MA448 – Classwork 0.
Euler's Method for Initial Value Problems
Date: 08/29/2019

Purpose: To implement the Euler method $Y_{n+1} = Y_n + hf(t_n, Y_n)$ for initial value problem:

$$\begin{aligned} y'(t) &= f(t, y(t)), \quad t_0 \le t \le t_{\max} \\ y(t_0) &= y_0 \end{aligned}$$

and investigate the order of convergence numerically. Your code requires input: t_0, $t_{\max}$, y_0, and N (the total number of time-steps to be executed). At the end, print out the final n, t_n, Y_n (appropriately labeled).

1. Below is a Python code `cw0.py` to solve the initial value problem $y'(t) = t/y,\ 0 \le t \le 5,\ y(0) = 1$.

```
def f(t,y):
    return t/y

def euler(t0,tmax,y0,N):
    t,dt=np.linspace(t0,tmax,N,retstep=True)
    y=np.zeros(N)
    y[0]=y0
    for n in range(N-1):
        y[n+1] = y[n] + dt * f(t[n],y[n])
    return t,y

t0=0
tmax=5
y0=1
N=10

[t,y]=euler(t0,tmax,y0,N)

plt.plot(t,y,'o-')
plt.title('Numerical Approximation to the IVP')
plt.ylabel('x(t)')
plt.xlabel('t')
plt.show()

print("==================================================================")
print("  n          tn                        yn  ",end='\n')
for n in range(len(y)):
    print('{0:3d}    {1:0.15f}    {2:0.15f}  '.format(n, t[n], y[n]))
print("==================================================================")
```

2. Since we know the exact solution ($y = \sqrt{t^2+1}$), we can compare the numerical solution y_n with the exact solution. Compute the exact solution at the point t_n and plot both numerical and exact solution together. Also, modify your output to print out ($n \quad t_n \quad y_n \quad |y_n - exact(t_n)|$) at every step, and maximum error at the end of the run.

```
exact=np.sqrt(t^2+1)
abs_err=abs(exact-y)
plt.plot(t,exact,'-',t,y,'d-')
plt.title('Numerical Approximation to the IVP')
plt.legend(['exact_sol','euler_approx'],loc='best')
plt.ylabel('x(t)')
plt.xlabel('t')
plt.show()
```

3. Numerically investigate the the rate of convergence of Euler's method to the following initial value problem.

$$\frac{dy}{dt} = 1 + \frac{y}{t},\ 1 \le t \le 6,\ y(1) = 1$$

4. Confirm the global error bound for Euler's method for the initial value problem from **3**.

$$|y(t_n) - Y_n| \le \frac{hM}{2L}\left(e^{L(t_n - t_0)} - 1\right)$$