Comparing greedy strategies against greedy strategies with broader strategy spaces in simple games of network formations

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1 Abstract

We explore a network formation game in which players take turns forming or removing links with the goal of becoming the most central player. We endow players with greedy strategies, and sufficiently limit and randomize the game so as to promote network structure away from the player nodes and avoid stalemates. We vary the power of a greedy strategy by endowing it with extra actions with some probability. Via numerical experiments we see that slight edges in luck usually produce consistent win advantages, but we also observe cases where the disadvantaged player achieves a stable advantage, contrary to initial hypothesis.

2 Definition and settings

We consider a competitive game consisting of 1 or 2 players (agents) and N nodes. We call the graph G with edge set E. The player's goal is to maximize its betweenness centrality. In the two player version, which is a binary win/loss game, they maximize the difference in betweenness centrality. If the player has no connections, we consider their centrality to be -0.001, so as to incentivize forming the first connection (because having one neighbor does not give a node any betweenness.)

All players play greedily with respect to their available actions, which we will modulate. The available actions will generally be to add or remove an arbitrary edge in the graph. Therefore, we are not considering a social game, as the non player nodes have no will or utility of any sort associated with them. There are $N(N-1)-|E|+1$ add options available, and $|E|+1$ remove options available, assuming players are allowed to do nothing. The basic strategy space we will consider and modify is the add-remove strategy, of size $N(N-1)+1$.

3 One-player games

For the simplest setting possible, we consider an N node graph with one player. Play continues indefinitely, until the node wants to do nothing further, or until a cycle is observed. The player would like to become the most central node in the graph. The ideal network for player 1 is a star with itself at the center, for a betweenness centrality of 1, normalized.

Our first (easy) question is: starting from an empty graph, can Player 1 achieve the optimal star graph by playing the greedy add-only strategy? If a player with no connections has centrality 0, as in the usual definition, they will experience indifference between all options available. There is no reason to believe they will work toward forming a star at this point, so the answer is no. The greedy algorithm cannot lookahead to the second connection. But if we modify the scoring metric so having one connection is worth 0.001 more than zero connections, the answer is yes. From here we assume that betweenness is endowed with some nonzero penalty for having no connections. (Note that the number of shortest paths between two nodes can be exponential, as in the opposite corners of a grid, but we use 0.001 for experimentation).

Proposition 1. *In a one-player game, starting from the empty graph plus one additional connection from the player to any node, the player becomes the central player in a star graph using the add-only strategy, in* $N-1$ *turns.*

Add-only is not powerful enough on an arbitrary graph, e.g. on a clique, where it has no options. However, what is true is that remove-only strategy works on a clique:

Proposition 2. *In a one-player game, starting from a clique, the player becomes the central player in a star graph using the remove-only strategy, in* $(N-1)^2$ *turns.*

Removing a link between any two other players opens up a shortest path through $P1$; it is a path of length 2 along with many other paths of length 2. But this is an increase in centrality as $P1$ was not on any shortest paths between these links before. This is true at any step until the graph is a star.

Add-remove is powerful enough most of the time.

Proposition 3. *The add-remove strategy produces the star graph, as long as the player does not have only one neighbor (at start or after forming one connection), which is a connected to every other node.*

If the player is connected to a node that connects to every node, the player cannot attain betweenness by forming another connection. Else, the player benefits by becoming direct connections with all non-connections of the first node it is connected to. The player then connects to all non-neighbors of the first node, since it will between those two nodes. Then the player deletes all connections from the first node to its neighbors, so it can monopolize those routes. The only stopping process for this condition is for all neighbors to have no outgoing connections, and for the player to be connected to all the nodes those neighbors are not connected to (i.e. all of them), which is the star.

This demonstrates some of the basic expressiveness of the add and remove strategies, which we use as the basis for future study.

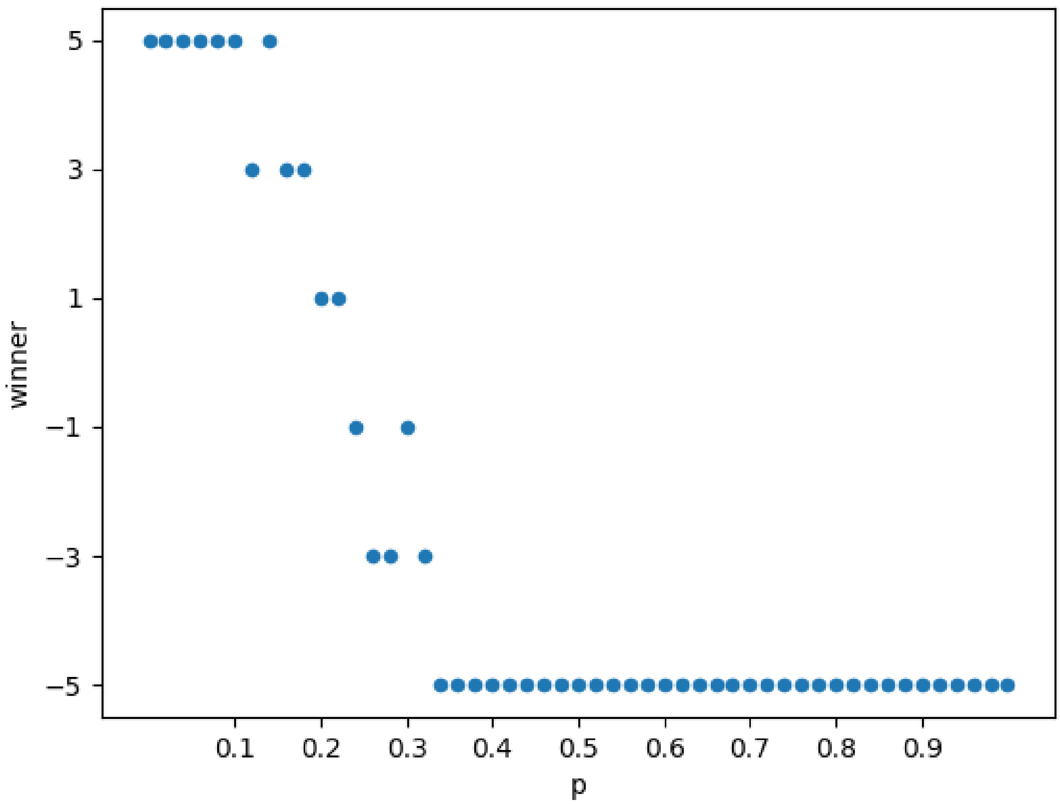
3.1 Add-only vs. remove-only strategies in one player games

Now we ask the question, is there a threshold in number of connections in a graph where add-only strategies win against remove-only strategies and vice versa? Since we have seen neither strategy by itself can conquer the graph, and the asymmetry may lead to complex cooperation effects, it is reasonable to compare one-player games on the same graph.

We run experiments to find that for $N = 20$, there is a threshold around $0.2 < p < 0.3$ where add-only strategies win for small p, and remove-only strategies win for large p. That produces some evidence there is a threshold function, and furthermore suggests a value of p for us to use when further experimenting with ER graphs: namely, $p \approx 0.25$ gives some balance between add and remove strategies.

Unfortunately it is quite difficult to calculate this (several hours on my personal laptop for 250 trials). Betweenness centrality may be estimated by sampling but for small N this may not be worth the accuracy cost. (Sampling is accurate according to the Chernoff bounds, but one must consider that for the games considered in this paper mistakes may lead to blunders.)

We conjecture that as $N \to \infty$ there is a threshold function. Wins minus losses for the add strategy are as follows:



4 Comparing greedy strategies in multiplayer competitive play

4.1 Game design

We would like to consider add-remove greedy strategies, where the players are equipped with extra abilities under some conditions. The abilities will simply be additional actions (e.g. add two edges instead of just one) that can be evaluated for betweenness score, and the player will use whichever action, the usual ones or the additional ones, is greedily optimal.

We consider 20 nodes with the following mechanic: each player is presented with 5 random nodes and must add or remove an edge between them, for a total of 20 options (plus any additional actions). This randomness solves three important problems in game setup: firstly, it breaks the tendency for back-and-forth, where player 2's best action is just to undo player 1's, and it does so statelessly, i.e. we do not need to consider game modifications like immunizing a connection for some number of turns. Secondly, it encourages network modification away from the players. Experimental data shows degree of player nodes tend to be around 4, which is promising in the sense that it is not very close to 0 or 19. Three, it randomizes. Four, it is far more computationally tractable than computing all possible actions, because betweenness centrality matrices are actually quite expensive to compute.

We make the additional actions available at some probability rate p. This aids experiment setup in a two ways. Firstly, it is stateless. If we offered a player a fixed number of bonus actions, presumably, if the actions are powerful, the player would use them all right away, as a greedy algorithm has no sense of supply management. Secondly, it is parametrized. We now have a quantified way to talk about how much of one bonus is roughly as powerful as an amount of another bonus. We focus our experiments on this.

We score based on number of half-turns winning. Naively, a player is expected to be winning after they play. So it is an upset if the other player is winning. This metric can capture phenomena like where one strategy is a bit more powerful even though the other strategy has lucky breaks. It also can capture situations where somehow the graph becomes more deeply favorable to one player, despite wins for another.

We largely have focused on 5 or 10 games of 400 or 1000 rounds. It is unclear if many, short games exposes a strategy's strength better, or several long games (although we tested the latter). Our main hypothesis is as follows:

Conjecture 1. *Any two bonus strategy spaces, possibly equal, above a base strategy space, exhibit an increasing threshold function $f(p) = q$ for which, if player 1 has access to strategy 1 at rate p, they will be evenly matched against player 2 with access to strategy 2 at rate $q = f(p)$.*

One way this conjecture might fail is if there were a particular action that is greedily optimal but harms the structure of the graph more deeply for the player in question. We have no evidence such an action exists. We furthermore conjecture transitivity.

4.2 Comparing double-play and assassination

Double-play is defined as offering the player every combination of 2 actions. This is more powerful than sequential as the player computes utility on pairs. Assassination is defined as removing all edges for one node. In either case, they are restricted to the 5 nodes offered the player.

We run 5 trials at 400 nodes with the following results:

p1—p2	0.00	0.05	0.10	0.20	0.50	1.00
0.00	1	-3	-5	-5	-5	-5
0.05	1	1	-3	-5	-5	-5
0.10	3	1	-3	-3	-5	-5
0.20	5	5	3	-1	-5	-5
0.50	5	5	5	5	-3	-5
1.00	5	5	5	5	5	5

Here is the same table in just net wins / net losses for player 1:

p1—p2	0.00	0.05	0.10	0.20	0.50	1.00
0.00	1	0	0	0	0	0
0.05	1	1	0	0	0	0
0.10	1	1	0	0	0	0
0.20	1	1	1	0	0	0
0.50	1	1	1	1	0	0
1.00	1	1	1	1	1	1

These results supports one aspect of our hypothesis and dispute another. The supported conjecture is that a threshold will emerge, and in fact for this table it is supported even when win or loss is not fully predicted. The disputed conjecture is that one strategy will win with near certainty. We have certainly witnessed contended games (bear in mind "close" games are close for hundreds of rounds). Luck is possible in this game.

5 Further research

The support of one conjecture and dispute of another suggests two research paths. One is to identify the aberrant games. The most unusual experiment showed a 2% double-play strategy losing to a basic strategy for thousands of rounds on end. The basic strategy seemed to have some early luck, which is not surprising, but somehow this luck was stable against double play every 50 rounds. The assassination strategy may be able to avoid this trap since assassinations can affect many edges, but we have also seen that 100% assassination consistently loses against 100% double edge.

So the first research area would be to investigate what is going on in graphs where an underdog strategy takes over. The greedy strategy is creating a durable lead.

The second research area is to polish the mathematical setting necessary to validate the conjecture. Many variables were tweaked, including the number of rounds to play and win condition as a function thereof, that compromised computational ability to run a large number of games. Large N went unexplored, and to do so would require a computationally tractable approach to betweenness. Perhaps one can consider commodity flow where betweenness only need be measured for the commodities. More specific conjectures may be formalizable. Simpler games than network strategy may elucidate the necessary convexity criteria to support the game that took place.

6 Availability of resources

The programming is Python with networkx, numpy and pandas. Code is available at https://github.com/djechlin/ieor8100-project. Bluntly put, an effort was made to make code modular and testable, but not well-documented. It is nearly possible to write a new strategy, a win rate, and plug it in. The data has a logging bug (now fixed in code) invalidating all of player 2 rows (the cumulative win count was unaffected and drove the above research).