Chapter 16

Green's Function

16.1 Introduction

The method of Green's functions is an alternative method for solving partial differential equations with general forcing. (Reference: Haberman (2004): *Applied Partial Differential Equations*, 4th ed.)

16.2 Green's functions for ODEs

Let L be the Sturm-Liouville operator:

$$L \equiv \frac{d}{dx}(p(x)\frac{d}{dx}) + q(x).$$

Consider the nonhomogeneous ODE

$$Lu = f(x), \quad a < x < b \tag{16.1}$$

subject to two homogeneous boundary conditions. Instead of solving it with the general forcing $f(x)$, we consider instead the more specific problem

$$LG = \delta(x - \xi), \text{ with the same boundary conditions} \tag{16.2}$$

where $G = G(x, \xi)$ describes the response to a concentrated source located at ξ. $G(x, \xi)$ is called the Green's function for the original problem (16.1), whose solution $u(x)$ is then recovered from

$$u(x) = \int_a^b f(\xi)G(x, \xi)d\xi \tag{16.3}$$

To verify that (16.3) satisfies (16.1), we see that

$$\begin{aligned} Lu &= \int_a^b f(\xi) LG(x,\xi) d\xi \\ &= \int_a^b f(\xi)\delta(x-\xi)d\xi = f(x), \end{aligned}$$

because of a fundamental property of the delta function.

The Green's function is symmetrical with respect to its two arguments, i.e.

$$G(\xi, x) = G(x, \xi) \tag{16.4}$$

The result (16.4) is called Maxwell's reciprocity. Although mathematically it is easily seen from the equation defining $G(x,\xi)$ and the fact that $\delta(x-\xi) = \delta(\xi-x)$, physically Maxwell's reciprocity is not obvious, as it states that the response at x due to a concentrated source at ξ is the same as the response at ξ due to a concentrated source at x.

16.2.1 Jump conditions

Since the Green's function is forced by a delta function, one needs to worry about its continuity. If G is discontinuous, i.e. it has a finite jump at $x = \xi$, then $\frac{d}{dx}G$ has a delta function singularity at the same point and $\frac{d^2}{dx^2}G$ on the left-hand side of (16.2) is more singular than the delta function on the right-hand side. It then follows that G must be continuous at $x = \xi$, but its first derivative must have a jump at $x = \xi$. Its second derivative is then a delta function, consistent with the right-hand side of (16.2).

Integrating (16.2) across $x = \xi$, from $x = \xi^-$ to $x = \xi^+$ ($\xi^- = \xi - \epsilon$, $\xi^+ = \xi + \epsilon$, $\epsilon > 0$ and vanishingly small), we get:

$$p(x)\frac{d}{dx}G \Big|_{x=\xi^-}^{x=\xi^+} = \int_{\xi^-}^{\xi^+} \delta(x-\xi)d\xi = 1, \tag{16.5}$$

This jump condition is supplemented by the matching condition argued earlier:

$$G\Big|_{x=\xi^-} = G\Big|_{x=\xi^+} \tag{16.6}$$

16.2.2 Green's formula

For any two differentiable functions $u(x)$ and $v(x)$

$$\begin{aligned} uLv - vLu &= u\frac{d}{dx}(p\frac{dv}{dx}) + uqv - v\frac{d}{dx}(p\frac{du}{dx}) - vqu \\ &= u\frac{d}{dx}(p\frac{dv}{dx}) - v\frac{d}{dx}(p\frac{du}{dx}). \end{aligned}$$

so

$$uLv - vLu = \frac{d}{dx}[p(u\frac{dv}{dx} - v\frac{du}{dx})]. \tag{16.7}$$

(16.7) is known as Lagrange's identity. Integrating,

$$\int_a^b [uLu - vLu]dx = p(u\frac{dv}{dx} - v\frac{du}{dx}) \Big|_a^b . \tag{16.8}$$

This is known as Green's formula.

If u and v are any two functions satisfying the same set of homogeneous boundary conditions, or the periodic condition, or $p(x) = 0$ at the boundaries, then the right-hand side of (16.8) vanishes, and therefore

$$\int_a^b [uLv - vLu]dx = 0 \tag{16.9}$$

An operator L satisfying (16.9) is called a self-adjoint operator.

The multi-dimensional counterparts to these formulae can be easily derived. For $L = \nabla^2$, we note that

$$\begin{aligned} \nabla \cdot (u\nabla v) &= u\nabla^2 v + \nabla u \cdot \nabla v \\ \nabla \cdot (v\nabla u) &= v\nabla^2 u + \nabla v \cdot \nabla u \end{aligned}$$

Subtracting, we get the multidimensional counterpart to (16.7)

$$u\nabla^2 v - v\nabla^2 u = \nabla \cdot (u\nabla v - v\nabla u). \tag{16.10}$$

In three dimensions, the domain is a volume V. The boundary of the volume is denoted by ∂V, which is a closed surface containing V. Integrating (16.10) over V, we get

$$\begin{aligned} &\iiint_V [u\nabla^2 v - v\nabla^2 u]dV \\ &\quad = \iiint_V \nabla \cdot (u\nabla v - v\nabla u)dV \\ &\quad = \iint_{\partial V} (u\nabla u - v\nabla u) \cdot \mathbf{n}dS \end{aligned} \tag{16.11}$$

The last step utilizes the divergence theorem, with $\hat{\mathbf{n}}$ being the outward unit normal.

In two dimensions, we have instead

$$\iint_A [u\nabla^2 u - v\nabla^2 u] dA = \int_{\partial A} (u\nabla u - v\nabla u)\cdot \hat{\mathbf{n}} ds$$

where ∂A is the closed curve bounding the area A.

16.2.3 Nonhomogeneous boundary conditions

The Green's function defined for homogeneous boundary conditions is also used to solve problems with nonhomogeneous boundary conditions

$$\begin{aligned} Lu &= f(x), \quad a < x < b \\ u(a) &= u_a, \quad u(b) = u_b. \end{aligned} \tag{16.12}$$

We always define the Green's function to be solution to

$$\begin{aligned} LG &= \delta(x-\xi) \\ G(a,\xi) &= 0, \quad G(b,\xi) = 0 \end{aligned} \tag{16.13}$$

Now we use Green's formula (16.8) with $v(x) = G(x,\xi)$ and $u(x)$ the solution to (16.12)

$$\begin{aligned} &\int_a^b [u(x)LG(x,\xi) - G(x,\xi)Lu]dx \\ &\quad = [u(x)p(x)\frac{d}{dx}G(x,\xi) - G(x,\xi)p(x)\frac{d}{dx}u(x)]_a^b \end{aligned}$$

Using the equation (16.12) for $u(x)$ and (16.13) for $G(x,\xi)$, we have

$$\begin{aligned} &\int_a^b u(x)\delta(x-\xi)dx - \int_a^b G(x,\xi)f(x)dx \\ &\quad = u(b)p(b)\frac{d}{dx}G(x,\xi)\Big|_{x=b} - u(a)p(a)\frac{d}{dx}G(x,\xi)\Big|_{x=a} \end{aligned}$$

The first term on the left-hand side of the above equation is $u(\xi)$. Therefore if we interchange x and ξ, we will get

$$u(x) = \int_a^b G(x,\xi)f(\xi)d\xi + u_b p(b)\frac{d}{d\xi}G(x,\xi)\Big|_{\xi=b} - u_a p(a)\frac{d}{d\xi}G(x,\xi)\Big|_{\xi=a} \tag{16.14}$$

16.2.4 Example

$$\frac{d^2}{dx^2}u = f(x), \quad 0 < x < L$$
$$u(0) = 0, \quad u(L) = 0$$

The Green's function satisfies

$$\frac{d^2}{dx^2}G = \delta(x - \xi)$$
$$G(0, \xi) = 0, \qquad G(L, \xi) = 0$$

For $x < \xi$:

$$\frac{d^2}{dx^2}G = 0, \qquad \text{subject to } G(0, \xi) = 0.$$
$$G(x, \xi) = Ax + B = Ax$$

($B = 0$ because $G(0, \xi) = 0$).

For $x > \xi$:

$$\frac{d^2}{dx^2}G = 0, \qquad \text{subject to } G(L, \xi) = 0$$
$$G(x, \xi) = C(x - L).$$

Continuity at $x = \xi$ implies

$$A\xi = C(\xi - L)$$

The jump condition at $x = \xi$ implies

$$C - A = 1.$$

Solving for C and A from these two relationships, we get

$$G(x, \xi) = \begin{cases} -\frac{x}{L}(L - \xi), & x < \xi \\ -\frac{\xi}{L}(L - x), & x > \xi \end{cases}$$

The solution to the original ODE with $f(x)$ as the general forcing term is

$$\begin{aligned} u(x) &= \int_0^L f(\xi)G(x, \xi)d\xi \\ &= \int_0^x -\frac{\xi}{L}(L - x)f(\xi)d\xi + \int_x^L -\frac{x}{L}(L - \xi)f(\xi)d\xi \\ &= \frac{x}{L}\int_0^L \xi f(\xi)d\xi - \int_0^x \xi f(\xi)d\xi - x\int_x^L f(\xi)d\xi. \end{aligned}$$

To verify that this is indeed the solution, we differentiate it twice:

$$\begin{aligned}
\frac{d}{dx}u &= \frac{1}{L}\int_0^L \xi f(\xi)d\xi - xf(x) - \int_x^L f(\xi)d\xi + xf(x) \\
&= \frac{1}{L}\int_0^L \xi f(\xi)d\xi - \int_x^L f(\xi)d\xi \\
\frac{d^2}{dx^2}u &= f(x),
\end{aligned}$$

which is the original OdE. It is easily shown that the boundary conditions are satisfied.

16.2.5 Example: Nonhomogeneous boundary condition

$$\frac{d^2}{dx^2}u = f(x),$$

$$u(0) = a, \quad u(L) = b$$

We always require our Green's function to satisfy the homogeneous boundary conditions. So $G(x,\xi)$ satisfies

$$\frac{d^2}{dx^2}G = \delta(x-\xi)$$

$$G(0,\xi) = 0, \quad G(L,\xi) = 0.$$

Thus $G(x,\xi)$ is the same as found in the previous example.

$$G(x,\xi) = \begin{cases} -\frac{x}{L}(L-\xi), & x < \xi \\ -\frac{\xi}{L}(L-x), & x > \xi \end{cases}$$

$$\frac{d}{d\xi}G(x,\xi) = \begin{cases} x/L, & x < \xi \\ -(L-x)/L, & x > \xi \end{cases}$$

Therefore, the solution from (16.14) is

$$\begin{aligned}
u(x) &= \int_0^L G(x,\xi)f(\xi)d\xi + b(x/L) + a(L-x)/L \\
&= \frac{x}{L}\int_0^L \xi f(\xi)d\xi - \int_0^x \xi f(\xi)d\xi - x\int_x^L f(\xi)d\xi + bx/L + a(L-x)/L.
\end{aligned}$$

16.3 Green's Function for Poisson's Equation

Poisson's equation is Laplace's equation with the addition of a forcing, $f(\mathbf{x})$:

$$\nabla^2 u = f(\mathbf{x}) \tag{16.15}$$

It describes, for example, the electrostatic potential due to a given distributing of electric charges. Or it can describe the gravitational potential due to some given mass distribution.

The Green's function, $G(\mathbf{x}, \boldsymbol{\xi})$, solves the problem for one concentrated source at $\mathbf{x} = \boldsymbol{\xi}$:

$$\nabla^2 G = \delta(\mathbf{x} - \boldsymbol{\xi}). \tag{16.16}$$

subject to *homogeneous* boundary condition. The multidimensional delta function is defined as: in 3-D:

$$\delta(\mathbf{x} - \boldsymbol{\xi}) = \delta(x - \xi)\delta(y - \eta)\delta(z - \xi),$$

and in 2-D:

$$\delta(\mathbf{x} - \boldsymbol{\xi}) = \delta(x - \xi)\delta(y - \eta),$$

where

$$\mathbf{x} = (x, y, z), \quad \boldsymbol{\xi} = (\xi, \eta, \xi) \text{ in 3-D}$$

and

$$\mathbf{x} = (x, y), \quad \boldsymbol{\xi} = (\xi, \eta) \text{ in 2-D.}$$

Using the Green's formula (16.14), and letting $u(\mathbf{x})$ be the solution to (16.15) and $G(\mathbf{x}, \boldsymbol{\xi})$ its Green's function, we have, in 3D:

$$\iiint_V [u\nabla^2 G - G\nabla^2 u] dx\ dy\ dz = \iint_{\partial V} (u\nabla G - G\nabla u) \cdot \hat{\mathbf{n}} dS \tag{16.17}$$

The right-hand side contains the boundary value terms and vanishes if $u(\mathbf{x})$ satisfies homogeneous boundary conditions. For this case, (16.17) becomes

$$\iiint_V [u(x)\delta(\mathbf{x} - \boldsymbol{\xi}) - Gf(\mathbf{x})] dx\ dy\ dz = 0,$$

which is

$$u(\mathbf{x}) = \iiint_V G(\mathbf{x}, \boldsymbol{\xi}) f(\boldsymbol{\xi}) d\xi\ d\eta\ d\xi \tag{16.18}$$

after switching $\mathbf{x}$ with $\boldsymbol{\xi}$. The Green's function formula is a simple extension of the 1-D case. Nonhomogeneous boundary conditions for $u(\mathbf{x})$ can be handled the same way as was done in the 1-D case.

In 2-D, the integral in (16.18) should be replaced by integration over the area.

16.3.1 3D Poisson's equation in infinite domain

$$\nabla^2 G = \delta(\mathbf{x} - \boldsymbol{\xi}) \tag{16.19}$$

$$G(\mathbf{x}, \boldsymbol{\xi}) \text{ bounded as } \mathbf{x} \to \infty.$$

G represent the response at $\mathbf{x}$ to a point source located at $\mathbf{x} = \boldsymbol{\xi}$. Let

$$\mathbf{r} \equiv \mathbf{x} - \boldsymbol{\xi}$$

be the distance measured from the source. In terms of $\mathbf{r}$ there is no preference in the angular direction in the definition of this problem. So we seek a solution which depends on $r = |\mathbf{r}|$ only. We let

$$G(\mathbf{x}, \boldsymbol{\xi}) = G(r).$$

In spherical coordinates,

$$\nabla^2 G(\mathbf{x}, \boldsymbol{\xi}) = \nabla^2 G(r) = \frac{1}{r^2}\frac{d}{dr}(r^2 \frac{d}{dr} G)$$

For $r \neq 0$, the delta function is zero, and we have

$$\frac{d}{dr}(r^2 \frac{d}{dr} G) = 0$$

Integrating

$$r^2 \frac{d}{dr} G = A$$

or

$$\frac{d}{dr} G = A/r^2.$$

Integration again:

$$G(r) = -A/r + B \tag{16.20}$$

We cannot impose the boundedness condition at $r = 0$ because the equation for this G is not valid at $r = 0$. We need to derive the matching condition for $G(r)$ as $r \to 0^+$.

Integrating Eq. (16.19) over a sphere of radius r

$$\iiint \nabla^2 G dV = \iiint \delta(\mathbf{x} - \boldsymbol{\xi}) dV = 1.$$

The left-hand side is, by the divergence theorem

$$\iiint_V \nabla \cdot (\nabla G) dV = \iint_{\partial V} \nabla G \cdot \hat{\mathbf{n}} dS,$$

where $\hat{\mathbf{n}} = \mathbf{r}/r$ is pointing radially, and so

$$\nabla G \cdot \hat{\mathbf{n}} = \frac{\partial}{\partial r} G.$$

$$\begin{aligned} 1 = \iint_{\partial V} \nabla G \cdot \hat{\mathbf{n}} dS &= \int_0^{2\pi} \int_0^{\pi} (\frac{\partial}{\partial r} G) r^2 \sin\theta \ d\theta \ d\varphi \\ &= 4\pi r^2 \frac{\partial}{\partial r} G \end{aligned}$$

From this we obtain the matching condition

$$\lim_{r \to 0^+} r^2 \frac{\partial}{\partial r} G = \frac{1}{4\pi}$$

From (16.20), which is valid for $r > 0$

$$\begin{aligned} \frac{\partial}{\partial r} G(r) &= A/r^2 \\ r^2 \frac{\partial}{\partial r} G &= A = \frac{1}{4\pi} \end{aligned}$$

Finally, we have

$$G(\mathbf{x}, \boldsymbol{\xi}) = G(r) = -\frac{1}{4\pi r} + B$$

This describes, for example, the electrostatic potential at (x, y, z) due to a point electrical charge located at (ξ, η, ζ).

The constant B is arbitrary since a potential is determined only up to an arbitrary constant. We set $B = 0$

$$G(\mathbf{x}, \boldsymbol{\xi}) = -\frac{1}{4\pi} [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{-1/2}.$$

16.3.2 2D Poisson's equation in an infinite domain

Again we let $\mathbf{r} = \mathbf{x} - \boldsymbol{\xi}$, except now

$$\begin{aligned} r &= ((x-\xi)^2 + (y-\eta)^2)^{1/2}, \text{ and} \\ \nabla^2 &= \frac{1}{r} \frac{d}{dr} (r \frac{d}{dr}) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \end{aligned}$$

in 2D polar coordinates. Since $G(\mathbf{x}, \boldsymbol{\xi})$ depends on r only, we write

$$G(\mathbf{x}, \boldsymbol{\xi}) = G(r)$$

and

$$\nabla^2 G(r) = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}G)$$

For $r \neq 0$,

$$\nabla^2 G = 0$$

so

$$\frac{\partial}{\partial r}(r\frac{\partial}{\partial r}G) = 0.$$

Integrating

$$r\frac{\partial}{\partial r}G = A, \quad \frac{\partial}{\partial r}G = A/r$$
$$G(r) = A\ell n r + B.$$

From

$$\iint \nabla^2 G dS = \iint \delta(\mathbf{x} - \boldsymbol{\xi}) dx\ dy = 1$$

and

$$\iint_A \nabla^2 G dS = \iint_A \nabla \cdot (\nabla G) dS = \int_{\partial A} \nabla G \cdot \hat{\mathbf{n}} ds$$

where A is a circular disk of radius r and ∂A a circle of radius r; $ds = r d\theta$.

$$\int_{\partial A} \nabla G \cdot \hat{\mathbf{n}} ds = \int_0^{2\pi} \frac{\partial}{\partial r} G r d\theta = 2\pi r \frac{\partial}{\partial r} G$$

Therefore,

$$2\pi r \frac{\partial}{\partial r} G = 1, \quad r > 0$$

resulting in the matching condition

$$\lim_{r \to 0^+} r\frac{\partial}{\partial r} G = \frac{1}{2\pi}$$

Since

$$G(r) = A\ell n r + B, \quad r > 0$$
$$\frac{\partial}{\partial r} G = A/r$$
$$r\frac{\partial}{\partial r} G = A.$$

Therefore

$$A = \frac{1}{2\pi}.$$

Finally, setting the arbitrary additive constant potential B to zero for convenience, we have

$$\begin{aligned} G(\mathbf{x}, \boldsymbol{\xi}) &= \frac{1}{2\pi}\ell n r, \quad r > 0 \\ &= \frac{1}{2\pi}\ell n[(x-\xi)^2 + (y-\eta)^2]^{1/2}. \end{aligned}$$

16.3.3 Poisson's equation in a finite domain

Green's function in infinite domains, as obtained in the previous section, are actually quite simple. We would like to utilize them to obtain the Green's function for the finite domain problem.

The idea is to treat the solution we obtained for the infinite domain as a particular solution to the problem in the finite domain, and add a homogeneous solution. That is, to solve

$$\nabla^2 G(\mathbf{x}, \boldsymbol{\xi}) = \delta(\mathbf{x} - \boldsymbol{\xi})$$

in a finite domain, we write, in 3D

$$G(\mathbf{x}, \boldsymbol{\xi}) = -\frac{1}{4\pi r} + v(\mathbf{x}, \xi)$$

where $v(\mathbf{x}, \xi)$ satisfies the homogeneous PDE

$$\nabla^2 v(\mathbf{x}, \boldsymbol{\xi}) = 0.$$

We choose the constants in v so that $G(\mathbf{x}, \xi)$ satisfies the boundary condition in the finite domain.

In 2-D, we try

$$G(\mathbf{x}, \xi) = \frac{1}{2\pi}\ell n r + v(\mathbf{x}, \boldsymbol{\xi})$$

where

$$\nabla^2 v(\mathbf{x}, \xi) = 0.$$

We will leave the consideration of these finite domain problems in Exercises.

16.4 Green's Function for the Wave Equation

We are interested in solving the wave equation in the presence of a source $Q(x, t)$:

$$\frac{\partial^2}{\partial t^2}u - c^2\nabla^2 u = Q(\mathbf{x}, t) \tag{16.21}$$

subject to two initial conditions

$$u(\mathbf{x},0) = f(\mathbf{x})$$
$$\frac{\partial}{\partial t}u(\mathbf{x},0) = g(\mathbf{x})$$

We define the Green's function $G(\mathbf{x},t;\boldsymbol{\xi},\tau)$ as the solution due to a concentrated source at $\mathbf{x} = \boldsymbol{\xi}$ acting only at $t = \tau$:

$$\frac{\partial^2}{\partial t^2}G - c^2\nabla^2 G = \delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-\tau). \tag{16.22}$$

subject to homogeneous boundary conditions and zero initial conditions. Then it is physically obvious that before the source acts at $t = \tau$, the response would be identically zero, i.e.

$$G(\mathbf{x},t;\boldsymbol{\xi},\tau) \equiv 0 \text{ for } t < \tau.$$

This is known as the casuality condition and will be imposed as a mathematical "initial" condition on G.

Because of the delta function in t in (16.22), the forcing term is actually zero for $t > \tau$. Thus the Green's function satisfied the following homogeneous system for $t > \tau$:

$$\frac{\partial^2}{\partial t^2}G - c^2\nabla^2 G = 0, \quad t > \tau. \tag{16.23}$$

subject to homogeneous boundary conditions. The "initial condition" at $t = \tau$ is given by

$$G = 0 \text{ at } t = \tau$$
$$\frac{\partial}{\partial t}G = \delta(\mathbf{x}-\boldsymbol{\xi}) \text{ at } t = \tau \tag{16.24}$$

The second condition is obtained by integrating Eq. (16.22) from $t = \tau^-$ to $t = \tau^+$.

16.4.1 Example: 1-D wave equation in infinite domain

Instead of solving

$$\frac{\partial^2}{\partial t^2}G - c^2\frac{\partial^2}{\partial x^2}G = \delta(x-\xi)\delta(t-\tau), \quad t > 0,$$

we note that the above equation is equivalent to

$$\frac{\partial^2}{\partial t^2}G - c^2\frac{\partial^2}{\partial x^2}G = 0, \quad t > \tau$$

$$G = 0, \quad \text{at } t = \tau$$
$$\frac{\partial}{\partial t} G = \delta(x - \xi) \text{ at } t = \tau$$

The solution to the homogeneous wave equation was discovered by d'Alembert (see Chapter 17) to be of the form

$$G = R(x - c(t - \tau)) + L(x + c(t - \tau))$$

where R and L are arbitrary differentiable functions, to be determined by initial conditions. You can verify that it satisfies the wave equation by differentiation.

Since $G = 0$ at $t = \tau$, we must have

$$R(x) + L(x) = 0.$$

Thus

$$G = R(x - c(t - \tau)) - R(x + c(t - \tau)).$$

Since

$$\begin{aligned}\left.\frac{\partial}{\partial t} G\right|_{t=\tau} &= -c\, R'(x - c(t - \tau))\big|_{t=\tau} - cR'(x + c(t - \tau))\big|_{t=\tau} \\ &= -2cR'(x)\end{aligned}$$

the second initial condition:

$$\frac{\partial}{\partial t} G \,\big|_{t=\tau} = \delta(x - \xi),$$

becomes

$$-2cR'(x) = \delta(x - \xi)$$

Integrating

$$\begin{aligned} R(x) &= -\frac{1}{2c} \int_{-\infty}^{x} \delta(x' - \xi) dx' + B \\ = -\frac{1}{2c} H(x - \xi) &= \begin{cases} 0 & \text{if } x < \xi \\ -\frac{1}{2c} & \text{if } x > \xi \end{cases}\end{aligned}$$

where $H(x)$ is the Heaviside step function: $H(x) = 0$ for $x < 0$, $H(x) = 1$ for $x > 0$.

The required Green's function is

$$G(x, t; \xi, \tau) = \frac{1}{2c} \left\{ (H(x - \xi) + c(t - \tau)) - H((x - \xi) - c(t - \tau)) \right\}.$$

The constant B cancels out.

The Green's function represents a rectangular pulse expanding from $x = \xi$ to the left and to the right each with speed c. Ahead of the expanding fronts, $G = 0$.

16.4.2 Example: 3-D wave equation in infinite domain

$$\frac{\partial^2}{\partial t^2}G - c^2\nabla^2 G = \delta(\mathbf{r})\delta(t-\tau), \quad \mathbf{r} = \mathbf{x} - \boldsymbol{\xi}$$

$$\nabla^2 G = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}G),$$

since G is independent of the angles.

We solve the following equivalent system for $t > \tau$:

$$\frac{\partial^2}{\partial t^2}G - c^2\nabla^2 G = 0$$

$$G = 0 \text{ at } t = \tau$$

$$\frac{\partial}{\partial t}G = \delta(\mathbf{r}) \text{ at } t = \tau$$

Writing $G \equiv U/r$,

$$\frac{\partial}{\partial r}G = \frac{1}{r}\frac{\partial}{\partial r}U - \frac{1}{r^2}U,$$

$$\frac{\partial}{\partial r}(r^2\frac{\partial}{\partial r}G) = \frac{\partial}{\partial r}(r\frac{\partial}{\partial r}U) - \frac{\partial}{\partial r}U = r\frac{\partial^2}{\partial r^2}U,$$

we obtain a simpler equation for U:

$$\frac{\partial^2}{\partial t^2}U - c^2\frac{\partial^2}{\partial r^2}U = 0, \quad t > \tau$$

Its solution is the d'Alembert's solution discussed for the 1-D wave equation

$$U = R(r - c(t-\tau)) - R(r + c(t-\tau))$$

$$\frac{\partial}{\partial t}G = \frac{\partial}{\partial t}U/r\,\Big|_{t=\tau} = -\frac{2c}{r}R'(r) = \delta(\mathbf{r})$$

For $r \neq 0$, $R'(r) = 0$ and so $R(r) = B$, a constant, which we set to zero. Near $r = 0$, $R(r)$ may have a singularity. To find out, we integrate over a sphere of radius ϵ:

$$
\begin{aligned}
1 &= -2c \int_0^\epsilon \frac{1}{r} R'(r) 4\pi r^2 dr \\
&= -8\pi c \int_0^\epsilon r R'(r) dr = -8\pi c [r R(r)|_0^\epsilon - \int_0^\epsilon R(r) dr] \\
&= 8\pi c \int_0^\epsilon R(r) dr
\end{aligned}
$$

From this we see that $R(r)$ must be a delta function

$$
R(r) = \frac{1}{4\pi c} \delta(r),
$$

since

$$
\int_0^\epsilon \delta(r) dr = \frac{1}{2} \int_{-\epsilon}^\epsilon \delta(r) dr = \frac{1}{2}.
$$

Finally,

$$
\begin{aligned}
G(\mathbf{x}, t; \boldsymbol{\xi}, \tau) &= \frac{1}{4\pi c} \frac{[\delta(r - c(t-\tau)) - \delta(r + c(t-\tau))]}{r} \\
&= \frac{1}{4\pi c} \frac{\delta(r - c(t-\tau))}{r} \quad \text{for } r > 0, \quad t > \tau,
\end{aligned}
$$

where

$$
r = [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2} = |\mathbf{x} - \boldsymbol{\xi}|.
$$

This represents a concentrated impulse spreading out from the source in a spherial shell with a radial velocity of c, while its amplitude decays as the distance from the source increases.

16.4.3 Example: 2-D wave equation in infinite domain

The 2-D problem turns out to be more difficult than even the 3-D problem. This problem becomes simpler if we treat the 2-D case as a 3-D problem but with no z-dependence of the source function.

$$
\text{3-D} : (\frac{\partial^2}{\partial t^2} - c^2 \nabla^2) G_{3D} = \delta(x-\xi)\delta(y-\eta)\delta(z-\zeta)\delta(t-\tau)
$$

$$
\text{2-D} : (\frac{\partial}{\partial t^2} - c^2 \nabla^2) G_{2D} = \delta(x-\xi)\delta(y-\eta)\delta(t-\tau)
$$

The 2-D forcing can be obtained by integrating the 3-D forcing from $\zeta = -\infty$ to $\zeta = +\infty$. It then follows that the 2-D Green's function can be obtained from 3-D Green function, integrated from $\zeta = -\infty$ to $\zeta = \infty$.

The 3-D Green's function was found to be:

$$
\begin{aligned}
G_{3D}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) &= \frac{\delta(r - c(t-\tau))}{4\pi c\, r} \\
G_{2D}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) &= \int_{-\infty}^{\infty} \frac{\delta(r - c(t-\tau))}{4\pi c\, r} d\zeta,
\end{aligned}
$$

where $r = [(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2]^{1/2}$.

Let

$$
\begin{aligned}
z' &= \zeta - z, \quad d\zeta = dz' \\
\rho &= [(x-\xi)^2 + (y-\eta)^2]^{1/2}, \\
r^2 &= \rho^2 + {z'}^2, \quad 2z'dz' = 2rdr
\end{aligned}
$$

$$
\begin{aligned}
G_{2D}(\mathbf{x}, t; \boldsymbol{\xi}, \tau) &= \int_{-\infty}^{\infty} \frac{\delta(r - c(t-\tau))dr}{4\pi c z'} \\
&= \frac{2}{4\pi c} \int_0^{\infty} \frac{\delta(r - c(t-\tau))dr}{\sqrt{r^2 - \rho^2}} = \frac{2}{\sqrt{c^2(t-\tau)^2 - \rho^2}} \int_0^{\infty} \delta(r - c(t-\tau))dr \\
&= \begin{cases} \frac{(1/2\pi c)}{\sqrt{c^2(t-\tau)^2 - \rho^2}}, & \rho < c(t-\tau) \\ 0, & \rho > c(t-\tau) \end{cases}
\end{aligned}
$$

Unlike the 3-D case, the effect at $t - \tau$ due to an impulsive source is spread over the entire regions $\rho < c(t-\tau)$, instead of concentrated at $\rho = c(t-\tau)$.

16.4.4 The solution to the nonhomogeneous wave equation

After finding the Green's function, we now return to finding the solution u to the original problem (16.21), which is rewritten as

$$
\mathcal{L}u = Q(\mathbf{x}, t), \tag{16.25}
$$

$$
\mathcal{L} \equiv \frac{\partial^2}{\partial t^2} - c^2 \nabla^2,
$$

subject to homogeneous boundary conditions and initial conditions.

The solution is

$$u(\mathbf{x},t) = \int_0^\infty \iiint_V G(\mathbf{x},t;\boldsymbol{\xi},\tau)Q(\boldsymbol{\xi},\tau)d\xi d\eta d\zeta d\tau, \tag{16.26}$$

where the Green's function satisfies (16.22) and homogeneous boundary and zero initial conditions.

$$\begin{aligned}\mathcal{L}u(\mathbf{x},t) &= \int_0^\infty \iiint_V \mathcal{L}G(\mathbf{x},t;\boldsymbol{\xi},\tau)Q(\boldsymbol{\xi},\tau)d\xi d\eta d\zeta d\tau \\ &= \int_0^\infty \iiint_V \delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-\tau)Q(\boldsymbol{\xi},\tau)d\xi d\eta d\zeta d\tau \\ &= Q(\mathbf{x},t),\end{aligned}$$

which is (16.25).

If instead of the above, $u(\mathbf{x},t)$ satisfies nonzero initial conditions:

$$\begin{aligned}u(\mathbf{x},0) &= f(\mathbf{x}) \\ \frac{\partial}{\partial t}u(\mathbf{x},0) &= g(\mathbf{x}),\end{aligned}$$

(16.26) should then be viewed only as a particular solution, and a homogeneous solution, $v(\mathbf{x},t)$ satisfying

$$Lv = 0$$

subject to homogeneous boundary conditions, but

$$\begin{aligned}v(\mathbf{x},0) &= f(\mathbf{x}) \\ \frac{\partial}{\partial t}v(\mathbf{x},0) &= g(\mathbf{x}),\end{aligned} \tag{16.27}$$

should be added to (16.26).

Alternatively, the same Green's function (satisfying zero initial conditions) can also be used to construct the solution for nonzero initial conditions. This is because of the Green's formula, which for the operator $\mathcal{L}$ involving space and time derivatives, is, with t_i and t_f being any "initial" and "final" times:

$$\begin{aligned}&\int_{t_i}^{t_f} \iiint_V [u\mathcal{L}v - v\mathcal{L}u]dVdt \\ &\quad = \iint_V \left(u\frac{\partial v}{\partial t} - v\frac{\partial u}{\partial t}\right)\Big|_{t_i}^{t_f} dV - c^2 \int_{t_i}^{t_f} \iint_{\partial V} (u\nabla v - v\nabla u)\cdot\hat{\mathbf{n}}dS\end{aligned}$$

We let $v = G$ be the Green's function satisfying (16.22) with homogeneous boundary conditions and zero initial conditions and u the required solution satisfying nonzero initial conditions and (16.25). The left-hand side of above formula becomes

$$\begin{aligned}
&\int_{t_i}^{t_f} \iiint_V [u\mathcal{L}G - G\mathcal{L}u] dV dt \\
=&\int_{t_i}^{t_f} \iiint_V [u\delta(\mathbf{x}-\boldsymbol{\xi})\delta(t-\tau) - G \cdot Q(\mathbf{x},t)] dV dt \\
=&u(\boldsymbol{\xi},\tau) - \int_{t_i}^{t_f} G(\mathbf{x},t;\boldsymbol{\xi},\tau)Q(\mathbf{x},t) dV dt
\end{aligned}$$

The first term on the right-hand side of the above Green's formula is

$$\begin{aligned}
&\iiint_V (u\frac{\partial G}{\partial t} - G\frac{\partial u}{\partial t})\Big|_{t_i}^{t_f} dV \\
=&-\iiint_V [f(\mathbf{x})\frac{\partial}{\partial t}G\Big|_{t=0} - g(\mathbf{x})G\Big|_{t=0}] dV
\end{aligned}$$

after we evaluate at $t_i = 0$ and $t_f = \infty$.

After switching $\boldsymbol{\xi}$ with $\mathbf{x}$, and τ with t, we obtain the general formula for finding $u(\mathbf{x},t)$:

$$\begin{aligned}
u(\mathbf{x},t) =& \int_0^{\infty} \iiint_V Q(\boldsymbol{\xi},\tau)G(\mathbf{x},t;\boldsymbol{\xi},\tau) d\xi\, d\eta\, d\zeta\, d\tau \\
&- \iiint_V [f(\boldsymbol{\xi})\frac{\partial}{\partial \tau}G\Big|_{\tau=0} - g(\boldsymbol{\xi})G\Big|_{\tau=0}] d\xi\, d\eta\, d\zeta \\
&- c^2 \int_0^{\infty} \iint_{\partial V} [u\nabla G - G\nabla u] \cdot \hat{\mathbf{n}} dS
\end{aligned}$$

The first term on the right-hand side deals with the effect of the forcing Q on the solution, the second term the effect of the initial conditions and the third term that of the nonhomogeneous boundary conditions. (The integral is on the boundary of V and the variable of integration and gradient is with respect to $\boldsymbol{\xi}$. It vanishes for homogeneous boundary conditions on u.) It is amazing that the same Green's function, which satisfies zero initial conditions and homogeneous boundary conditions, is all that is needed to construct the general solution.

Example

Solve

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)u = Q(x,t), \quad -\infty < x < \infty, \ \ t > 0$$

subject to initial conditions

$$\begin{aligned} u(x,0) &= f(x) \\ \frac{\partial u}{\partial t}(x,0) &= g(x), \qquad -\infty < x < \infty. \end{aligned}$$

Method 1

We first find the solution to the homogeneous PDE satisfying the correct initial condition, i.e. find $v(x,t)$ satisfying

$$\left(\frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2}\right)v = 0$$

$$v(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

d'Alembert's solution is of the form

$$v(x,t) = R(x-ct) + L(x+ct).$$

To satisfy initial conditions:

$$\begin{aligned} v(x,0) = f(x) &= R(x) + L(x) \\ \frac{\partial}{\partial t}v &= -cR'(x-ct) + cL'(x+ct) \\ g(x) &= \frac{\partial}{\partial t}v(x,0) = -cR'(x) + cL'(x) \end{aligned}$$

Integrating:

$$L(x) - R(x) = \frac{1}{c}\int_0^x g(\bar{x})d\bar{x} + K,$$

where K is an arbitrary constant of integration. Adding it to

$$L(x) + R(x) = f(x)$$

we get

$$L(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(\bar{x})d\bar{x} + \frac{K}{2}.$$

Subtracting, we get

$$R(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(\bar{x})d\bar{x} - \frac{K}{2}$$

Finally,

$$\begin{aligned} v(x,t) &= R(x-ct) + L(x+ct) \\ &= \frac{1}{2}f(x-ct) + \frac{1}{2}f(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})d\bar{x}. \end{aligned}$$

The constant K cancels out.

The particular solution is, from (16.26) and an earlier section (16.4.1):

$$\begin{aligned} u_p(x,t) &= \int_0^\infty \int_{-\infty}^\infty G(x,t;\xi,\tau)Q(\xi,\tau)d\xi d\tau \\ &= \frac{1}{2c}\int_0^\infty \int_{-\infty}^\infty \{H((x-\xi)+c(t-\tau)) - H((x-\xi)-c(t-\tau))\}Q(\xi,\tau)d\xi d\tau \\ &= \frac{1}{2c}\int_0^\infty \int_{x-c(t-\tau)}^{x+c(t-\tau)} Q(\xi,\tau)d\xi\ d\tau \end{aligned}$$

The full solution is

$$u(x,t) = v(x,t) + u_p(x,t).$$

Method 2

Using the Green's function exclusively, the solution is

$$\begin{aligned} u(x,t) = &\int_0^\infty \int_{-\infty}^\infty G(x,t;\xi,\tau)Q(\xi,\tau)d\xi d\tau \\ &- \int_{-\infty}^\infty [f(\xi)\frac{\partial}{\partial\tau}G\Big|_{\tau=0} - g(\xi)G\Big|_{\tau=0}]d\xi \end{aligned}$$

Since

$$\begin{aligned} G(x,t;\xi,\tau) &= \frac{1}{2c}\{H((x-\xi)+c(t-\tau)) - H((x-\xi)-c(t-\tau))\} \\ G|_{\tau=0} &= \frac{1}{2c}\{H((x-\xi)+ct) - H((x-\xi)-ct)\} \\ \frac{\partial}{\partial\tau}G\Big|_{\tau=0} &= -\frac{1}{2}\{\delta((x-\xi)+ct) + \delta((x-\xi)-ct)\} \end{aligned}$$

The last integral for $u(x,t)$ is

$$\begin{aligned}
&-\int_{-\infty}^{\infty} -\frac{1}{2}[f(\xi)\{\delta((x-\xi)+ct)+\delta((x-\xi)-ct)\} \\
&- g(\xi)\frac{1}{2c}\{H((x-\xi)+ct)-H((x-\xi)-ct)\}]d\xi \\
&= \frac{1}{2}f(x-ct)+\frac{1}{2}f(x+ct)+\frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)d\xi.
\end{aligned}$$

This is the same as the homogeneous solution found in Method 1, showing that the two methods yield the same solution.

16.4.5 Example: In 3-D infinite space

$$\frac{\partial^2}{\partial t^2}u - c^2\nabla^2 u = Q(\mathbf{x},t)$$

subject to zero initial conditions, the solutiono is,

$$\begin{aligned}
u(\mathbf{x},t) &= \int_0^{\infty} d\tau \iiint_V G(\mathbf{x},t;\boldsymbol{\xi},\tau)Q(\boldsymbol{\xi},\tau)d\xi d\eta d\zeta \\
&= \int_0^{t} d\tau \iiint_V G(\mathbf{x},t;\boldsymbol{\xi},\tau)Q(\boldsymbol{\xi},\tau)d\xi d\eta d\zeta
\end{aligned}$$

since $G(\mathbf{x},t;\boldsymbol{\xi},\tau)\equiv 0$ for $\tau > t$.

Using the Green's function derived in the previous section, we have

$$u(\mathbf{x},t) = \frac{1}{4\pi c}\int_0^{t} d\tau \iiint_V \frac{1}{r}\delta(r-c(t-\tau))Q(\boldsymbol{\xi},\tau)d\xi d\eta d\zeta$$

where

$$r = [(x-\xi)^2+(y-\eta)^2+(z-\zeta)^2]^{1/2}$$

Because of the presence of the delta function in the integrand, the response, $u(\mathbf{x},t)$, at $\mathbf{x}$ is a superposition of the sources which satisfy

$$|\mathbf{x}-\boldsymbol{\xi}| = c(t-\tau)$$

divided by the distance between $\boldsymbol{\xi}$, the source location, and $\mathbf{x}$, where the response is measured.

16.5 Green's Function for the Heat Equation

We are interested in solving for the temperature $u(\mathbf{x}, t)$ in the presence of a heat source distribution $Q(\mathbf{x}, t)$. u satisfies the following nonhomogeneous PDE:

$$\frac{\partial}{\partial t}u = \alpha^2\nabla^2 u + Q(\mathbf{x}, t) \tag{16.28}$$

We first consider the case where u satisfies homogeneous boundary conditions and zero initial condition. The Green's function is defined by

$$\frac{\partial}{\partial t}G = \alpha^2\nabla^2 G + \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t - \tau), \tag{16.29}$$

subject to the same homogeneous initial and boundary conditions. The original solution is then constructed using

$$u(\mathbf{x}, t) = \int_0^\infty d\tau \iiint_V G(\mathbf{x}, t; \boldsymbol{\xi}, \tau)Q(\boldsymbol{\xi}, \tau)d\xi d\eta d\zeta \tag{16.30}$$

This can be verified by substituting (16.30) into (16.28)

$$\begin{aligned}(\frac{\partial}{\partial t}u - \alpha^2\nabla^2 u) &= \int_0^\infty d\tau \iiint_V (\frac{\partial}{\partial t} - \alpha^2\nabla^2)G(\mathbf{x}, t; \boldsymbol{\xi}, \tau)Q(\boldsymbol{\xi}, \tau)d\xi d\eta d\zeta \\ &= \int_0^\infty d\tau \iiint_V \delta(\mathbf{x} - \boldsymbol{\xi})\delta(t - \tau)Q(\xi, \tau)d\xi d\eta e\zeta \\ &= Q(\mathbf{x}, t).\end{aligned}$$

Next we consider the case of $u(\mathbf{x}, t)$ satisfying (16.28) and homogeneous boundary conditions, but nonzero initial condition

$$u(\mathbf{x}, 0) = f(\mathbf{x}).$$

We still use the same Green's function defined in (16.29) satisfying zero initial condition, but add an extra term in the formula (16.30):

$$\begin{aligned}u(\mathbf{x}, t) = &\int_0^\infty d\tau \iiint_V G(\mathbf{x}, t; \boldsymbol{\xi}, \tau)Q(\boldsymbol{\xi}, \tau)d\xi d\eta d\zeta \\ &+ \iiint_V G(\mathbf{x}, t; \boldsymbol{\xi}, 0)f(\boldsymbol{\xi})d\xi d\eta d\zeta.\end{aligned} \tag{16.31}$$

Since $\left(\frac{\partial}{\partial t} - \alpha^2\nabla^2\right)G(\mathbf{x}, t; \boldsymbol{\xi}, 0) = 0$ for $t > 0$, the last term added in (16.31) satisfies the homogeneous heat equation.

Since the Green's function should be zero before the source is tunred on at $t = \tau$, we have

$$G(\mathbf{x}, t; \boldsymbol{\xi}, \tau) \equiv 0, \quad \text{for } t < \tau,$$

and so the upper limit in the τ-integration in (16.31) can be replaced by t, i.e.

$$u(\mathbf{x}, t) = \int_0^t d\tau \iiint_V G(\mathbf{x}, t; \boldsymbol{\xi}, \tau) Q(\boldsymbol{\xi}, \tau) d\xi d\eta d\zeta \\ + \iiint_V G(\mathbf{x}, t; \boldsymbol{\xi}, 0) f(\xi) d\xi d\eta d\zeta \tag{16.32}$$

Therefore,

$$\lim_{t\to 0^+} u(\mathbf{x}, t) = 0 + \lim_{t\to 0^+} \iiint_V G(\mathbf{x}, t; \xi, 0) f(\xi) d\xi d\eta d\zeta \\ = f(\mathbf{x}).$$

This is because integrating (16.29) in t:

$$G(\mathbf{x}, t; \boldsymbol{\xi}, 0) = \alpha^2 \int_0^t \nabla^2 G(\mathbf{x}, t; \boldsymbol{\xi}, 0) dt \\ + \delta(\mathbf{x} - \boldsymbol{\xi}) \int_0^t \delta(t) dt \\ = \delta(\mathbf{x} - \boldsymbol{\xi}) \text{ as } t \to 0^+.$$

Solution in infinite space

Taking advantage of the fact that

$$G(\mathbf{x}, t; \boldsymbol{\xi}, \tau) = 0 \text{ for } t < \tau$$

and

$$G(\mathbf{x}, t; \xi, \tau) = \delta(\mathbf{x} - \boldsymbol{\xi}) \text{ for } t = \tau$$

we can obtain the Green's function without the forcing term on the right-hand side of (16.29) by solving the following "initial value" problem

$$\frac{\partial}{\partial t} G = \alpha^2 \nabla^2 G, \quad t > \tau \\ G = \delta(\mathbf{x} - \boldsymbol{\xi}) \quad \text{at } t = \tau \tag{16.33}$$

In one-dimension, this problem in the infinite domain is the same as the Drunken Sailor problem solved previously using Fourier transforms. The solution is

$$G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi\alpha^2(t-\tau)}} \exp\left\{-\frac{(x-\xi)^2}{4\alpha^2(t-\tau)}\right\} \tag{16.34}$$

In n-dimensions, the infinite space Green's function for the heat equation is

$$G(\mathbf{x},t;\boldsymbol{\xi},\tau) = [\frac{1}{4\pi\alpha^2(t-\tau)}]^{\frac{n}{2}} \exp\{-\frac{|\mathbf{x}-\boldsymbol{\xi}|^2}{4\alpha^2(t-\tau)}\}.$$

This is because, in Fourier transform solution, each dimension involves its own Fourier integral and the n-dimensional problem is simply n one-dimensional problem.