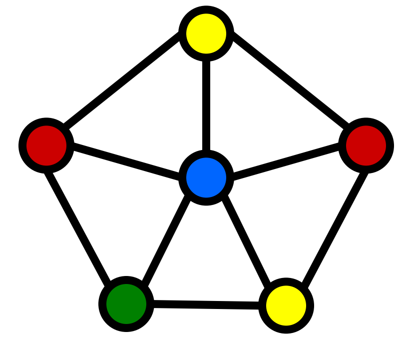
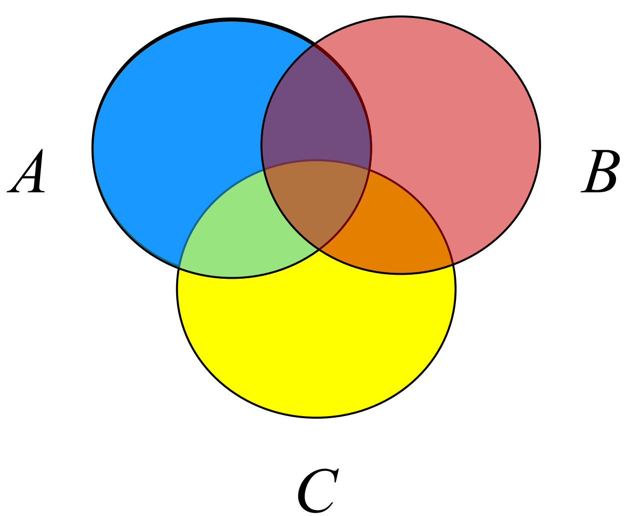
CS222: Intro to Discrete Structures



L10: Discrete Probability



Probability

- What is the probability that a dice comes up an odd number?

- What is the probability to win a lottery?

History of Probability Theory

Dates back to 1526 when the Italian mathematician, physician, and gambler Girolamo Cardano wrote the first known systematic treatment of the subject in his book Liber de Ludo Aleae (Book on Games of Chance).

In the seventeenth century the French mathematician Blaise Pascal determined the odds of winning some popular bets based on the outcome when a pair of dice is repeatedly rolled.

In the eighteenth century, the French mathematician Laplace, who also studied gambling, defined the probability of an event as the number of successful outcomes divided by the number of possible outcomes

Finite Probability

An experiment is a procedure that yields one of a given set of possible outcomes.

The sample space of the experiment is the set of possible outcomes.

An event is a subset of the sample space.

If S is a finite nonempty sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the *probability* of E is $p(E) = \frac{|E|}{|S|}$.

According to Laplace's definition, the probability of an event is between 0 and 1

Examples

A box contains four blue balls and five red balls. What is the probability that a ball chosen at random from the box is blue?

Sample space: 9 possible outcomes
Event: 4 of these possible outcomes produce a blue ball
Probability: 4/9

What is the probability that when two dice are rolled, the sum of the numbers on the two dice is 7?

Sample space: 36 possible outcomes (6*6)
Event: 6 of these possible outcomes produce 7
(1,6), (6,1), (2,5), (5,2), (3,4), (4,3),
Probability: 1/6

Examples

In a lottery, players win a large prize when they pick four digits (0~9) that match, in the correct order, four digits selected by a random mechanical process. A smaller prize is won if only three digits are matched.

What is the probability that a player wins the large prize? What is the probability that a player wins the small prize?

Sample space: 10^4 = 10000
Event for big: 1
Probability: 1/10000

Event for small: 36
You need 3 correct digits and 1 incorrect digits
First digit incorrect : 9 choices
Second digit incorrect :9 choices
Same as the third and forth.
Probability: 36/10000

Examples

There are many lotteries now that award enormous prizes to people who correctly choose a set of six numbers out of the first n positive integers, where n is usually between 30 and 60.

What is the probability that a person picks the correct six numbers out of 1~49?

Sample space: $\binom{49}{6} = 49!/((49-6)!\,6!)$=13983816

Event for win: 1

Probability: 1/13983816

Probabilities of Complements and Unions of Events

Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E).$$

There are many lotteries now that award enormous prizes to people who correctly choose a set of six numbers out of the first n positive integers, where n is usually between 30 and 60.

What is the probability that a person does not picks the correct six numbers out of 49?

1-1/13983816

Probabilities of Complements and Unions of Events

Let E be an event in a sample space S. The probability of the event $\overline{E} = S - E$, the complementary event of E, is given by

$$p(\overline{E}) = 1 - p(E).$$

A sequence of 10 bits is randomly generated. What is the probability that at least one of these bits is 0?

What is the complementary event?

- All 10 bits are 1, i.e., only one case 1111111111

The probability is 1-1/2^10

Probabilities of Complements and Unions of Events

Let E_1 and E_2 be events in the sample space S. Then

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

Let E_1 be the event that the integer selected at random from 1 to 100 is divisible by 2, and let E_2 be the event that it is divisible by 5.

Then $E_1 \cup E_2$ is the event that it is divisible by either 2 or 5.

Also, $E_1 \cap E_2$ is the event that it is divisible by both 2 and 5, or equivalently, that it is divisible by 10.

Because $|E_1| = 50$, $|E_2| = 20$, and $|E_1 \cap E_2| = 10$, it follows that

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$$
$$= \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}.$$

Probabilistic Reasoning

Suppose you are a game show contestant. You have a chance to win a large prize. You are asked to select one of three doors to open; the large prize is behind one of the three doors and the other two doors are losers. Once you select a door, the game show host, who knows what is behind each door, does the following.

First, whether or not you selected the winning door, he opens one of the other two doors that he knows is a losing door (selecting at random if both are losing doors).

Then he asks you whether you would like to switch doors. Which strategy should you use? Should you change doors or keep your original selection, or does it not matter?

Probabilistic Reasoning

The probability you select the correct door (before the host opens a door and asks you whether you want to change) is 1/3, because the three doors are equally likely to be the correct door. The probability this is the correct door does not change once the game show host opens one of the other doors, because he will always open a door that the prize is not behind.

The probability that you selected incorrectly is the probability the prize is behind one of the two doors you did not select. Consequently, the probability that you selected incorrectly is 2/3. If you selected incorrectly, when the game show host opens a door to show you that the prize is not behind it, the prize is behind the other door. You will always win if your initial choice was incorrect and you change doors. So, by changing doors, the probability you win is 2/3

In other words, you should always change doors when given the chance to do so by the game show host. This doubles the probability that you will win.

Assigning Probabilities

Let S be the sample space of an experiment with a finite outcomes. We assign a probability p(s) to each outcome s. We require that two conditions be met:

$$(i) \quad 0 \leq p(s) \leq 1 \text{ for each } s \in S$$

and

$$(ii) \quad \sum_{s \in S} p(s) = 1.$$

Condition (i) states that the probability of each outcome is a nonnegative real number no greater than 1.

Condition (ii) states that the sum of the probabilities of all possible outcomes should be 1; that is, when we do the experiment, it is a certainty that one of these outcomes occurs.

Assigning Probabilities

When there are n possible outcomes, x1 ,x2 ,...,xn , the two conditions to be met are

$$(i) \quad 0 \leq p(x_i) \leq 1 \text{ for } i = 1, 2, \ldots, n$$

and

$$(ii) \quad \sum_{i=1}^{n} p(x_i) = 1.$$

The function p from the set of all outcomes of the sample space S is called a probability distribution

Assigning Probabilities

What probabilities should we assign to the outcomes H (heads) and T (tails) when a fair coin is flipped?

p(H)=p(T)=1/2

What probabilities should be assigned to these outcomes when the coin is biased so that heads comes up twice as often as tails?

Suppose that S is a set with n elements. The *uniform distribution* assigns the probability $1/n$ to each element of S.

The *probability* of the event E is the sum of the probabilities of the outcomes in E. That is,

$$p(E) = \sum_{s \in E} p(s).$$

p(H)= 2/3 p(T)=1/3

Assigning Probabilities

Suppose that a dice is biased (or loaded) so that 3 appears twice as often as each other number but that the other five outcomes are equally likely. What is the probability that an odd number appears when we roll this dice ?

We want to find the probability of the event $E = \{1,3,5\}$

Since we have $p(1) = p(2)=p(6) = p(4)=p(5)$, denote it as $p(x)$, then $p(3)=2p(x)$

$5p(x)+2p(x)=1 \Rightarrow p(x)=1/7$

$P(odd)=p(1)+p(3)+p(5)=4/7$

Expected Payout

Given a set of events, e1, e2, … , ek, and associated gain or loos, c1, c2, …, ck, the Expected Payout, EP = p(e1)*c1+p(e2)*c2+…+p(ek)*ck

Example: In a lottery, players win a large prize when they pick four digits that match, in the correct order, four digits selected by a random mechanical process. A smaller prize is won if only three digits are matched.

Suppose big prize is $3,000, small prize is $200, what is the expected payout for this lottery?

We already calculated p(big) = 1/10000, and p(small) = 36/10000

EP = 1/10000 * 3000+36/10000 * 200 = $1.02

Conditional Probability

Let E and F be events with $p(F) > 0$. The *conditional probability* of E given F, denoted by $p(E \mid F)$, is defined as

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}.$$

Suppose that we flip a coin three times, and all eight possibilities are equally likely

Moreover, suppose we know that the event F, that the first flip comes up tails, occurs.

Given this information, what is the probability of the event E, that an odd number of tails appears?

Because the first flip comes up tails, there are only two possible outcomes for an odd number of tails: TTT and THH, i.e., $E \cap F = \{TTT, THH\}$

TTT, THH -> E, $p(E|F) = \frac{p(TTT)}{p(F)} + \frac{p(THH)}{p(F)} = \frac{1/8}{1/2} + \frac{1/8}{1/2} = 1/2$

Conditional Probability

What is the conditional probability that a family with two children has two boys, given they have at least one boy? Assume that each of the possibilities BB, BG, GB, and GG is equally likely, where B represents a boy and G represents a girl.

(Note that BG represents a family with an older boy and a younger girl while GB represents a family with an older girl and a younger boy.)

Let E be the event that a family with two children has two boys, and let F be the event that a family with two children has at least one boy. It follows that E = {BB}, F = {BB, BG, GB}, and E ∩ F = {BB}.

Because the four possibilities are equally likely, it follows that p(F) = 3/4 and p(E ∩ F) = 1/4. We conclude that

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Independent Events

When two events are independent, the occurrence of one of the events gives no information about the probability that the other event occurs.

E.g., the results of flipping two coins

- The events E and F are independent if and only if $p(E \cap F) = p(E)p(F)$.

- Example: Suppose you flip a coin twice, what is the probability that the first outcome is head and the second outcome is tail?

- $P(1^{st}\text{-head}) = 1/2$
- $P(2^{nd}\text{-tail}) = 1/2$
- $P(1^{st}\text{-head} \;\&\; 2^{nd}\text{-tail}) = P(1^{st}\text{-head})P(2^{nd}\text{-tail}) = 1/4$

The events $E_1, E_2, \ldots, E_n$ are *pairwise independent* if and only if $p(E_i \cap E_j) = p(E_i)p(E_j)$ for all pairs of integers i and j with $1 \leq i < j \leq n$. These events are *mutually independent* if $p(E_{i_1} \cap E_{i_2} \cap \cdots \cap E_{i_m}) = p(E_{i_1})p(E_{i_2}) \cdots p(E_{i_m})$ whenever i_j, $j = 1, 2, \ldots, m$, are integers with $1 \leq i_1 < i_2 < \cdots < i_m \leq n$ and $m \geq 2$.

More Examples about Probability

What is the conditional probability that exactly four heads appear when a fair coin is flipped five times, given that the first flip came up heads?

Given the first one is head, there are 4 outcomes for 4 heads:

i.e. $E \cap F$ ={HHHHT, HHHTH, HHTHH, HTHHH}

There are 16 outcomes in total when first flip is head.

Thus, $p(E \cap F) = 4 * \frac{1}{2\text{^}5} = \frac{1}{8}$

p(1st-H) = 1/2

p= p(4Hs&1st-H) / p(1st-H) = (1/8)/1/2=1/4

More Examples about Probability

Which is more likely: rolling a total of 9 when two dice are rolled or rolling a total of 9 when three dice are rolled?

For two dice:

4 cases to be 9, (3,6) (6,3) (4,5) (5,4)

P=4/(6*6) = 1/9

For three dice:

25 cases to be 9

P=25/(6*6*6) = 25/216

Three dice is more possible.

More Examples about Probability

In roulette, we have a wheel with 38 numbers. Of these, 18 are red, 18 are black, and 2 are green. The probability that when the wheel lands on any particular number is 1/38.

a) What is the probability that the wheel lands on a red number?

b) What is the probability that the wheel lands on a black number twice in a row?

c) What is the probability that the wheel lands on green?

d) What is the probability that in five spins the wheel never lands on either green?

e) What is the probability that the wheel lands on one of the first six integers on one spin, but does not land on any of them on the next spin?

More Examples about Probability

If you flip a coin ten times, what is the probability that you will get exactly 4 heads and 6 tails?

When heads happens four times, we have $\binom{10}{4}$ possible cases.

For each case (4 heads and 6 tails), we have a probability 0.5^4*0.5^6=0.0009765625.

So the total probability is 210*0.0009765625=0.205078125