Math 104

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Contents

1 Natural Numbers

Lecture 1
August 27^{th}, 2015

1.1 definition. Peano axioms for the set of natural numbers:

(N1) $1 \in \mathbb{N}$

(N2) $n \in \mathbb{N} \Rightarrow \exists n+1 \in \mathbb{N}$, called the **successor** of n

(N3) 1 is not the successor of any element of $\mathbb{N}$

(N4) $n+1 = m+1 \Rightarrow n = m$

(N5) A subset of $\mathbb{N}$ containing 1 and containing $n+1$ whenever it contains n must be the entire set $\mathbb{N}$.

Lecture 2
September 1^{st}, 2015

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There are some intuitions about the natural numbers which are not represented directly by these axioms. For example, we know that any natural number which is not 1 is the successor of some natural number.

1.2 theorem. $\forall n \in \mathbb{N} : n \neq 1 \Rightarrow \exists m \in \mathbb{N} : n = m + 1$

Proof. Let $n \in N$ s.t. $n \neq 1$. Suppose $\forall m \in \mathbb{N}$, $n \neq m + 1$. Let $S = \mathbb{N} \setminus \{n\}$. Let $q \in S$. Then $q \in \mathbb{N}$ and $q \neq n$. Since $q + 1 \in \mathbb{N}$ by N2 and $q + 1 \neq n$ (since n is not the successor of any natural number), then $q + 1 \in S$. Since $n \neq 1$, $1 \in S$. Therefore $S = \mathbb{N}$ by N5. But $n \in \mathbb{N}$ and $n \notin S$. Contradiction. $\square$

1.3 theorem (Well-Ordering Principle). *Any subset of the natural numbers admits a "least element." Logically,*

$$\forall S \subseteq \mathbb{N} : \exists n_0 \in S : \forall n \in S : n_0 \leq n + 1$$

TODO: Proof of WOP based on these Peano postulates

1.4 definition. For some $S \subseteq \mathbb{N}$, if

1. $1 \in S$
2. Whenever $\{1, 2, \ldots, n\} \subset S$, then $n + 1 \in S$

then $S = \mathbb{N}$. This is called **strong induction**.

2 Real Numbers

2.1 *Ordered Fields*

Lecture 3
September 3rd, 2015

Nicholas Bourbaki: school of thought putting forth that there are three main types of structures in mathematics:

- Algebraic structures $\xrightarrow{\text{binary operations}}$ Algebra
- Order structures $\xrightarrow{\text{inequalities}}$ Analysis
- Topological structures $\xrightarrow{\text{continuums, stretches}}$ Geometry/Topology

Goal: identify the "optimal" sets of axioms (related to the above three structures) which will uniquely determine the set of real numbers.

Lecture 4
September 8th, 2015

TODO: Incorporate notes from when I left early (on groups and fields)

2.1 definition. An **ordered field** is a tuple $(F, +, \cdot, \leq)$ with axioms:

- $(F, +)$ is an abelian group.
- $(F \setminus \{0\})$ is an abelian group.
- $\cdot$ distributes over $+$.
- $\leq$ is a total ordering on F (i.e. it is reflexive, antisymmetric, transitive, and total).
- $\forall a, b, c \in F : a \leq b \implies a + c \leq b + c$.
- $\forall a, b, c \in F : a \leq b \wedge 0 \leq c \implies a \cdot c \leq b \cdot c$

2.2 fact. $\mathbb{C}$ cannot be an ordered field for any ordering $\leq$.

The following are true in any ordered field:

1) $\forall a, b \in F : a \leq b \implies -b \leq -a$.
2) $\forall a, b \in F : a \leq b \wedge c \leq 0 \implies b \cdot c \leq a \cdot c$.

2.3 definition. Let F be a field. For all $a \in F$, define

$$|a| := \begin{cases} a & a \geq 0 \\ -a & a < 0 \end{cases}$$

This is called the **absolute value** of a.

Properties of the absolute value:

(i) $\forall a \in F, |a| \geq 0$, and $|a| > 0$ if and only if $a \neq 0$.

(ii) $\forall a, b \in F, |a \cdot b| = |a| \cdot |b|$.

(iii) $\forall a, c \in F$ with $c \geq 0$, $|a| \leq c \iff -c \leq a \leq c$.

(iv) $\forall a, b \in F, |a + b| \leq |a| + |b|$. This is the **triangle inequality**.

A useful consequence of the triangle inequality:

$$||a| - |b|| \leq |a - b|$$

2.4 definition. We say a is a **maximum** for $A \subset F$ if and only if $a \in A$ and $\forall x \in A, x \leq a$. A **minimum** is defined similarly.

2.5 definition. $a \in F$ is an **upper bound** for the set $A \subset F$ if and only if $\forall x \in A, x \leq a$. If such a bound exists, we say A is **bounded above**. Definitions for **lower bound** and **bounded below** are similar.

2.6 definition. $s \in F$ is a **supremum** (least upper bound) for the set A if and only if

(i) s is an upper bound for A.

(ii) For all upper bounds a for A, we have $s \leq a$.

We then say $s = \sup F$.

An **infimum** of F is defined identically as the greatest lower bound. If i is an infimum of F we say $i = \inf F$.

2.2 *Properties of the Real Numbers*

2.7 definition (Completeness Axiom). Any nonempty subset $A \subseteq \mathbb{R}$ which is bounded above admits a supremum in $\mathbb{R}$.

Lecture 5
September 10th, 2015

$\mathbb{R}$ is identified as the only possible "complete" ordered field. Any other ordered field that satisfies the completeness axiom is isomorphic to $\mathbb{R}$.

2.8 proposition. *Consider the open interval $S = (-3, 2]$ in $\mathbb{R}$. No minimum exists in S.*

Proof. Assume $a = \min S$. Then $-3 < a \leq 2 \implies 0 < a + 3$. $1 < 1 + 1 = 2 \implies 1^{-1} > 2^{-1} \implies \frac{1}{2} < 1 \implies \frac{a+3}{2} < a + 3$. Let $b = a - \frac{a+3}{2} = \frac{a-3}{2}$. Then $-3 < b < a$. □

TODO: Why does that last sentence hold?

2.9 example. Let $A = \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$. $\max A = 1$ and $\min A$ does not exist. The set of upper bounds $U = [1, +\infty)$. The set of lower bounds $L = (-\infty, 0]$. The last of these needs justification, which we'll see later.

2.10 definition. If A admits no upper bound, we say A is **unbounded above** and if A admits no lower bound, we say A is **unbounded below**.

2.11 notation. If $A \neq \emptyset$ and A is unbounded above, we write that $\sup A = +\infty$. This does not mean that $+\infty$ is a number nor that $\sup A$ exists. If $A \neq \emptyset$ is unbounded below, we write $\inf A = -\infty$. We also write that $\sup \emptyset = -\infty$ and $\inf \emptyset = +\infty$.

2.12 theorem (Archimedean Principle). *$\mathbb{N}$ is unbounded above in $\mathbb{R}$.*

Proof. TODO □

2.13 theorem. *If $\sup A$ exists, it is unique. Same for $\inf A$.*

Proof. Assume $s_1 = \sup A$ and $s_2 = \sup A$. Fix some $\varepsilon > 0, \varepsilon \in \mathbb{R}$. Then $s_2 - \varepsilon < s_2 \implies s_2 - \varepsilon$ is not an upper bound so there exists some element $x \in A$ such that $x > s_2 - \varepsilon \implies s_2 < x + \varepsilon \leq s_1 + \varepsilon$ because s_1 is an upper bound. Since $\forall \varepsilon > 0$, $s_2 < s_1 + \varepsilon$, $\forall \varepsilon > 0, s_2 - s_1 < \varepsilon \implies s_2 - s_1 \leq 0$. This argument is symmetric w.r.t. s_1 and s_2, so we also have that $s_1 - s_2 \leq 0 \implies s_2 = s_1$. Thus $\sup A$ is unique. The same argument can be applied for $\inf A$. □

TODO: Get that guy's name Here is an alternate, cleaner proof of the uniqueness of $\sup A$, courtesy of []:

Proof. □

2.14 theorem (Existence of $\sqrt{2}$). *There exists $s_0 \in \mathbb{R}, s_0 > 0$ such that $s_0^2 = 2$.*

Proof. Let $A = \{x \in \mathbb{R} : x > 0, x^2 < 2\}$. The A is bounded above because 2 is an upper bound for A. (If $x \in A$ and $x > 2$, then $x^2 > 2 \cdot 2 = 2 + 2 > 2$. This is a contradiction because $x \in A$ implies $x^2 < 2$.) By the completeness axiom, this means that $\sup A$ must exist. Let $s_0 = \sup A$.

Since s_0 is an upper bound, $\forall x \in A$, $s_0 \geq x > 0 \implies \forall x \in A$, $s_0^2 \geq x^2$. Suppose $s_0^2 < 2$. Let $x = (s_0 + \varepsilon)$ for some $\varepsilon > 0$. Then $x^2 = s_0^2 + 2\varepsilon s_0 + \varepsilon^2 = s_0^2 + (2s_0 + \varepsilon)\varepsilon$. Choose $\varepsilon < \min\{1, \frac{2-s_0^2}{2s_0+1}\}$. Then $s_0^2 + (2s_0^2 + \varepsilon)\varepsilon < s_0^2 + (2s_0 + 1)\varepsilon \leq s_0^2 + (2s_0 + 1)\frac{2-s_0^2}{2s_0+1} = 2 \implies x^2 < 2 \implies x \in A$. But $x = s_0 + \varepsilon > s_0$, which is an upper bound for A. Contradiction. So $s_0^2 \not< 2 \implies s_0^2 \geq 2$.

Since s_0 is the smallest upper bound for A, $\forall \varepsilon > 0$, $s_0 - \varepsilon$ is not an upper bound and there exists some $x \in A$ such that $x > s_0 - \varepsilon$. $0 < s_0 < x + \varepsilon \implies s_0^2 < (x + \varepsilon)^2 = x^2 + 2\varepsilon x + \varepsilon^2 = x^2 + \varepsilon(2x + \varepsilon) < 2 + \varepsilon(4 + \varepsilon)$. So for all $0 < \varepsilon < 1$, $s_0^2 < 2 + 5\varepsilon$. Therefore $s_0^2 \leq 2 + 0 = 2$. □