MA448 – Project 2.

Boundary Value Problems in Matlab/Python

Due: Tuesday 11/05/2019

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1. Linear algebraic equations can arise in the solution of differential equations. Consider the following second order ODE describing a heat balance in a long thin rod.

$$\frac{d^2T}{dx^2} + h(T_a - T) = 0,\ T(0) = 40,\ T(10) = 200 \tag{1}$$

Here $T(x)$ is the temperature of the rod ($°C$) at the position x (m), h is the heat transfer coefficient between the rod and the ambient air (m^{-2}), and T_a is the temperature of the surrounding air ($°C$). The equation (1) can be transformed into a set of linear algebraic equations by using a finite divided difference approximation for the second derivative as

$$\frac{d^2T}{dx^2} \approx \frac{T_{i+1} - 2T_i + T_{i-1}}{\Delta x^2}$$

where T_i designates the temperature at node x_i. Substituting this approximation in (1) yields the following discrete equation.

$$-T_{i-1} + (2 + h\Delta x^2)T_i - T_{i+1} = h\Delta x^2 T_a \tag{2}$$

a. Solve the ODE (1) for a 10m long rod with $T_a = 20$ and $h = 0.002$.

b. Implement the Shooting method algorithm presented in class to solve the above BVP (1).

c. Solve the BVP (1) using the Matlab's built-in function `bvp4c` or SciPy's `solve_bvp`.

d. Take $n = 100$. Then $\Delta x = \frac{10}{100} = 0.1$. There will be 101 nodes x_i, $i = 0, \cdots, 100$. The temperature is known at the boundary nodes $T_0 = T(x_0) = 40$ and $T_{100} = T(x_{100}) = 200$. Using the relation (2) write a system of 99 linear equations for T_1, T_2, T_3, $\cdots$, T_{99} (the temperature at the internal nodes) in a matrix form $A\mathbf{T} = \mathbf{b}$. Solve the linear system in Matlab/Python.

e. Plot the numerical solutions from (b), (c) and (d) together with the analytical solution (a).

2. Heat transfer through a long thin rod can be described by the following partial differential equation

$$\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)\ ,\quad 0 < x < 1\ ,\ 0 < t \leq T.$$

At the boundary the temperature is kept constant

$$u(0,t) = u(1,t) = 0\ ,\quad 0 < t < T,$$

and the initial temperature profile is given as

$$u(x,0) = 10\sin(\pi x)\ ,\quad 0 \leq x \leq 1.$$

A discretization with respect to x yields the following system of ordinary differential equations.

$$\frac{d\mathbf{v}}{dt} = A\mathbf{v}(t), \qquad v_i(0) = 10\sin(\pi x_i),\ i = 1, \cdots, N-1$$

where

$$A = \frac{1}{\Delta x^2}\begin{pmatrix} -2 & 1 & 0 & \cdots & \cdots & \cdots \\ 1 & -2 & 1 & 0 & \cdots & \cdots \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ \vdots & & & & & \vdots \\ \cdots & 0 & 1 & -2 & 1 & 0 \\ \cdots & \cdots & 0 & 1 & -2 & 1 \\ \cdots & \cdots & \cdots & 0 & 1 & -2 \end{pmatrix}, \qquad \mathbf{v}(t) = (v_1(t), v_2(t), \cdots, v_{N-1}(t))^T,$$

$$\Delta x = 1/N,\ 0 < x_1 = \Delta x < x_2 = 2\Delta x < \cdots < x_{N-1} = (N-1)\Delta x < 1.$$

a. Apply the explicit Euler method

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \Delta t A\mathbf{v}_n$$

to find the solution at time $t = 0.1$, $t = 0.5$ and $t = 1$ for $N = 21$. Note: You need to use small time-step Δt for the stability of the explicit Euler method.

b. Apply the implicit Euler method

$$\mathbf{v}_{n+1} = \mathbf{v}_n + \Delta t A\mathbf{v}_{n+1}$$

to find the solution at time $t = 0.1$ for $N = 21$.

c. Run your code for 100 time-steps with $\Delta t = 0.001$ and plot the numerical results (the temperature $u(x, t)$ at the grids (x_i, t_i).