Dr. H. Khanal MA448, Fall 2019

Project 1. Numerical Methods for ODEs

Due: 10/18/2019

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1. Compare the approximate solutions of the following initial value problems

(a) $y'(t) = 3 + 5\sin t + 0.2y,\ y(0) = 0$ (b) $y'(t) = -1000y - e^{-t},\ y(0) = 0$

obtained using (a) the explicit Euler method, (b) the implicit Euler method, (c) the trapezoidal method, (d) the fourth-order classical RK method, (e) the fourth-order Adams-Bashforth-Moulton method, (g) MATLAB function `ode45` (or SciPy function `solve_ivp`).

2. Write a code in Matlab/Python to implement the *Adams-Bashforth-Moulton method of fourth order* for the autonomous system of ODEs.

$$\tilde{\mathbf{Y}}_{n+1} = \mathbf{Y}_n + \frac{h}{24}\left[55\mathbf{F}(\mathbf{Y}_n) - 59\mathbf{F}(\mathbf{Y}_{n-1}) + 37\mathbf{F}(\mathbf{Y}_{n-2}) - 9\mathbf{F}(\mathbf{Y}_{n-3})\right]$$

$$\mathbf{Y}_{n+1} = \mathbf{Y}_n + \frac{h}{24}\left[9\mathbf{F}(\tilde{\mathbf{Y}}_{n+1}) + 19\mathbf{F}(\mathbf{Y}_n) - 5\mathbf{F}(\mathbf{Y}_{n-1}) + \mathbf{F}(\mathbf{Y}_{n-2})\right]$$

Use it to solve the following well known *Lorenz problem* that arises in the study of dynamical systems

$$\begin{aligned}
\frac{dy_1}{dt} &= 10(y_2 - y_1) \\
\frac{dy_2}{dt} &= y_1(28 - y_3) - y_2 \\
\frac{dy_3}{dt} &= y_1 y_2 - \tfrac{8}{3} y_3
\end{aligned}$$

with initial conditions $y_1(0) = 15,\ y_2(0) = 15,\ y_3(0) = 36$. Plot the solution curves for $0 \le t \le 20$.

In problems 3-7 first, use your code RK4/RKF45/ABM4, and also explore some MATLAB functions (`ode23, ode23s, ode23tb, ode45`, etc.) or SciPy function `solve_ivp` (selecting different methods).

3. The following system is a classic example of stiff ODEs that can occur in the solution of chemical reaction kinetics.

$$\begin{aligned}
\frac{dy_1}{dt} &= -0.013y_1 - 1000y_1y_3 \\
\frac{dy_2}{dt} &= -2500y_2y_3 \\
\frac{dy_3}{dt} &= -0.013y_1 - 1000y_1y_3 - 2500y_2y_3
\end{aligned}$$

Solve these equations from $t = 0$ to 50 with initial conditions $y_1(0) = y_2(0) = 1$ and $y_3(0) = 0$. Present your results in graphical form.

4. S. Rinaldi ("Laura and Petrarch: An Intriguing Case of Cyclical Love Dynamics", *SIAM J. Appl. Math*, **58**, pp. 1205-1221, 1998) presents the following model for emotional and inspirational cycle of the fourth-century Italian poet petrarch:

$$\begin{aligned}\frac{dL}{dt} &= -3.6L + 1.2\left(P(1-P^2) - 1\right)\\ \frac{dP}{dt} &= -1.2P + 6\left(L + \frac{2}{1+Z}\right)\\ \frac{dZ}{dt} &= -1.2Z + 12P\end{aligned}$$

Here, L represents the love of Laura (a beautiful woman who was Petrarch's inspiration) for Petrarch, P represents the magnitude of Petrarch's love for Laura, and Z represents the poet's inspiration level. The time is measured in years. Starting from the initial conditions

$$L(0) = P(0) = Z(0) = 0,$$

simulate 21 years of Petrarch's emotional cycle. Display your results as functions of time and in the $P - L$ and $Z - P$ phase planes.

5. D. Winter ("On the Stem Curve of a Tall Palm in a Strong Wind," *SIAM Review*, **35**, pp. 567-579, 1993) develops the following model for the stem curve of a palm tree subject to wind loading:

$$\begin{aligned}\frac{d^2\theta}{ds^2} &= \frac{W_s}{EI}\left(1 - \frac{s}{L} + \frac{W_c}{W_s}\right)\sin\theta + \frac{D}{EI}\cos\theta\\ \frac{dx}{ds} &= \sin\theta\\ \frac{dz}{ds} &= \cos\theta.\end{aligned}$$

The variables in the problem are the angle of the stem relative to the vertical position, θ, the arc length measured along the stem, s, the horizontal displacement of the stem, x, and height of a location along the stem, z. Both x and z are treated as functions of s. The parameters are the total stem weight $W_s = 22700\,N$, the Young's modulus of the stem $E = 6 \times 10^9\,N/m^2$, the moment of inertia of the stem $I = 5.147 \times 10^{-4}\,m^4$, the length of stem $L = 30\,m$, the total canopy weight $W_c = 1385.5\,N$, and the wind drag force on the canopy $D = 4.135\,U^2 N$, where U is the wind speed in m/s. With initial conditions, $\theta(0) = \theta'(0) = x(0) = z(0) = 0$, simulate the stem curve of a palm tree subject to a wind speed of $18\,m/s$.

6. In 1940, the third-largest suspension bridge in the world collapsed in a high wind. The following system of differential equations is a mathematical model that attempts to explain how twisting oscillations can be magnified and cause such a calamity.

$$\begin{aligned}y'' &= -dy' - (\frac{K}{ma}\left[e^{\alpha(y-l\sin\theta)} - 1 + e^{\alpha(y+l\sin\theta)} - 1\right] + 0.2W\sin\omega t\\ \theta'' &= -d\theta' + \frac{3\cos\theta}{l}\frac{K}{ma}\left[e^{\alpha(y-l\sin\theta)} - e^{\alpha(y+l\sin\theta)}\right]\\ \texttt{ICs} &: \; y(0)=0,\, y'(0)=0,\, \theta(0)=\theta_0,\, \theta'(0)=0\end{aligned}$$

The last term in the y equation is the forcing term for the wind W, which adds a strictly vertical oscillation to the bridge. Here, the roadway has width $2l$ hanging between two suspended cables, y is the current distance from the center of the roadway as it hangs below its equilibrium point, and θ is the angle the roadway makes with the horizontal. Also, Newton's Law $F = ma$ is used and Hooke's constant K. Explore how ODE solvers (MATLAB function `ode45` or SciPy function `solve_ivp`) can be used to generate numerical trajectories for various parameter settings. Illustrate different types of phenomena that are available in this model. (`a sample:` $m = 2500Kg$, $K = 1000N$, $l = 12m$, $d = 0.01$, $\alpha = 0.2$, $\omega = 2\pi(38/60)$, $\theta_0 = 0.001$).