MA345 Differential Equations & Matrix Method

Lecture: 03

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Definition 1.1.1 Differential Equation

An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a **differential equation (DE)**.

In order to talk about them, we will classify a differential equation by **type**, **order**, and **linearity**.

Classification by Type If a differential equation contains only ordinary derivatives of one or more functions with respect to a *single* independent variable it is said to be an **ordinary differential equation (ODE)**. An equation involving only partial derivatives of one or more functions of two or more independent variables is called a **partial differential equation (PDE)**.

Definition 2.3.1 Linear Equation

A first-order differential equation of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x) \tag{1}$$

is said to be a **linear equation** in the dependent variable y.

When $g(x) = 0$, the linear equation (1) is said to be **homogeneous**; otherwise, it is **nonhomogeneous**.

A second-order ODE is called **linear** if it can be written

$$y'' + p(x)y' + q(x)y = r(x) \tag{1}$$

and **nonlinear** if it cannot be written in this form.

The distinctive feature of this equation is that it is *linear in y and its derivatives,* whereas the functions p, q, and r on the right may be any given functions of x. If the equation begins with, say, $f(x)y''$, then divide by $f(x)$ to have the **standard form** (1) with y'' as the first term.

Explicit Solution

Definition 1. A function $\phi(x)$ that when substituted for y in equation (1) [or (2)] satisfies the equation for all x in the interval I is called an **explicit solution** to the equation on I.

Definition 1.1.2 Solution of an ODE

Any function ϕ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the interval.

Example 1 Show that $\phi(x) = x^2 - x^{-1}$ is an explicit solution to the linear equation

$$\frac{d^2y}{dx^2} - \frac{2}{x^2}y = 0, \qquad (3)$$

but $\psi(x) = x^3$ is not.

$f(x) = x^2 - x^{-1}$

$f'(x) = 2x + x^{-2}$

$f''(x) = 2 - 2x^{-3}$

$(2 - 2x^{-3}) - \frac{2}{x^2}(x^2 - x^{-1}) =$

$x \neq 0$

$= (2 - 2x^{-3}) - (2 - 2x^{-3}) = 0$

$x \neq 0$

$g(x) = x^3$ is not

$g' = 3x^2$

$g'' = 6x$

$6x - \frac{2}{x^2} \cdot x^3 = 4x \neq 0$

Example 2 Show that for *any* choice of the constants c_1 and c_2, the function

$$\phi(x) = c_1 e^{-x} + c_2 e^{2x}$$

is an explicit solution to the linear equation

$$y'' - y' - 2y = \underline{0}\,. \tag{4}$$

$y = f(x) = c_1 e^{-x} + c_2 e^{2x}$

$y' = f'(x) = -c_1 e^{-x} + 2c_2 e^{2x}$

$y'' = f''(x) = c_1 e^{-x} + 4c_2 e^{2x}$

$$\overbrace{(c_1 e^{-x} + 4c_2 e^{2x})}^{y''} - \overbrace{(-c_1 e^{-x} + 2c_2 e^{2x})}^{y'} - 2\overbrace{(c_1 e^{-x} + c_2 e^{2x})}^{y} =$$

$$= \underbrace{(c_1 + c_1 - 2c_1)}_{=0} e^{-x} + \underbrace{(4c_2 - 2c_2 - 2c_2)}_{0} e^{2x} = \underline{0}$$

$(-\infty, \infty)$

Implicit Solution

Definition 2. A relation $G(x, y) = 0$ is said to be an **implicit solution** to equation (1) on the interval I if it defines one or more explicit solutions on I.

Example 4 Show that

$$x + y + e^{xy} = 0 \tag{7}$$

is an implicit solution to the nonlinear equation

$$(1 + xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0\,. \tag{8}$$

$$\frac{dy}{dx} + \frac{1+ye^{xy}}{1+xe^{xy}} = 0$$

$$\frac{d}{dx}(x+y+e^{xy}) = 1 + \frac{dy}{dx} + ye^{xy} + xe^{xy}\frac{dy}{dx} = 0$$

$$\boxed{(1+xe^{xy})\frac{dy}{dx} + 1 + ye^{xy} = 0}$$

IVP - initial value problem

IC : initial condition

$$y(x_0)=y_0$$

$$y'(x_0)=y_1$$

$$y''(x_0)=y_2$$

Initial-Value Problem

In Section 1.2 we defined an initial-value problem for a general nth-order differential equation. For a linear differential equation, an ***n*th-order initial-value problem** is

$$\text{Solve:} \quad a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\text{Subject to:} \quad y(x_0) = y_0, \quad y'(x_0) = y_1, \ldots, \quad y^{(n-1)}(x_0) = y_{n-1}. \tag{1}$$

Boundary-Value Problem

Another type of problem consists of solving a linear differential equation of order two or greater in which the dependent variable y or its derivatives are specified at *different points*. A problem such as

$$\text{Solve:} \quad a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\text{Subject to:} \quad y(a) = y_0, \quad y(b) = y_1$$

is called a **two-point boundary-value problem**, or simply a **boundary-value problem (BVP)**. The prescribed values $y(a) = y_0$ and $y(b) = y_1$ are called **boundary conditions (BC)**. A solution of the foregoing problem is a function satisfying the differential equation on some interval I, containing a and b, whose graph passes through the two points (a, y_0) and (b, y_1).

Existence and Uniqueness of Solution

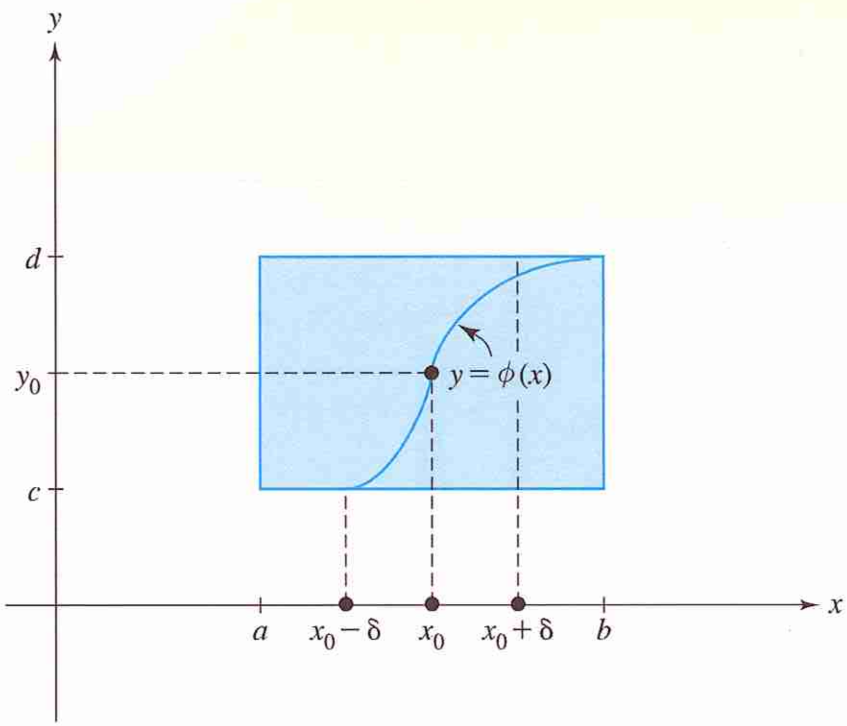
Theorem 1. Consider the initial value problem

$$\frac{dy}{dx} = f(x, y) \ , \qquad y(x_0) = y_0 \ .$$

If f and $\partial f/\partial y$ are continuous functions in some rectangle

$$R = \{(x, y) : a < x < b, c < y < d\}$$

that contains the point (x_0, y_0), then the initial value problem has a unique solution $\phi(x)$ in some interval $x_0 - \delta < x < x_0 + \delta$, where δ is a positive number.†



$$\frac{dy}{dx} = g(x)$$

$$y = \int g(x)\,dx = G(x) + C$$

$$\frac{dy}{dx} = 1 + e^{2x}$$

$$y = \int 1 + e^{2x}\,dx = x + \frac{1}{2}e^{2x} + \underline{\underline{C}}$$

$$y = x + \frac{1}{2}e^{2x} + C$$

SOLUTION BY INTEGRATION Consider the first-order differential equation $dy/dx = f(x, y)$. When f does not depend on the variable y, that is, $f(x, y) = g(x)$, the differential equation

$$\frac{dy}{dx} = g(x) \tag{1}$$

can be solved by integration. If $g(x)$ is a continuous function, then integrating both sides of (1) gives $y = \int g(x)\, dx = G(x) + c$, where $G(x)$ is an antiderivative (indefinite integral) of $g(x)$. For example, if $dy/dx = 1 + e^{2x}$, then its solution is $y = \int (1 + e^{2x})\, dx$ or $y = x + \frac{1}{2}e^{2x} + c$.

DEFINITION 2.2.1 Separable Equation

A first-order differential equation of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separable variables.**

$$\frac{dy}{dx} = y^2 \cdot x \cdot e^{3x+4y} = (x e^{3x})(y^2 e^{4y})$$

$$e^{3x+4y} = e^{3x} \cdot e^{4y}$$

$$\frac{dy}{dx} = (xe^{3x})(y^2e^{4y})$$

$$\int \frac{1}{y^2 e^{4y}}\, dy = \int (x e^{3x})\, dx$$

$$\frac{dy}{dx} = y + \sin x$$

Method for Solving Separable Equations

To solve the equation

$$\frac{dy}{dx} = g(x)p(y) \tag{2}$$

multiply by dx and by $h(y) := 1/p(y)$ to obtain

$$h(y)\,dy = g(x)\,dx\,.$$

Then integrate both sides:

$$\int h(y)\,dy = \int g(x)\,dx\,,$$

$$H(y) = G(x) + C\,, \tag{3}$$

where we have merged the two constants of integration into a single symbol C. The last equation gives an implicit solution to the differential equation.

Example 1 Solve the nonlinear equation

$$\frac{dy}{dx} = \frac{x-5}{y^2}.$$

$$y^2 \frac{dy}{dx} = (x-5)$$

$$y^2\,dy = (x-5)\,dx$$

$$\int y^2\,dy = \int (x-5)\,dx$$

$$\frac{y^3}{3} = \frac{x^2}{2} - 5x + C$$

$$y^3 = \frac{3}{2}x^2 - 15x + 3C$$

$$y = \sqrt[3]{\frac{3}{2}x^2 - 15x + K}$$

EXAMPLE 1 Solving a Separable DE

Solve $(1 + x)\,dy - y\,dx = 0.$

$$\int \frac{dy}{y} = \int \frac{dx}{1+x}$$

$$\ln|y| = \ln|1+x| + C$$

$$\underbrace{e^{\ln|y|}} = e^{\ln|1+x| + C} = \underbrace{e^{\ln|1+x|}} \cdot e^{C}$$

$$y = |1+x|\, e^{C}$$

$$\begin{cases} |1+x| = 1+x & x \geq -1 \\ |1+x| = -(1+x) & x \leq -1 \end{cases}$$

$$y = \pm e^{C}(1+x)$$

$$y = k(1+x)$$

Solve the initial-value problem $\dfrac{dy}{dx} = -\dfrac{x}{y}, \quad y(4) = -3.$

$x=4 \quad y=-3$

$$\int y\,dy = \int -x\,dx$$

$$\frac{y^2}{2} + M = -\frac{x^2}{2} + N$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + C$$

$$\frac{y^2}{2} = -\frac{x^2}{2} + \underbrace{(N-M)}_{C}$$

$$y^2 = -x^2 + 2C$$

$$x^2 + y^2 = 2C$$

$$x^2 + y^2 = M^2$$

$$16 + 9 = M^2 = 25$$

$$\boxed{x^2 + y^2 = 25}$$

EXAMPLE 3 Initial Value Problem (IVP). Bell-Shaped Curve

Solve $y' = -2xy$, $y(0) = 1.8$.

$$\int \frac{dy}{y} = \int dx\,(-2x)$$

$$\ln y = -x^2 + C_1$$

$$y = e^{-x^2+C_1} = e^{-x^2} \cdot \underbrace{e^{C_1}}_{=c}$$

general solution: $y = c \cdot e^{-x^2}$

IC: $x=0$ $y=1.8$

$$1.8 = c \cdot e^0 = c$$

$$c = 1.8$$

particular solution: $\boxed{y = 1.8\, e^{-x^2}}$

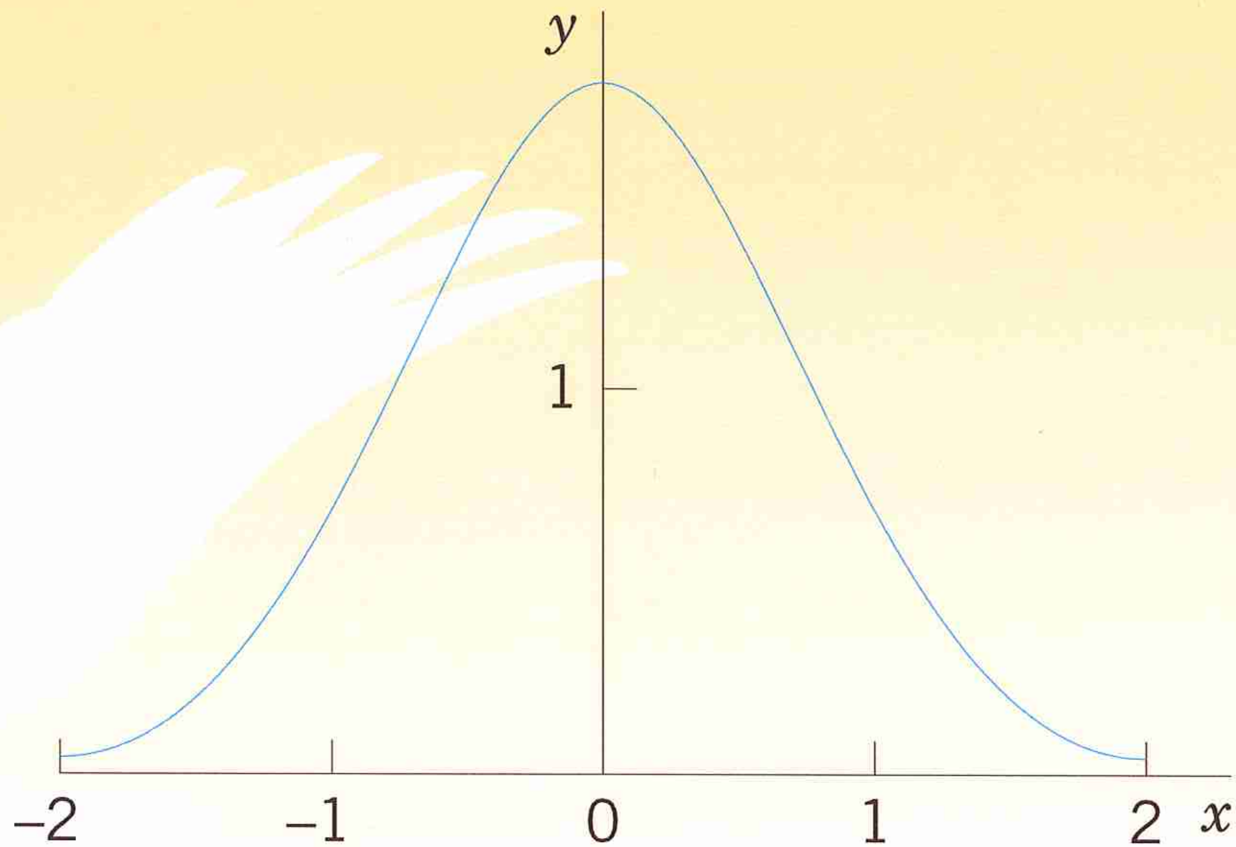


Fig. 10. Solution in Example 3 (bell-shaped curve)

Example 3 Solve the nonlinear equation

$$\frac{dy}{dx} = \frac{6x^5 - 2x + 1}{\cos y + e^y}. \tag{9}$$

$$\int (\cos y + e^y)\,dy = \int (6x^5 - 2x + 1)\,dx$$

$$\sin y + e^y + C_1 = x^6 - x^2 + x + C_2$$

$$\boxed{\sin y + e^y = x^6 - x^2 + x + C}$$

Implicit solution

$$y' = y^2 e^{-x}$$

$y \neq 0$

$$\frac{dy}{dx} = y^2 e^{-x}$$

$$\int \frac{1}{y^2}\, dy = \int e^{-x}\, dx$$

$$-\frac{1}{y} = -e^{-x} + C$$

$$\boxed{y = \frac{1}{e^{-x} - C}}$$ general solution

$\underline{y = 0}$ -> singular solutions