Beta-binomial and Dirichlet-multinomial models

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Intended learning outcomes

- Recognize and apply the beta-binomial model
- Recognize and apply the Dirichlet-multinomial model

Solving for the posterior

Given observation X and parameter θ, solve for $p\left(\theta \mid x\right)$.

$$
\begin{aligned}
p\left(x \mid \theta\right)\, p\left(\theta\right) &= h(x)\, g(x,\theta). \\
p\left(x\right) = \int_{\Theta} p\left(x \mid \theta\right)\, p\left(\theta\right)\, d\theta &= h(x) \int_{\Theta} g(x,\theta)\, d\theta.
\end{aligned}
$$

Therefore, the posterior is

$$
\begin{aligned}
p\left(\theta \mid x\right) &= \frac{p\left(x \mid \theta\right)\, p\left(\theta\right)}{p\left(x\right)} = \frac{h(x)\, g(x,\theta)}{h(x) \int_{\Theta} g(x,\theta)\, d\theta} \\
&= \frac{g(x,\theta)}{\int_{\Theta} g(x,\theta)\, d\theta} \\
&= c(x)^{-1} g(x,\theta),
\end{aligned}
$$

where $c(x) = \int_{\Theta} g(x,\theta)\, d\theta$.

Solving for the posterior, simplified

Given observation X and parameter θ, solve for $p\left(\theta \mid x\right)$.

The posterior is

$$\begin{aligned} p\left(\theta \mid x\right) &\propto p\left(x \mid \theta\right)\, p\left(\theta\right) \\ &= h(x)\, g(x, \theta) \\ &\propto g(x, \theta) \end{aligned}$$

Therefore,

$$p\left(\theta \mid x\right) = c(x)^{-1} g(x, \theta),$$

where $c(x) = \int_{\Theta} g(x, \theta)\, d\theta$.

If $p\left(\theta\right)$ is a **conjugate prior**, then $c(x)$ can be solved easily.

Conjugate prior

If $p\left(\theta\right)$ is conjugate to $p\left(x \mid \theta\right)$, then $p\left(\theta \mid x\right)$ will be in the same probability distribution family.

In this case, we call $p\left(\theta\right)$ a **conjugate prior**. For example,

$p\left(\theta\right)$	$p\left(x \mid \theta\right)$
beta	Bernoulli
beta	binomial
Dirichlet	categorical
Dirichlet	multinomial
gamma	Poisson
normal-gamma	univariate normal

Beta-binomial model

$$X \sim \mathsf{Binomial}\left(N, \theta\right), \quad \theta \sim \mathsf{Beta}\left(a, b\right)$$

$$\mathsf{Binomial}\left(x \mid N, \theta\right) = \binom{N}{x} \theta^{x} \left(1 - \theta\right)^{N-x}$$

$$\mathsf{Beta}\left(x \mid \alpha, \beta\right) = B(\alpha, \beta)^{-1} x^{\alpha-1} (1 - x)^{\beta-1}$$

Dirichlet-multinomial model

$$\boldsymbol{X} \sim \text{Multinomial}\left(N, \boldsymbol{\theta}\right), \quad \boldsymbol{\theta} \sim \text{Dirichlet}\left(\boldsymbol{d}\right)$$

$$\text{Multinomial}\left(\boldsymbol{x} \mid N, \boldsymbol{\theta}\right) = \frac{\Gamma(N+1)}{\prod_k \Gamma(x_k+1)} \prod_k \theta_k^{x_k}$$

$$\text{Dirichlet}\left(\boldsymbol{x} \mid \boldsymbol{\alpha}\right) = \frac{1}{B\left(\boldsymbol{\alpha}\right)} \prod_k^K x_k^{\alpha_k - 1}$$

Summary

Conjugate priors facilitate fast Bayesian updates.

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