Operations Research

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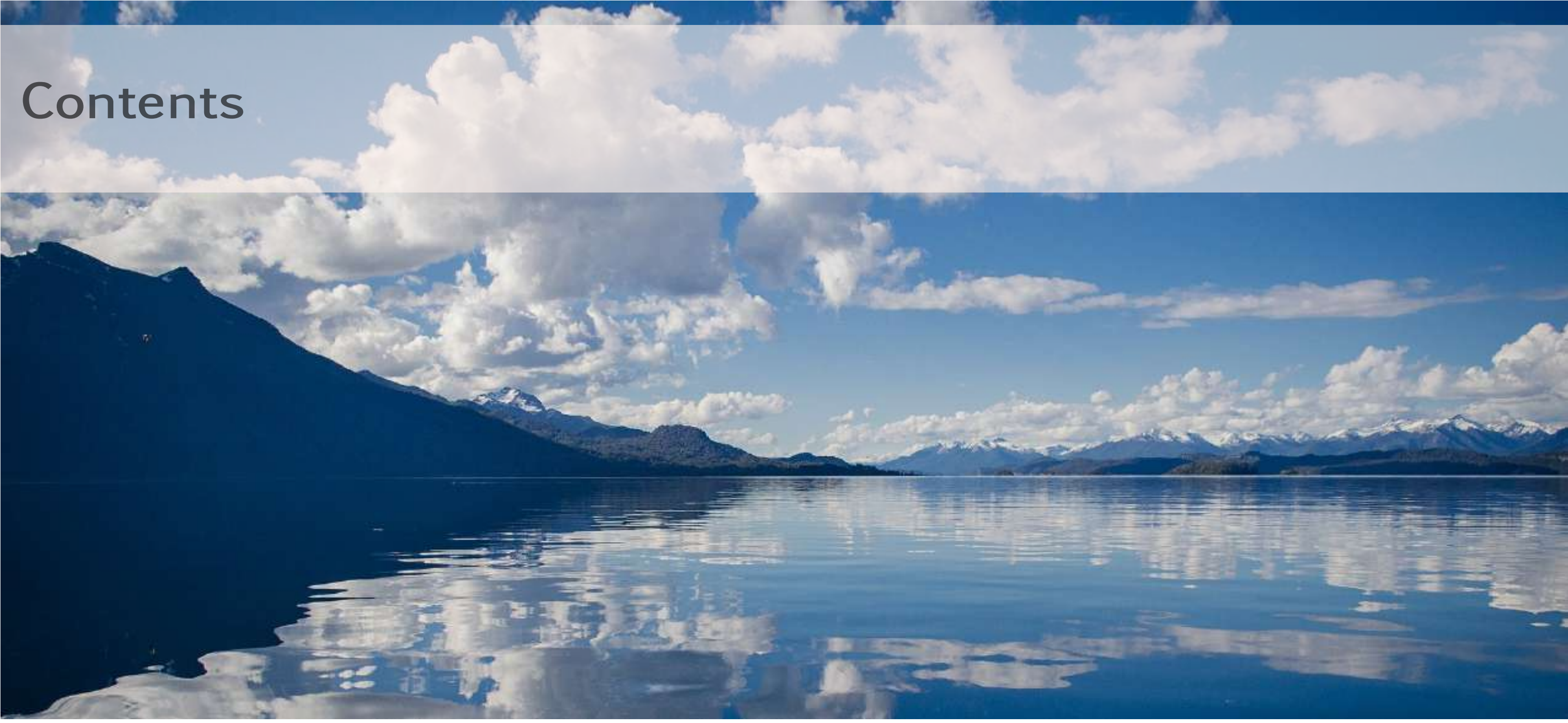

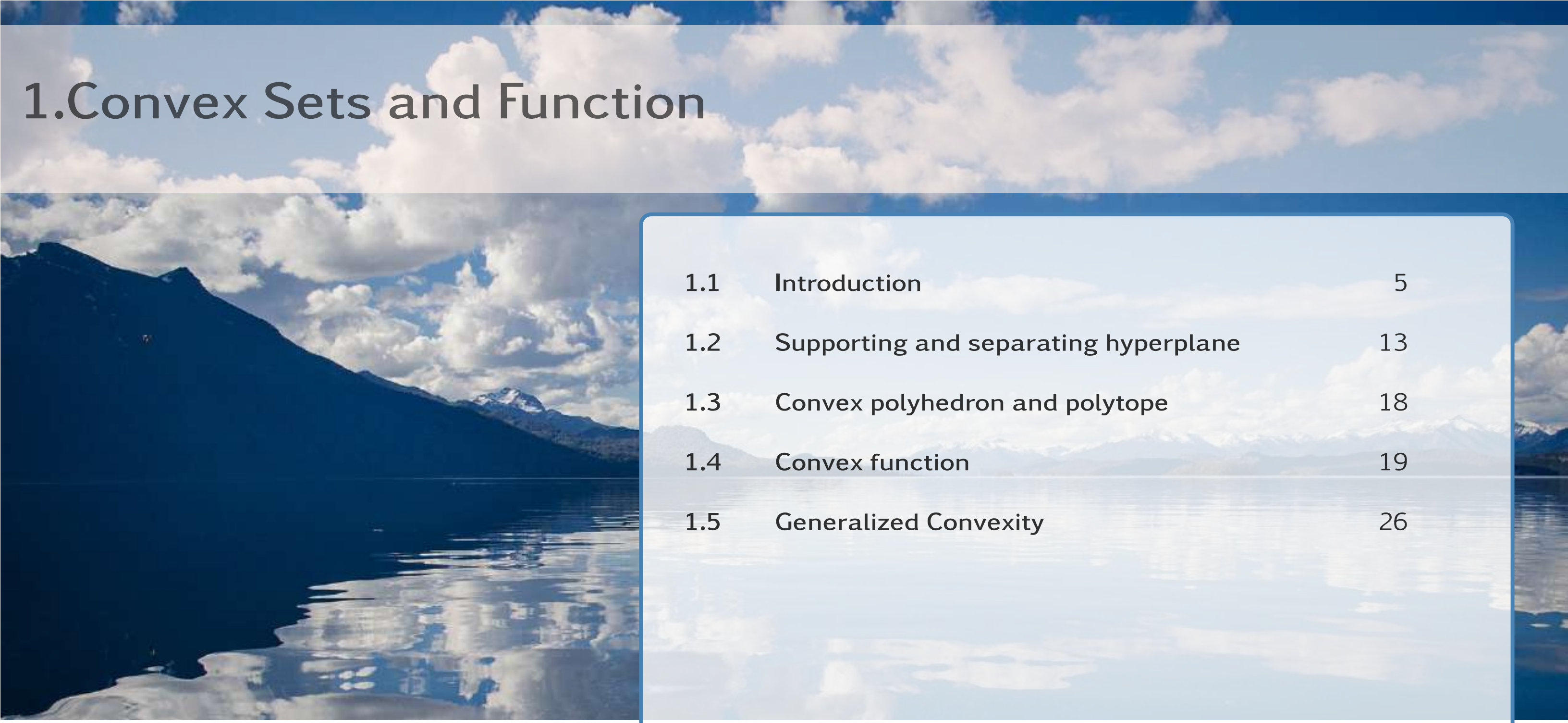

1.Convex Sets and Function

In this chapter, we have discussed the concept convex set and convex functions. These are help us to understand the important aspects of optimization techniques. These are useful for studying the optimum of function over the convex region, duality theory etc. Probably, we have to study the existence of solution of linear programming problems over the convex region. So it is need to study the properties of convex sets and convex functions.

1.1 Introduction

Definition 1.1 The line segment joining the points $x_1, x_2 \in \mathbb{R}^n$ is the set of points (Vectors) given by

$$\{x : x = \lambda x_1 + (1-\lambda)x_2, 0 \le \lambda \le 1\}.$$

Definition 1.2 The line joining the points x_1 and x_2 is the set

$$\{x : x = \lambda x_1 + (1-\lambda)x_2, \lambda \in \mathbb{R}\}.$$

Definition 1.3 A vector $x \in \mathbb{R}^n$ is called a linear combination of vector $x_1,\ x_2,\cdots,x_m$ in $\mathbb{R}^n$ if there exists numbers $\lambda_i, i = 1,2,\cdots,m$ such that $x = \sum_{i=1}^{m} \lambda_i x_i$.

Definition 1.4 A vector $x \in \mathbb{R}^n$ is called a convex combination of vector $x_1,\ x_2,\cdots,x_m$ if there exists number λ_i satisfying $\lambda_i \geq 0\ (i = 1,2.\cdots,m)$, $\sum_{i=1}^{m} \lambda_i = 1$ such that $x = \sum_{i=1}^{m} \lambda_i x_i$.

Definition 1.5 The set of all convex combination of vectors $x_1,\ x_2,\cdots,x_m$ in $\mathbb{R}^n$ is the set of points

$$\left\{ x : x = \sum_{i=1}^{m} \lambda_i x_i,\ \ \lambda_i \geq 0, i = 1,2,\cdots,m\ \ xand \sum_{i=1}^{m} \lambda_i = 1 \right\}.$$

Definition 1.6 A set $S \subseteq \mathbb{R}^n$ is called a convex set if $x_1, x_2 \in S$ then $\lambda x_1 + (1-\lambda)x_2 \in S$ for all $0 \leq \lambda \leq 1$.

Remarks.

1. Empty set and singleton set are trivially convex sets.
2. A set S is convex if the line segment joining any two points of S lies in S.
3. Number of points in a convex set are zero, one or infinite.

Definition 1.7 Given a point $x_0 \in \mathbb{R}^n$ and a non zero vector $d \in \mathbb{R}^n$ the set $\{x_0 + \lambda d : \lambda \geq 0\}$ is called a ray in $\mathbb{R}^n$.

Here, the point x_0 is the vertex of the rays and the vector d of the direction of the ray.

Definition 1.8 A set $S \subseteq \mathbb{R}^n$ is called a linear variety if $x_1, x_2 \in S$ then $\lambda x_1 + (1-\lambda)x_2 \in S$ for all $\lambda \in R$.

Example 1.1 1. Parabola, hyperbola, ellipse and circles are not convex sets.

2. Then intervals (0, 1), [0,1], (0,1],[0,1) are convex sets.
3. The set $\mathbb{R}^n$ is a convex set.
4. The set $\left\{(x_1, x_2) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}$ it is a convex set.
5. The straight line is a convex set.
6. The Solid sphere is a convex set.
7. Triangle, square and rectangle are not convex sets.
8. The set $\mathbb{R}^2 - \{\mathbf{0}\}$ is not a convex set.

Definition 1.9 Let $c \in R$, $a \in \mathbb{R}^n$, $a \neq 0$. Then the set $H = \left\{x : a^T x = c\right\}$ is said to be a hyperplane in $\mathbb{R}^n$.

The nature of hyperplane in $\mathbb{R}$ is singleton set, in $\mathbb{R}^2$ it is a straight line and in $\mathbb{R}^3$ it is a plane.

Definition 1.10 Let $a \in \mathbb{R}^n$. The sets $H_+ = \left\{x \in \mathbb{R}^n | a^T x \geq c\right\}$, $H_- = \left\{x \in \mathbb{R}^n | a^T x \leq c\right\}$ are called the closed half spaces generated by the hyperplane H. The set H_+ is known as the positive closed half space. The set H_- is known as the negative closed half space. The sets $H_+^0 = \left\{x : a^T x > c\right\}$ and $H_-^0 = \left\{x : a^T x < c\right\}$ are called the positive and negative half space generated by H respectively. Clearly H_+, H_- are closed sets and H_+^0, H_-^0 are open sets.

Example 1.2 The hyperplane H is convex set.

Solution. Let $H = \left\{x : a^T x = c\right\}$ where $x, a \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$. Let $x, y \in H \Longrightarrow a^T x = c$

and $a^T y = c$. To show that H is convex set. Let $0 \le \lambda \le 1$. Now

$$\begin{aligned} a^T(\lambda x + (1-\lambda)y) &= \lambda a^T x + (1-\lambda)a^T y \\ &= \lambda c + (1-\lambda)c \\ &= c. \end{aligned}$$

Hence, $\lambda x + (1-\lambda)y \in H$. Thus H is a convex set.

Example 1.3 The hyperplane H_+ and H_- are convex sets.

Solution. Let $H_+ = \left\{x : a^T x \ge c\right\}$ where $x, a \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$. Let $x, y \in H_+ \Longrightarrow a^T x \ge c$ and $a^T y \ge c$. To show that H_+ is convex set. Let $0 \le \lambda \le 1$. Consider

$$\begin{aligned} a^T(\lambda x + (1-\lambda)y) &= \lambda a^T x + (1-\lambda)a^T y \\ &\ge \lambda c + (1-\lambda)c \\ &= c. \end{aligned}$$

i.e. $a^T(\lambda x + (1-\lambda)y) \ge c$. Hence, $\lambda x + (1-\lambda)y \in H_+$. Similarly we can show that H_- is convex set.

Example 1.4 The hyperplane H_+^0 and H_-^0 are convex sets.

Solution. Let $H_+^0 = \left\{x : a^T x > c\right\}$, where $x, a \in \mathbb{R}^n$ and $c \in \mathbb{R}^n$. Let $x, y \in H_+ \Longrightarrow a^T x > c$ and $a^T y > c$. To show that H_+^0 is convex set. Let $0 \le \lambda \le 1$. Consider

$$\begin{aligned} a^T(\lambda x + (1-\lambda)y) &= \lambda a^T x + (1-\lambda)a^T y \\ &> \lambda c + (1-\lambda)c \\ &= c. \end{aligned}$$

i.e. $a^T(\lambda x + (1-\lambda)y) > c$. Hence, $\lambda x + (1-\lambda)y \in H_+^0$. Similarly we can show that H_-^0 is convex set.

Example 1.5 A ray in $\mathbb{R}^n$ is a convex set.

Solution. Let $x_0 \in \mathbb{R}^n, d \ne 0 \in \mathbb{R}^n x\ and\ \ \lambda x{\ge}x0$. Let $A = \{x : x = x_0 + \lambda d,\ \ \lambda \ge 0\}$ is a ray in $\mathbb{R}^n$. Let $x, y \in H$, then $x = x_0 + \lambda_1 d,\ \ y = x_0 + \lambda_2 d$. Let $0 \le \mu \le 1$. Consider

$$\begin{aligned}\mu x + (1-\mu)y &= \mu(x_0 + \lambda_1 d) + (1-\mu)(x_0 + \lambda_2 d)\\ &= \mu x_0 + \mu\lambda_1 d + x_0 + \lambda_2 d - \mu x_0 - \mu\lambda_2 d\\ &= x_0 + (\mu\lambda_1 + \lambda_2 - \mu\lambda_2)d\\ &= x_0 + \lambda_3 d\end{aligned}$$

where $\lambda_3 = x_0 + \lambda d \ge 0$ is real number, since $0 \le \lambda_1,\ \lambda_2 \le 1$ and $0 \le \mu \le 1$ are real numbers. Hence $\mu x + (1-\mu)y \in A$. Thus a ray in $\mathbb{R}^n$ is a convex set.

Example 1.6 Show that open ball $B(x_0,\ r)$ is a convex set.

Solution. Let $x, y \in B(x_0, r)$ where $x_0 \in \mathbb{R}^n$ and $r \ge 0$. Then $\|x - x_0\| \le r$ and $\|y - x_0\| \le r$. Let $0 \le \lambda \le 1$. To show that $\lambda x + (1-\lambda)y \in B(x_0, r)$.

$$\begin{aligned}\|\lambda x + (1-\lambda)y - x_0\| &= \|\lambda x - \lambda x_0 + (1-\lambda)y - x_0 + \lambda x_0\|\\ &= \|\lambda(x - x_0) + (1-\lambda)(y - x_0)\|\\ &\le |\lambda|\|x - x_0\| + |1-\lambda|\|y - x_0\|\\ &= \lambda r + r - \lambda r\\ &= r.\end{aligned}$$

This shows that $\lambda x + (1-\lambda)y \in B(x_0, r)$.

Example 1.7 A closed ball in $\mathbb{R}^n$ is a set of type $\{x \in \mathbb{R}^n : \|x - x_0\| \le r\}$, where $r > 0$ is a convex set.

Solution. Let $S = \{x : \|x - x_0\| \le r\}$. Let $x, y \in S$ and $0 \le \lambda \le 1$. To show that $\lambda x + (1-\lambda)y \in S$. Consider

$$\begin{aligned}\left\|\lambda x + (1-\lambda)y - x_0\right\| &= \left\|\lambda x - \lambda x_0 + \lambda x_0 + y - \lambda y - x_0\right\|\\ &= \left\|\lambda(x - x_0) + (1-\lambda)(y - x_0)\right\|\\ &= \left\|\lambda(x - x_0) + (1-\lambda)(y - x_0)\right\|\\ &\le \lambda r + (1-\lambda)r = r.\end{aligned}$$

It shows that

$$\Rightarrow \left\|\lambda x + (1-\lambda) y - x_0\right\| \leq r \Rightarrow \lambda x + (1-\lambda) y \in S.$$

Theorem 1.1 If C is a convex set then λC is convex set.

Proof. Let C is a convex set. To show λC is convex set. Let $\lambda C = \{\lambda c : c \in X\}$. Let $x, y \in \lambda C$.Then $x = \lambda c_1$ and $y = \lambda c_2$ for some $c_1, c_2 \in C$. Let $0 \leq \mu \leq 1$. To show that $\mu x + (1-\mu) y \in \lambda C$. Now

$$\begin{aligned} \mu x + (1-\mu) y &= \mu\lambda c_1 + \lambda c_2 - \mu\lambda\, c_2 \\ &= \lambda(\mu c_1 + (1-\mu) c_2) \\ &= \lambda c_3 \end{aligned}$$

where $c_3 = \mu c_1 + (1-\mu) c_2 \in C$. This show that $\mu x + (1-\mu) y \in \lambda C$. Thus λC is a convex set.

Theorem 1.2 If C and D are convex sets, then $C + D$ is also convex set.

Proof. Let C and D be convex sets.Then $C + D = \{x + y : x \in C, y \in D\}$. Let $x, y \in C + D$ and $0 \leq \mu \leq 1$. Then $x = x_1 + y_1$ and $y = x_2 + y_2$. To show $\mu x + (1-\mu) \in C + D$. Consider

$$\begin{aligned} \mu x + (1-\mu) y &= \mu(x_1 + y_1) + (1-\mu)(x_2 + y_2) \\ &= \mu x_1 + \mu y_1 + (1-\mu) x_2 + (1-\mu) y_2 \\ &= \mu\, x_1 + (1-\mu) x_2 + \mu y_1 + (1-\mu) y_2 \\ &= x_3 + y_3. \end{aligned}$$

where $x_3 = \mu x_1 + (1-\mu) x_2 \in C$ and $y_3 = \mu y_1 + (1-\mu) y_2 \in D$, so $x_3 + y_3 \in C + D$ and hence $\mu x + (1-\mu) y \in C + D$. Thus C + D are convex set.

Theorem 1.3 The intersection of any convex sets is a convex set.

Proof. Let $A = \bigcap_\alpha S_\alpha$where each S_α is convex set. To show that A is a convex set. Let $0 \leq \lambda \leq 1$ and $x, y \in A$. Then $x, y \in S_\alpha$ for each α. Since each S_α is convex set implies $\lambda x + (1-\lambda) y \in S_\alpha$ and hence $\lambda x + (1-\lambda) y \in \bigcap_\alpha S_\alpha = A$. Thus intersection of any convex set is convex set.

Theorem 1.4 If $C \subseteq \mathbb{R}^n$ is convex, then $Cl(C)$, the closure of C, is also convex.

Proof. Suppose $x,\ y \in Cl(C)$. Then there exist sequences $\{x_n\}$ and $\{y_n\}$ in C such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Let $0 \leq \lambda \leq 1$. Consider $z_n = \lambda x_n + (1-\lambda) y_n$. Then, by convexity of C, $z_n \in C$. Moreover

$$\lim_{n\to\infty} z_n = \lim_{n\to\infty} (\lambda x_n + (1-\lambda) y_n) = \lambda x + (1-\lambda) y.$$

Hence $\lambda x + (1-\lambda) y \in Cl(C)$.

Theorem 1.5 A set $S \subseteq \mathbb{R}^n$ is convex if and only if every convex combination of any finite number of points of S is contained in S.

Proof. Assume that every convex combination of any finite number of points of S is contained in S. To show that the set $S \subseteq \mathbb{R}^n$ is convex set. Let $x_1, x_2 \in S$ and $\lambda_1 + \lambda_2 = 1$. Then by hypothesis $\lambda_1 x_1 + \lambda_2 x_2 \in S$. Thus S is a convex set.

Conversely, suppose that S is a convex set. To prove that every convex combination of any finite number points of S is a point of S. We prove this result by mathematical induction. For $n = 2$, clearly the result is true, since the convex combination of two points of S is contained in S.

Suppose, the result is true for $n = k$, i.e. the convex combination of k points of S is contained in S. Now, we show that the result is true for $n = k+1$. Let $x_1, x_2, \cdots, x_{k+1} \in S$ and $0 \leq \lambda_1, \lambda_2, \cdots, \lambda_{k+1} \leq 1$ such that $\sum_{i=1}^{k+1} \lambda_i = 1$. Suppose $0 < \lambda_{k+1} < 1$. Now consider

$$\begin{aligned}\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_k x_k + \lambda_{k+1} x_{k+1} &= \sum_{i=1}^{k} \lambda_i x_i + \lambda_{k+1} x_{k+1} \\ &= (1-\lambda_{k+1}) \sum_{i=1}^{k} \frac{\lambda_i x_i}{(1-\lambda_{k+1})} + \lambda_{k+1} x_{k+1}.\end{aligned}$$

Clearly $\displaystyle\sum_{i=1}^{k} \frac{\lambda i}{(1-\lambda_{k+1})} = 1$ and by assumption $\displaystyle\sum_{i=1}^{k} \frac{\lambda_i}{(1-\lambda_{k+1})} x_i \in S$. Since S is convex, then $(1-\lambda_{k+1}) x_i + \lambda_{k+1} x_{k+1} \in S$ implies $\lambda_1 x_1 + \lambda_2 x_2 + \cdots + \lambda_{k+1} x_{k+1} \in S$. Thus every convex combination of any finite number of points of S is contained in S.

Definition 1.11 Let $S \subseteq \mathbb{R}^n$ be a convex set. A point $x \in S$ is called an extreme point or vertex of S if there exists no points $x_1$and x_2 in S such that $x = \lambda x_1 + (1-\lambda)x_2, 0 < \lambda < 1$.

Definition 1.12 Let $S \subseteq \mathbb{R}^n$. The intersection of all the convex set containing the set S is called the convex hull of S and it is denoted by $CO(S)$.

Definition 1.13 Let $S = \{x, y\} \subseteq R^2$. Then $CO(S)$ is the line segment joining x and y.

Example 1.8 Let $S = \{x, y, z\} \subseteq R^2$. Convex hull of these three points is the solid part of triangle

Example 1.9 If S is convex set then convex hull of this set is itself i.e. $CO(S) = S$.

Example 1.10 If $S = \{x \in \mathbb{R} : \ |x| \ = 5\}$ then $CO(s)$ is the line segment joining -5 to 5.

Example 1.11 If $S = \{x : |x| > 5\} \subseteq \mathbb{R}^2$ then $CO(S) = \mathbb{R}^2$.

Remark. $CO(S)$ is a convex set. The convex hull $CO(S)$ is actually the smallest convex set in $\mathbb{R}^n$ containing S.

Theorem 1.6 The convex hull of the set S is the set of all convex combination of the point in S i.e.

$$CO(S) = \left\{ x : x = \sum_{i=1}^{k} \lambda_i x_i, x_i \in S, 0 \le \lambda_i \le 1, \sum_{i=1}^{k} \lambda_i = 1 \right\}.$$

1.2 Supporting and separating hyperplane

Definition 1.14 A hyperplane H containing a convex set $S \subseteq \mathbb{R}^n$ in one of its closed half spaced H_+ or H_- and a boundary point of S said to be supporting hyperplane of S. if the boundary point w of S lies in the supporting hyperplane H, then H is supporting hyperplane of S at w. Let w be a boundary point S, then $a^T x = c$ is a supporting hyperplane of S at w if $a^T w = c$ and either $a^T x \geq c$ for $x \in S$ or $a^T x \leq c$ for all $x \in S$.

Theorem 1.7 If $S \subseteq \mathbb{R}^n$ be a convex set and y be a boundary point exterior to the Closure of $\overline{S}$, then there exist a vector $a \neq 0 \in \mathbb{R}^n$ such that

$$a^T y < \inf_{x \in S} \left(a^T x \right).$$

Proof. Denote $\overline{S}$ be the closure of S. Define δ by

$$\delta = \inf_{x \in S} |x - y|.$$

Clearly $\delta > 0$, since $y \notin \overline{S}$. Let $B_{2\delta}(y) = \{x : |x - y| < 2\delta\}$, then

$$\delta = \inf_{x \in S} |x - y| = \inf_{x \in \overline{S} \cap B_{2\delta}} |x - y|.$$

Define $f : \overline{S} \cap B_{2\delta} \to R$ by $f(x) = |x - y|$. Then f is continuous function on closed set $\overline{S} \cap B_{2\delta}$. Therefore f attains its minimum value on $\overline{S} \cap B_{2\delta}$. Thus there exist a point $x_0 \in \overline{S} \cap B_{2\delta}$ such that

$$\delta = \inf_{x \in \overline{S} \cap B2\delta} |x - y| = |x_0 - y|.$$

Clearly, x_0 is a boundary point of $\overline{S}$. We claim that $a = x_0 - y$ will satisfies required condition. Let $x, x_0 \in \overline{S}$. Then for $0 \le \lambda \le 1$, $\lambda x + (1-\lambda)x_0 \in \overline{S}$ and therefore

$$\begin{aligned}
|\lambda x + (1-\lambda)x_0 - y| &\ge |x_0 - y| \\
\Rightarrow |(x_0 - y) + \lambda(x - x_0)| &\ge |x_0 - y| \\
\Rightarrow |(x_0 - y) + \lambda(x - x_0)|^2 &\ge |x_0 - y|^2 \\
\Rightarrow [(x_0 - y) + \lambda(x - x_0)]^T [(x_0 - y) + \lambda(x - x_0)] &\ge (x_0 - y)^T (x_0 - y) \\
\Rightarrow \left[(x_0 - y)^T + \lambda(x - x_0)^T\right][(x_0 - y) + \lambda(x - x_0)] &\ge (x_0 - y)^T (x_0 - y) \\
\Rightarrow (x_0 - y)(x_0 - y)^T + \lambda(x - x_0)^T (x - x_0) + \lambda(x - x_0)^T (x_0 - y) + \lambda^2 (x - x_0)(x_0 - y) & \\
\ge (x_0 - y)^T (x_0 - y) & \\
\Rightarrow 2\lambda(x_0 - y)^T (x - x_0) + \lambda^2 |x - x_0|^2 \ge 0. &
\end{aligned}$$

Letting $\lambda \to 0$, we have

$$\begin{aligned}
&(x_0 - y)^T (x - x_0) \ge 0 \\
&\Rightarrow (x_0 - y)^T x \ge (x_0 - y)^T x_0 \\
&\Rightarrow a^T x \ge (x_0 - y)^T (x_0 - y + y) \\
&\Rightarrow a^T x \ge (x_0 - y)^T (x_0 - y) + (x_0 - y)^T y \\
&\Rightarrow a^T x \ge (x_0 - y)^T + |x_0 - y|^2 \\
&\Rightarrow a^T x \ge (x_0 - y)^T + \delta^2 \text{ for all } x \in S \\
&\Rightarrow a^T x > a^T y \quad \text{for all } x \in S, \text{ since } \delta > 0 \\
&\Rightarrow a^T y < \inf_{x \in S} a^T x.
\end{aligned}$$

Theorem 1.8 If $S \subseteq \mathbb{R}^n$ be a convex set and y be the boundary point of S, then there is a supporting hyperplane of S at y.

Proof. Let $S \subseteq \mathbb{R}^n$ and y be a boundary point of S. Let $\{y_n\}$ be a sequence in exterior to $\overline{S}$ converging to y. Then by Theorem 1.7, there exists a sequence $\{a_n\}$ of non zero vectors a_n such that $a_n{}^T y_n < a_n{}^T x$ for all $x \in S$. We can normalize $\{a_n\}$ with $|a_n| = 1$.

Thus $a_n{}^T y + a_n{}^T y_n - a_n{}^T y < a_n{}^T x$ for all $x \in S$. But $\{y_n\}$ converges to y, so for large n, we have $a_n{}^T y < a_n{}^T x$ for all $x \in S$.

Since $\{a_n\}$ is a bounded sequence, so there exists a sub sequence$\{a_{nk}\}$ of $\{a_n\}$ which is bounded such that $a_{nk} \to a$ as $k \to \infty$. Now

$$a^T y = \lim_{k \in K} a_n^T y \leq \lim_{k \in K} a_n^T x = a^T x.$$

Thus, $a^T y \leq a^T x$ for all $x \in S$. Hence the hyperplane $H = \left\{x : a^T x = a^T y\right\}$ is supporting hyperplane to S at y.

Theorem 1.9 Let $S \subseteq \mathbb{R}^n$ be a convex set. H is a supporting hyperplane of S and $T = S \cap H$. Then every extreme point of T is an extreme point of S.

Proof. We will prove this by method of contradiction. Suppose $x_0 \in T$ is an extreme point of T but not of S. This means that there exists $x_1, x_2 \in S$ and $0 < \lambda < 1$ such that $x_0 = \lambda x_1 + (1-\lambda) x_2$. Let $H = \left\{x : a^T x = c\right\}$ be supporting hyperplane of S and we assume that S is contained in the negative closed half space H of S. i.e. $a^T x \leq c$ for all $x \in S$. Now $x_1, x_2 \in S \Rightarrow a^T x_1 \leq cx$ and $a^T x_2 \leq c$. Moreover $x_0 \in T \Rightarrow x_0 \in H \Rightarrow a^T x_0 = c$ and

$$\begin{aligned}
&a^T (\lambda x_1 + (1-\lambda) x_2) = c \\
\Rightarrow\ &a^T \lambda x_1 + a^T (1-\lambda) x_2 = c \\
\Rightarrow\ &a^T x_1 + a^T x_2 - \lambda a^T x_2 = c.
\end{aligned}$$

Since $0 < \lambda < 1$, we must have $a^T x_1 = c$ and $a^T x_2 = c$. $\Rightarrow x_1, x_2 \in H \Rightarrow x_1, x_2 \in T$ and $x_0 = \lambda x_1 + (1-\lambda) x_2$, $0 < \lambda < 1$. This implies that x_0 is not extreme point of T. It is a contradiction to our assumption. Hence, every extreme point of T is a extreme point of S.

Theorem 1.10 Every closed bounded convex set in $\mathbb{R}^n$ in equal to the closed convex hull of the extreme point of S.

Proof. For $S = \phi$, then there is nothing to prove. We assume that $S \neq \phi$. We will give proof by induction on dimension on the spaces $\mathbb{R}^n$. For $n = 1$, S is a closed bounded interval $[c, d]$. We know that $[c, d]$ is a closed convex hull of it's extreme points c and d. Thus, the result is true for $n = 1$.

Now, assume that the result is true for the dimension $n-1$. We will show it for dimension n. Suppose K (of dimension n) be the closed convex hull of the extreme point of S. We claim that that $K = S$. Clearly $K \subseteq S$. Suppose $S \not\subseteq K$. Then there is a $y \in S$ but $y \notin K$. But y is exterior to K, by Theorem 1.7 there exists $a \neq 0$ such that

$$a^T y < \inf_{x \in K} (a^T x) \tag{1.1}$$

Let $s_0 = \inf_{x \in S} a^T x$. Since the function $a^T x$ is continuous on compact set S, then the function $a^T x$ attains its minimum value at $x_0 \in S$ with

$$s_0 = \inf_{x \in S} a^T x = \min_{x \in S}(a^T x) = a^T x. \tag{1.2}$$

It gives that

$$a^T x_0 \leq a^T x \text{ for all } x \in S. \tag{1.3}$$

Then (1.1) and (1.2) implies that, the hyperplane $H = \left\{x : a^T x = s_0\right\}$ is a supporting hyperplane to S at $x_0 \in S$. Using relation (1.2) and (1.3), we have

$$y \in S \Rightarrow a^T x_0 \leq a^T y < \inf_{x \in K}(a^T x).$$

Since $K \subseteq S$, $x_0 \notin K$ and H is a supporting hyperplane to S at x_0. Then the sets H and K are disjoint. Let $T = H \cap S$. Then T is a closed bounded subset of H and it is a space of dimension $(n-1)$. Since $x_0 \in S$, $x_0 \in H$ then $x_0 \in T$. This means that T is a non-empty closed bounded subset of $\mathbb{R}^{n-1}$. Hence by induction hypothesis, T is a closed convex hull of extreme point of T, i.e. T contains extreme points. By using repeated use of this Theorem, we an prove all other extreme point of T are also the extreme point of S. Thus, we found x_0 that lies in the convex hull of some extreme point of S and $x_0 \notin K$. It is a contradiction to that K is a closed convex hull of the extreme points of S, so, we have $K = S$.

Definition 1.15 Let S and T be two non empty subset of $\mathbb{R}^n$ then a hyperplane$H$ is said to be separate S and T if S is contained in one of the closed half spaces generated by H and T is contained in the other closed half space. The hyperplane$H$ in this case is called a separating hyperplane.

Definition 1.16 A hyperplane H strictly separates S and T if S is contained in one of the open half spaces generated by H and T is contained in other half plane.

Theorem 1.11 If $S \subseteq \mathbb{R}^n$ is non empty convex set and $0 \notin S$, then there exists a hyperplane separating S and 0.

Proof. We will give proof in two different situations.

1. Suppose 0 lies in an exterior S. Then by Theorem 1.7, there exists a vector $0 \neq a \in \mathbb{R}^n$ such that $0 < a^T x$ for $x \in S$. So, the hyperplane $H = \left\{x : a^T x = c\right\}$, where $0 < c < a^T x$ separate S and 0.
2. Suppose $0 \in \overline{S}$, by Theorem, there is a supporting hyperplane $H = \left\{x : a^T x = 0\right\}$ to S at the 0 and it separates S and 0.

Theorem 1.12 Let S and T be two non empty disjoint convex sets in $0 < a^T x$ for $x \in S$. Then exists a hyperplane that separates S and T.

Proof. By theorem, $S - T$ is convex and $0 \notin S \cap T$, because $S \cap T = \phi$. Hence, there exists a vector a such that $a^T x \geq 0$ for all $x \in S - T$. It means that for all $u \in S$ and $v \in T$, we have $a^T (u - v) \geq 0$. So, there exist a number c satisfying.

$$\begin{aligned}
a^T u - a^T v &\geq 0 \\
\inf\left(a^T u - a^T v\right) &\geq 0 \\
\inf a^T u - \sup a^T v &\geq 0 \\
\inf a^T u \geq c &\geq \sup a^T v.
\end{aligned}$$

This implies that the hyperplane $H = \left\{x : a^T x = 0\right\}$ separate S and T.

Theorem 1.13 Let S be a non empty closed convex set in $\mathbb{R}^n$ not containing 0. Then there exists a hyperplane that strictly separates S and the 0.

Proof. Let S is closed set, so $\overline{S} = S$ and $0 \notin S$ i.e. 0 is the exterior to S. Hence by Theorem 1.7, there exist $a \neq 0 \in \mathbb{R}^n$ such that $0 < \inf_{x \in S} a^T x$. Now, we choose a real number c such that $0 < c < \inf_{x \in S} a^T x$. Clearly the hyperplane $H = \left\{x : a^T x = c\right\}$ strictly separates 0 and S.

1.3 Convex polyhedron and polytope

Definition 1.17 The convex hull of a finite (non zero) number of points is called convex polytope spanned by these points.

Let $S = \{x_1, x_2, \cdots, x_m\}$ where $x_i \in \mathbb{R}^n$ then the convex polytope spanned by the points of S is the convex set

$$\mathrm{CO}(s) = \left\{x : x = \sum_{i=1}^{m} \lambda_i x_i, \lambda_1 \geq 0, \sum_{i=1}^{m} \lambda_i = 1\right\}.$$

Clearly a convex polytope is a non-empty convex set.

Theorem 1.14 The set of vertices of a convex polytope is a subset of the set of spanning points of the polytope.

Proof. Let V be the set of vertices of the convex polytope $\mathrm{CO}(s)$ spanned by the points of the set $S = \{x_1, x_2, \cdots, x_m\}$. Clearly the result is true when $m = 1$. Now suppose $V \not\subset S$. i.e. there exist $x \in V$ such that $x \notin S$. Since $x \in V \Rightarrow x \in \mathrm{CO}(S)$. Therefore $x = \sum_{i=1}^{m} \mu_i x_i$. Where $\mu_i \geq 0$, $i = 1, \cdot, m$ and $\sum_{i=1}^{m} \mu_i = 1$. Since $x \notin S$, it follows that $\mu_i > 0$ and $\mu_i \neq 1$, $i = 1, 2, \cdots, mx$. Hence $x = \sum_{i=1}^{m} \mu_i x_i$ implies that there exists $\mu_i\,(0 < \mu_i < 1)$. Let it be μ_1

$$x = \mu_1 x_1 + \sum_{i=2}^{m} \mu_i x_i = \mu_1 x_1 + (1-\mu_1) \sum_{i=2}^{m} \frac{\mu_i}{(1-\mu_1)}\, x_i = \mu_1 x_1 + (1-\mu_1)\, x_i$$

where $y = \sum_{i=2}^{m} \frac{\mu i}{(1-\mu_1)} x_i$. Clearly $y = \sum_{i=2}^{m} \left(\frac{\mu_i}{(1-\mu_1)}\right) x_i$, $0 < \mu_1 < 1$. This implies that $y \in \mathrm{CO}(S)$. Hence x is not a extreme (vertex) of $CO(S)$. It is a contradiction. Hence, we must have $V \subseteq S$.

Theorem 1.15 Let $K = \{x : Ax = b,\ x \geq 0\}$ be a non-empty polyhedral set. Then the set of extreme points of K is non empty and has a finite number of points.

1.4 Convex function

Definition 1.18 A function f defined on a set $T \subseteq \mathbb{R}^n$ is said to be convex at $x_0 \in T$ if $x_1 \in T$, $0 \leq \lambda \leq 1$, $\lambda x_0 + (1-\lambda)x_1 \in T$, then

$$f(\lambda x_0 + (1-\lambda)x_1) \leq \lambda f(x_0) + (1-\lambda)f(x_1).$$

Definition 1.19 A function f is said to be convex on T if it is convex at every point of T.

Domain of f necessary to be a convex set. Therefore other way convex set is defined as follows.

Definition 1.20 If T is convex set then f is said to be convex on T if $x_1,\ x_2 \in T$, $0 \leq \lambda \leq 1 \Longrightarrow$

$$f(\lambda x_0 + (1-\lambda)x_1) \leq \lambda f(x_0) + (1-\lambda)f(x_1).$$

In geometrical point of view, a function $y = f(x)$ defined on a convex set T is convex if the chord joining, any two points on the graph of f lies on or above the graph.

Definition 1.21 A function f defined on a set $T \subseteq \mathbb{R}^n$ is said to be strictly convex at $x_0 \in T$ if $x_1 \in T$, $0 < \lambda < 1$, $x_0 \neq x_1$, $\lambda x_0 + (1-\lambda)x_1 \in T$, then

$$f(\lambda x_0 + (1-\lambda)x_1) < \lambda f(x_0) + (1-\lambda)f(x_1).$$

Definition 1.22 A function f is said to be concave at $x_0 \in T$ if $-f$ is convex at $x_0 \in T$.

Example 1.12 Show that the linear function $f(x) = c^T x + d$ is both convex and convex on $\mathbb{R}^n$.

Solution. Clearly, $\mathbb{R}^n$ is a convex set. Let $x_1,\ x_2 \in \mathbb{R}^n$ and $0 \le \lambda \le 1$. Consider

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &= c^T(\lambda x_1 + (1-\lambda)x_2) + d \\ &= \lambda c^T x_1 + \lambda d + (1-\lambda)c^T x_2 + d - \lambda d \\ &= \lambda\left(c^T x_1 + d\right) + (1-\lambda)\left(c^T x_2 + d\right) \\ &= c^T(\lambda x_1 + (1-\lambda)x_2) + d \end{aligned}$$

$\Rightarrow f$ is convex on $\mathbb{R}^n$. Similarly, we can show that $-f$ is convex function.

Example 1.13 Let f be a convex function on a convex set $T \subseteq \mathbb{R}^n$. Then for every $k \in \mathbb{R}$. Show that $T_k = \{x : x \in T,\ f(x) \le k\}$ is convex set.

Solution. Let $x_1, x_2 \in T_K$ and $0 \le \lambda \le 1$. Then $x_1,\ x_2 \in T$. Since T is convex $\Rightarrow \lambda x_1 + (1 - \lambda x_2) \in T$ and, $f(x_1) \le k,\ \ f(x_2) \le k$. Given that f is convex on T, we have

$$\begin{aligned} f(\lambda x_1 + (1-\lambda)x_2) &\le f(x_1) + (1-\lambda)f(x_2) \\ &\le \lambda k x + (1-\lambda)k = k \\ &\Rightarrow \lambda x_1 + (1-\lambda)x_2 \in T_k. \end{aligned}$$

Theorem 1.16 Let f and g be convex function and let μ be a non negative number. Then μf and $f+g$ are convex functions. Further if $\mu > 0$ and f is strictly convex then μf is strictly convex.

Proof. Let f and g be convex function on convex set T. Let $x_1,\ x_2 \in T,\ 0 \le \lambda \le 1$,

$$\begin{aligned} \mu f(\lambda x_1 + (1-\lambda)x_2) &= \mu(f(\lambda x_1 + (1-\lambda)x_2)) \\ &\le \ \mu(\lambda f(x_1) + (1-\lambda)f(x_2)) \\ &= \lambda \mu f(x_1) + (1-\lambda)\mu f(x_2) \\ &= \lambda(\mu f)(x_1) + (1-\lambda)(\mu f)(x_2). \end{aligned}$$

$\Rightarrow \mu f$ is a convex function. Now

$$\begin{aligned}(f+g)(\lambda x_1+(1-\lambda)\ x_2) &= \ f(\lambda x_1+(1-\lambda)\ x_2)+g(\lambda x_1+(1-\lambda)\ x_2)\\ &\leq \ \lambda f(x_1)+(1-\lambda)f(x_2)+\lambda g(x_1)+(1-\lambda)g(x_2)\\ &= \lambda(f(x_1)+g(x_1))+(1-\lambda)(f(x_2)+g(x_2))\\ &= \lambda(f+g)(x_1)+(1-\lambda)(f+g)(x_2).\end{aligned}$$

It shows that $f\ +\ g$ is a convex function.

If $\mu>0$ and f is strictly convex, then

$$\begin{aligned}(\mu f)(\lambda x_1+(1-\lambda)x_2) &= \mu(f(\lambda x_1+(1-\lambda)x_2)) < \mu(\lambda f(x_1)+(1-\lambda)f(x_2))\\ &= \lambda\mu f(x_1)+(1-\lambda)\mu f(x_2) = \lambda(\mu f)(x_1)+(1-\lambda)(\mu f)(x_2)\end{aligned}$$

Thus μf is strictly convex

Theorem 1.17 Every linear combination $\sum_{i=1}^{k}\mu_i f_i$ of convex function f_i where $\mu_i\geq 0,\ \ i=1,2,\cdots,k$ is a convex function. Also, such a combination is strictly convex if at last one $\mu_i>0,\ i=1,2,\cdots,k$ and the corresponding f_i strictly convex.

Proof. Repeated use of Theorem 1.16 follows proof.

Theorem 1.18 Let h be a non decreasing convex function on $\mathbb{R}$ and f be a convex function on a convex set $T\subseteq\mathbb{R}^n$. Then the composite function $h\circ f$ is convex on T.

Proof. Let $h: R\rightarrow Rx\ and\ fx{:}Tx{\rightarrow}R$ be two convex functions. To show that $h\circ f$ is a convex on T. Let $x_1,x_2\in Tx\ and\ 0\leq\lambda\leq 1$. Since f is convex on T

$$f(\lambda x_1+(1-\lambda)x_2)\leq\lambda f(x_1)+(1-\lambda)f(x_2). \tag{1.4}$$

Since h is non decreasing, we have

$$h(f(\lambda x_1+(1-\lambda)x_2))\leq h(\lambda f(x_1)+(1-\lambda)f(x_2)). \tag{1.5}$$

Also h is convex function

$$h(\lambda f(x_1)+(1-\lambda)f(x_2))\leq\lambda h(f(x_1))+(1-\lambda)h(f(x_2)) \tag{1.6}$$

From (1.5) and (1.6)

$$h(f(x_1+(1-\lambda)x_2)) \leq \lambda h(f(x_1)) + (1-\lambda)h(f(x_2))$$

$\Rightarrow h \circ f$ is convex function on T.

Theorem 1.19 Let f be differentiable on an open convex set $T \subseteq \mathbb{R}^n$. Then f is convex on T iff

$$f(x_2) \geq f(x_1) + \nabla f(x_1)^T (x_2 - x_1) \tag{1.7}$$

For all $x_1, x_2 \in T$. Further f is strictly convex on T if and on if the inequality is strict (>) for all $x_1, x_2 \in T, x_1 \neq x_2$.

Proof. Let f be differentiable on open convex set $T \subseteq \mathbb{R}^n$. Suppose f is convex on T. Let $x_1, x_2 \in T$ and $0 \leq \lambda \leq 1$,

$$f(\lambda x_2 + (1-\lambda)x_1) \leq \lambda f(x_2) + (1-\lambda) f(x_1)$$

It implies that

$$f(x_1 + \lambda\ (x_2 - x_1)) \leq f(x_1) + \ \lambda\ (f(x_2) - f(x_1)).$$

Using differentiability of f, we have

$$f(x_1) + \lambda \nabla f(x_1)^T (x_2 - x_1) + \alpha |x_2 - x_1| \leq f(x_1) + \ \lambda\ (f(x_2) - f(x_1)).$$

It implies that

$$\lambda(f(x_1) - f(x_2)) \leq f(x_1) - \left[f(x_1) + \lambda \nabla f(x_1)^T (x_2 - x_1) + \alpha |x_2 - x_1| \right].$$

Canceling λ on both sides and letting $\lambda \to 0$ ($\alpha \to 0$), we have

$$\begin{aligned} f(x_1) - f(x_2) &\leq \nabla f(x_1)^T (x_2 - x_1) \\ \Rightarrow f(x_2) &\geq f(x_1) + \nabla f(x_1)^T (x_2 - x_1). \end{aligned}$$

Hence the inequality (1.7).

Now, suppose (1.7) is true. Let $x_1, x_2 \in T$ and $0 \leq \lambda \leq 1$ with $x_3 = \lambda x_1 + (1 - \lambda x_2)$. Using (1.7) for $x_1,\ x_3$ and x_1, x_2, we have

$$f(x_1) \geq f(x_3) + \nabla f(x_3)^T (x_1 - x_3) \tag{1.8}$$