AppMath 503
Homework 1

Dan Jinguji
7339426

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1:

The cable equation:

$$\frac{\partial}{\partial t}v = \gamma\frac{\partial^2}{\partial x^2}v - \alpha v, \qquad \text{with}\, \alpha, \gamma > 0$$

also known as the lossy heat equation, was derived by the nineteenth-century Scottish physicist William Thomson to model propagation of signals in transatlantic cables.

(a)

Show that the general solution for this equation is given by: $v(x,t) = e^{-\alpha t}u(x,t)$, where $u(x,t)$ solves the heat equation.

$$\begin{aligned}
\frac{\partial}{\partial t}u(x,t) &= \gamma\frac{\partial^2}{\partial x^2}u(x,t) \qquad \text{given}\, v(x,t) = e^{-\alpha t}u(x,t) \\
\frac{\partial^2}{\partial x^2}v(x,t) &= e^{-\alpha t}\frac{\partial^2}{\partial x^2}u(x,t) \\
\frac{\partial}{\partial t}v &= (-\alpha)e^{-\alpha t}u(x,t) + e^{-\alpha t}\frac{\partial}{\partial t}u(x,t) \\
&= -\alpha e^{-\alpha t}u(x,t) + e^{-\alpha t}\gamma\frac{\partial^2}{\partial x^2}u(x,t) \\
&= -\alpha v(x,t) + \gamma\frac{\partial^2}{\partial x^2}v(x,t) \\
&= \gamma\frac{\partial^2}{\partial x^2}v(x,t) - \alpha v(x,t) \qquad \text{QED}
\end{aligned}$$

(b)

Find the Fourier series solution to this equation subject to:

$$v(0,t) = 0 = v(1,t), v(x,0) = f(x), 0 \leq x \leq 1, t > 0.$$

Does your solution approach equilibrium? How fast?

If we take our general solution from part (a), $v(x,t) = e^{-\alpha t}u(x,t)$. We know we can solve the heat equation using separation of variables:

$$\begin{aligned}
u(x,t) &= X(x)T(t) \\
\frac{\partial}{\partial t}u(x,t) &= \gamma\frac{\partial^2}{\partial x^2}u(x,t) \\
XT' &= \gamma X''T \\
\frac{T'}{\gamma T} &= \frac{X''}{X} = -\lambda^2 \\
X'' &= -\lambda^2 X \\
X(x) &= A\sin(\lambda x) + B\cos(\lambda x) \\
X(0) &= 0 \qquad \because v(0,t) = 0 \\
&= A\sin(0) + B\cos(0) \qquad \therefore B = 0 \\
X(x) &= A\sin(\lambda x) \\
X(1) &= 0 \qquad \because v(1,t) = 0 \\
&= A\sin(\lambda) \qquad \therefore \lambda = n\pi \\
X(x) &= A_n\sin(n\pi x) \\
\frac{T'}{\gamma T} &= -\lambda^2 \\
T' &= -\gamma(n\pi)^2 T \\
T(t) &= T_n(0)e^{-\gamma(n\pi)^2 t} \\
v(x,t) &= e^{-\alpha t}A_n\sin(n\pi x)T_n(0)e^{-\gamma(n\pi)^2 t} \\
v(x,0) &= f(x) = A_n\sin(n\pi x)T_n(0) \\
&= C_n\sin(n\pi x) \\
\int_0^1 f(x)\sin(m\pi x)dx &= \int_0^1 C_n\sin(m\pi x)\sin(n\pi x)dx
\end{aligned}$$

Our function $v(x,t)$ is well behaved given these BCs and ICs. It is dominated by the $e{-}\alpha t$ term, so will reach equilibrium faster given larger values of α, $\alpha > 0$ per the problem statement.

(c)

Redo part (b) but with the boundary condition:

$$\frac{\partial}{\partial x}v(0,t) = 0 = \frac{\partial}{\partial x}v(1,t)$$

This problem is ill-posed. Since the boundary conditions are both Neumann boundary conditions it does not sufficiently specify the behavior of the solution.

2:

Find the solution to the following wave problems in the form of a Fourier series:

(a)

$$\frac{\partial^2}{\partial t^2}u = \frac{\partial^2}{\partial x^2}u, u(0,t) = u(\pi,t) = 0, u(x,0) = 1, \frac{\partial}{\partial t}u(x,0) = 0, 0 < x < \pi$$

Assume separable function $u(x,t) = X(x)T(t)$.

$$\begin{aligned}
XT'' &= X''T \\
\frac{T''}{T} &= \frac{X''}{X} = -\lambda^2 \\
X(x) &= A\sin(\lambda x) + B\cos(\lambda x) \\
X(0) &= 0 \qquad \because u(0,t) = 0 \\
&= A\sin(0) + B\cos(0) \qquad \therefore B = 0 \\
X(x) &= A\sin(\lambda x) \\
X(\pi) &= 0 \qquad \because u(\pi,t) = 0 \\
&= A\sin(\lambda\pi) = 0 \qquad \therefore \lambda \in J \\
X(x) &= A_n\sin(nx) \\
T(t) &= \alpha\sin(nt) + \beta\cos(nt) \\
u(x,t) &= \Sigma_{n=1}^{\infty}\sin(nx)(\alpha_n\sin(nt) + \beta_n\cos(nt)) \\
\frac{\partial}{\partial t}u(x,t) &= \Sigma_{n=1}^{\infty}\sin(nx)(\alpha_n n\cos(nt) - \beta_n n\sin(nt)) \\
\frac{\partial}{\partial t}u(x,0) &= 0 = \Sigma_{n=1}^{\infty}\sin(nx)(\alpha_n n\cos(0) - \beta_n n\sin(0)) \qquad \therefore \alpha_n = 0
\end{aligned}$$

$$
\begin{aligned}
u(x,t) &= \Sigma_{n=1}^{\infty} \sin(nx)(\beta_n \cos(nt)) \\
u(x,0) &= 1 = \Sigma_{n=1}^{\infty} \sin(nx)(\beta_n \cos(0)) \\
1 &= \Sigma_{n=1}^{\infty} \sin(nx)\beta_n \\
\int_0^\pi \sin(mx)dx &= \beta_n \int_0^\pi \sin(mx)\sin(nx)dx \\
\beta_n &= 2/n \text{ where } n \in 1,3,5,7,9... \\
u(x,t) &= \Sigma_{n=1}^{\infty} \sin((2n-1)x)(2/(2n-1)\cos((2n-1)t))
\end{aligned}
$$

(b)

$$\frac{\partial^2}{\partial t^2}u = 2\frac{\partial^2}{\partial x^2}u, u(0,t) = u(\pi,t) = 0, u(x,0) = 0, \frac{\partial}{\partial t}u(x,0) = 1, 0 < x < \pi$$

Assume separable function $u(x,t) = X(x)T(t)$.

$$
\begin{aligned}
XT'' &= 2X''T \\
\frac{T''}{T} &= \frac{2X''}{X} = -\lambda^2 \\
X(x) &= A\sin(\sqrt{2}\lambda x) + B\cos(\sqrt{2}\lambda x) \\
X(0) &= 0 \qquad \because u(0,t) = 0 \\
&= A\sin(0) + B\cos(0) \qquad \therefore B = 0 \\
X(x) &= A\sin(\sqrt{2}\lambda x) \\
X(\pi) &= 0 \qquad \because u(\pi,t) = 0 \\
&= A\sin(\sqrt{2}\lambda\pi) \qquad \therefore \lambda \in \{\sqrt{2}/2, 2\sqrt{2}/2, 3\sqrt{2}/2, \dots\}
\end{aligned}
$$

(c)

$$\frac{\partial^2}{\partial t^2}u = 3\frac{\partial^2}{\partial x^2}u, u(0,t) = u(\pi,t) = 0, u(x,0) = \sin^3 x, \frac{\partial}{\partial t}u(x,0) = 0, 0 < x < \pi$$

Assume separable function $u(x,t) = X(x)T(t)$.

$$
\begin{aligned}
XT'' &= 3X''T \\
\frac{T''}{T} &= \frac{3X''}{X} = -\lambda^2 \\
X(x) &= A\sin(\sqrt{3}\lambda x) + B\cos(\sqrt{3}\lambda x) \\
X(0) &= 0 \qquad \because u(0,t) = 0 \\
0 &= A\sin(\sqrt{3}\lambda 0) + B\cos(\sqrt{3}\lambda 0) \qquad \therefore B = 0
\end{aligned}
$$

(d)

$$\frac{\partial^2}{\partial t^2}u = 4\frac{\partial^2}{\partial x^2}u, u(0,t) = u(\pi,t) = 0, u(x,0) = x, \frac{\partial}{\partial t}u(x,0) = -x, 0 < x < \pi$$

Assume separable function $u(x,t) = X(x)T(t)$.

$$\begin{aligned}
XT'' &= 4X''T \\
\frac{T''}{T} &= \frac{4X''}{X} = -\lambda^2 \\
X(x) &= A\sin(2\lambda x) + B\cos(2\lambda x) \\
X(0) &= 0 \qquad \because u(0,t) = 0 \\
&= A\sin(0) + B\cos(0) \qquad \therefore B = 0 \\
X(x) &= A\sin(2\lambda x) \\
X(\pi) &= 0 \qquad \because u(\pi,t) = 0 \\
&= A\sin(2\lambda\pi) \qquad \therefore \lambda = n/2 \\
&= A_n\sin(\tfrac{1}{2}nx) \\
T(t) &= \alpha\sin(\lambda t) + \beta\cos(\lambda t) \\
&= \alpha\sin(\tfrac{1}{2}nt) + \beta\cos(\tfrac{1}{2}nt) \\
u(x,t) &= \Sigma_{n=1}^{\infty}\sin(\tfrac{1}{2}nx)(\alpha_n\sin(\tfrac{1}{2}nt) + \beta_n\cos(\tfrac{1}{2}nt)) \\
u(x,0) &= \Sigma_{n=1}^{\infty}\sin(\tfrac{1}{2}nx)(\alpha_n\sin(0) + \beta_n\cos(0)) \\
x &= \Sigma_{n=1}^{\infty}\sin(\tfrac{1}{2}nx)\beta_n
\end{aligned}$$

$$\begin{aligned}
\int_0^\pi x\sin(\tfrac{1}{2}mx)dx &= \Sigma_{n=1}^{\infty}\beta_n\int_0^\pi \sin(\tfrac{1}{2}mx)\sin(\tfrac{1}{2}nx)dx \\
\tfrac{4}{m^2}\sin(\tfrac{1}{2}\pi m) - \tfrac{2\pi}{m}\cos(\tfrac{1}{2}\pi m) &= \beta_m(\tfrac{1}{2}\pi - \tfrac{1}{2m}\sin(\pi m)) \\
\frac{\partial}{\partial t}u(x,t) &= \Sigma_{n=1}^{\infty}\sin(\tfrac{1}{2}nx)(\tfrac{1}{2}\alpha_n n\cos(\tfrac{1}{2}nt) - \tfrac{1}{2}\beta_n n\sin(\tfrac{1}{2}nt)) \\
\frac{\partial}{\partial t}u(x,0) &= -x = \Sigma_{n=1}^{\infty}\sin(\tfrac{1}{2}nx)\tfrac{1}{2}(\alpha_n n\cos(0) - \beta_n n\sin(0)) \qquad \therefore
\end{aligned}$$

(e)

$$\frac{\partial^2}{\partial t^2}u = \frac{\partial^2}{\partial x^2}u, u(0,t) = 0, \frac{\partial}{\partial x}u(\pi,t) = 0, u(x,0) = 1, \frac{\partial}{\partial t}u(x,0) = 0, 0 < x < \pi$$

Assume separable function $u(x,t) = X(x)T(t)$.

$$
\begin{aligned}
XT'' &= X''T \\
\frac{T''}{T} &= \frac{X''}{X} = -\lambda^2 \\
X(x) &= A\sin(\lambda x) + B\cos(\lambda x) \\
X(0) &= 0 \qquad \because u(0,t) = 0
\end{aligned}
$$

(f)

$$\frac{\partial^2}{\partial t^2}u = 2\frac{\partial^2}{\partial x^2}u, \frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(2\pi,t) = 0, u(x,0) = -1, \frac{\partial}{\partial t}u(x,0) = 1, 0 < x < 2\pi$$

Assume separable function $u(x,t) = X(x)T(t)$.

$$
\begin{aligned}
XT'' &= 2X''T \\
\frac{T''}{T} &= \frac{2X''}{X} = -\lambda^2 \\
X(x) &= A\sin(\lambda x) + B\cos(\lambda x) \\
X(0) &= 0 \qquad \because u(0,t) = 0
\end{aligned}
$$

(g)

$$\frac{\partial^2}{\partial t^2}u = \frac{\partial^2}{\partial x^2}u, \frac{\partial}{\partial x}u(0,t) = \frac{\partial}{\partial x}u(1,t) = 0, u(x,0) = x(1-x), \frac{\partial}{\partial t}u(x,0) = 0, 0 < x < 1$$

Assume separable function $u(x,t) = X(x)T(t)$.

$$
\begin{aligned}
XT'' &= X''T \\
\frac{T''}{T} &= \frac{X''}{X} = -\lambda^2 \\
X(x) &= A\sin(\lambda x) + B\cos(\lambda x) \\
X' &= A\lambda\cos(\lambda x) - B\lambda\sin(\lambda x) \\
X'(0) &= 0 \qquad \because \frac{\partial}{\partial x}u(0,t) = 0 \ \therefore A = 0
\end{aligned}
$$