.1 Let the constraint set T be non empty closed and bounded then an optimal solution to the linear program (LP) exists and it is attained at a vertex of T.

Proof. Since, T is non empty, closed and bounded, i.e. T is compact and $z = c^T x$ is continuous on the compact set T, hence the minimum of z exist i.e. optimal solution exists over T. We know that the set of vertices of the convex polyhedron T is finite. Let the vertices of T be $x_1, x_2, , x_k$ where all $x_i \in \mathbb{R}^n$. By Theorem, the set T is equal to the convex hull of the points $x_1, x_2, \cdots, x_k$. Thus, any feasible point $x \in T$ can be written as

$$x = \sum_{i=1}^{k} \lambda_i x_i, \; where \; \sum_{i=1}^{k} \lambda_i = 1, \; \lambda_i \geq 0. \; (i = 1, 2, \cdots, k)$$

Let $z_0 = \min\{c^T x_i \, : \, i = 1, 2, \cdots, k\}$. Then for any $x \in T$, we have

$$z = c^T x = c^T \left(\sum_{i=1}^{k} \lambda_i x_i \right) = \sum_{i=1}^{k} \lambda_i c^T x_i \geq \sum_{i=1}^{k} \lambda_i z_0 = z_0 \sum_{i=1}^{k} \lambda_i = z_0.$$

Thus, $z \geq z_0$. Hence the minimum value of z $= c^T x$ over T is z_0 and it is attained at a vertex of T.

Theorem 2.2 Let $C \subseteq \mathbb{R}^n$ be a convex set and $f(x)$ be a non constant concave function of C. Then any optimal solution x^0 to the problem

$$Minimum \; f(x) \; for \; x \in C \tag{2.3}$$

if it exists must be a boundary point of C.

Proof. Assume that x^0 is an optimal solution of (2.3). Given that $f(x)$ is a non constant on C. There exists a point $x \in C$ such that

$$f(x) > f(x^0) \tag{2.4}$$

Suppose z be any interior point of C. We choose $\lambda(0 \le \lambda \le 1)$ such that $y \in C$, and $z = \lambda y + (1-\lambda)x \in C$. Since, $f(x)$ is concave, then

$$f(z) = f(\lambda y + (1-\lambda)x) \ge f(y) + (1-\lambda) f(x) > f(x^0) + (1-\lambda) f(x^0) = f(x^0).$$

Thus, $f(z) > f(x^0)$. Hence, the interior point z of C cannot be optimal solution of (2.3).

Theorem 2.3 Let $f(x)$ be a concave function on a bounded convex set $C \subseteq \mathbb{R}^n$. Then if f has a minimum over C, it is achieved at an extreme point of C.

Proof. By Theorem 2.2, the optimal solution x^0 of (2.3), if it exists, then it must attained at boundary point of C. If this boundary point is an extreme point of C, then, the proof is complete.

Now, suppose that the boundary point x^0 is not an extreme point of C. Let H be a supporting hyperplane to C at x^0 and $C_1 = C \cap H$. Clearly $\dim(C_1) \le n-1$. Since C_1 is closed and bounded convex set. Moreover, the global minimum of f over C_1 is equal to $f(x^0)$. By Theorem 2.2, this minimum must be attained at a boundary point $\underline{x}_1$ of C_1. If $\underline{x}_1$ is an extreme point of C_1, then it is also extreme point of C, so the proof is complete.

If x_1 is not extreme point of C_1, let $C_2 = C_1 \cap H_1$ where, H_1 is a supporting hyperplane to C_1 at $\underline{x}_1$. Clearly C_2 is closed bounded and convex subset of C and $\dim(C_2) \leq n-2$. In this case, the global minimum of f over C_2 is $f(x_1) = f(x^0)$ and it is attained of a bounded point x_2 of C_2. Now. if $\underline{x}_2$ also at an extreme point of C_2 then $\underline{x}_2$ also at an an extreme point of C_1 and therefore of C and the result follows. This process is continued for a maximum of n times until a set C_n of dimension zero, consisting a single point is obtained. Obviously, this single point is an extreme point of C_n. Hence it is an extreme point of C. The proof is complete.

2.2 Graphical solution of LPP

Graphical method is used for finding the solution of a linear programming problem which contains two or three variables.

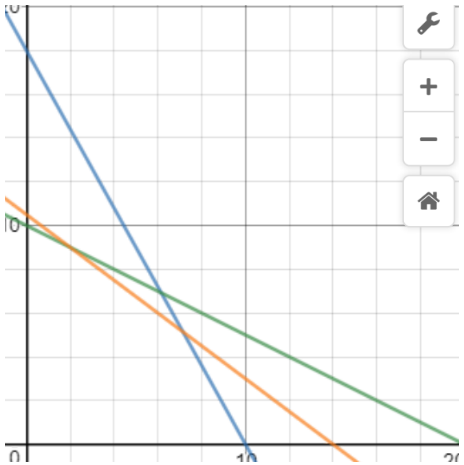
Example 2.1 Find the solution of following LLP by graphically.

$$minimum\ z = 5x + 8y$$

subject to

$$\begin{aligned} 18x + 10y &\leq 180 \\ 10x + 20y &\leq 200 \\ 15x + 20y &\leq 210 \\ x, y &\geq 0. \end{aligned}$$

Solution.



Consider the LPP

$$minimumxz = 5x + 8y$$

Consider, the equality of the constraints

$$\begin{aligned} 18x+10y &= 180 \\ 10x+20y &= 200 \\ 15x+20y &= 210 \\ x, y &\geq 0. \end{aligned}$$

By observing the graph feasible region is OABCD. We know that the optimal solution of LPP exists at extreme points. Clearly $O(0,0), A(0,10), B(2,9), C(50/7, 36/7), D(10,0)$ are extreme points of the convex polyhedron.

$$\begin{aligned} z_o &= 5xx + 8xy = 0, \\ z_A &= 5\times 0 + 8\times 10 = 80, \\ z_B &= 5\times 2 + 8\times 9 = 10 + 72 = 82, \\ z_c &= 5\times \frac{50}{7} + 8\times \frac{36}{7} = \frac{250+288}{7} = \frac{538}{7} = 76.85, \\ z_D &= 5\times 10\times 8\times 0 = 50. \end{aligned}$$

z_B is maximum value, so the optimal solution is $xx = 2$ and $xy = 9$.

2.3 Standard Linear Program

A program of the form

(LP1)

$$Minimize\, z = c^T x \tag{2.5}$$

subject to

$$Ax = b \tag{2.6}$$

$$x \geq 0. \tag{2.7}$$

where $b \geq 0$ is said to be a linear program in the standard form.In this program, $A = (a_{i,j})$ is $m \times n$ coefficients matrix of the equality constraints, $b = (b_1, b_2, \cdots, b_m)^T$ is the vector of constants, $c = (c_1, c_2, \cdots, c_n)^T$ be cost factors in objective function and $x = (x_1, x_2, \cdots, x_n)^T$ is a decision variable.

Reduction to the standard form

1. Slack variable: The non-negative variable which is added to LHS of the constraint to convert the inequality $\leq$ into an equation is called slack variable.

$\sum_{j=1}^{n} a_{ij}x_i + s_i = b_i (i = 1, 2, \cdots, m)$ where s_i are called slack variables.

2. The non-negative variable which is subtracted from the LHS of the constraint to convert the inequality $\geq$ into an equation is called surplus variable.

   $\sum_{j=1}^{n} a_{ij}x_i - s_j = b_i (i = 1, 2, \cdots, m)$ where s_i are called surplus variables.

3. Making all variables non-negative: If some variable is of the form $x_i \leq 0$, then put $x_i' = -x_i$, so that $x_i' \geq 0$. If x_i is unrestricted in sign (free variable), then we put $x_i = u - v$, where $u \geq 0, v \geq 0$.

4. the constant of the each constraint i.e. b must be ≥ 0. If not then multiplying by -1, we can make it non negative

Basic solution

Suppose the constraint $Ax = b$ are constraints and $\text{rank}(A) = m (\leq n)$. Let B be any non singular $m \times m$ sub-matrix made up of the columns of A. further suppose that x_B

is the vector of variables associated with column B. Then (2.6) can be written

$$[B \;\; R]\begin{bmatrix} x_B \\ x_{NB} \end{bmatrix} = b$$

$$Bx_B + Rx_{NB} = b$$

The general solution is

$$\begin{aligned} Bx_B &= b - Rx_{NB} \\ x_B &= B^{-1}b - B^{-1}Rx_{NB} \end{aligned}$$

where, the $(n-m)$ variable x_{NB} can be assign arbitrary values. So in particular $x_B = B^{-1}b,\ x_{NB} = 0$ is called the basic solution to the system $Ax = b$ with respective to the matrix B. The variable x_{NB} are known as non basic variables. The variables x_B are said to be basic variables.

Definition 2.3 A basic solution is called a basic feasible solution if it satisfies non negativity constraints i.e. if all the basic variables are non negative.

Definition 2.4 A basic solution is a non degenerate basic solution if all the basic variables are non zero.

Definition 2.5 A basic solution that is not non degenerate is known as degenerate basic solution.

A basic feasible solution i.e. non degenerate is referred as non degenerate basic feasible solution No. of basic feasible solution ≤ no. of basic solutions ≤ $n\,C_n$.

Example 2.2 Find the basic and non basic solutions of following LPP.

$$\text{Minimize } z = x_1 + 2x_3 - 2x_4$$

Subject to

$$\begin{aligned} 2x_1' + x_3 - x_4 &\geq 2 \\ -x_1' + x_3 + 3x_4 &\leq 3, \\ x_3 \geq 0, x_4 &\geq 0. \end{aligned}$$

Solution. The given problem we can write in standard form as follows. Given that x_1' is unrestricted and hence, we put $x_1' = x_1 - x_2$. Then

$$\text{Minimize } z = x_1 - x_2 + 2x_3 - 2x_1$$

subject

$$2x_1 - 2x_2 + x_3 - x_4 - x_5 = 2$$
$$-x_1 + x_2 + x_3 + 3x_4 + x_6 = 3$$

where z_5 is surplus variable and x_6 is slack variable and all $x_i \geq 0$ or $i = 1, 2, \cdots, 6$.

$$\begin{bmatrix} 2 & -2 & 1 & -1 & -1 & 0 \\ 1 & 1 & 1 & 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}.$$

This system has at most 15 basic solutions. Some of them are (i) Consider the sub matrix $B_1 = \begin{bmatrix} 2 & 2 \\ -1 & -1 \end{bmatrix}$ and the determinant $|B_1| = 0$. So x_1 and x_2 does not forms a basic solution.

(ii) Consider $B_2 = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$. Then $|B_2| = \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} = -1 \neq 0$, So x_2 and x_3 forms a basic solution. Set the value of non-basic variables $x_1 = x_4 = x_5 = x_6 = 0$. We get $-2\underline{x}_2 + x_3 = 2$

and $\underline{x}_2 + x_3 = 3$. On solving, $\underline{x}_2 = 1/3$ and $x_3 = 8/3$. Hence $\underline{x}_1 = 0, \underline{x}_2 = 1/2, x_3 = 8/3, x_4 = 0, x_5 = 0, x_6 = 0)$ is a non-degenerate basic feasible. It is basic because each variable $x_i \geq 0,\ i = 1,2.$ and it is non-degenerate because the basic variable $\underline{x}_2$ and x_3 are both $\neq 0$.

(iii) Consider $B_3 = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$ and the determinant $|B_3| = 3 \neq 0$. Therefor x_4 and x_5 constructs the basic solution. The non-basic variables $\underline{x}_1 = \underline{x}_2 = x_3 = x_6 = 0$. Then, we have system of equation $x_4 - x_5 = 2$ and $3x_4 = 3$. It implies that $x_4 = 1$ and $x_5 = -3$. Thus the basic solution is $\underline{x}_1 = 0, \underline{x}_2 = 0, x_3 = 0, x_4 = 1, x_5 = -3, x_6 = 0$. It is not feasible since $x_5 = -3 < 0$. Also, it is non-degenerate because the basic variable $x_4 \neq 0$ and $x_3 \neq 0$.

(iv) Consider $B_4 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and the determinant $|B_4| = -1 \neq 0$. Therefor x_5 and x_6 constructs the basic solution. The non-basic variables $\underline{x}_1 = \underline{x}_2 = x_3 = x_4 = 0$. Then, we have $x_5 = 2$ and $x_6 = 3$. Thus the basic solution is $\underline{x}_1 = 0, \underline{x}_2 = 0, x_3 = 0, x_4 = 0, x_5 = 2, x_6 = 3$. It is a feasible and non-degenerate.

(v) Consider $B_5 = \begin{bmatrix} -1 & -1 \\ 3 & 0 \end{bmatrix}$ and the determinant $|B_3| = 3 \neq 0$. Therefor x_4 and x_5 constructs the basic solution. The non-basic variables $\underline{x}_1 = \underline{x}_2 = x_3 = x_6 = 0$. Then, we have system of equation $x_4 - x_5 = 2$ and $3x_4 = 3$. It implies that $x_4 = 1$ and $x_5 = -3$. Thus the basic solution is $\underline{x}_1 = 0, \underline{x}_2 = 0, x_3 = 0, x_4 = 1, x_5 = -3, x_6 = 0$. It is not feasible since $x_5 = -3 < 0$. Also, it is non-degenerate because the basic variable $x_4 \neq 0$ and $x_3 \neq 0$. Similarly, we can find the remaining basic solutions.

Theorem 2.4 If a standard linear program with the constraints $Ax = b$ and $x \geq 0$ where A is an $m \times n$ matrix of rank m has a feasible solution then it also has a basic feasible solution.

Proof. Let $A = (a_1, a_2, \cdots, a_n)$ where the vector a_i is the i^{th} column of matrix A. Suppose that $x = (x_1, x_2, \cdots, x_n)^T$ a feasible solution. Then

$$x_1 a_1 + x_2 a_2 + \cdots + x_n a_n = b$$
$$x_i \geq 0, \ i = 1, 2, \cdots, n.$$

Now, let the feasible vector x have $p(\leq n)$ positive variables. Let these variable be $x_1, x_2, \cdots, x_p$. Therefore

$$x_1 a_1 + x_2 a_2 + \cdots + x_p a_p = b, \tag{2.8}$$

where, $x_i > 0, i = 1, 2, \cdots, P$ and $x_i = 0, i = p+1, \cdots, n$. The columns vectors $a_1, a_2, \cdots, a_p$

associated with the positive variables may be linearly independent or linearly dependent.

(i) Suppose $a_1, a_2, \cdots, a_p$ be linearly independent, then $p \le m$. If $p = m$ then, the feasible solution x is a non degenerate basic feasible solution. If $p < m$, then we assign the value zero to the $(m-p)$ variables corresponding to the selected $(m-p)$ column vector of A. These $(m-p)$ variables including $x_1, x_2, \cdots, x_p$ form a degenerate basic feasible solution.

(ii) Suppose $(a_1,\ a_2, \cdots, a_p)$ be linearly dependent. This means that there exists constant $y_1, y_2, \cdots, y_P$ not all zero such that

$$y_1 a_1 + y_2 a_2 + \cdots + y_p a_p = 0. \tag{2.9}$$

It means that there is at least one of constant out of $y_1, y_2, \cdots, y_p$ to be positive. From the assumed feasible solution with p positive variables, we can construct a feasible solution with at most $(p-1)$ positive variables. Since, the set $\{a_1, a_2, \cdots, a_p\}$ is linearly dependent obviously $p > m$. let $\{\alpha_1,\ \alpha_2, \cdots, \alpha_p\}$ be set of constants (not all zero) such that $\sum_{j=1}^{p} \alpha_j a_j = 0$. Suppose that for any index $r, \alpha_r \neq 0$. Then $a_r = -\sum_{j=1, j\neq r}^{p} \frac{\alpha_j}{\alpha_r} a_j$. Substituting in the relation $\sum_{j=1}^{p} a_j x_j = b$, we get

$$\sum_{j=1, j\neq r}^{p} a_j x_j + \left(- \sum_{j=1, j\neq r}^{p} \frac{\alpha_j}{\alpha_r} a_j \right) x_r = b$$
$$\Rightarrow \sum_{j=1, j\neq r}^{p} \left(x_j - x_r \frac{\alpha_j}{\alpha_r} \right) a_j = b.$$

Thus, we have a solution with not more that $p-1$ non zero components to assume that these are positive, we shall have a_r in such a way that

$$x_j - x_r \frac{\alpha_j}{\alpha_r} \geq 0 \quad for\ all\ j \neq r.$$

This requires that either $\alpha_j = 0$ or $\frac{x_j}{\alpha_j} \geq \frac{x_r}{\alpha_r}$ if $\alpha_j > 0$ and $\frac{x_j}{\alpha_j} \leq \frac{x_r}{\alpha_r}$ if $\alpha_j < 0$. Thus if we select a_r such that

$$\frac{x_r}{\alpha_r} = \min_j \left\{ \frac{x_j}{\alpha_j}, \quad \alpha_j > 0 \right\}.$$

Then each of the $p-1$ variable $\left(x_j - x_r \frac{\alpha_j}{\alpha_r} \right)$ is non negative. So, we have feasible solution not more than $p-1$ non zero components. If the corresponding set $a_1, a_2, \cdots, a_{p-1}$

is linearly independent, then it is required basic feasible solution. If the set $a_1, a_2, \cdots, a_{p-1}$ is linearly dependent, then we repeat the case ii) up to get a basic feasible solution.

Example 2.3 The column vector [1, 1, 1] is feasible solution to the system of solution

$$x_1 + x_2 + 2x_3 = 4$$
$$2x_1 - x_2 + x_3 = 2$$

Reduce the given feasible solution to a basic feasible solution

Solution. We have the system of equation is

$$\begin{bmatrix} 1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

i.e. $Ax = b$ with rank $(A) = 2$.

Therefore, there are only two linearly independent columns of A. Let the column vectors of A be denoted by a_1, a_2, a_3 respectively. Thus

$$\alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 = 0. \tag{2.10}$$

Not all α_j are zero $(1 = 1, 2, 3)$. Since

$$\alpha_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \alpha_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \alpha_3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

It gives that

$$\begin{aligned} \alpha_1 + \alpha_2 + 2\alpha_3 &= 0 \\ 2\alpha_1 - \alpha_2 + \alpha_3 &= 0. \end{aligned}$$

Taking the addition of above equations, we get $\alpha_1 + 3\alpha_3 = 0 \Rightarrow \alpha_1 = -\alpha_3$. Choosing arbitrary $\alpha_3 = 1$, we get $\alpha_1 = -1$ and $\alpha_2 = -1$. The equation (2.9) becomes

$$-\alpha_1 - \alpha_2 + \alpha_3 = 0.$$

Thus, the given system of equation is

$$\underline{x}_1 a_1 + \underline{x}_2 a_2 + x_3 a_3 = b$$

where $\underline{x}_1 = \underline{x}_2 = x_3 = 1$. In order to reduce the number of positive variables. The vector to be removed as follows. Now

$$\frac{x_r}{\alpha_r} = \min\left\{\frac{x_i}{\alpha_j},\ a_j > 0,\ j = 1,2,3\right\}$$
$$\frac{x_r}{\alpha_r} = \min\left\{\frac{1}{1}\right\} = 1.$$

The vector for which $\dfrac{x_r}{\alpha_r} = 1$ is a α_3, so we can remove the vector α_3 and obtain a new solution

$$\hat{x}_j = x_j - \frac{x_r}{\alpha_r}\alpha_j,\ j = 1,2,3.$$
$$\underline{\hat{x}}_1 = 1-(-1) = 1+1 = 2$$
$$\underline{\hat{x}}_2 = 1-1(-1) = 1+1 = 2.$$

Then [2, 2, 0] is also a feasible solution. It contains only non zero components and clearly, the corresponding a_1, a_2 is linearly independent. Hence, it is a basic feasible solution.

Theorem 2.5 Let K be the convex polyhedron consisting all vectors $x \in \mathbb{R}^n$ satisfying the system $Ax = b, x \geq 0$ where A is an $m \times n$ matrix of rank m. Then x is an extreme point K if and only if x is a basic feasible solution to the system.

Proof: Suppose x is a basic feasible solution to $Ax = b$. Let B be the basic matrix corresponding to x and let

$$x = \begin{bmatrix} x_B \\ 0 \end{bmatrix} = B^{-1}b.$$

Further, suppose

$$x = \lambda x' + (1-\lambda)x'',$$

where $0 < \lambda < 1$ and $x',\ x'' \in K$. In order to show that x is an extreme point of K, it is sufficient to show that $x' = x''$. Let

$$x' = \begin{bmatrix} x'_B \\ x'_{NB} \end{bmatrix} \ \ and\ \ x'' = \begin{bmatrix} x''_B \\ x''_{NB} \end{bmatrix}.$$

It should be note that $x'_{NB} \geq 0$ and $x''_{NB} \geq 0$. Now

$$x = \lambda x' + (1-\lambda)x'', \quad yields$$

$$\begin{bmatrix} x_B \\ 0 \end{bmatrix} = \lambda \begin{bmatrix} x'_B \\ x'_{NB} \end{bmatrix} + (1-\lambda) \begin{bmatrix} x''_B \\ x''_{NB} \end{bmatrix}$$

It gives that

$$\begin{aligned} x_B &= \lambda x'_B + (1-\lambda)x''_B \\ 0 &= \lambda x'_{NB} + (1-\lambda)x''_{NB} \end{aligned} \tag{2.11}$$

Since $x'_{NB} \geq 0$, $x''_{NB} \geq 0$ and $0 < \lambda < 1$. Then (2.11) implies

$$x'_{NB} = x''_{NB} = 0. \tag{2.12}$$

Moreover, $x \in K$ yield

$$b = Ax' = Bx'_B + Bx'_{NB} = Bx'_B.$$

It gives that

$$x'_B = B^{-1}b = x_B. \tag{2.13}$$

Similarly, we can show

$$x''_B = x_B. \tag{2.14}$$

From (2.13) and (2.14), we have

$$x_B = x'_B = x''_B.$$

Hence, x is an extreme point of K. Conversely, suppose x be an extreme point of the polyhedron K. Assume that the first $k(\leq n)$ components of x are non-zero. Then

$$\underline{x}_1 a_1 + \underline{x}_2 a_2 + \cdots + x_k a_k = b \tag{2.15}$$

where, $x_i > 0, i = 1, \cdots, k$ To prove that x is a basic feasible solution, we have to show the vectors $a_1, a_2, \cdots, a_k$ are linearly independent.

Suppose $a_1, a_2, \cdots, a_k$ are linearly dependent. Then there exists constant $y_1, y_2, \cdots, y_k$ not all zero such that $y_1 a_1 + y_2 a_2 + \cdots + y_k a_k = 0$. Now let $y = \big(y_1, y_2, \cdots, y_k, 0, 0, \cdots, 0\big) \in \mathbb{R}^n$. Since $x_i > 0 (i = 1, \cdots, k)$. It is possible to select on $\epsilon > 0$ such that

$$x + \epsilon y \geq 0, x - \alpha y \geq 0. \tag{2.16}$$

In fact, the inequality (2.16) is true if $0 < \epsilon < \min\left\{\frac{x_i}{|y_i|}, y_i \neq 0, i = 1, 2, \cdots, k\right\}$. since, at least one $y_i \neq 0$. These $i = 1, 2, \ldots.$ k exists an $\epsilon > 0$ satisfying (2.16). Now

$$x = \frac{1}{2}(x + \epsilon y) + \frac{1}{2}(x - \epsilon y).$$

Clearly $x + \epsilon y$ and $x - \epsilon y$ are feasible point so, x is a convex combination of $x + \epsilon y$ and $x - \epsilon y$. This implies x is not an extreme point. This is a contradiction, so We must have $a_1, a_2, \cdots, a_k$ are linearly independent. Thus $x = (x_1, x_2, \cdots, x_k, 0, \cdots, 0)$ be a required basic feasible solution.

Example 2.4 Consider the system of constraint

$$x_1 + x_2 \leq 1$$

$$x_1 \leq 0$$

$$x_1 \geq 0, x_2 \geq 0.$$

Find the extreme points of the feasible region and the basic feasible solution also set up the correspondence between the two.

Solution. Clearly the feasible region is given by $x_1 = 0$, and $0 \le x_2 \le 1$ and is the line segment joining the points $(0,0)$ and $(0,1)$ in the x_1, x_2 plane. These two point are extreme points of the convex set of feasible solution. Now, consider constraints as equality, we have

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ x_1 + x_4 &= 0 \\ x_1 \ge 0, x_2 &\ge 0. \end{aligned}$$

On comparing with $Ax = b$, we have

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \; x = (\underline{x}_1, \underline{x}_2, x_3, x_4)^T, \; b = (1,0)^T.$$

We find the basic feasible solution of given system.

(i) Consider $B_1 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $|B_1| = -1 \neq 0$, so the corresponding variables $\underline{x}_1$ and $\underline{x}_2$ forms the basic solution. Put $x_3 = x_4 = 0$ (non-basic variables). Then, we get $\underline{x}_1 = 0$ and $\underline{x}_2 = 1$. It is feasible solution since $\underline{x}_1 \ge 0, \underline{x}_2 \ge 0$. It is degenerate basic feasible solution, since $\underline{x}_1 = 0$. The solution $x^T = (0,1,0,0)^T$ corresponds with the extreme point $(0,1)$.

(ii) Consider $B_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $|B_2| = -1 \neq 0$, so the corresponding variables $\underline{x}_1$ and x_3 forms the basic solution. Put $\underline{x}_2 = x_4 = 0$ (non-basic variables). Then, we get $\underline{x}_1 = 0$ and $x_3 = 1$. It is feasible solution since $\underline{x}_1 \geq 0, x_3 \geq 0$. It is degenerate basic feasible solution, since $\underline{x}_1 = 0$. The solution $x^T = (0,0,1,0)^T$ corresponds with the extreme point $(0,0)$.

(iii) Consider $B_3 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$ and $|B_3| = 1 \neq 0$, so the corresponding variables $\underline{x}_1$ and x_4 forms the basic solution. Put $\underline{x}_2 = x_3 = 0$ (non-basic variables). Then, we get $\underline{x}_1 = 1$ and $x_4 = -1$. It is not feasible solution since $x_4 < 0$. It is non-degenerate basic solution. Since it is not feasible, so it does not corresponds with any extreme point in $\underline{x}_1$-$\underline{x}_2$ plane.

(iv) Consider $B_4 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ and $|B_4| = 0$, so the corresponding variables $\underline{x}_2$ and x_3 do not forms the basic solution.

(v) Consider $B_5 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $|B_5| = 1 \neq 0$, so the corresponding variables $\underline{x}_2$ and x_4 forms the basic solution. Put $\underline{x}_1 = x_3 = 0$ (non-basic variables). Then, we get $\underline{x}_2 = 1$ and

$x_4 = 0$. It is feasible solution since $\underline{x}_2 \geq 0, x_4 \geq 0$. It is degenerate basic feasible solution, since $x_4 = 0$. The solution $x^T = (0,1,0,0)^T$ corresponds with the extreme point $(0,1)$.

(vi) Consider $B_6 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $|B_6| = 1 \neq 0$, so the corresponding variables x_3 and x_4 forms the basic solution. Put $\underline{x}_1 = \underline{x}_2 = 0$ (non-basic variables). Then, we get $x_3 = 1$ and $x_4 = 0$. It is feasible solution since $x_3 \geq 0, x_4 \geq 0$. It is degenerate basic feasible solution, since $x_4 = 0$. The solution $x^T = (0,0,1,0)^T$ corresponds with the extreme point $(0,0)$.

Theorem 2.6 The polyhedron $k = \{x : x \in \mathbb{R}^n,\ Ax = b, x \geq 0\}$ possesses at the most a finite number of extreme points.

Proof. Since the number of basic feasible solutions to problem (LP1) is finite, the result follows by Theorem.

Theorem 2.7 If the polyhedron $K = \{x : x \in \mathbb{R}^n,\ Ax = b, x \geq 0\}$ is non empty then it has at least one extreme point.

Proof. Since k is non empty, there is at least one feasible solution to problem (LP1) and therefor there exists at least one basic feasible solution to (LP1). Hence it follows from Theorem2.6 that there is at least one extreme point of the polyhedron K.

Theorem 2.8 Let the polyhedron $k = \{x : x \in Rn, Ax = b, x \geq 0\}$ be non empty and bounded. Then an optimal solution to the linear program (LP) exists and is obtained at a basic feasible solution to (LPP).

2.4 Simplex method

Example 2.5 Find the solution of

$$Minimize\ z = -6x_1 - 5x_2$$

subject to

$$x_1 + x_2 \leq 5,$$
$$3x_1 + 2x_2 \leq 12,$$
$$x_1, x_2 \geq 0.$$

Solution. We first write the given problem in standard form as follows

$$Minimize\ z = -6x_1 - 5x_2$$

subject to

$$x_1 + x_2 + x_3 = 5,$$
$$3x_1 + 2x_2 + x_4 = 12,$$
$$x_1, x_2 \geq 0.$$

First iteration: We start with a basic feasible solution with x_3 and x_4 as a basic variable. We write the basic variables in terms of non basic variables as

$$x_3 \;=\; 5 - x_1 - x_2 \tag{2.17}$$

$$x_4 = 12 - 3x_1 - 2x_2 \tag{2.18}$$

The present solution has values $z = 0$, since x_1 and x_2 has values zero. We want to decrease the value of z and this is possible by increasing x_1 or x_2. We choose to increase x_1 by bringing it to the basic, because it has highest rate of decrease.

Consider the equation (2.17), $\underline{x}_1$ can be increase up to 5, beyond this x_3 will be negative and infeasible and considering the equation (2.18), $\underline{x}_1$ can be increase to 4, beyond this x_4 will become negative. The limiting value of $\underline{x}_1$ is 4.

Second iteration: Since variable x_1 becomes basic based on the following equation

$$
\begin{aligned}
x_4 &= 12 - 3x_1 - 2x_2 \\
\Rightarrow 3x_1 &= 12 - 2x_2 - x_4 \\
\Rightarrow x_1 &= 4 - \frac{2}{3}x_2 - \frac{1}{3}x_4.
\end{aligned} \tag{2.19}
$$

Substituting the value of $\underline{x}_1$ in the (2.17), we have

$$
\begin{aligned}
x_3 &= 5 - 4 + \frac{2}{3}x_2 + \frac{1}{3}x_4 - \underline{x}_2 \\
&= 1 - \frac{1}{3}\underline{x}_2 + \frac{1}{3}x_4.
\end{aligned} \tag{2.20}
$$

Now, our modified objective function is

$$
\begin{aligned}
z &= -6(4 - \frac{2}{3}x_2 - \frac{1}{3}x_4) - 5x_2 \\
&= -24 + 4x_2 + \frac{1}{3}x_4 - 5x_2 \\
&= -24 - x_2 + 2x_4.
\end{aligned} \tag{2.21}
$$

In order to minimize z and this can be achieved by increasing x_2 or decreasing x_4 is not possible because x_4 is at zero and decreasing it will become infeasible. So, increase in $\underline{x}_2$ is only possible, since it is at zero.

From equation (2.19) we observe that x_2 can be increase to 6 beyond which variable $\underline{x}_1$ would become negative and infeasible. From equation (2.20), we observe that $\underline{x}_2$ can be increase up to 3 beyond this X_3 will be negative. The limiting value is 3 and the variable $\underline{x}_2$ replaces the variable X_3 in the basis.

Iteration 3: Rewriting (2.20) in terms of $\underline{x}_2$, we get

$$x_2 = 3 - 3x_3 + x_4. \tag{2.22}$$

Substituting $\underline{x}_2$ in equation (2.19), we have

$$\begin{aligned}\underline{x}_1 &= 4 - \frac{2}{3}x_2 - \frac{1}{4}x_4 \\ &= 4 - \frac{2}{3}(3 - 3x_3 + x_4) - \frac{1}{3}x_4 \\ &= 2 + 2x_3 - x_4. \end{aligned} \tag{2.23}$$

The value of z is

$$\begin{aligned} z &= -24 - x_2 + 2x_4 \\ &= -24 - (3 - 3x_3 + x_4) + 2x_4 \\ &= -24 - 3 + 3x_3 - x_4 + 2x_4 \\ &= -27 + 3x_3 + x_4. \end{aligned}$$

Now, we would still like to decrease z and this is possible only if we can decrease the value of x_3 and x_4. Since both have a positive coefficients in z both are non basic at zero and will only yield infeasible solution if we decrease then we observe that the optimum is reached. Since there is no entering variable. Thus the optimal solution is given by $\underline{x}_1 = 2, \underline{x}_2 = 3, x_3 = 0, x_4 = 0$ with $z = -27$.

Example 2.6 Find the optimal solution of the LPP

Minimize $z = -6\underline{x}_1 - 5\underline{x}_2$

subject to

$$\begin{aligned} \underline{x}_1 + \underline{x}_2 &\leq 5, \\ 3\underline{x}_1 + 2\underline{x}_2 &\leq 12 \\ \underline{x}_1,\ \underline{x}_2 &\geq 0. \end{aligned}$$

Solution. We first write the given problem in standard form of given problem is

Minimize $z = -6\underline{x}_1 - 5\underline{x}_2$

subject to

$$
\begin{aligned}
\underline{x}_1 + \underline{x}_2 + x_3 &= 5\\
3\underline{x}_1 + 2\underline{x}_2 + x_4 &= 12\\
\underline{x}_1,\ \underline{x}_2,\ x_3,\ x_4 &\geq 0.
\end{aligned}
$$

Initially, we choose x_3 and x_4 as a basic variable. We have simplex table as follows

		-6	-5	0	0		
CB	BV	x_1	x_2	x_3	x_4	b	Min + ve ratio
0	x_3	1	1	1	0	5	5
0	$\leftarrow x_4$	3	2	0	1	12	4
	$\overline{z_j}$	$-6\uparrow$	-5	0	0	0	
0	x_3	0	$\frac{1}{3}$	1	$-\frac{1}{3}$	1	3
-6	x_1	1	$\frac{2}{3}$	0	$\frac{1}{3}$	4	6
	$\overline{z_j}$	0	$-1\uparrow$	0	2	-24	
-5	x_2	0	1	3	-1	3	
-6	x_1	1	0	-2	1	2	
	$\overline{z_j}$	0	0	3	1	-27	

We observe that each $\overline{z_j} \geq 0$, thus the simplex table is terminated. The optimal solution of given LPP is $\underline{x}_1 = 2,\ \underline{x}_2 = 3,\ x_3 = 0,\ x_4 = 0$ with minimum value $z = -27$.

Example 2.7 Solve the following LPP

$$\text{Minimize } z = -\underline{x}_1 - \underline{x}_2$$

subject to

$$\begin{aligned} \underline{x}_1 + \underline{x}_2 &\leq 2 \\ \underline{x}_1 - \underline{x}_2 &\leq 1 \\ \underline{x}_2 &\leq 1 \\ \underline{x}_1, \underline{x}_2 &\geq 0. \end{aligned}$$

Solution. The standard form of given problem is

$$\text{Minimize } z = -\underline{x}_1 - \underline{x}_2$$

subject to

$$\begin{aligned} \underline{x}_1 + \underline{x}_2 + x_3 &= 2 \\ \underline{x}_1 - \underline{x}_2 + x_4 &= 1 \\ \underline{x}_2 + x_5 &= 1 \\ \underline{x}_1, \underline{x}_2, x_3, x_4, x_5 &\geq 0. \end{aligned}$$

Initially we take x_3, x_4 and x_4 as a basic variable. The simplex table is

		-1	-1	0	0	0		
CB	B V	x_1	x_2	x_3	x_4	x_5	b	Min + ve ratio
0	x_3	1	1	1	0	0	2	2
0	$\leftarrow x_4$	1	-1	0	1	0	1	1
0	x_5	0	1	0	0	1	1	$-$
	$\overline{z_j}$	$-1\uparrow$	-1	0	0	0	0	
0	$\leftarrow x_3$	0	2	1	-1	0	1	$\frac{1}{2}$
-1	x_1	1	-1	0	1	0	1	$-$
0	x_5	0	1	0	0	1	1	1
	$\overline{z_j}$	0	$-2\uparrow$	0	1	0	-1	
-1	x_2	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	$\frac{1}{2}$	
-1	x_1	1	0	$\frac{1}{2}$	$\frac{1}{2}$	0	$\frac{3}{2}$	
0	x_5	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1	$\frac{1}{2}$	
	$\overline{z_j}$	0	0	1	0	0	-2	

We observe the simplex table and each $\overline{z_j} \geq 0$. We reached at the optimality test. The optimal solution is $\underline{x}_1 = \frac{3}{2}, \underline{x}_2 = \frac{1}{2}, x_3 = 0,\ x_4 = 0,\ \ x_5 = \frac{1}{2}$ with minimum value $z\ =\ -2$.

Example 2.8 Solve the LPP by simplex method

$$\text{Minimize } z = -5\underline{x}_1 - 2\underline{x}_2$$

Subject to

$$\underline{x}_1 + 4\underline{x}_2 \leq 4$$
$$5\underline{x}_1 + 2\underline{x}_2 \leq 10$$
$$\underline{x}_1,\ \underline{x}_2 \geq 0.$$

Solution. The standard form is

$$\text{Minimize } z = -5\underline{x}_1 - 2\underline{x}_2$$

Subject to

$$\underline{x}_1 + 4\underline{x}_2 + x_3 = 4$$
$$5\underline{x}_1 + 2\underline{x}_2 + x_4 = 10$$
$$\underline{x}_1,\ \underline{x}_2,\ x_3,\ x_4 \geq 0.$$

The simplex table is

		-5	-2	0	0		
CB	BV	x_1	x_2	x_3	x_4	b	Min +ve ratio
0	x_3	1	4	1	0	4	4
0	$\leftarrow x_4$	5	2	0	1	10	2
	$\overline{z_j}$	$-5\uparrow$	-2	0	0	0	
0	x_3	0	$\frac{18}{5}$	1	$-\frac{1}{5}$	2	
-5	x_1	1	$\frac{2}{5}$	0	$\frac{1}{5}$	2	
	$\overline{z_j}$	0	0	0	1	-10	

By observing the simplex table, we have all $\overline{z_j} \geq 0$. Thus, the optimal solution is $\underline{x}_1 = 2$, $\underline{x}_2 = 0$, $x_3 = 2$ and $x_4 = 0$ with minimum value $z = -10$.

2.5 Two-Phase Method

Example 2.9 Use the simplex method to solve the problem

$$\text{Minimize } z = -2\underline{x}_1 - \underline{x}_2$$

Subject to

$$\underline{x}_1 + \underline{x}_2 \geq 2$$
$$\underline{x}_1 + \underline{x}_2 \leq 4$$
$$\underline{x}_1, \underline{x}_2 \geq 0.$$

Solution. The standard form is

$$\text{Minimize } z = -2\underline{x}_1 - \underline{x}_2$$

Subject to

$$\underline{x}_1 + \underline{x}_2 - x_3 = 2$$
$$\underline{x}_1 + \underline{x}_2 + x_4 = 4$$
$$\underline{x}_1, \underline{x}_2, x_3, x_4 \geq 0.$$

Phase I: In order to obtain an initial feasible solution, we need to add the artificial variable w_1 in the first constraint. Let us consider the following problem

$$\text{Minimize w} = w_1$$

Subject to

$$\underline{x}_1 + \underline{x}_2 - x_3 + w_1 = 2$$

$$\underline{x}_1 + \underline{x}_2 + x_4 = 4$$

$$\underline{x}_1, \underline{x}_2, x_3, x_4, \ w_1 \geq 0.$$

		0	0	0	0	1		
CB	BV	x_1	x_2	x_3	x_4	w_1	b	Min + ve ratio
1	$\leftarrow w_1$	1	1	-1	0	1	2	2
0	x_4	1	1	0	1	0	4	4
	$\overline{w_j}$	-1 ↑	-1	1	0	0	2	
0	x_1	1	1	-1	0	1	2	
0	x_4	0	0	1	1	-1	2	
	$\overline{w_j}$	0	0	0	0	1		

By observing each $\overline{w_j} \geq 0$. The artificial variable w_1 is removed from the phase I, so, we proceed to phase II.

Phase II: For the phase II, consider the objective function $z = -2\underline{x}_1 - \underline{x}_2$.

		-2	-1	0	0		
C_B	BV	x_1	x_2	x_3	x_4	b	Min +ve ratio
-2	x_1	1	1	-1	0	2	–
0	$\leftarrow x_4$	0	0	1	1	2	2
	$\overline{z_j}$	0	1	$-2\uparrow$	0	-4	
-2	x_1	1	1	0	1	4	
0	x_3	0	0	1	1	2	
	$\overline{z_j}$	0	1	0	2	-8	

Since, each $\overline{z_j} \geq 0$. The optimal solution is $\underline{x}_1 = 4,\ \underline{x}_2 = 0,\ x_3 = 2,\ x_4 = 0$ with $z = -8$.

Example 2.10 Use the simplex method to solve the problem

$$\text{Minimize } z = \underline{x}_1 + \underline{x}_2$$

subject to

$$\underline{x}_1 + 2\underline{x}_2 \le 2$$
$$3x_1 + 5\underline{x}_2 \ge 15$$
$$\underline{x}_1,\ \underline{x}_2 \ge 0.$$

Solution. The standard form of the LPP is

$$\text{Minimize } z = \underline{x}_1 + \underline{x}_2$$

subject to

$$\underline{x}_1 + 2\underline{x}_2 + x_3 = 2$$
$$3x_1 + 5\underline{x}_2 - x_4 = 15$$
$$\underline{x}_1,\ \underline{x}_2,\ x_3,\ x_4 \ge 0.$$

Phase I: To obtain an initial feasible solution, we need to add an artificial variable w_1 in the second constraint equation and consider the following LPP for Phase I

$$\text{Minimize } w = w_1$$

Subject to

$$\underline{x}_1 + 2\underline{x}_2 + x_3 = 2$$
$$3x_1 + 5\underline{x}_2 - x_4 + w_1 = 15$$
$$\underline{x}_1,\ \underline{x}_2,\ x_3,\ x_4,\ w_1 \geq 0.$$

The simplex table is

		0	0	0	0	1		
CB	BV	x_1	x_2	x_3	x_4	w_1	b	Min +ve ratio
0	$\leftarrow x_3$	1	2	1	0	0	2	1
1	w_1	3	5	0	-1	1	15	3
	$\overline{w_j}$	-3	$-5\uparrow$	0	1	0	15	
0	$\leftarrow x_2$	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	1	2
1	w_2	$\frac{1}{2}$	0	$-\frac{5}{2}$	-1	1	10	20
	$\overline{w_j}$	$-\frac{1}{2}\uparrow$	0	$\frac{5}{2}$	1	0	10	
0	$\leftarrow x_1$	1	2	1	0	0	2	
1	w_1	0	-1	-3	-1	1	9	
	$\overline{w_j}$	0	1	3	1	0	9	

Clearly $\overline{w_j} \geq 0$ and the artificial variables w_1 present in the basic solution. Hence, the given constraints are inconsistent. Thus, the problem does not have solution.

Example 2.11 Use the two-phase method to show that the LPP

$$\text{Minimize } z = -\underline{x}_1 + 2\underline{x}_2$$

subject to

$$\begin{aligned} \underline{x}_1 + 2\underline{x}_2 &\geq 1 \\ -\underline{x}_1 + \underline{x}_2 &\leq 1 \\ \underline{x}_1, \underline{x}_2 &\geq 0. \end{aligned}$$

has an unbounded solution.

Solution: The standard form of given problem is

$$\text{Minimize } z = -\underline{x}_1 + 2\underline{x}_2 +$$

subject to

$$\underline{x}_1 + 2\underline{x}_2 - x_3 = 1$$
$$-\underline{x}_1 + \underline{x}_2 + x_4 = 1$$
$$\underline{x}_1, \underline{x}_2, \ x_3, \ x_4 \geq 0.$$

Phase I: We add an artificial variable w_1 and consider the problem

$$\text{Minimize } w = w_1$$

subject to

$$\underline{x}_1 + 2\underline{x}_2 - x_3 + w_1 = 1$$
$$-\underline{x}_1 + \underline{x}_2 + x_4 = 1$$
$$\underline{x}_1, \underline{x}_2, \ x_3, \ x_4, \ w_1 \geq 0.$$

		0	0	0	0	1		
CB	BV	x_1	x_2	x_3	x_4	w_1	b	Min + ve ratio
1	$\leftarrow w_1$	1	2	-1	0	1	1	$\frac{1}{2}$
0	x_4	-1	1	0	1	0	1	1
	$\overline{w_j}$	1	$2\uparrow$	1	0	1	1	
0	x_2	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$	
0	x_4	$-\frac{3}{2}$	0	$\frac{1}{2}$	1	$-\frac{1}{2}$	$\frac{1}{2}$	
	$\overline{w_j}$	0	0	0	0	1	0	

By observing each $\overline{w_j} \geq 0$, and w_1 is removed from the basic variables so we proceed to phase II.

Phase II: The object function is minimize $z = -\underline{x}_1 + 2\underline{x}_2$

		-1	2	0	0		
CB	BV	x_1	x_2	x_3	x_4	b	Min +ve ratio
2	$\leftarrow x_2$	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
0	x_4	$-\frac{3}{2}$	0	$\frac{1}{2}$	1	$\frac{1}{2}$	–
	$\overline{z_j}$	$-2 \uparrow$	0	1	0	1	
-1	x_1	1	2	-1	0	1	–
0	x_4	0	3	-1	1	2	--
	$\overline{z_j}$	0	4	$-1 \uparrow$	0	-1	

Here, we cannot find the new outgoing vector. so, it concludes that the given LPP has an unbounded solution

Example 2.12 Find the solution of the LPP

$$\text{Minimize } z = 2 - \underline{x}_2$$

subject to

$$\underline{x}_1 - \underline{x}_2 = 4$$
$$-\underline{x}_2 - x_3 = 0$$
$$\underline{x}_1,\ \underline{x}_2, x_3 \geq 0.$$

Solution: The given problem is in the standard form. We treat $\underline{x}_1$ as a slack variable and x_3 as a surplus variable. We add an artificial variable w_1 in view of initial basic feasible solution.

Phase I: consider the problem

$$\text{Minimize } w = w_1$$

subject to

$$\begin{aligned} \underline{x}_1 - \underline{x}_2 &= 4 \\ -\underline{x}_2 - x_3 + w_1 &= 0 \\ \underline{x}_1,\ \underline{x}_2, x_3, w_1 &\geq 0. \end{aligned}$$

		0	0	0	1	
C_B	BV	x_1	x_2	x_3	w_1	b
0	x_1	1	-1	0	0	4
1	w_1	0	-1	-1	1	0
	$\overline{w_j}$	0	1	1	0	0
0	x_1	1	-1	0	0	4
1	$\leftarrow w_1$	0	1	1	-1	0
	$\overline{w_j}$	0	$-1 \uparrow$	-1	2	
0	x_1	1	0	0	-1	
0	x_2	0	1	1	-1	
	$\overline{w_j}$	0	0	0	1	

All $\overline{z_j} \geq 0$ and w_1 is removed from the basic variable. So, we proceed for the phase II.

Phase II: The objective function is $z = 2 - \underline{x}_2$

		0	-1	0	
CB	BV	x_1	x_2	x_3	b
0	x_1	1	0	1	4
-1	x_2	0	1	1	0
	$\overline{z_j}$	0	0	1	2

All $\overline{z_j} \geq 0$. Thus, the optimal solution is $\underline{x}_1 = 4$, $\underline{x}_2 = 0$, $x_3 = 0$ and minimum $z = 2$.

Example 2.13 Solve the LPP by two-phase method

$$\text{Maximize } z = -3\underline{x}_1 + 7\underline{x}_2$$

Subject to

$$\underline{x}_1 + 4\underline{x}_2 \geq 4$$
$$5\underline{x}_2 + 2\underline{x}_2 \geq 10$$
$$4\underline{x}_1 + 5\underline{x}_2 \leq 20$$
$$\underline{x}_1, \underline{x}_2 \geq 0.$$

Solution: The given problem, we can write in the standard form as follows.

$$\text{Maximize } z = -3\underline{x}_1 + 7\underline{x}_2$$

Subject to

$$
\begin{aligned}
\underline{x}_1 + 4\underline{x}_2 - x_3 &= 4 \\
5\underline{x}_2 + 2\underline{x}_2 - x_4 &= 10 \\
4\underline{x}_1 + 5\underline{x}_2 + x_5 &= 20 \\
\underline{x}_1, \underline{x}_2,\ x_3,\ x_4,\ x_5 &\geq 0.
\end{aligned}
$$

Phase I:

		0	0	0	0	0	1	1		
CB	BV	x_1	x_2	x_3	x_4	x_5	w_1	w_2	b	Min + ratio
1	$\leftarrow w_1$	1	4	-1	0	0	1	0	4	1
1	w_2	5	2	0	-1	0	0	1	10	5
0	x_5	4	5	0	0	1	0	0	20	4
	$\overline{w_j}$	-6	$-6\uparrow$	1	1	0	0	0	14	
0	x_2	$\frac{1}{4}$	1	$-\frac{1}{4}$	0	0	$\frac{1}{4}$	1	1	4
1	$\leftarrow w_2$	$\frac{9}{2}$	0	$\frac{1}{2}$	-1	1	$-\frac{1}{2}$	1	8	$\frac{16}{9}$
0	x_5	$\frac{1}{4}$	0	$\frac{5}{4}$	0	0	$-\frac{5}{4}$	0	15	$\frac{60}{11}$
	$\overline{w_j}$	$-\frac{9}{2}\uparrow$	0	$-\frac{1}{2}$	1	-1	$\frac{3}{2}$	0	8	
0	x_2	0	1	$-\frac{5}{8}$	$\frac{1}{18}$	0	$\frac{5}{18}$	$-\frac{1}{18}$	$\frac{5}{9}$	
0	x_1	1	0	$\frac{1}{9}$	$-\frac{2}{9}$	0	$-\frac{1}{9}$	$\frac{2}{9}$	$\frac{16}{9}$	
0	x_5	0	0	$\frac{17}{18}$	$\frac{11}{18}$	1	$-\frac{18}{18}$	$-\frac{1}{18}$	$\frac{11}{9}$	
	$\overline{w_j}$	0	0	0	0	0	1	1	0	

We need to add two artificial variables in first and second constraints.

$$\text{Maximize } w = w_1 + w_2$$

Subject to

$$\begin{aligned} \underline{x}_1 + 4\underline{x}_2 - x_3 + w_1 &= 4 \\ 5\underline{x}_2 + 2\underline{x}_2 - x_4 + w_2 &= 10 \\ 4\underline{x}_1 + 5\underline{x}_2 + x_5 &= 20 \\ \underline{x}_1, \underline{x}_2,\ x_3,\ x_4,\ x_5,\ w_1,\ w_2 &\geq 0. \end{aligned}$$

All $\overline{w_j} \geq 0$ and $w_1,\ w_2$ are removed from the basic variables. We have we go to the phase II.

Phase II: The objective function for this phase is minimum $z = -3\underline{x}_1 + 7\underline{x}_2$