Interpolation

May 22, 2021

Introduction

We have to arrays of numbers X and Y. Array X contains independent data points. Array Y contains dependent data points $y_i, i = 1, \ldots, m$.

We want to find a function $\hat{y}(x)$, which gets the exact same value with given points.

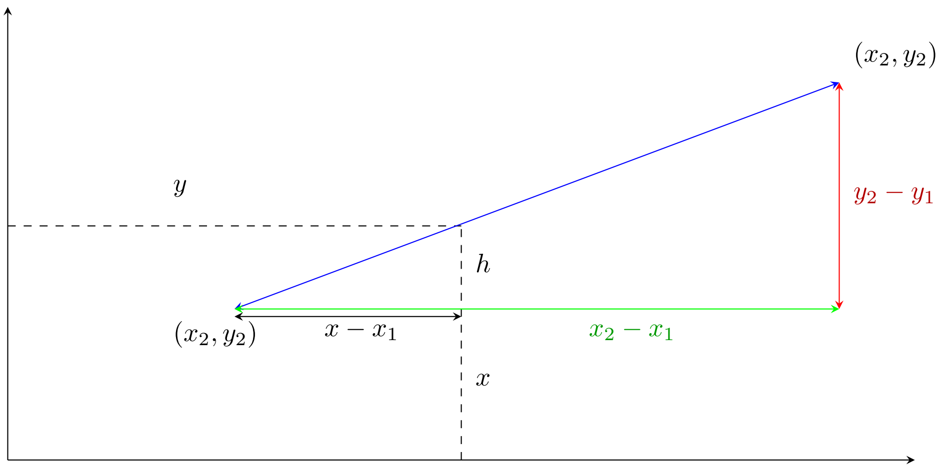
Linear Interpolation

Linear interpolation is achieved by connecting two data points with a straight line.

For $x_i < x < x_{i+1}$:

$$\hat{y}(x) = y_i + \frac{(y_{i+1} - y_i)(x - x_i)}{(x_{i+1} - x_i)}.$$

Derivation



$$\alpha = \frac{y_2 - y_1}{x_2 - x_1}$$

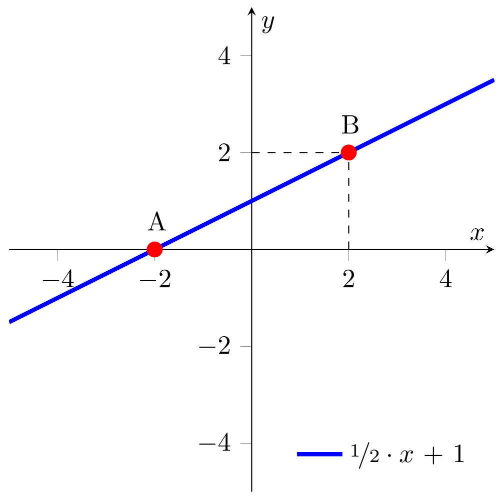
$$h = \alpha \cdot (x - x_1)$$

$$y = y_1 + h$$

$$y = y_1 + (x - x_1) \cdot \frac{y_2 - y_1}{x_2 - x_1}$$

Example

We are given two points A(-2, 0) and B (2, 2).



Let's try to evaluate the value of the function at $x = 1$

$$\hat{y}(x) = y_i + \frac{(y_{i+1} - y_i)(x - x_i)}{(x_{i+1} - x_i)} = 0 + \frac{(2 - 0)(1 - (-2))}{(2 - (-2))} = 1.5$$

Cubic Spline

The interpolating function in cubic spline interpolation is a set of piecewise cubic functions.

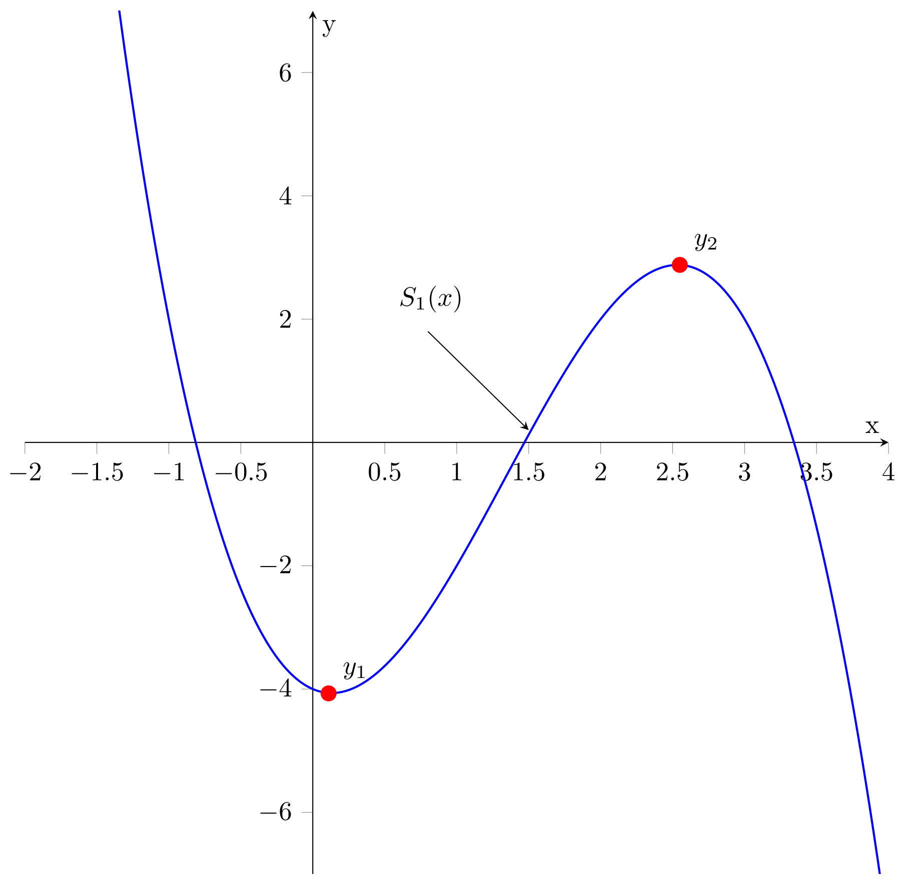
For $x_i < x < x_{i+1}$:

We have two points (x_i, y_i) and (x_{i+1}, y_{i+1}) joined with a cubic polynomial:

$$S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$$

For n points, there are $n-1$ cubic functions to find, and each cubic function requires four coefficients (a_i, b_i, c_i, d_i).

There are $4(n-1)$ unknowns to find.



Derivation

We are trying to find a function $S_i(x) = a_i x^3 + b_i x^2 + c_i x + d_i$ going trough both points: (x_i, y_i) and x_{i+1}, y_{i+1}.

$$S_i(x_i) = y_i, \quad i = 1, \dots, n-1, \tag{1}$$

$$S_i(x_{i+1}) = y_{i+1}, \quad i = 1, \dots, n-1, \tag{2}$$

Smoothness condition:

$$S_i'(x_{i+1}) = S_{i+1}'(x_{i+1}), \quad i = 1, \dots, n-2, \tag{3}$$

$$S_i''(x_{i+1}) = S_{i+1}''(x_{i+1}), \quad i = 1, \dots, n-2, \tag{4}$$

Boundry condition: The curve is a “straight line” at the end points:

$$S_1''(x_1) = 0 \tag{5}$$

$$S_{n-1}''(x_n) = 0 \tag{6}$$

Let $h_i = x_i - x_{i-1}$
Let $S_i^{''}(x_i) = S_i^{''}(x_{i+1}) = M_i$
$S_1^{''}(x_1) = M_0 = 0$ and $S_n^{''}(x_n) = M_n = 0$
Other M_i are unknown.
By Lagrange interpolation, we can interpolate each S_i'' on $[x_{i-1}, x_i]$:

$$S_i''(x) = M_{i-1}\frac{x_i - x}{h_i} + M_i\frac{x - x_{i-1}}{h_i} \quad for \quad x \in [x_{i-1}, x_i]$$

Integrating the above equation twice and using the condition that $C_i(x_{i-1}) = y_{i-1}$ and $C_i(x_i) = y_i$ to determine the constants of integration, we have.

$$S_i(x) = M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} + \left(y_{i-1} - \frac{M_{i-1}h_i^2}{6}\right)\frac{x_i - x}{h_i} + \left(y_i - \frac{M_i h_i^2}{6}\right)\frac{x - x_{i-1}}{h_i}$$

$$\text{for} \quad x \in [x_{i-1}, x_i]$$

This expression gives us the cubic spline S(x) if $M_i, i = 0, 1, \cdots, n$ can be determined.

$$S_{i+1}'(x) = -M_i\frac{(x_{i+1} - x)^2}{2h_{i+1}} + M_{i+1}\frac{(x - x_i)^2}{2h_{i+1}} + \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{M_{i+1} - M_i}{6}h_{i+1}$$

$$S'_{i+1}(x_i) = -M_i\frac{h_{i+1}}{2} + \frac{y_{i+1}-y_i}{h_{i+1}} - \frac{M_{i+1}-M_i}{6}h_{i+1}$$

Similarly, when $x \in [x_{i-1}, x_i]$, we can shift the index to obtain

$$S'_i(x) = -M_{i-1}\frac{(x_i-x)^2}{2h_i} + M_i\frac{(x-x_{i-1})^2}{2h_i} + \frac{y_i-y_{i-1}}{h_i} - \frac{M_i-M_{i-1}}{6}h_i \quad (7)$$

$$S'_i(x_i) = M_i\frac{h_i}{2} + \frac{y_i-y_{i-1}}{h_i} - \frac{M_i-M_{i-1}}{6}h_i$$

Since $S'_{i+1}(x_i) = S'_i(x_i)$, we can derive:

$$\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i \quad \text{for} \quad i = 1, 2, \cdots, n-1,$$

$$\mu_i = \frac{h_i}{h_i+h_{i+1}}, \quad \lambda_i = 1-\mu_i = \frac{h_{i+1}}{h_i+h_{i+1}}, \quad \text{and} \quad d_i = 6f[x_{i-1}, x_i, x_{i+1}]$$

and $f[x_{i-1}, x_i, x_{i+1}]$ is a divided difference.

According to different boundary conditions, we can solve the system of equations above to obtain the values of M_i's.

$S'_1(x_0) = f'_0$ and $S'_n(x_n) = f'_n$. According to equation (7), we can obtain:

$$S'_1(x_0) = -M_0\frac{(x_1-x_0)^2}{2h_1} + M_1\frac{(x_0-x_0)^2}{2h_1} + \frac{y_1-y_0}{h_1} - \frac{M_1-M_0}{6}h_1$$

$$\Rightarrow f'_0 = -M_0\frac{h_1}{2} + f[x_0, x_1] - \frac{M_1-M_0}{6}h_1$$

$$\Rightarrow 2M_0 + M_1 = \frac{6}{h_1}(f[x_0, x_1] - f'_0) = 6f[x_0, x_0, x_1]$$

Analogously:

$$S'_n(x_n) = -M_{n-1}\frac{(x_n-x_n)^2}{2h_n} + M_n\frac{(x_n-x_{n-1})^2}{2h_n} + \frac{y_n-y_{n-1}}{h_n} - \frac{M_n-M_{n-1}}{6}h_n$$

$$M_{n-1} + 2M_n = \frac{6}{h_n}(f'_n - f[x_{n-1}, x_n]) = 6f[x_{n-1}, x_n, x_{n+1}]$$

Let:
$\lambda_0 = \mu_n = 1$,
$d_0 = 6f[x_0, x_0, x_1]$ and
$d_n = 6f[x_{n-1}, x_n, x_n]$

$$
\begin{bmatrix}
2 & \lambda_0 & & & & & \\
\mu_1 & 2 & \lambda_1 & & & & \\
 & \ddots & \ddots & \ddots & & & \\
 & & \ddots & \ddots & \ddots & & \\
 & & & \ddots & \ddots & \ddots & \\
 & & & & \mu_{n-1} & 2 & \lambda_{n-1} \\
 & & & & & \mu_n & 2
\end{bmatrix}
\begin{bmatrix}
M_0 \\ M_1 \\ \vdots \\ \vdots \\ \vdots \\ M_{n-1} \\ M_n
\end{bmatrix}
=
\begin{bmatrix}
d_0 \\ d_1 \\ \vdots \\ \vdots \\ \vdots \\ d_{n-1} \\ d_n
\end{bmatrix}
$$

Lagrange Polynomial Interpolation

Lagrange polynomial interpolation gives us a single polynomial that connects all of the data points.

That polynomial is denoted as $L(x)$. It is true that $L(x_i) = y_i$ for all points (x_i, y_i) .

$$L(x) = \sum_{i=1}^{n} y_i P_i(x).$$

Each polynomial appearing in the sum is called a Lagrange basis polynomials, $P_i(x)$.

$$P_i(x) = \prod_{j=1, j\neq i}^{n} \frac{x - x_j}{x_i - x_j},$$

Example

We are given three points A(-1, 1), B(2, 3) and C(3,5).

$$P_1(x) = \frac{(x - x_2)(x - x_3)}{(x_1 - x_2)(x_1 - x_3)} = \frac{(x - 2)(x - 3)}{(-1 - 2)(-1 - 3)} = \frac{1}{12}(x^2 - 5x + 6)$$

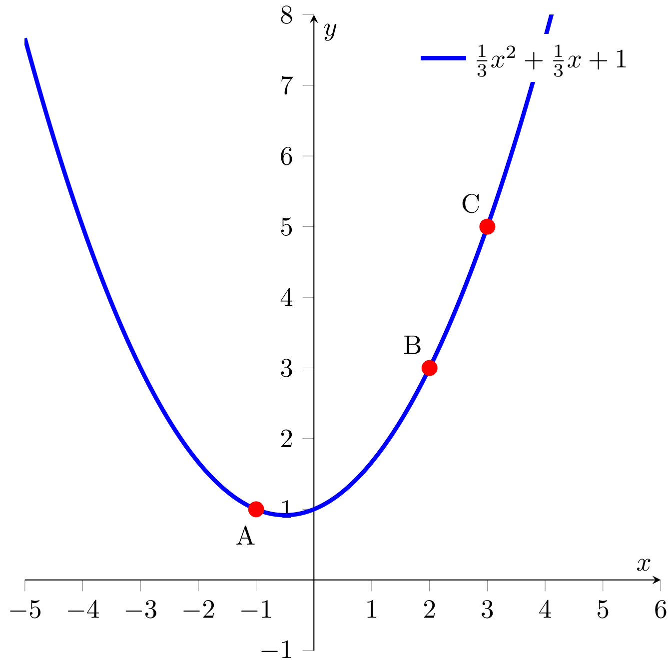
$$P_2(x) = \frac{(x - x_1)(x - x_3)}{(x_2 - x_1)(x_2 - x_3)} = \frac{(x + 1)(x - 3)}{(2 + 1)(2 - 3)} = -\frac{1}{3}(x^2 - 2x - 3)$$

$$P_3(x) = \frac{(x - x_1)(x - x_2)}{(x_3 - x_1)(x_3 - x_2)} = \frac{(x + 1)(x - 2)}{(3 + 1)(3 - 2)} = \frac{1}{4}(x^2 - x - 2)$$

$$L(x) = 1 \cdot P_1(x) + 3 \cdot P_2(x) + 5 \cdot P_3(x)$$

$$L(x) = 1 \cdot P_1(x) + 3 \cdot P_2(x) + 5 \cdot P_3(x)$$

$$L(x) = \frac{1}{3}x^2 + \frac{1}{3}x + 1$$



$\frac{1}{3}x^2 + \frac{1}{3}x + 1$
y
x
A
B
C
8
7
6
5
4
3
2
1
−1
−5
−4
−3
−2
−1
1
2
3
4
5
6