Poisson distribution

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Intended learning outcomes

- Recognize and apply Poisson distributions in statistical models
- Derive the Poisson distributions

Recall: Binomial distribution

Definition

A random variable X has a binomial distribution if

$$P(X = x \mid N, \theta) = \binom{N}{x} \theta^x (1-\theta)^{N-x}, \quad x \in \{0, 1, \dots, N\}$$

where $\theta \in [0, 1]$ is the probability of success.

Definition

e^x is defined by the product limit:

$$e^x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n$$

Counting events within a unit interval

Suppose we expect $\lambda > 0$ events to occur within a unit interval $[0, 1)$.

What is the distribution of the number of event occurrences X?

Split the interval into N segments and regard each segment as a Bernoulli trial.

N has to be sufficiently large so that at most one event is observed per segment.

What happens as $N \to \infty$?

What happens to Binomial(N, θ) as $N \to \infty$ while λ is fixed?

$$\text{Binomial}\,(x \mid N, \theta) = \binom{N}{x} \theta^x \,(1 - \theta)^{N-x}$$

Substituting $\theta = \frac{\lambda}{N}$,

$$\begin{aligned}
\text{Binomial}\left(x \mid N, \frac{\lambda}{N}\right) &= \frac{N!}{x!\,(N-x)!} \left(\frac{\lambda}{N}\right)^x \left(1 - \frac{\lambda}{N}\right)^{N-x} \\
&= \frac{\lambda^x}{x!} \frac{N!}{(N-x)!N^x} \left(1 - \frac{\lambda}{N}\right)^N \left(1 - \frac{\lambda}{N}\right)^{-x}
\end{aligned}$$

Take the limit of each term separately:

$$\lim_{N\to\infty} \frac{N!}{(N-x)!N^x} = 1$$

$$\lim_{N\to\infty} \left(1 - \frac{\lambda}{N}\right)^N = e^{-\lambda} \quad \text{(def'n of } e\text{)}$$

$$\lim_{N\to\infty} \left(1 - \frac{\lambda}{N}\right)^N = 1$$

Therefore,

$$\lim_{N\to\infty} \text{Binomial}\left(x \mid N, \frac{\lambda}{N}\right) = \frac{\lambda^x}{x!} e^{-}\lambda$$

Poisson distribution

Definition

A random variable X has a Poisson distribution if

$$P\left(X = x \mid \mu\right) \;=\; \frac{\mu^x}{x!}e^{-\mu}, \quad x \in \{0, 1, ...\}$$

where $\mu > 0$ is the constant rate of event occurrences per unit interval.
X represents count of event occurrences within the unit interval.

Is Poisson$(x \mid \mu)$ proper?

Non-negativity

Unit measure

We need to prove

$$\sum_{x=0}^{\infty} \text{Poisson}\left(x \mid \mu\right) = 1$$

$$\sum_{x=0}^{\infty} e^{\mu} \frac{\mu^x}{x!} = 1$$

In other words, we need to prove

$$\sum_{x=0}^{\infty} \frac{\mu^x}{x!} = e^{\mu}$$

Recall the Taylor expansion:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Apply Taylor expansion on e^{μ} completes the proof.

Counting events within a variable interval

Let X be the event count within a unit interval ($t = 1$).

Then, we showed that

$$X \sim \mathsf{Poisson}(\mu).$$

If $t \neq 1$, then

$$X \sim \mathsf{Poisson}\left(t\mu\right).$$

Recall: Geometric distribution

A random variable X has a geometric distribution if

$$P\left(X = x \mid \theta\right) \;=\; \theta(1-\theta)^{x-1}, \quad x \in \{1, 2, ...\},$$

where θ is the probability of success, and X represents the total number trials required for one success.

Recall the cumulative distribution function

$$P(X \leq x \mid \theta) = \sum_{k=0}^{x} P(X = k \mid \theta)$$

For the geometric distribution,

$$P(X \leq x \mid \theta) = \sum_{k=1}^{x} \theta(1-\theta)^{k-1} = 1-(1-\theta)^{x} \quad \text{(geometric series).}$$

Therefore, for a geometric random variable X,

$$P(X > x \mid \theta) \ = \ 1 - P(X \leq x \mid \theta) \ = \ (1-\theta)^{x}.$$

Waiting time until an event

Suppose we expect to wait μ amount of time until an event.

What the distribution of the waiting time T until the event?

1. Find $P\left(T > t\right)$
2. Find $F_T(t) = P\left(T \leq t\right)$
3. Find $f_T(t) = \frac{d}{dt} F_T(t)$

Divide waiting time into small intervals with size τ.

Let X represent the number of trials before event occurs. Then,

$$t = x\tau, \quad \theta = \frac{\tau}{\mu},$$

where θ is the probability of event occurrence within a trial.

$$\begin{aligned} P(X > x) &= P\left(X > \frac{t}{\tau}\right) \\ &= \left(1 - \frac{\tau}{\mu}\right)^{t\tau^{-1}} \\ &= \left(1 - \frac{\lambda}{m}\right)^{tm}, \quad m = \tau^{-1},\ \lambda = \mu^{-1} \end{aligned}$$

As $\tau \to 0$, $m \to \infty$ and

$$\lim_{m\to\infty}\left(1-\frac{\lambda}{m}\right)^{tm} = e^{-\lambda t} \quad \text{(Product limit def'n of } e\text{)}.$$

Therefore,

$$P\left(T > t\right) = \lim_{\tau\to 0} P\left(X\tau > t\right) = e^{-\lambda t}$$

$$F_T(t) = P\left(T \leq t\right) = 1 - P\left(T > t\right) = 1 - e^{-\lambda t}$$

$$f_T(t) = \frac{d}{dt}F_T(t) = \lambda e^{-\lambda t}.$$

Exponential distribution

A random variable $X > 0$ has an exponential distribution if

$$f_X(x) = \lambda e^{-\lambda x},$$

where $\lambda > 0$ is the rate parameter.

$\mathsf{E}(X) = \frac{1}{\lambda}$.

If X is event count that follows

$$X \sim \text{Poisson}\left(\mu\right),$$

then the time T between events follows

$$T \sim \text{Exponential}\left(\frac{1}{\mu}\right).$$

Gamma distribution

A random variable $X > 0$ has a Gamma distribution if

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x},$$

where $\lambda > 0$ is the rate parameter and $\alpha > 0$ is a shape parameter.

Gamma distribution is a conjugate prior to the Poisson likelihood.

Summary

"La vie n'est bonne qu'à deux choses: à faire des mathématiques et à les professer."
- Siméon Poisson

Casella & Berger 2002, sections 3.2, pages 92-94, 99-102.

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