Eigenvalues and Eigenvectors

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The power method

Let us have a square matrix A. It's size is nxn.
It has a number of independet real eigenvalues: $\lambda_1, \lambda_2, \ldots, \lambda_n$.
Eigenvectors corresponding to the eigenvalues are: $v_1, v_2, \ldots, v_n$.
One condition: $|\lambda_1| > |\lambda_2|$.

The eigenvectors are independent, which means that they are as well basis vectors. This implies that any vector in the same space can be written as a linear combination of the eigenvectors.

$$x_0 = c_1 v_1 + c_2 v_2 + \cdots + c_n v_n$$

where $c_1 \neq 0$

After multypling both sides by matrix A:

$$Ax_0 = c_1 A v_1 + c_2 A v_2 + \cdots + c_n A v_n$$

From the defintion $Av_i = \lambda v_i$, so:

$$Ax_0 = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_n \lambda_n v_n$$

$$Ax_0 = c_1 \lambda_1 [v_1 + \frac{c_2}{c_1} \frac{\lambda_2}{\lambda_1} v_2 + \cdots + \frac{c_n}{c_1} \frac{\lambda_n}{\lambda_1} v_n] = c_1 \lambda_1 x_1$$

This was the first iteration. To begin the second iteration, we multiply A by x_1:

$$Ax_1 = \lambda_1 v_1 + \frac{c_2}{c_1} \frac{\lambda_2^2}{\lambda_1} v_2 + \cdots + \frac{c_n}{c_1} \frac{\lambda_n^2}{\lambda_1} v_n$$

$$Ax_1 = \lambda_1 [v_1 + \frac{c_2}{c_1} \frac{\lambda_2^2}{\lambda_1^2} v_2 + \cdots + \frac{c_n}{c_1} \frac{\lambda_n^2}{\lambda_1^2} v_n] = \lambda_1 x_2$$

After k iterations we have:

$$Ax_{k-1} = \lambda_1 [v_1 + \frac{c_2}{c_1} \frac{\lambda_2^k}{\lambda_1^k} v_2 + \cdots + \frac{c_n}{c_1} \frac{\lambda_n^k}{\lambda_1^k} v_n] = \lambda_1 x_k$$

For large k, $(\frac{\lambda_n}{\lambda_1})^k = 0$

We have now the largest eigenvalue and its corresponding eigenvector:

$$Ax_{k-1} = \lambda_1 v_1$$

The inverse power method

The reciprocals of the eigenvalues of A are the eigenvalues of it's inverse matrix A^{-1}.
This will help us to find the smallest eigenvalue of A.
Instead of multiplying A as in power method, we multiply it's inverse to find it's largest value.

QR method

The QR method is used to find all eigenvalues of a matrix, without finding the eigenvectors at the same time.

1. The eigenvalues and corresponding eigenvectors of similar matrices are the same.

Two square matrices A and B are similar if:

$$A = C^{-1}BC$$

where C is an invertible matrix.

2. You can present any matrix as a product of two other matrices.

$$A = QR \tag{1}$$

Here we want to get an orthogonal matrix Q and an upper triangular matrix R.

A matrix M is an orthogonal matrix if: $M^{-1} = M^T$. Thus $M^*M = I$.

Let us rewrite equation (1):

$$RQ = Q^*AQ$$

$$RQ = Q^{-1}AQ$$

RQ has the same eigenvalues as A.

Compute a QR factorization and reverse the order of multiplcation of Q and R.

$A_0 = A$

$$A_k = R_kQ_k = Q_k^{-1}A_kQ_k$$

$$A_{k-1} = Q_kR_k$$

We will finally converge to an upper triangular matrix form as the iteration progresses:

$$A_k = R_kQ_k = \begin{bmatrix} \lambda_1 & X & \dots & X \\ 0 & \lambda_2 & \dots & X \\ & & \dots & \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}$$