Series Solutions of Linear Differential Equations

5.1.2 Power Series Solutions

A Definition Suppose the linear second-order differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \tag{5}$$

is put into standard form

$$y'' + P(x)y' + Q(x)y = 0 \tag{6}$$

by dividing by the leading coefficient $a_2(x)$. We make the following definition.

Analytic at a Point A function f is analytic at a point a if it can be represented by a power series in $x - a$ with a positive radius of convergence.

Definition 5.1.1 Ordinary and Singular Points

A point x_0 is said to be an **ordinary point** of the differential equation (5) if both $P(x)$ and $Q(x)$ in the standard form (6) are analytic at x_0. A point that is not an ordinary point is said to be a **singular point** of the equation.

Theorem 5.1.1 Existence of Power Series Solutions

If $x = x_0$ is an ordinary point of the differential equation (5), we can always find two linearly independent solutions in the form of a power series centered at x_0; that is, $y = \sum_{n=0}^{\infty} c_n(x - x_0)^n$. A series solution converges at least on some interval defined by $|x - x_0| < R$, where R is the distance from x_0 to the closest singular point.

$$\frac{1}{1-x} = \sum_{m=0}^{\infty} x^m = 1 + x + x^2 + \longrightarrow \qquad (|x| < 1, \textbf{ geometric series})$$

$$e^x = \sum_{m=0}^{\infty} \frac{x^m}{m!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \longrightarrow$$

$$\cos x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m}}{(2m)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - + \longrightarrow$$

$$\sin x = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+1}}{(2m+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - + \longrightarrow .$$

$$y'' + y = 0$$

about $x_0 = 0$

$$y = A\cos x + B\sin x$$

$$y = \sum_{n=0}^{\infty} a_n x^n \qquad y'' = \sum_{n=2}^{\infty} a_n (n)(n-1) x^{n-2}$$

$$\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2} + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\begin{bmatrix} k = n-2 \\ n = k+2 \end{bmatrix}$$

$$\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) \underbrace{x^k} + \sum_{n=0}^{\infty} a_n \underbrace{x^n} = 0$$

$$\sum_{n=0}^{\infty} a_{n+2} (n+2)(n+1) x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=0}^{\infty} \left(a_{n+2} (n+2)(n+1) + a_n \right) x^n = 0$$

$$a_{n+2} (n+2)(n+1) + a_n = 0$$

$$a_{n+2} = -\frac{a_n}{(n+2)(n+1)}$$

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$n=0$ $\quad a_2 = \frac{-a_0}{2}$ $\qquad$ $n=1$ $\quad a_3 = \frac{-a_1}{3\cdot 2}$

$n=2$ $\quad a_4 = -\frac{a_2}{4\cdot 3} = \frac{a_0}{4\cdot 3\cdot 2} = \frac{a_0}{4!}$ $\qquad$ $n=3$ $\quad a_5 = -\frac{a_3}{5\cdot 4} = \frac{a_1}{5\cdot 4\cdot 3\cdot 2} = \frac{a_1}{5!}$

$n=4$ $\quad a_6 = -\frac{a_4}{6\cdot 5} = -\frac{a_0}{6!}$ $\qquad$ $n=5$ $\quad a_7 = \frac{-a_5}{7\cdot 6} = \frac{-a_1}{7!}$

$$a_{2n} = \frac{(-1)^n a_0}{(2n)!} \qquad a_{2n+1} = \frac{(-1)^n a_1}{(2n+1)!}$$

$$y = a_0 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}}_{\cos x} + a_1 \underbrace{\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}}_{\sin x}$$

$$y = a_0 \cos x + a_1 \sin x$$

$$(1-x^2)y'' - 2xy' + 12y = 0\ ; \quad y(0)=0 \quad y'(0)=1$$

$a_0 = 0 \qquad a_1 = 1$

1) Check if there is power series solution!

in the standard form:

$$y'' - \underbrace{\frac{2x}{1-x^2}}_{P(x)} y' + \underbrace{\frac{12}{1-x^2}}_{Q(x)} y = 0$$

$$\frac{1}{1-x^2} = \frac{1}{1-x} + \frac{1}{1+x}$$

$$\sum x^n$$

$$|x|<1$$

P(x) and Q(x) have power series expansion about $x_0 = 0$ with $R=1$

$$(1-x^2)\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2} - 2x \sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1} + 12 \cdot \sum_{n=0}^{\infty} a_n x^n = 0$$

$$\sum_{n=2}^{\infty} a_n \cdot n \cdot (n-1) x^{n-2} - \sum_{n=2}^{\infty} n \cdot (n-1) \cdot a_n x^n - 2\sum_{n=1}^{\infty} a_n \cdot n \cdot x^n + 12\sum_{n=0}^{\infty} a_n x^n = 0$$

$k = n-2$

$n = k+2$

$$\sum_{n=0}^{\infty} a_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty} n(n-1) \cdot a_n x^n - 2\sum_{n=1}^{\infty} a_n \cdot n \cdot x^n + 12\sum_{n=0}^{\infty} a_n x^n = 0$$

take out $n=0$ and $n=1$

$$\underbrace{\left(a_2\cdot 2\cdot 1\cdot x^0 + 12\cdot a_0 x^0\right)}_{n=0} + \underbrace{\left(a_3\cdot 3\cdot 2\cdot x^1 - 2\cdot a_1\cdot 1\cdot x^1 + 12 a_1 x^1\right)}_{n=1} +$$

$$+\sum_{n=2}^{\infty}\left((n+2)(n+1)a_{n+2} - n(n-1)a_n - 2\cdot a_n\cdot n + 12 a_n\right)x^n = 0$$

$$\underbrace{(2a_2 + 12a_0)} + \underbrace{(6a_3 x - 2a_1 x + 12 a_1 x)} +$$

$$+\sum_{n=2}^{\infty}\underbrace{\left[(n+2)(n+1)a_{n+2} + (-n^2+n-2n+12)a_n\right]}x^n = 0$$

$$a_2 = -6a_0$$

from IC: $a_0 = 0$
$a_1 = 1$

$$a_3 = \frac{2a_1 - 12a_1}{6} = \frac{-10a_1}{6} = -\frac{5}{3}a_1$$

$a_0 = 0 \Rightarrow$

$a_1 = 1$

$a_2 = 0$

$a_3 = -\frac{5}{3}$

$a_4 = 0 \Rightarrow$ all further even $= 0$

$$(n+2)(n+1)a_{n+2} + (-n^2 - n + 12)a_n = 0$$

$$(n+2)(n+1)a_{n+2} - ((n-3)(n+4))a_n = 0$$

$$\boxed{a_{n+2} = \frac{(n-3)(n+4)}{(n+2)(n+1)}a_n}$$

$$(n=3) \Rightarrow a_5 = \frac{0\cdot 7}{5\cdot 4}a_3 = 0 \Rightarrow \text{all further odd} = 0$$

$$y = a_1 x + a_3 x^3 = x - \frac{5}{3}x^3$$