Series Solutions of Linear Differential Equations

5.1.2 Power Series Solutions

☐ **A Definition** Suppose the linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{5}$$

is put into standard form

$$y'' + P(x)y' + Q(x)y = 0 \tag{6}$$

by dividing by the leading coefficient $a_2(x)$. We make the following definition.

Analytic at a Point A function f is analytic at a point a if it can be represented by a power series in $x - a$ with a positive radius of convergence.

Definition 5.1.1 Ordinary and Singular Points

A point x_0 is said to be an **ordinary point** of the differential equation (5) if both $P(x)$ and $Q(x)$ in the standard form (6) are analytic at x_0. A point that is not an ordinary point is said to be a **singular point** of the equation.

Theorem 5.1.1 Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of the differential equation (5), we can always find two linearly independent solutions in the form of a power series centered at x_0; that is, $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$. A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

ex) $y'' + e^x y' + \ln(x)\, y = 0$

$x = 0$ is singular as $\ln x$ is not defined at $x=0$

ex) $(x^2-1)\, y'' + 2x y' + 6 = 0$

$$y'' + \frac{2x}{(x^2-1)} y' + \frac{6}{(x^2-1)} = 0$$

2 singular points: $x = \pm 1$, all other points are ordinary

Idea:

$$y=\sum_{n=0}^{\infty} a_n (x-x_0)^n$$

$y=a_0+a_1(x-x_0)^1+a_2(x-x_0)^2+\ldots$

$a_2(x)y''+a_1(x)y'+a_0(x)y=0$

ordinary point

$$y'=\sum_{n=1}^{\infty} a_n\cdot n\cdot(x-x_0)^{n-1}$$

$\cancel{a_1(x-x_0)^0+a_2\cdot 2(x-x_0)^1}\ldots$

$$y''=\sum_{n=2}^{\infty} a_n\cdot n(n-1)(x-x_0)^{n-2}$$

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ex: $y''-2xy'+8y=0$ (about $x_0=0$)

$$\sum_{n=2}^{\infty} a_n(n\cdot(n-1))(x)^{n-2}-2x\cdot\sum_{n=1}^{\infty} a_n\cdot n(x)^{n-1}+8\sum_{n=0}^{\infty} a_n(x)^n=0$$

$x^{n-1}\cdot x=x^n$ we look for same param

$N=n-2$

$n=N+2$

$$\sum_{N=0}^{\infty} a_{N+2}(N+2)(N+1)x^N-2\sum_{n=1}^{\infty} a_n n x^n+8\sum_{n=0}^{\infty} a_n x^n=0$$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n$$