Aggregation of results from stochastic reserving methods: Worked example

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Abstract

This article describes a worked example of the aggregation of results produced using stochastic reserving methods across classes of business. It is designed to demonstrate the impact on results of using different assumptions for each class of business and to explore how different correlation assumptions affect, for example, the diversification benefits produced when the results are aggregated across classes. Results are shown using a simple "Variance/Covariance" approach as well as different copula approaches. The web-based software application that accompanies the book entitled "Claims Reserving in General Insurance", by David Hindley, includes a module that allows the user to reproduce the results in this article, and to explore the impact on results of alternative assumptions. This is available at www.claimsreserving.com - selecting the "Reserving book /app" option. All references to "Sections" in this article are to sections in that book.

1 Assumptions

For this simplified example, it is assumed that there are three classes of business where a mathematical distribution has been selected for the total future claims across all cohorts.

In practice, if, for example, bootstrapping had been used for each class of business, then the bootstrap simulation results could be combined instead to produce the aggregate distribution, using, for example, a copula re-sorting approach, as outlined in Section 4.9.1. If this re-sorting approach was applied, for example, to the distribution of total future claims across all cohorts, then the resulting sort order could also be used for other components of each simulation that might be of interest, such as the individual cohorts or CDR, thus implying the same dependency between classes as for the total across cohorts.

The use of mathematical distributions, rather than bootstrapping in this example makes the implementation of the aggregation straightforward using the R software. This then enables the main focus of the worked example to be on the conclusions that can be drawn from the results using different correlation and copula assumptions. The same observations would apply equally well if bootstrap simulation results had been aggregated.

The distributional and associated parameter assumptions for each of the three classes are shown in Table 1. It is assumed that both parameter and process error have been allowed for in these distributions, and hence the standard deviation assumption is referred to as the Prediction Error or "PE". Class A is based on the same Taylor and Ashe data that has been used in the worked examples in the reserving book, and has the same reserve as derived by applying Mack's method to this data, including a tail factor, as shown in Table **??**. The CV has been selected as 16%, which was derived by applying the ODP Bootstrap (with constant scale parameter) to the data, as shown in Reserving book Table 4.20 [1].

Table 1
Aggregation example: Summary of individual classes of business

Class	Reserve	Pred'n Error	CV	Fitted Dist'n	Parameters
A	20,219	3,235	16%	Lognormal	$\mu = 9.9017; \sigma = 0.1589$
B	21,250	1,630	8%	Gamma	$\beta = 125; \gamma = 170$
C	18,606	4,725	25%	Lognormal	$\mu = 9.8; \sigma = 0.25$
Total	60,075				

Class B and C have been chosen to have lower and higher CV's respectively, when compared to Class A. The relative variability of the future claims distribution for each class can also be seen in Figure 1, which shows the distribution of the three classes. The lower variability of Class B is clearly evident from this figure.

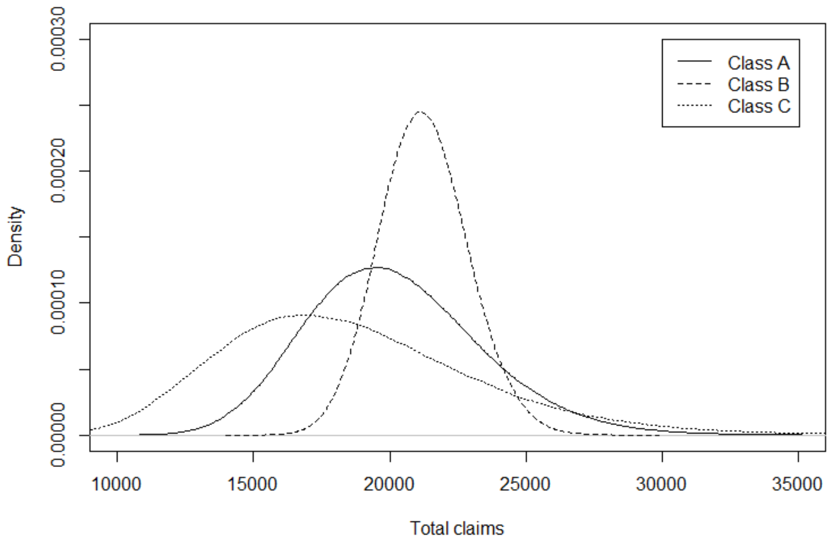
Another way of comparing the three distributions is to compare the percentile values. Table 2 shows the percentile values and the implied ratio of these values to the mean reserve for each class. The total column in this table assumes that the future claims for the three classes of business are fully correlated - i.e. that there is no diversification benefit between them. Hence, the total percentile values are the sum of those for the individual classes. The PE is the same as the fully correlated value in the Variance-Covariance approach, as explained in that subsection below.

For Class A, the ratio of the values at different percentile levels to the mean, as shown in Table 2, are very similar[2] to those produced from the ODP Bootstrap process for the Taylor and Ashe data, as shown in Reserving book Table 4.43, suggesting that the Lognormal is a reasonable fit to the bootstrap data.

[1] The use of 16% rather than the value produced from the Mack method (of 13%) is just to demonstrate that in some cases, judgement might be used to blend the results from different methods.

[2] In fact, they are identical at all the stated percentiles below 99.5^{th}, and only marginally different at that level.

Figure 1
Aggregation example:Distribution of three classes



For this example, to understand the impact on the aggregrate distribution of different dependency assumptions between the classes, results have been derived using the following:

- Case 1: Simple linear correlation using the Variance/Covariance approach.
- Case 2: Full independence.
- Case 3: Gaussian copula.
- Case 4: t-copula with 1 degree of freedom[3].
- Case 5: t-copula with 4 degrees of freedom.

To demonstrate the interaction of choice of copula and level of correlation, as well as Case 2, with no correlation, a High and a Low scenario for the correlation between the classes has been assumed for each of the other cases:

$$\text{High} = \begin{array}{cc} \text{Class} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1.0 & 0.8 & 0.6 \\ 0.8 & 1.0 & 0.5 \\ 0.6 & 0.5 & 1.0 \end{bmatrix} \end{array} ; \text{Low} = \begin{array}{cc} \text{Class} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1.0 & 0.1 & 0.2 \\ 0.1 & 1.0 & 0.1 \\ 0.2 & 0.1 & 1.0 \end{bmatrix} \end{array} \tag{1}$$

[3]Note that the $t-$copula with 1 degree of freedom is equivalent to the Cauchy copula.

Table 2
Aggregation example: Distributions of individual classes

Item	Class A	Class B	Class C	Total[a]
Mean	20,219	21,250	18,606	60,075
Pred' Error	3,235	1,630	4,725	9,590
CV	16%	8%	25%	16%

	Value for Class:			
Percentile	A	B	C	Total[a]
50.00%	19,965	21,208	18,034	59,207
75.00%	22,225	22,325	21,346	65,896
90.00%	24,477	23,364	24,844	72,685
95.00%	25,932	23,400	27,206	76,538
99.50%	30,069	25,682	34,336	90,087
99.90%	32,632	26,644	39,048	98,324

	Ratio to mean for Class:			
Percentile	A	B	C	Total[a]
50.00%	99%	100%	97%	99%
75.00%	110%	105%	115%	110%
90.00%	121%	110%	134%	121%
95.00%	128%	110%	146%	127%
99.50%	149%	121%	185%	150%
99.90%	161%	125%	210%	164%

[a] Undiversified

For the purpose of this simplified example, it is assumed that these are linear correlations. In practice, if rank correlations had been determined, then they can be transformed into linear correlations, as referred to in the Gaussian copula procedure given in Section 4.9.1.

1.1 Case 1: Variance/Covariance approach

Here, the distribution assumptions in Table 1 are not needed, as the Prediction Errors are just combined using the procedure outlined in Section 4.9.2. The $\boldsymbol{S}$ matrix referred to in that section will include the Prediction Errors from Table 1 on its diagonal, that is:

$$\boldsymbol{S} = \begin{pmatrix} 3,235 & 0 & 0 \\ 0 & 1,630 & 0 \\ 0 & 0 & 4,725 \end{pmatrix}$$

Then, for the Low correlation example, this is combined with the correlation matrix above, which is denoted by $\boldsymbol{V}$, to give:

$$\boldsymbol{SVS} = \begin{pmatrix} 10,465,225 & 527,240 & 3,057,195 \\ 527,240 & 2,656,250 & 770,111 \\ 3,057,195 & 770,111 & 22,327,373 \end{pmatrix} \tag{2}$$

The square root of the sum of these values then represents the estimated Prediction Error for the aggregate reserve. This is 6,645, which gives a CV of 11% using the mean reserve of 60,075. The same matrix calculation is done using the High correlation matrix, as well as with the correlations all set to one, and the opposite - fully independent - is calculated by taking the square root of the sum of the individual PE's. A summary of the results is given in Table 3. For

Table 3
Aggregation example: Variance/Covariance approach - Summary of Standard Errors of Aggregate distribution

Correlation	PE	CV
Zero	5,954	9.9%
Low	6,645	11.1%
High	8,362	13.9%
Full	9,590	16.0%

illustration purposes only, if it is assumed that the Lognormal distribution is appropriate for the aggregate distribution of claims across all classes combined, then each of these PEs (along with the mean of 60,075) can be used to estimate the Lognormal parameters and then the reserves at selected percentile values (using the approach given in Section 6.9.4). These are shown in the "Case 1 - VCV" columns in Table 4, which compares the results for all the cases and correlation assumptions.

1.2 Cases 2 to 4: Independence and Copula approach

To produce the results for these cases, the R software package was used (see **?**), as it has very convenient copula functions, as well as the ability to model a wide range of marginal distributions for each reserving category. Other software packages could of course also be used. The relevant R code is included in the software application referred to in the abstract. Rather than provide the detailed workings here, which would be impractical for such complex calculations, a summary of the algorithm that is used in R is outlined below.

- Define the three marginal distributions for the aggregate claims of each class and their associated parameters, as per Table 1.
- Calculate summary statistics and percentiles for these three distributions, as shown in Table 2.

- If required, draw a plot of the distributions, as shown in Figure 1 just to check that they look reasonable.
- Specify the correlation matrices, as per those shown in the Variance/ Covariance case.
- Specify the types of copulas that will be used (these will be Independent, Gaussian, $t-$copula 1df and $t-$ copula 4df in this example).
- Create the aggregate (or "joint") distribution using the selected marginal distributions along with each combination of copula type and correlation assumption. In this example, this was implemented using the "copula" package in the R software.
- Simulate from each of these joint distributions (100,000 values were simulated for this example).
- If required, draw scatter-plots of the simulated results, showing, for selected pairs of classes, how they are related under each dependency assumption (e.g. to compare tail-dependency).
- Determine statistics for these joint distributions, including percentile values.
- If required, draw plots of the different aggregate cumulative distributions to compare results graphically.

The numerical results are summarised in Table 4. "Case 0" in this table is simply the result of adding the values for each class, without any diversification credit - as per the Total column in Table 1. The diversification credit implied by the Gaussian copula and the two $t-$copulas is summarised in Table 5, which shows the reduction in the reserve value, compared to the undiversified results (as a % of the undiversified results of Case 0). Figures 2 and 3 show some example scatter-plots for the different copula and correlation assumptions - in each case for the pair of classes A and B.

Several observations can be made regarding the numerical results. Some of these will apply in most situations and are very intuitive and somewhat obvious, but they are stated anyway for completeness. Others may only apply to this specific example.

- The mean values for the reserves are very slightly different to the theoretical mean for the aggregate distribution (of 60,075) in some cases, which is due to the fact that simulation is used to produce the distribution of the aggregate claims (using the relevant copula functionality in the R software to impose the required input correlations between the classes). If bootstrap results were being re-sorted instead, then the means would all be the same.

Table 4
Aggregation example: Comparison of results for different dependency approaches

	Case 0	Case 1 - VCV				Case 2 - Indep'	Case 3 - Gaussian copula		
Corr'n:	Undiversified	Zero	Low	High	Full	Copula	Zero	Low	High
Mean	60,075	60,075	60,075	60,075	60,075	60,049	60,077	60,077	60,072
PE	9,590	5,954	6,645	8,362	9,590	5,953	5,968	6,649	8,349
CV	16.0%	9.9%	11.1%	13.9%	16.0%	9.9%	9.9%	11.1%	13.9%
50.00%	59,207	59,782	59,711	59,501	59,324	59,642	59,621	59,560	59,392
75.00%	65,896	63,905	64,322	65,329	66,023	63,759	63,765	64,166	65,197
90.00%	72,685	67,857	68,775	71,061	72,697	67,780	67,869	68,778	71,038
95.00%	76,538	70,339	71,586	74,729	77,010	70,471	70,588	71,826	74,865
99.50%	90,087	77,120	79,326	85,015	89,266	78,022	78,413	80,569	85,872
99.90%	98,324	81,144	83,956	91,295	96,855	83,003	83,301	85,891	92,751

	Case 4 - t copula 1df			Case 5 - t copula 4df		
Corr'n:	Zero	Low	High	Zero	Low	High
Mean	60,032	60,032	60,021	60,083	60,084	60,084
PE	5,994	6,590	8,171	6,015	6,684	8,375
CV	10.0%	11.0%	13.6%	10.0%	11.1%	13.9%
50.00%	59,603	59,557	59,464	59,622	59,557	59,403
75.00%	62,860	63,254	64,345	63,482	63,895	64,993
90.00%	66,979	67,870	70,266	67,542	68,456	70,811
95.00%	70,430	71,617	74,640	70,497	71,704	74,863
99.50%	82,252	84,069	87,947	80,464	82,569	87,571
99.90%	90,518	92,319	95,893	87,630	90,152	96,064

Table 5

Aggregation example:Summary of diversification benefits

	Gaussian			$\boldsymbol{t}-$copula 1df			$\boldsymbol{t}-$copula 4df		
Percentile	Zero	Low	High	Zero	Low	High	Zero	Low	High
50.00%	-1%	-1%	0%	-1%	-1%	0%	-1%	-1%	0%
75.00%	3%	3%	1%	5%	4%	2%	4%	3%	1%
90.00%	7%	5%	2%	8%	7%	3%	7%	6%	3%
95.00%	8%	6%	2%	8%	6%	2%	8%	6%	2%
99.50%	13%	11%	5%	9%	7%	2%	11%	8%	3%
99.90%	15%	13%	6%	8%	6%	2%	11%	8%	2%

Figure 2
Aggregation example: Scatter plots of Total reserves for different copula

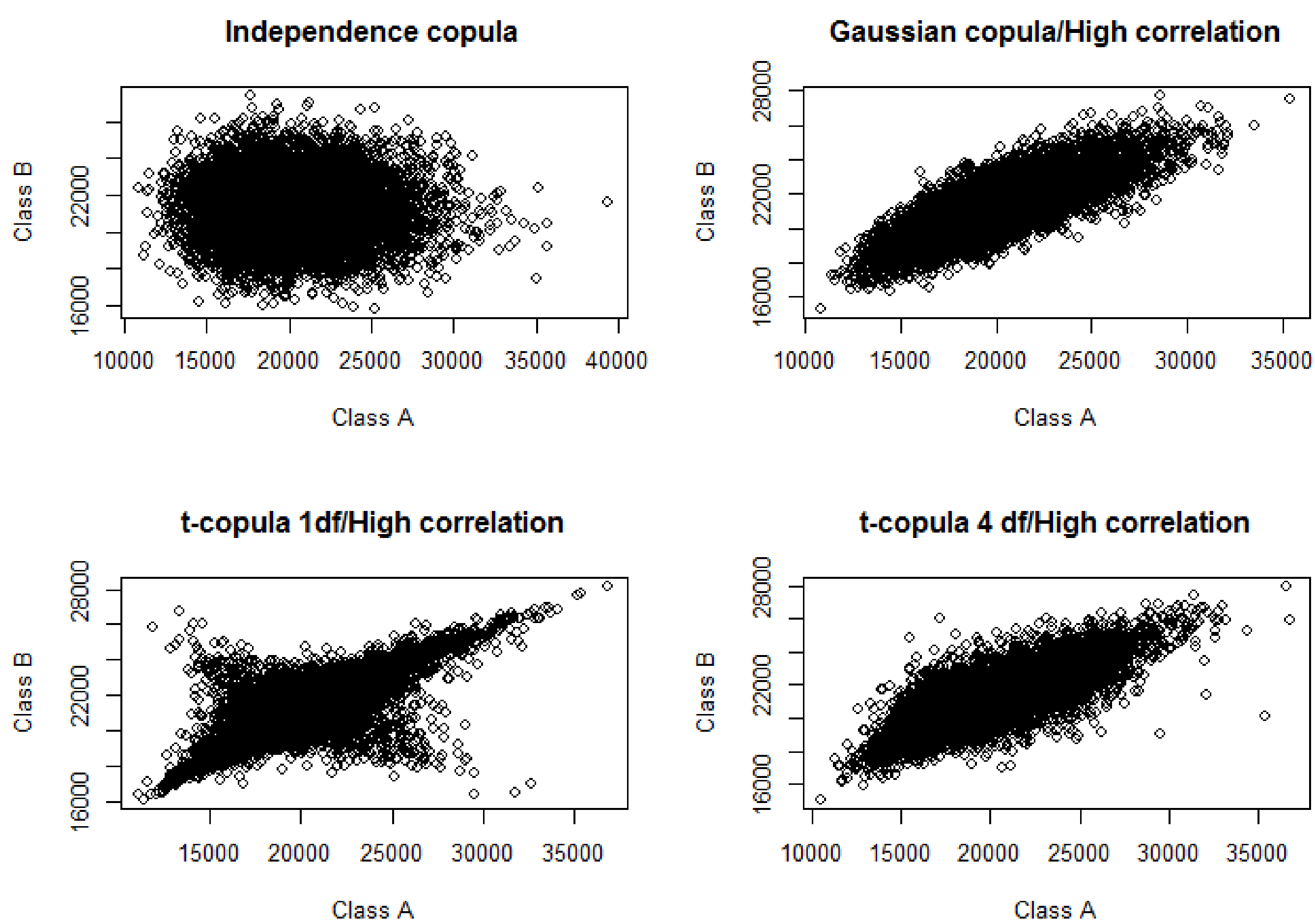
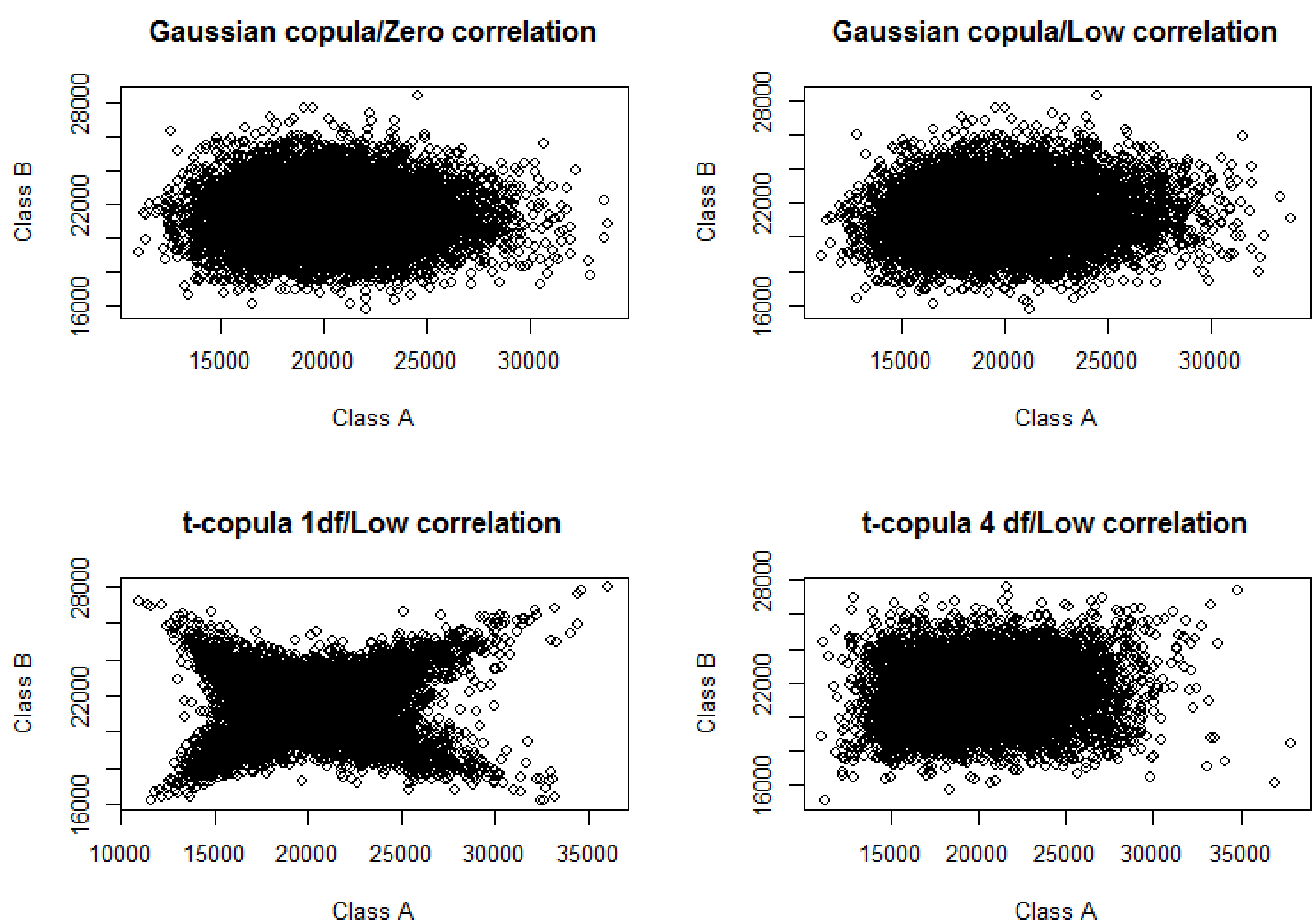


Figure 3
Aggregation example: Scatter plots of Total reserves for different copula - Continued



- For all percentiles above 50%, the higher the correlation, the higher the estimated future claims. In other words, the higher the correlation, the lower the diversification benefit.

- For each correlation and dependency assumption, the higher the percentile, the higher the proportionate diversification benefit.

- The variability(as measured by the CV) increases as the level of correlation increases.

- The VCV approach produces a reasonable approximation to the Gaussian copula, for each correlation assumption, although in this case it seems to slightly overstate the results for high and full correlation, but slightly understate the results at the very high percentiles for the zero and low correlation assumptions, and so should be used with caution.

- The results for the Gaussian copula with zero correlation are very close to the results for the Independence copula. In theory they should be the same, but simulation will cause some differences. In practice, the Gaussian copula with zero correlation is often used to produce results assuming that all classes are independent, leading to an effective lower bound for the reserve values at higher percentiles.

- At the higher percentiles, the t-copulas produce higher values than than the Gaussian copula, reflecting the effect of tail-dependency in the t-copulas. In other words, the tail-dependency of the t-copulas reduces the diversification benefit compared to the Gaussian copula with the same correlation assumptions.

- The increase, compared to the Gaussian copula, caused by the tail-dependency of the t copulas is reduced the higher the correlations. In other words, if there is low correlation between the classes, the impact of using the t-copula instead of the Gaussian copula, will be proportionately greater than with higher correlation.

- For the t-copulas, even with zero correlation, the values are higher than the Independence copula (and Gaussian with zero correlation) reflecting the fact that there is still some tail-dependency introduced through the use of the t-copula.

- For the t-copulas, the lower degrees of freedom in Case 4 produces higher results than those produced using the higher degrees of freedom in Case 5. In other words, Case 4 produces less diversification benefit than Case 5, as the degrees of freedom are lower.

- The 1df t-copula results with Low correlation give only slightly more diversification benefit than the Gaussian copula with High correlation. Hence, if in a particular situation, for whatever reason, it was only possible to produce results using the Gaussian copula, then a very approximate proxy for the impact of tail-dependency could be achieved by increasing the correlation assumptions above what were the baseline assumptions. This should obviously be used with caution, and where tail-dependency needs to be allowed for, it would normally be preferable to find a way to allow for it explicitly through for example the t-copula or through other copulas.

From the scatter-plots given in Figures 2 and 3 it can be seen that:

- The independence copula in Figure 2 is very similar, as expected, to the Gaussian copula with zero correlation in Figure 3, confirming the equivalence of these two approaches.
- The impact of high correlation, compared with low correlation, is seen clearly by comparing the corresponding copula graphs in the two figures.
- The star shape of the $t-$copula can be a feature of this type of copula when the degrees of freedom are low, depending on the selected marginal distributions. The implied relationship between the variables is somewhat complex and may not be a desirable feature.
- With low correlation, the Gaussian copula is very close to the zero correlation plot, which is consistent with the relative diversification credit seen in Table 5.
- With low correlation, the t-copula with higher degrees of freedom is reasonably similar to the Gaussian copula with low correlation, indicating the lessening impact of tail-dependency as the degrees of freedom increases.