extreme point of S. We claim that that $K = S$. Clearly $K \subseteq S$. Suppose $S \nsubseteq K$. Then there is a $y \in S$ but $y \notin K$. But y is exterior to K, by Theorem 1.7 there exists $a \neq 0$ such that

$$a^T y < \inf_{x \in K} (a^T x) \tag{1.1}$$

Let $s_0 = \inf_{x \in S} a^T x$. Since the function $a^T x$ is continuous on compact set S, then the function $a^T x$ attains its minimum value at $x_0 \in S$ with

$$s_0 = \inf_{x \in S} a^T x = \min_{x \in S}(a^T x) = a^T x. \tag{1.2}$$

It gives that

$$a^T x_0 \leq a^T x \text{ for all } x \in S. \tag{1.3}$$

Then (1.1) and (1.2) implies that, the hyperplane $H = \left\{x : a^T x = s_0\right\}$ is a supporting hyperplane to S at $x_0 \in S$. Using relation (1.2) and (1.3), we have

$$y \in S \Rightarrow a^T x_0 \leq a^T y < \inf_{x \in K}(a^T x).$$

Since $K \subseteq S$, $x_0 \notin K$ and H is a supporting hyperplane to S at x_0. Then the sets H and K are disjoint. Let $T = H \cap S$. Then T is a closed bounded subset of H and it is a space of dimension $(n-1)$. Since $x_0 \in S$, $x_0 \in H$ then $x_0 \in T$. This means that

T is a non-empty closed bounded subset of $\mathbb{R}^{n-1}$. Hence by induction hypothesis, T is a closed convex hull of extreme point of T, i.e. T contains extreme points. By using repeated use of this Theorem, we an prove all other extreme point of T are also the extreme point of S. Thus, we found x_0 that lies in the convex hull of some extreme point of S and $x_0 \notin K$. It is a contradiction to that K is a closed convex hull of the extreme points of S, so, we have $K = S$.

Definition 1.15 Let S and T be two non empty subset of $\mathbb{R}^n$ then a hyperplane$H$ is said to be separate S and T if S is contained in one of the closed half spaces generated by H and T is contained in the other closed half space. The hyperplane$H$ in this case is called a separating hyperplane.

Definition 1.16 A hyperplane H strictly separates S and T if S is contained in one of the open half spaces generated by H and T is contained in other half plane.

Theorem 1.11 If $S \subseteq \mathbb{R}^n$ is non empty convex set and $0 \notin S$, then there exists a hyperplane separating S and 0.

Proof. We will give proof in two different situations.

1. Suppose 0 lies in an exterior S. Then by Theorem 1.7, there exists a vector $0 \neq a \in \mathbb{R}^n$ such that $0 < a^T x$ for $x \in S$. So, the hyperplane $H = \left\{x : a^T x = c\right\}$, where $0 < c < a^T x$ separate S and 0.
2. Suppose $0 \in \overline{S}$, by Theorem, there is a supporting hyperplane $H = \left\{x : a^T x = 0\right\}$ to S at the 0 and it separates S and 0.

Theorem 1.12 Let S and T be two non empty disjoint convex sets in $\mathbb{R}^n$. Then there exists a hyperplane that separates S and T.

Proof. Clearly, $S - T$ is convex and $0 \notin S \cap T$, because $S \cap T = \phi$. So, there exists a vector a such that $a^T x \geq 0$ for all $x \in S - T$. It means that for all $u \in S$ and $v \in T$, we have $a^T (u - v) \geq 0$. So, there exist a number c satisfying.

$$a^T u - a^T v \geq 0$$

$$\inf\left(a^T u - a^T v\right) \geq 0$$

$$\inf a^T u - \sup a^T v \geq 0$$

$$\inf a^T u \geq c \geq \sup a^T v.$$

This implies that the hyperplane $H = \left\{x : a^T x = 0\right\}$ separate S and T.

Theorem 1.13 Let S be a non empty closed convex set in $\mathbb{R}^n$ not containing 0. Then there exists a hyperplane that strictly separates S and the 0.

Proof. Let S is closed set, so $\overline{S} = S$ and $0 \notin S$ i.e. 0 is the exterior to S. Hence by Theorem 1.7, there exist $a \neq 0 \in \mathbb{R}^n$ such that $0 < \inf_{x \in S} a^T x$. Now, we choose a real number c such that $0 < c < \inf_{x \in S} a^T x$. Clearly the hyperplane $H = \left\{x : a^T x = c\right\}$ strictly separates 0 and S.

1.3 Convex polyhedron and polytope

Definition 1.17 The convex hull of a finite (non zero) number of points is called convex polytope spanned by these points.

Let $S = \{x_1, x_2, \cdots, x_m\}$ where $x_i \in \mathbb{R}^n$ then the convex polytope spanned by the points of S is the convex set

$$\mathrm{CO}(s) = \left\{ x : x = \sum_{i=1}^{m} \lambda_i x_i, \lambda_1 \geq 0, \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Clearly a convex polytope is a non-empty convex set.

Theorem 1.14 The set of vertices of a convex polytope is a subset of the set of spanning points of the polytope.

Proof. Suppose V is the set of vertices of the convex polytope $\mathrm{CO}(s)$ spanned by the points of the set $S = \{x_1, x_2, \cdots, x_m\}$. It is clear that the result is true when $m = 1$. Now, assume contrary that $V \not\subset S$. Then, there exist $x \in V$ such that $x \notin S$. Since $x \in V \Rightarrow x \in \mathrm{CO}(S)$. Therefore $x = \sum_{i=1}^{m} \mu_i x_i$, Where $\mu_i \geq 0,\ i = 1, \cdots, m$ and $\sum_{i=1}^{m} \mu_i = 1$. Since $x \notin S$, it follows that $\mu_i > 0$ and $\mu_i \neq 1,\ i = 1, 2, \cdots, m.$. Hence $x = \sum_{i=1}^{m} \mu_i x_i$ implies that there exists $\mu_i\,(0 < \mu_i < 1)$. Let it be μ_1

$$x = \mu_1 x_1 + \sum_{i=2}^{m} \mu_i x_i = \mu_1 x_1 + (1 - \mu_1) \sum_{i=2}^{m} \frac{\mu_i}{(1 - \mu_1)} x_i = \mu_1 x_1 + (1 - \mu_1) x_i,$$

where $y = \sum_{i=2}^{m} \frac{\mu i}{(1 - \mu_1)} x_i$. Clearly $y = \sum_{i=2}^{m} \left(\frac{\mu_i}{(1 - \mu_1)} \right) x_i$, $0 < \mu_1 < 1$. This implies that

$y \in \mathrm{CO}(S)$. Hence x is not a extreme (vertex) of $CO(S)$. It is a contradiction. Hence, we must have $V \subseteq S$.

Theorem 1.15 Let $K = \{x : Ax = b,\ x \geq 0\}$ be a non-empty polyhedral set. Then the set of extreme points of K is non empty and has a finite number of points.

1.4 Convex function

Definition 1.18 A function f defined on a set $T \subseteq \mathbb{R}^n$ is said to be convex at $x_0 \in T$ if $x_1 \in T$, $0 \leq \lambda \leq 1$, $\lambda x_0 + (1-\lambda)x_1 \in T$, then

$$f(\lambda x_0 + (1-\lambda)x_1) \leq \lambda f(x_0) + (1-\lambda) f(x_1).$$

Definition 1.19 A function f is said to be convex on T if it is convex at every point of T.

$x_0, x_1 \in T,$

Domain of f necessary to be a convex set. Therefore other way convex set is defined as follows.

Definition 1.20 If T is convex set then f is said to be convex on T if $x_1, x_2 \in T, 0 \leq \lambda \leq 1 \Rightarrow$

$$f(\lambda x_0 + (1-\lambda)x_1) \leq \lambda f(x_0) + (1-\lambda) f(x_1).$$

In geometrical point of view, a function $y = f(x)$ defined on a convex set T is convex if the chord joining, any two points on the graph of f lies on or above the graph.

Definition 1.21 A function f defined on a set $T \subseteq \mathbb{R}^n$ is said to be strictly convex at $x_0 \in T$ if $x_1 \in T$, $0 < \lambda < 1$, $x_0 \neq x_1$, $\lambda x_0 + (1-\lambda)x_1 \in T$, then

$$f(\lambda x_0 + (1-\lambda)x_1) < \lambda f(x_0) + (1-\lambda) f(x_1).$$

Definition 1.22 A function f is said to be concave at $x_0 \in T$ if $-f$ is convex at $x_0 \in T$.

$f: \mathbb{N} \longrightarrow \mathbb{N}$ & $f(x) = x^2$.
is f convex.

Example 1.12 Show that the linear function $f(x) = c^T x + d$ is both convex and convex on $\mathbb{R}^n$.

$f(x) = c^T x + d$. $\quad x' \quad \lambda x_1 + (1-\lambda)x_2$

Solution. Clearly, $\mathbb{R}^n$ is a convex set. Let $x_1,\ x_2 \in \mathbb{R}^n$ and $0 \leq \lambda \leq 1$. Consider

$f(\lambda x_1 + (1-\lambda x_2 \in$

$$
\begin{aligned}
f(\lambda x_1 + (1-\lambda)x_2) &= c^T(\lambda x_1 + (1-\lambda)x_2) + d \\
&= \lambda c^T x_1 + \lambda d + (1-\lambda)c^T x_2 + d - \lambda d \\
&= \lambda\left(c^T x_1 + d\right) + (1-\lambda)\left(c^T x_2 + d\right) \\
&\cancel{= c^T(\lambda x_1 + (1-\lambda)x_2) + d}
\end{aligned}
$$

$\leq$

$f(x) = c^T x + d$

$= \lambda f(x_1) + (1-\lambda) f(x_2)$

$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$

$\Rightarrow f$ is convex on $\mathbb{R}^n$. Similarly, we can show that $-f$ is convex function.

Example 1.13 Let f be a convex function on a convex set $T \subseteq \mathbb{R}^n$. Then for every $k \in \mathbb{R}$. Show that $T_k = \{x : x \in T,\ f(x) \leq k\}$ is convex set.

Solution. Let $x_1, x_2 \in T_K$ and $0 \leq \lambda \leq 1$. Then $x_1,\ x_2 \in T$. Since T is convex $\Rightarrow \lambda x_1 + (1 - \lambda x_2) \in T$ and, $f(x_1) \leq k,\ \ f(x_2) \leq k$. Given that f is convex on T, we have

$$
\begin{aligned}
f(\lambda x_1 + (1-\lambda)x_2) &\leq f(x_1) + (1-\lambda) f(x_2) \\
&\leq \lambda k x + (1-\lambda)k = k \\
&\Rightarrow \lambda x_1 + (1-\lambda)x_2 \in T_k.
\end{aligned}
$$

2. f convex $\Rightarrow -f$ concave $f+g$, $f-g$, $f\cdot g$, f/g.

(−1).

Theorem 1.16 Let f and g be convex function and let μ be a non negative number. Then μf and $f+g$ are convex functions. Further if $\mu > 0$ and f is strictly convex then μf is strictly convex.

μf, $f+g$.

Proof. Let f and g be convex function on convex set T. Let $x_1, x_2 \in T$, $0 \le \lambda \le 1$,

f is convex on T

$(\mu f)(x) = \mu(f(x))$

$\mu f(\lambda x_1 + (1-\lambda)x_2)$

$$
\begin{aligned}
\mu f(\lambda x_1 + (1-\lambda)x_2) &= \mu(f(\lambda x_1 + (1-\lambda)x_2)) \\
&\le \mu(\lambda f(x_1) + (1-\lambda)f(x_2)) \\
&= \lambda \mu f(x_1) + (1-\lambda)\mu f(x_2) \\
&= \lambda(\mu f)(x_1) + (1-\lambda)(\mu f)(x_2).
\end{aligned}
$$

$\Rightarrow \mu f$ is a convex function. Now

$(f+g)(x) = f(x) + g(x)$

$$
\begin{aligned}
(f+g)(\lambda x_1 + (1-\lambda)\ x_2) &= f(\lambda x_1 + (1-\lambda)\ x_2) + g(\lambda x_1 + (1-\lambda)\ x_2) \\
&\le \lambda f(x_1) + (1-\lambda) f(x_2) + \lambda g(x_1) + (1-\lambda) g(x_2) \\
&= \lambda(f(x_1) + g(x_1)) + (1-\lambda)(f(x_2) + g(x_2)) \\
&= \lambda(f+g)(x_1) + (1-\lambda)(f+g)(x_2).
\end{aligned}
$$

It shows that $f + g$ is a convex function.

$\Rightarrow f+g$ — convex function

If $\mu > 0$ and f is strictly convex, then

$$
\begin{aligned}
(\mu f)(\lambda x_1 + (1-\lambda)x_2) &= \mu(f(\lambda x_1 + (1-\lambda)x_2)) < \mu(\lambda f(x_1) + (1-\lambda) f(x_2)) \\
&= \lambda \mu f(x_1) + (1-\lambda)\mu f(x_2) = \lambda(\mu f)(x_1) + (1-\lambda)(\mu f)(x_2)
\end{aligned}
$$

$\mu_1 \quad \mu_2 \quad \mu_n > 0$

$\sum_{i=1}^{k} \mu_i f_i \quad f_1, f_2, \ldots, f_n$

Thus μf is strictly convex

Theorem 1.17 Every linear combination $\sum_{i=1}^{k} \mu_i f_i$ of convex function f_i where $\mu_i \geq 0$, $i = 1, 2, \cdots, k$ is a convex function. Also, such a combination is strictly convex if at last one $\mu_i > 0$, $i = 1, 2, \cdots, k$ and the corresponding f_i strictly convex.

Proof. Repeated use of Theorem 1.16 follows proof.

Theorem 1.18 Let h be a non decreasing convex function on $\mathbb{R}$ and f be a convex function on a convex set $T \subseteq \mathbb{R}^n$. Then the composite function $h \circ f$ is convex on T.

Proof. Let $h : R \to R$ and $f x{:}T{\to}R^n$ be two convex functions. To show that $h \circ f$ is a convex on T. Let $x_1, x_2 \in T textand\ 0 \leq \lambda \leq 1$. Since f is convex on T

$$
f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2). \tag{1.4}
$$

Since h is non decreasing, we have

$$h(f(\lambda x_1 + (1-\lambda)x_2)) \leq h(\lambda f(x_1) + (1-\lambda)f(x_2)). \tag{1.5}$$

Also h is convex function

$$h(\lambda f(x_1) + (1-\lambda)f(x_2)) \leq \lambda h(f(x_1)) + (1-\lambda)h(f(x_2)) \tag{1.6}$$

From (1.5) and (1.6)

$$h(f(x_1 + (1-\lambda)x_2)) \leq \lambda h(f(x_1)) + (1-\lambda)h(f(x_2))$$

$\Rightarrow h \circ f$ is convex function on T.

Theorem 1.19 Let f be differentiable on an open convex set $T \subseteq \mathbb{R}^n$. Then f is convex on T iff

$$f(x_2) \geq f(x_1) + \nabla f(x_1)^T (x_2 - x_1) \tag{1.7}$$

For all $x_1, x_2 \in T$. Further f is strictly convex on T if and on if the inequality is strict ($>$) for all $x_1, x_2 \in T, x_1 \neq x_2$.

Proof. Let f be differentiable on open convex set $T \subseteq \mathbb{R}^n$. Suppose f is convex

$\lambda x_1 + (1-\lambda)x_2$ $\quad$ $\lambda x_2 + (1-\lambda)x_1$

on T. Let $x_1, x_2 \in T$ and $0 \le \lambda \le 1$,

$$f(\lambda x_2 + (1-\lambda)x_1) \le \lambda f(x_2) + (1-\lambda) f(x_1).$$

$\lambda x_2 + x_1 - \lambda x_1$

$x_1 + \lambda(x_2 - x_1)$

It implies that

$$f(x_1 + \lambda(x_2 - x_1)) \le f(x_1) + \lambda(f(x_2) - f(x_1)).$$

Using differentiability of f, we have

$$f(x_1) + \lambda \nabla f(x_1)^T (x_2 - x_1) + \alpha |x_2 - x_1| \le f(x_1) + \lambda(f(x_2) - f(x_1)).$$

It implies that

$$\lambda(f(x_1) - f(x_2)) \le f(x_1) - \left[f(x_1) + \lambda \nabla f(x_1)^T (x_2 - x_1) + \alpha |x_2 - x_1|\right].$$

Canceling λ on both sides and letting $\lambda \to 0$ $(\alpha \to 0)$, we have

$$f(x_1) - f(x_2) \le \nabla f(x_1)^T (x_2 - x_1)$$
$$\Rightarrow f(x_2) \ge f(x_1) + \nabla f(x_1)^T (x_2 - x_1).$$

$\Rightarrow f(x_1) + \nabla f(x_1)^T (x_2 - x_1) \le f(x_2)$

Hence the inequality (1.7).

Now, suppose (1.7) is true. Let $x_1, x_2 \in T$ and $0 \le \lambda \le 1$ with $x_3 = \lambda x_1 + (1 - \lambda x_2)$. Using (1.7) for $x_1,\ x_3$ and x_1, x_2, we have

$$f(x_1) \ge f(x_3) + \nabla f(x_3)^T (x_1 - x_3) \tag{1.8}$$

$f(x_2) \geq f(x_3) + f'(x_3)(x_2 - x_3)$

$$f(x_2) \geq f(x_3) + \nabla f(x_3)^T (x_2 - x_3) \tag{1.9}$$

Multiplying (1.8) by λ and (1.9) by $1-\lambda$ and adding these, we get

$$\begin{aligned}
&\lambda f(x_1) + (1-\lambda) f(x_2) \\
&\geq \lambda f(x_3) + \lambda \nabla f(x_3)^T (x_1 - x_3) + (1-\lambda) f(x_3) + (1-\lambda) \nabla f(x_3)^T (x_1 - x_3) \\
&= \lambda f(x_3) + f(x_3) - \lambda f(x_3) + \nabla f(x_3)^T (\lambda (x_1 - x_3) + (1-\lambda)(x_1 - x_3)) \\
&= f(x_3) + \nabla f(x_3)^T (\lambda x_1 - \lambda x_3 + x_2 - x_3 - \lambda x_2 + \lambda x_3) \\
&= f(x_3) + \nabla f(x_3)^T (\lambda x_1 + (1-\lambda) x_2 - x_3) \\
&= f(x_3) + \nabla f(x_3)^T (x_3 - x_3) = f(x_3).
\end{aligned}$$

$B(x_0, r) \subseteq \mathbb{R}^n$
$= \{x \in \mathbb{R}^n : d(x, x_0) < r\}$

$f : T =$

It implies that

$$f(x_1 + (1-\lambda) x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2).$$

f' f'

Thus f is convex on T.

Similarly we can prove the result when f is strictly convex.

Theorem 1.20 Let f be a differentiable function of one variable defined on an open interval $T \subseteq \mathbb{R}^n$. Then f is convex (strictly convex) on T if and only if f' the derivative of f is non decreasing (strictly increasing) on T.

f'

$x_1 \leq x_2 \implies f'(x_1) \leq f'(x_2)$

Proof. Let f be convex on T and let $x_1 < x_2$, $x_1, x_2 \in T$. By Theorem 1.19, we have

$\nabla = \frac{\partial}{\partial x}$

$$f(x_2) \geq f(x_1) + f^{1}(x_1)(x_2 - x_1)$$

$f(x_2) - f(x_1) \geq f'(x_1)(x_2 - x_1)$

$$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq f'(x_1).$$

$\Rightarrow \frac{f(x_2) - f(x_1)}{x_2 - x_1} \geq f'(x_1)$

Again we have,

$$f(x_1) \geq f(x_2) + f'(x_2)(x_1 - x_2).$$

$x_2 \leq x_1$ $\quad$ $x_1 < x_2$

It implies

$$f(x_1) - f(x_2) \geq f'(x_2)(x_1 - x_2).$$

$x_2 - x_1 \geq 0$

Hence

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} \geq f'(x_2)$$

$x_1 - x_2 < 0$

$\Rightarrow f'(x_1) \leq \frac{f(x_1) - f(x_2)}{x_1 - x_2} \leq f'(x_2)$

From above inequalities, we have,

$x_1 \leq x_2 \Rightarrow$

$$f'(x_1) \leq f'(x_2).$$

Conversely, Let $x_1, x_2 \in T$, $x_1 < x_2$ and $x_3 = \lambda x_1 + (1-\lambda)x_2$, $0 < \lambda < 1$. By mean value theorem

$$f'(\eta_2) = \frac{f(x_3) - f(x_1)}{x_3 - x_1} x \; for \; x_1 < \eta_2 < x_3.$$

$x_1 < x_2$

$f(x_3) = f(\lambda x_1 + (1-\lambda) x_2) \le \lambda f(x_1) + (1-\lambda) f(x_2)$

$x_3 = \lambda x_1 + (1-\lambda) x_2$

$$\Rightarrow f(x_3) = f(x_1) + f'(n_2)(x_3 - x_1)$$
$$= f(x_1) + f'(n_2)(\lambda x_1 + (1-\lambda)x_2 - x_1)$$
$$= f(x_1) + (1-\lambda)(x_2 - x_1) f'(n_2)$$

$f : [a,b]$

(a,b)

$a < c < b$ s.t.

$f'(c) = \dfrac{f(b) - f(a)}{b-a}$

$f'(\eta_2) = \dfrac{f(x_3) - f(x_1)}{(x_2 - x_1)(1-\lambda)}$

$x_1 < \eta_2 < x_2$

Also,

$$f'(\eta_1) = \frac{f(x_2) - f(x_3)}{x_2 - x_3} x\ for\ x_3 < \eta_1 < x_2.$$

$f'(\eta_2) = \dfrac{f(x_3) - f(x_1)}{x_3 - x_1}$

$f'(\eta_2) \cdot (x_3 - x_1) = f(x_3) - f(x_1)$

$\Rightarrow f(x_3) = f(x_1) + f'(\eta_2)(x_3 - x_1)$

It can be written as

$$f(x_3) = f(x_2) + f'(\eta_1)(x_2 - x_3)$$
$$= f(x_2) + f'(\eta_1)(x_2 - (\lambda x_1 + (1-\lambda)x_2))$$
$$\Rightarrow f(x_3) = f(x_2) + f'(\eta_1)\lambda(x_2 - x_1).$$

$\eta_2 < \eta_2$

If f' is non decreasing and $\eta_2 < \eta_1$, we have

$$
\begin{aligned}
& f'(\eta_2) \le f'(\eta_1) \\
\Rightarrow f'(\eta_2) = \frac{f(x_3) - f(x_1)}{(1-\lambda)(x_2 - x_1)} &\le \frac{f(x_2) - f(x_3)}{\lambda(x_2 - x_1)} = f'(\eta_1) \\
\Rightarrow \frac{f(x_3) - f(x_1)}{(1-\lambda)(x_2 - x_1)} &\le \frac{f(x_2) - f(x_3)}{\lambda(x_2 - x_1)} \\
\Rightarrow \lambda f(x_3) - \lambda f(x_1) &\le f(x_2) - f(x_3) - \lambda f(x_2) + \lambda f(x_3) \\
\Rightarrow f(x_3) &\le f(x_2) + \lambda f(x_1) - \lambda f(x_2) \\
\Rightarrow f(x_3) &\le \lambda f(x_1) + (1-\lambda) f(x_2).
\end{aligned}
$$

$f(x_3) \le \lambda f(x_1)$ $f(-x)$ $f(x_2)$

Hence proved

$\Rightarrow$ f is a convex

Hence f is convex function on T. Similarly we can prove the result when f is strictly convex.

$g = -f \Rightarrow R^0 = \{x : g\, f(x)\}$ $f(x) < 0$

Example 1.14 Let $g(x)$ be a concave function on $R^0 = \{x : x \in \mathbb{R}^n, g(x) > 0\}$. Then show that the function $\dfrac{1}{g(x)}$ is convex on R^0.

$\rightarrow$ convex..

Solution. Put $h(x) = -\dfrac{1}{x}$ for $\{x : x < 0\}$ and $f(x) = -g(x)$. Since $x < 0$,

$$h'(x) = \frac{1}{x^2} > 0 \quad \text{and} \quad h(x) = -\frac{2}{x^3} > 0.$$

R

$x_1 < x_2 < 0$

$\frac{1}{x_1} > \frac{1}{x_2}$ $\quad -\frac{1}{x_1} < -\frac{1}{x_2} \Rightarrow h(x_1) < h(x_2)$ $(-\infty, 0]$

h h' $-f(x)$

Thus h is a non decreasing convex function on $\{x : x < 0\}$. Moreover, $f(x)$ is convex on R^0. Hence by Theorem 1.18 the composition

$$h \circ f = -\frac{1}{f(x)} = -\frac{1}{-g(x)} = \frac{1}{g(x)}$$

$$R_0 \xrightarrow{f} (-\infty, 0] \xrightarrow{h} \mathbb{R}$$

is convex on R^0.

$h \circ f : R_0 \longrightarrow \mathbb{R}$

$h(f(x)) = h(f(x)) = -\frac{1}{f(x)} = \frac{1}{g(x)}$ is convex function on R^0

Example 1.15 Show that the function

$$g(z) = \sum_{i=1}^{n} c_i \exp\left(\sum_{j=1}^{m} a_{i,j} z_j\right),$$

where $a_{i,j} \in \mathbb{R}$ and $c_i > 0$ is convex on $\mathbb{R}^m$.

Solution. Put

$$g_i(z) = \exp\left(\sum_{j=1}^{m} a_{i,j} z_j\right).$$

Since $h(x) = e^x$ is non a decreasing convex function on R and

$$f_i(z) = \sum_{j=1}^{m} a_{i,j} z_j$$

being a linear, it is convex on R^m. By Theorem 1.18,

$$g_i(z) = (h \circ f_i)(z) = \exp\left(\sum_{j=1}^{m} a_{i,j} z_j\right)$$

is convex on $\mathbb{R}^m$. Thus by Corollary, we have

$$g(z) = \sum_{i=1}^{m} c_i g_i(z) = \sum_{i=1}^{m} c_i exp(\sum_{j=1}^{m} a_{i,j} z_j)$$

is convex on R^m.

quasi-convex

1.5 Generalized Convexity

$x_1, x_2 \in T,\ 0 < \lambda < 1 \Rightarrow f(\lambda x_1 + (1-\lambda)x_2) \le \max\{f(x_1), f(x_2)\}$

Definition 1.23 A function f is said to be quasi convex on a convex set $T \subseteq \mathbb{R}^n$ if $x_1, x_2 \in T$, $0 < \lambda < 1 \Rightarrow f(\lambda x_1 + (1-\lambda)x_2) \le \max\{f(x_1), f(x_2)\}$.

$<$

Definition 1.24 A function f is said to be strictly quasiconvex of $T \subseteq \mathbb{R}^n$ if $x_1, x_2 \in T$, $x_1 \neq x_2$, $0 < \lambda < 1 \Rightarrow f(\lambda x_1 + (1-\lambda)x_2) < \max\{f(x_1), f(x_2)\}$.

Theorem 1.21 Let f be differentiable on an open convex set $T \subseteq \mathbb{R}^n$. Then f is quasi convex on T if and only if for all $x_1, x_2 \in T$ satisfying $f(x_1) \le f(x_2)$ the inequality

f is quasiconvex $\Longleftrightarrow$ $x_1, x_2 \in T$, $f(x_1) \le f(x_2)$

$$\nabla f(x_2)^T (x_1 - x_2) \le 0$$

$\Rightarrow \nabla f(x_2)^T (x_1 -$

holds.

Proof. Let f be quasiconvex and $x_1, x_2 \in T$ such that $f(x_1) \le f(x_2)$, then for $0 < \lambda < 1$, we have,

$$f(\lambda x_1 + (1-\lambda)x_2) \le \max\{f(x_1), f(x_2)\} = f(x_2) \Rightarrow f(\lambda x_1 + (1-\lambda)x_2) - f(x_2) \le 0.$$

$(\lambda x_1 + x_2 - \lambda x_1)$

That is

$f(x_2+h) = f(x_2) + h \cdot f'(x_2) + \frac{h^2}{2!} \cdot f''(x_2) + \cdots$

$$f(x_2 + \lambda(x_1 - x_2)) - f(x_2) \le 0$$

$x_2 + \lambda(x_1 - x_2)$

$$\frac{f(x_2 + \lambda(x_1 - x_2)) - f(x_2)}{\lambda} \le 0.$$

By Taylor's theorem, we have

$$\frac{f(x_2) + \lambda f(x_2)^T (x_1 - x_2) + \lambda^2 f(x_2)^{TT} (x_1 - x_2) + \cdots - f(x_2)}{\lambda} \le 0.$$

$\lambda \nabla f(x_2)^T (x_1 - x_2) + \lambda^2 \nabla^2 f(x_2)^T (x_1 - x_2)^2 + \cdots - f(x_2)$

Letting $\lambda \to 0^+$, we have

$$\nabla f(x_2)^T(x_1 - x_2) \le 0.$$

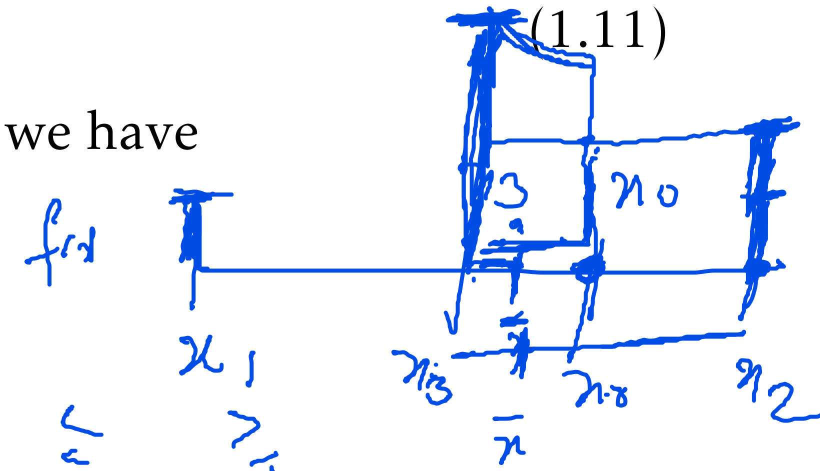
Converse, suppose $f(x_1) \le f(x_2)$ and $\nabla f(x_2)^T(x_1 - x_2) \le 0$ holds. Put $x_3 = \lambda x_1 + (1-\lambda)x_2, 0 < \lambda < 1$. We will show that

$$f(x_3) = f(\lambda x_1 + (1-\lambda)x_2) \le \max\{f(x_1), f(x_2)\} = f(x_2)$$

i.e to show $f(x_3) \le f(x_2)$. We assume contrary that $f(x_3) > f(x_2)$. Then $f(x_1) \le f(x_2) < f(x_3)$. By (1.21), we have

$$\nabla f(x_3)^T(x_2 - x_3) \le 0. \tag{1.10}$$

$$\nabla f(x_3)^T(x_1 - x_3) \le 0. \tag{1.11}$$



By changing x_3 by x_1 in (1.10) and x_3 by x_2 in (1.11), we have

$$\nabla f(x_3)^T(x_2 - x_1) \le 0.$$

$$\nabla f(x_3)^T(x_1 - x_2) \le 0.$$

It shows that

$$\nabla f(x_3)^T(x_1 - x_2) = 0. \tag{1.12}$$

Thus, we have shown that if $f(x_1) \le f(x_2) < f(x_3)$ and x_3 is a convex combination of x_1 and x_2 then the equation (1.12) is true. Since f is continuous so there exists an x_0 between x_3 and x_2 i.e. $x_0 = \mu x_2 + (1-\mu)x_3$ for some $0 \le \mu \le 1$ such that $f(x_2) < f(x)$ for all x between x_0 and x_3, and $f(x_2) \ge f(x_0)$. Now, by mean value theorem, we have

$$
\begin{aligned}
f(x_3) &= f(x_0) + \nabla f(\bar{x})^T (x_3 - x_0) \\
&= f(x_0) + \nabla f(\bar{x})^T (\lambda x_1 + (1-\lambda)x_2 - x_0) \\
&= f(x_0) + \nabla f(\bar{x})^T (\lambda x_1 + \lambda_2 - \lambda x_2 - x_0) \\
&= f(x_0) + \nabla f(\bar{x})^T (\lambda x_1 + x_2 - \lambda x_2 - \mu x_2 - (1-\mu)x_3) \\
&= f(x_0) + \nabla f(\bar{x})^T (\lambda(x_1 - x_2) + x_2 - \mu x_2 - (1-\mu)(\lambda x_1 + (1-\lambda)x_2) \\
&= f(x_0) + \nabla f(\bar{x})^T (\lambda x_1 - \lambda x_2 + x_2 - \mu x_2 - \cancel{\lambda x_1 + \mu\lambda x_1 + \lambda x_2}) \\
\Rightarrow f(x_3) &= f(x_0) + \nabla f(\bar{x})^T (\mu\lambda(x_1 - x_2).
\end{aligned}
$$

$\cancel{(-1+\mu)(\lambda x_1 + (1-\lambda)x_2)}$

$-\lambda x_1 - (1-\lambda)x_2 + \mu\lambda x_1 + (1-\lambda)\mu x_2$
$= -\lambda x_1 - x_2 + \lambda x_2 + \mu\lambda x_1 + \mu x_2 - \lambda\mu x_2$

Since $\bar{x}$ is a convex combination of x_1 and x_2, we have

$$\nabla f(\bar{x})^T (x_1 - x_2) = 0. \quad \text{Thus} f(x_3) = f(x_0).$$

$f(x_3) = f(x_0) \le f(x_2)$.

$$\Rightarrow f(x_3) = f(x_0) \le f(x_2).$$

It is contradiction to assumption. Hence the proof.

$f(x_1) < f(x_2)$.

Chap II.

profit maximal $\Longrightarrow$ $Z =$

min $\Longrightarrow$

$Z = \min$

Linear programming problem is one of rich developed optimization technique. Linear programming models arise in a several of decision problems in government, engineering, computer science, economics etc. These models is effective for taking a decisions in a critical positions. This technique also help for getting maximum benefit or reduce the time problems.

4 1 10 3 ≤ 8

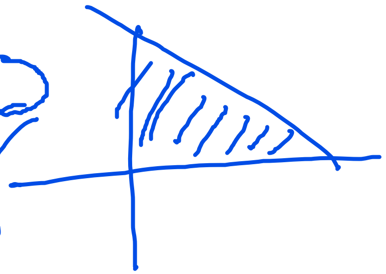
x_1, x_2, x_3, x_4.

$x_1 - x_2 + x_3 + x_4 \geq 3$

$x_1 + x_2 + x_3 + x_4 \leq 4$

$x_1 \quad x_2 \quad x_3 \quad x_4$

$4x_1 + x_2 + 10x_3 + 3x_4 \leq 18$



2.1 Linear Programming Model

The general problem of linear programming is to optimize a linear function subject to linear equality or inequality constraint. i.e. to determine the values of $x_1, x_2, \cdots, x_n$ that solve the problem (LP)

max

$$\text{Minimize } z = \sum_{i=1}^{n} c_i x_i \quad = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

$+x_2 \leq 2$

Subject to

$$\sum_{j=1}^{n} a_{i,j} x_j \ \{\leq, =, \geq\} \ b_i, \ i = 1, 2, \cdots, m. \tag{2.1}$$

$x \geq 0$

where one and only one of the sign $\leq, =, \geq$ holds for each constraint in (2.1) and this sign may varies from one constraint to another. Here c_i, b_i and $a_{i,j}$ are known real numbers.

> **Definition 2.1** A vector $(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ is called a feasible solution, to problem (LP) if constraint (2.1) is satisfied by the vector.