Math 104

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Contents

1 Defining Numbers

1.1 *Natural Numbers*

Lecture 1
August 27^{th}, 2015

1.1 definition. Peano axioms for the set of natural numbers:

(N1) $1 \in \mathbb{N}$

(N2) $n \in \mathbb{N} \Rightarrow \exists n + 1 \in \mathbb{N}$, called the **successor** of n

(N3) 1 is not the successor of any element of $\mathbb{N}$

(N4) $n + 1 = m + 1 \Rightarrow n = m$

(N5) A subset of $\mathbb{N}$ containing 1 and containing $n + 1$ whenever it contains n must be the entire set $\mathbb{N}$.

Lecture 2
September 1^{st}, 2015

There are some intuitions about the natural numbers which are not represented directly by these axioms. For example, we know that any natural number which is not 1 is the successor of some natural number.

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1.2 theorem. $\forall n \in \mathbb{N} : n \neq 1 \Rightarrow \exists m \in \mathbb{N} : n = m + 1$

Proof. Let $n \in N$ s.t. $n \neq 1$. Suppose $\forall m \in \mathbb{N}$, $n \neq m + 1$. Let $S = \mathbb{N} \setminus \{n\}$. Let $q \in S$. Then $q \in \mathbb{N}$ and $q \neq n$. Since $q + 1 \in \mathbb{N}$ by N2 and $q + 1 \neq n$ (since n is not the successor of any natural number), then $q + 1 \in S$. Since $n \neq 1$, $1 \in S$. Therefore $S = \mathbb{N}$ by N5. But $n \in \mathbb{N}$ and $n \notin S$. Contradiction. □

1.3 theorem (Well-Ordering Principle). *Any subset of the natural numbers admits a "least element." Logically,*

$$\forall S \subseteq \mathbb{N} : \exists n_0 \in S : \forall n \in S : n_0 \leq n + 1$$

TODO: Proof of WOP based on these Peano postulates

1.4 definition. For some $S \subseteq \mathbb{N}$, if

1. $1 \in S$
2. Whenever $\{1, 2, \ldots, n\} \subset S$, then $n + 1 \in S$

then $S = \mathbb{N}$. This is called **strong induction**.

Lecture 3
September 3rd, 2015

Nicholas Bourbaki: school of thought putting forth that there are three main types of structures in mathematics:

- Algebraic structures $\xrightarrow{\text{binary operations}}$ Algebra
- Order structures $\xrightarrow{\text{inequalities}}$ Analysis
- Topological structures $\xrightarrow{\text{continuums, stretches}}$ Geometry/Topology

Goal: identify the "optimal" sets of axioms (related to the above three structures) which will uniquely determine the set of real numbers.