Math 250A

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Contents

1 Group Actions

Lecture 1
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Let G be a group. There are two equivalent ways to formulate G as an action on a set S:

1. As a map $G \times S \to S$, $(g, s) \mapsto gs = g \cdot s$. Under this formulation, there are two axioms:
   - $e \cdot s = s$
   - $(gg') \cdot s = g \cdot (g' \cdot s)$
2. As a homomorphism from G to the symmetric group on S, Perm(S). It is defined as $(\phi(g))(s) = g \cdot s$. There are again two axioms:
   - $\phi(e) = \text{id}$

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- $\phi(gg') = \phi(g) \cdot \phi(g')$

One group action is just the trivial action, $g \cdot s = s$ or $g \mapsto \text{id}$.

G acts on itself by left translation. Here, $G = S$. $g \cdot s = gs$, and this is simply the group product. $(gg')s = g(g's)$, which is just the associative law. G acts on itself by conjugation. Again, $G = S$. $g \cdot s := gsg^{-1}$ and $(gg')s = (gg')s(gg')^{-1} = g(g'sg'^{-1})g^{-1}$.

G acts of the set of subgroups of G by conjugation.

$$g \cdot H := gHg^{-1} = \{ghg^{-1} | h \in H\}$$

Let $N \trianglelefteq G$. Then you can say that G acts on N:

$$g \cdot n := gng^{-1} \in N$$

1.1 EXAMPLE. Let V be a vector space over a field K. Let $G = GL(V)$ be the group of invertible linear maps $V \to V$. Then G acts on V. Let $L \in G$. Then $L \cdot v = L(v)$.

1.1 *Stabilizers and Orbits*

Suppose we have a group G acting on a set S. This defines a natural equivalence relation on S:

1.2 DEFINITION. Let $s, s' \in S$. Then $s \sim s' \Leftrightarrow s' = gs$ for some $g \in G$. The **orbit** of S is the equivalence class of s under this relation.

S can now be written as a disjoint union of orbits.

1.3 DEFINITION. Let G act on itself by conjugation. $S = G$. Then the orbit of $s \in S$ is $\{gsg^{-1} | g \in G\}$. This is called the **conjugacy class** of s.

1.4 NOTATION. The orbit of s is usually notated as $O(s)$ or $G \cdot s$.

1.5 DEFINITION. The **stabilizer** of s, G_s is $\{g \in G | g \cdot s = s\}$. Intuitively, it is the set of elements of the group which leave s alone.

One piece of intuition here is that an element with a large stabilizer should have a small orbit and vice-versa.

1.6 theorem. *Let s be an element of a group G. There is a natural bijection α between $O(s)$ and G/G_s where $\alpha(gG_s) = g \cdot s$.*

Lecture 2
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Therefore $\#O(s) = \#(G/G_s) = (G : G_s)$.

1.7 definition. Let Σ be a set of representatives for the equivalence relation given above (i.e. one point from each orbit). Then $\#S = \sum_{s \in \Sigma} \#O(s) = \sum_s (G : G_s)$. If G is finite, then $(G : G_s) = \frac{\#G}{\#G_s}$, so $\#S = \#G \sum_s \frac{1}{\#G_s}$. This is known as the **mass formula**.

1.8 corollary. $G_{s'} = G_{s.g} = gG_s g^{-1}$.

1.9 theorem. *$g \in G$ leaves all $s \in S$ fixed if and only if $g \in G_s$ for all s. This is the case if and only if $g \in \bigcap_{s \in S} G_s$.*

1.10 definition. The above expression $\bigcap_{s \in S} G_s$ is called the **kernel**.

1.11 claim. *Assume $x \in G_s$. Then $gxg^{-1} \in G_{gs}$. This shows that $gG_s g^{-1} \subseteq G_{gs}$.*

Proof. $(gxg^{-1})(gs) = gxs = gs$ since $x \in G_s$. □

1.2 *Applications of Group Actions*

In nearly all applications of group actions, we have a prime number p.

1.12 definition. A **p-group** is a finite group G with $\#G = p^n$ for some $n \in \mathbb{N}$.

1.13 theorem ("A p-group has a non-trivial center"). *Let Z be the subset of G such that the elements of Z commute with all elements of G. Then $Z = \{g \in G : gs = sg \forall s \in G\}$. Note*

1.14 notation. S^G refers to the set of fixed points for the action of G on S.

Let G be some p-group, let S be a finite set, and let $s \in S$. Then $\#O(s) = \frac{\#G}{\#G_s} = \frac{p^n}{p^k}$. There are two possible cases:

1. $\#O(s) = 1$, so s is fixed by S^G
2. $k < n$, so $\#(S)$ is divisible by p.

$\#S$ is the sum of the numbers of elements in the orbits. Considered modulo p, this is the number of orbits of size 1. This is equal to $\#(S^G)$.

TODO: Make sense of this and finish derivation

1.15 theorem. *Let G be a finite group and H be a subgroup of G. Assume that $(G : H) = p$ where p is prime and that p is the smallest prime number dividing $\#G$. Then $H \trianglelefteq G$.*

Proof. Consider the set $S = G/H = \{gH : g \in G\}$. Then $\#S = (G : H) = p$. Then we have an action of G on G/H by left translation. This action is a map $G \to \text{Perm}(S)$ which is the symmetric group S_p

Consider the stabilizer of H, $G_H = \{x \in G : xH = H\} = H$. So $G_{gH} = gHg^{-1}$. So the kernel K (the intersection of the stabilizers) is $\bigcap_{g \in G} gHg^{-1}$ which is the largest normal subgroup contained in H.

Reminder: First Isomorphism Theorem

$(G : K) = \#(G/K)$ divides the order of Perm(S), which is equal to $p!$.

$G \supseteq H \supseteq K$, $(G : K) = (G : H) \cdot (H : K)$. The LHS divides $p!$. $(G : H) = p$. WTF $(H : K)$. From above we know that $(H : K)$ divides $(p - 1)!$. We know it is relatively prime to $\#(G)$. Therefore $(H : K) = 1$, i.e. $H = K$. We know K is normal. Therefore so is H. □

1.3 *Sylow's Theorem*

1.16 theorem (Cauchy). *There is a natural way to embed G in Perm(G), written $G \overset{\varphi}{\hookrightarrow} Perm(G)$.*

Proof. Let $S = G$. The action $g : s \mapsto gs \in G$. This action has trivial kernel. $K = \{g \in G : gs = s \text{ for all } s\} = \{e\}$.

When a homomorphism has trivial kernel, it is one-to-one. Thus φ is an embedding. □

1.17 definition. Let G be a finite group of order n. Let V be a set of functions $G \xrightarrow{f} \mathbb{Z}$, $V \approx \mathbb{Z}^n$.

Linear maps $V \xrightarrow{L} V \leftrightarrow n \times n$ matrices over $\mathbb{Z}$

Invertible linear maps $V \leftrightarrow n \times n$ invertible matrices over $\mathbb{Z}$

This forms a group $GL(V) \approx GL(n, \mathbb{Z})$.

Embeddings: $G \subseteq GL(V)$. $G \subseteq GL(n, \mathbb{Z})$.