MA345 Differential Equations & Matrix Method

Lecture: 04

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COAS.301.12

$$\frac{dN}{dt} = kN$$

$$\frac{dN}{N} = k\,dt$$

$$\ln|N| = kt + C$$

$$N = e^{kt} \cdot e^{C} = A e^{kt}$$

Linear Equations

A type of first-order differential equation that occurs frequently in applications is the linear equation. Recall from Section 1.1 that a **linear first-order equation** is an equation that can be expressed in the form

$$\textbf{(1)} \qquad a_1(x)\frac{dy}{dx} + a_0(x)y = b(x) ,$$

where $a_1(x)$, $a_0(x)$, and $b(x)$ depend only on the independent variable x, not on y.

$$x^2 \sin x - (\cos x)y = \sin x \frac{dy}{dx}$$

$$\underbrace{\sin x}\underbrace{\frac{dy}{dx}} + \underbrace{(\cos x)}\underbrace{y} = x^2 \sin x$$

$$y\frac{dy}{dx} + (\sin x)y^3 = e^x + 1$$

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One can seldom rewrite a linear differential equation so that it reduces to a form as simple as (2). However, the form (3) can be achieved through multiplication of the original equation (1) by a well-chosen function $\mu(x)$. Such a function $\mu(x)$ is then called an "integrating fae tor" for equation (1). The easiest way to see this is first to divide the original equation (1) by $a_1(x)$ and put it into **standard form**

$$\text{(4)} \qquad \frac{dy}{dx} + P(x)y = Q(x)\,,$$

where $P(x) = a_0(x)/a_1(x)$ and $Q(x) = b(x)/a_1(x)$.

$$y' + y = x$$

$$g(x) = e^x$$

$$\underbrace{e^x y' + y e^x}_{\downarrow} = x e^x$$

$$\int (y e^x)' = \int x e^x \, dx \qquad \text{by parts}$$

$$u = x \quad dv = e^x$$

$$du = dx \quad v = e^x \qquad \int u\,dv = uv - \int v\,du$$

$$y e^x = x e^x - e^x + C$$

$$y \cdot e^x = x e^x - e^x + C$$

$$\boxed{y = x - 1 + Ce^{-x}}$$

$$\underbrace{y'} + p(x)\cdot \underbrace{y} = q(x)$$

$$\boxed{g(x) = e^{\int p(x)\,dx}}$$

$$g'(x) = p(x)\, e^{\int p(x)\,dx} = \underbrace{p(x)\, g(x)}$$

$$y' + p(x) y = q(x) \quad \Big|\times g(x)$$

$$g(x) y' + \underbrace{g(x) p(x)} y = g(x)\, q(x)$$

$$g(x) y' + g'(x) y = g(x)\cdot q(x)$$

$$\big(g(x) y\big)' = g(x)\, q(x)$$

$$g(x)\cdot y = \int g(x)\cdot q(x)\,dx + c$$

$$\boxed{y = \frac{1}{g(x)} \int g(x)\cdot q(x)\,dx + \frac{c}{g(x)}}$$

Method for Solving Linear Equations

(a) Write the equation in the standard form

$$\frac{dy}{dx} + P(x)y = Q(x) .$$

(b) Calculate the integrating factor $\mu(x)$ by the formula

$$\mu(x) = \exp\left[\int P(x)dx\right] .$$

(c) Multiply the equation in standard form by $\mu(x)$ and, recalling that the left-hand side is just $\frac{d}{dx}[\mu(x)y]$, obtain

$$\underbrace{\mu(x)\frac{dy}{dx} + P(x)\mu(x)y}_{\frac{d}{dx}[\mu(x)y]} = \mu(x)Q(x) ,$$

$$\frac{d}{dx}[\mu(x)y] = \mu(x)Q(x) .$$

(d) Integrate the last equation and solve for y by dividing by $\mu(x)$ to obtain (8).

$$y' = 3x^2 - \frac{y}{x} \qquad y(1) = 5.$$

$$y' + \boxed{\overset{p(x)}{\frac{1}{x}}}\, y = \boxed{\overset{q(x)}{3x^2}}$$

$$g(x) = e^{\int \frac{1}{x} dx} = e^{\ln x} = x \qquad x > 0$$

$$x y' + \frac{x}{\cancel{x}} y = 3x^3$$

$$x y' + y = 3x^3$$

$$(xy)' = 3x^3$$

$$xy = \frac{3}{4} x^4 + C$$

general. $\Rightarrow y = \frac{3}{4} x^3 + \frac{C}{x}$ $\qquad y = \frac{3}{4} x^3 + \frac{17}{4x}$

particular : IC $\quad y(1) = 5 \quad \Rightarrow \quad 5 = \frac{3}{4} + C$

$\boxed{\begin{matrix} x=1 \\ y=5 \end{matrix}}$ $\qquad \boxed{C = \frac{17}{4}}$

Example 1 Find the general solution to

$$\frac{1}{x}\frac{dy}{dx} - \frac{2y}{x^2} = x\cos x\,, \qquad x > 0\,. \tag{9}$$

$$y' - \frac{2}{x}y = x^2 \cos x$$

$$P(x) = -\frac{2}{x}$$

integrating factor: $g(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln|x|} = e^{\ln x^{-2}} = x^{-2}$

$$\boxed{x^{-2}y' - 2x^{-3}y = \cos x}$$

$$(x^{-2}y)' = \cos x$$

$$x^{-2}y = \sin x + C$$

$$\boxed{y = x^2 \sin x + Cx^2}$$