Probably the best tool for numerically approximating a function $f(x)$ near a particular point x_0 is the *Taylor polynomial.* The formula for the Taylor polynomial of degree n centered at x_0, approximating a function $f(x)$ possessing n derivatives at x_0, is given by

$$
\begin{aligned}
\text{(1)} \qquad p_n(x) &= f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\
&\quad + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n \\
&= \sum_{j=0}^{n} \frac{f^{(j)}(x_0)}{j!}(x - x_0)^j .
\end{aligned}
$$

Radius of Convergence

Theorem 1. For each power series of the form (1), there is a number ρ $(0 \le \rho \le \infty)$, called the **radius of convergence** of the power series, such that (1) converges absolutely for $|x - x_0| < \rho$ and diverges for $|x - x_0| > \rho$. (See Figure 8.2.)

If the series (1) converges for all values of x, then $\rho = \infty$. When the series (1) converges only at x_0, then $\rho = 0$.

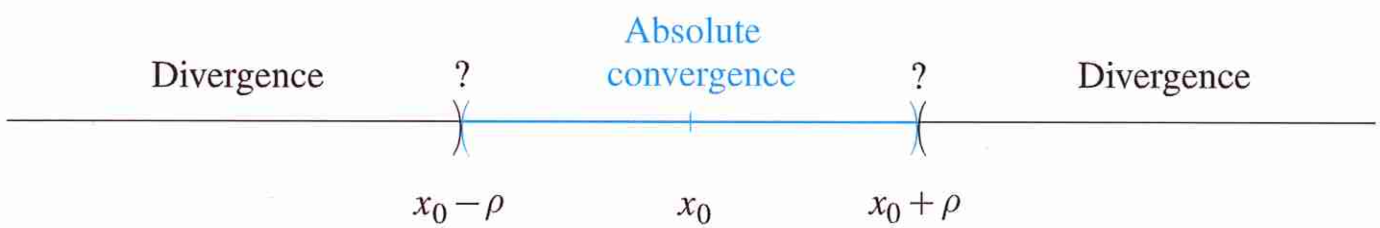

Figure 8.2 Interval of convergence

Ratio Test for Power Series

Theorem 2. If, for n large, the coefficients a_n are nonzero and satisfy

$$\lim_{n \to \infty} \frac{a_n}{a_{n+1}} = L \qquad (0 \le L \le \infty),$$

then the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n (x - x_0)^n$ is $\rho = L$.

Power Series

A **power series** about the point x_0 is an expression of the form

(1) $$\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots ,$$

where x is a variable and the a_n's are constants. We say that (1) **converges** at the point $x = c$ if the infinite series (of real numbers) $\sum_{n=0}^{\infty} a_n(c-x_0)^n$ converges; that is, the limit of the partial sums,

$$\lim_{N\to\infty} \sum_{n=0}^{N} a_n(c-x_0)^n ,$$

exists (as a finite number). If this limit does not exist, the power series is said to **diverge** at $x = c$. Observe that (1) converges at $x = x_0$, since

$$\sum_{n=0}^{\infty} a_n(x_0-x_0)^n = a_0 + 0 + 0 + \cdots = a_0 .$$

for all numbers x in the convergence interval. For example, the **geometric series** $\sum_{n=0}^{\infty} x^n$ has the radius of convergence $\rho = 1$ and the sum function $f(x) = 1/(1-x)$; that is,

$$\textbf{(3)} \qquad \frac{1}{1-x} = 1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n \quad \text{for} \quad -1 < x < 1.$$

Power Series Vanishing on an Interval

Theorem 3. If $\sum_{n=0}^{\infty} a_n (x - x_0)^n = 0$ for all x in some open interval, then each coefficient a_n equals zero.

Differentiation and Integration of Power Series

Theorem 4. If the series $f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$ has a positive radius of convergence ρ, then f is differentiable in the interval $|x - x_0| < \rho$ and termwise differentiation gives the power series for the derivative:

$$f'(x) = \sum_{n=1}^{\infty} n a_n (x - x_0)^{n-1} \qquad \text{for} \qquad |x - x_0| < \rho.$$

Furthermore, termwise integration gives the power series for the integral of f:

$$\int f(x)\,dx = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (x - x_0)^{n+1} + C \quad \text{for} \quad |x - x_0| < \rho.$$

Shifting the Summation Index

The index of summation in a power series is a dummy index just like the variable of integration in a definite integral. Hence,

$$\sum_{n=0}^{\infty} a_n(x-x_0)^n = \sum_{k=0}^{\infty} a_k(x-x_0)^k = \sum_{i=0}^{\infty} a_i(x-x_0)^i .$$

Example 3 Express the series

$$\sum_{n=2}^{\infty} n(n-1)a_n x^{n-2}$$

as a series where the generic term is x^k instead of x^{n-2}.

$k=n-2$

$n=k+2$

$$\sum_{k=0}^{\infty} (k+2)(k+2-1)a_{k+2}x^{k+2-2} =$$

$$= \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^{k}$$

Example 4 Show that

$$x^3\sum_{n=0}^{\infty} n^2(n-2)a_n x^n = \sum_{n=3}^{\infty}(n-3)^2(n-5)a_{n-3}x^n .$$

$$x^3\sum_{n=0}^{\infty} n^2(n-2)a_n x^n = \sum_{n=0}^{\infty} n^2(n-2)a_n x^{n+3} =$$

$k = n+3$

$n = k-3$

$$= \sum_{k=3}^{\infty}(k-3)^2(k-5)a_{k-3}x^k =$$

$$= \sum_{n=3}^{\infty}(n-3)^2(n-5)a_{n-3}x^n$$

Analytic Function

Definition 1. A function f is said to be **analytic at x_0** if, in an open interval about x_0, this function is the sum of a power series $\sum_{n=0}^{\infty} a_n(x - x_0)^n$ that has a positive radius of convergence.

Ordinary and Singular Points

Definition 2. A point x_0 is called an **ordinary point** of equation (2) if both p and q are analytic at x_0. If x_0 is not an ordinary point, it is called a **singular point** of the equation.

ex) $y'' - 2xy' + 8y = 0$ (about $x_0=0$)

$y = \sum_{n=0}^{\infty} a_n (x-x_0)^n$

$y' = \sum_{n=1}^{\infty} n \cdot a_n (x-x_0)^{n-1}$

$y'' = \sum_{n=2}^{\infty} n(n-1) a_n (x-x_0)^{n-2}$

$$\sum_{n=2}^{\infty} a_n \cdot n(n-1)(x)^{n-2} - 2x \sum_{n=1}^{\infty} a_n n x^{n-1} + 8 \sum_{n=0}^{\infty} a_n x^n = 0$$

shift

$k = n-2$

$n = k+2$

$$\sum_{k=0}^{\infty} a_{k+2}(k+2)(k+1) x^k - 2 \sum_{n=1}^{\infty} a_n n x^n + 8 \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1) x^n - 2 \sum_{n=1}^{\infty} a_n n x^n + 8 \sum_{n=0}^{\infty} a_n x^n = 0$$

pull out n=0 terms

$$a_2 \cdot 2 \cdot 1 + 8 \cdot a_0 \cdot 1 + \sum_{n=1}^{\infty} \left(a_{n+2}(n+2)(n+1) - 2a_n \cdot n + 8 \cdot a_n \right) x^n = 0$$

$2a_2 + 8a_0 = 0$

$a_{n+2}(n+2)(n+1) + a_n(8-2n) = 0$

$a_2 = -4a_0$

$$a_{n+2} = -\frac{(8-2n)}{(n+2)(n+1)} a_n$$

$$a_2 = -4a_0$$

$$a_{n+2} = \frac{2n-8}{(n+2)(n+1)} a_n$$

we let:

$n=1$ $$a_3 = \frac{-6}{6} a_1 = -a_1$$

$n=2$ $$a_4 = \frac{-4}{4 \cdot 3} a_2 = \frac{-4}{4 \cdot 3} \cdot -4a_0 = \frac{4}{3} a_0$$

$n=3$ $$a_5 = \frac{-2}{5 \cdot 4} a_3 = \frac{a_1}{10}$$

$n=4$ $$a_6 = \frac{0}{6 \cdot 5} = 0 \Rightarrow$$ all following even coef. will be 0

$n=5$ $$a_7 = \frac{2}{7 \cdot 6} a_5 = \frac{a_1}{210} \quad \ldots\ldots$$

Can we do more? $y'' - 2xy' + 8y = 0$ $\quad y(0) = 3 \quad y'(0) = 0$

$$y = \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \ldots$$

$$a_0 = 3$$

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + a_5 x^5 + a_6 x^6 + \ldots$$

$$a_1 = 0$$

all odd coef. are 0

$$y = a_0 + a_2 x^2 + a_4 x^4$$

$$\boxed{y = 3 - 12x^2 + 4x^4}$$

$\Rightarrow$ polynomial

Not always the case!!!