AMath 503
Homework 5

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1. Solving a Nonhomogeneous System

Consider the following nonhomogeneous system:

$$\begin{aligned}
&\text{PDE} : \frac{\partial^2}{\partial t^2}u = \frac{\partial^2}{\partial x^2}u + 1,\ 0 < x < 1 \\
&\text{BC} : u(0,t) = 0 = u(1,t) \\
&\text{IC} : u(x,0) = 0,\ \frac{\partial}{\partial t}u(x,0) = 0
\end{aligned}$$

Solve this problem in two ways.

(a) By eigenfunction expansion

By eigenfunction expansion, that is, expand the solution in the form of an infinite sum of eigenfunctions (in space) of the homogeneous system with unknown coefficient (which is a function of time) in front of each eigenfunction. Do the same for the forcing term, "1". Then solve an ODE in time.

We start by solving the homogeneous PDE using an eigenfunction expansion meets the boundary conditions.

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}u &= \frac{\partial^2}{\partial x^2}u \\
\text{Let } u(x,t) &= T(t)X(x) \\
T''X &= X''T \\
\frac{T''}{T} &= \frac{X''}{X} = \lambda^2
\end{aligned}$$

To meet the boundary conditions, $u(0,t) = u(1,t) = 0$, we choose the eigenfunction expansion as a sine series, $\lambda_n = n\pi$.

$$u(x,t) = \sum_{n=1}^{\infty} b_n(t)\sin(n\pi x)$$

Now, we express the forcing function in terms of this same eigenfunction expansion.

$$1 = \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x)$$

Substituting back into the original PDE,

$$\sum_{n=1}^{\infty} b_n''(t)\sin(n\pi x) = \sum_{n=1}^{\infty} -(n\pi)^2 b_n(t)\sin(n\pi x) + \sum_{n=1}^{\infty} f_n(t)\sin(n\pi x)$$

Based on the orthogonality of the sine series,

$$\begin{aligned} b_n''(t) &= f_n(t) - (n\pi)^2 b_n(t) \\ b_n''(t) + (n\pi)^2 b_n(t) &= f_n(t) \end{aligned}$$

Solving this second-order ODE using variation of parameters gives:

$$\begin{aligned} b_n(t) = {} & c_{1,n}\cos(n\pi t) + c_{2,n}\sin(n\pi t) + \\ & \sin(n\pi x)\int_1^x \frac{f_n(\xi)\cos(n\pi\xi)}{n\pi}d\xi + \cos(n\pi x)\int_1^x -\frac{f_n(\xi)\sin(n\pi\xi)}{n\pi}d\xi \end{aligned}$$

From our solution to Homework 2, also derivable from Equation (4.6) in the notes:

$$f_n(t) = \begin{matrix} 0 & n, \text{ even} \\ \frac{4}{n\pi} & n, \text{ odd} \end{matrix}$$

(b) By steady state

By first finding the steady state solution to the nonhomogeneous equation and then the transient solution; the latter is the difference between the true solution and the steady state solution and should satisfy a homogeneous equation.

Since the forcing function for this PDE does not depend on time, the solution can be written as the sum of an equilibrium term and a transient term.

$$u(x,t) = u_{\text{eq}}(x) + u_{\text{trans}}(x,t)$$

Considering the original PDE as $t \to \infty$, we have:

$$\frac{\partial^2}{\partial x^2} u_{\text{eq}} = -1$$

Integrating twice, we get:

$$u_{\text{eq}}(x) = -\frac{x^2}{2}$$

Substituting this back into the original PDE we have:

$$\frac{\partial^2}{\partial t^2} u_{\text{trans}} = \frac{\partial^2}{\partial x^2} u_{\text{trans}}, \ 0 < x < 1$$

with the same boundary conditions, $u_{\text{trans}}(0,t) = u_{\text{trans}}(1,t) = 0$.

Solving this homogeneous PDE, we have

$$u_{\text{trans}}(x,t) = T(t)X(t)$$
$$\frac{T''}{T} = \frac{X''}{X} = \lambda^2$$

To meet the boundary conditions, use sine series.

$$X(x) = \sum_{n=1}^{\infty} \sin(n\pi x)$$

$$T'' = \lambda^2 T$$
$$T(t) = \sum_{n=1}^{\infty} A_n \sin(n\pi t) + B_n \cos(n\pi t)$$
$$u_{\text{trans}}(x,t) = \sum_{n=1}^{\infty} (A_n \sin(n\pi t) + B_n \cos(n\pi t)) \sin(n\pi x)$$

However the initial conditions have changed to account for the steady-state solution.

$$
\begin{aligned}
\frac{\partial}{\partial t} u_{\text{trans}} &= 0 \\
\frac{\partial}{\partial t} u_{\text{trans}}(x,0) &= \sum_{n=1}^{\infty} n\pi A_n \sin(n\pi x) = 0 \\
\text{So, } A_n &= 0 \\
u_{\text{trans}}(x,0) &= u(x,0) - u_{\text{eq}}(x) \\
&= 0 + \frac{x^2}{2} = \frac{x^2}{2} \\
u_{\text{trans}}(x,0) &= \sum_{n=1}^{\infty} B_n \sin(n\pi x) = \frac{x^2}{2} \\
A_n &= 2\int_0^1 \frac{x^2}{2} \sin(n\pi x) dx \\
&= \frac{(2-\pi^2 n^2)\cos(n\pi) + 2n\pi \sin(n\pi) - 2}{\pi^3 n^3}
\end{aligned}
$$

2. Nonhomogeneous Wave Equation

Consider the following one-dimension nonhomogeneous wave equation (assume the boundary conditions are such that the solution is integrable):

$$\text{PDE} : \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = \delta(x - \xi)\delta(t - \tau),\ t > 0,\ -\infty > x > \infty,\ -\infty < \xi < \infty$$

subject to the zero initial conditions:

$$u = 0 \text{ and } \frac{\partial}{\partial t} u = 0,\ \text{at } t = 0.$$

For the solution to be integrable, the function $u \to 0$ as $x \to \pm\infty$.

(a) Equivalence to Homogeneous System

Show that the above problem is the same as the following homogeneous problem:

$$\text{PDE} : \left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = 0,\ t > \tau.$$

subject to the following "initial condition" at $t = \tau$:

$$u = 0 \text{ at } t = \tau,\ \text{and } \frac{\partial}{\partial t} u = \delta(x - \xi) \text{ at } t = \tau.$$

And $u \equiv 0$ for $t < \tau$.

The $\delta(t-\tau)$ factor in the "forcing" term means that we could consider the problem as broken into two segments: $t < \tau$ and $t \geq \tau$.

Given the initial conditions for the original problem, $u(x, 0) = 0$ and $\frac{\partial}{\partial t} u(x, 0) = 0$. The forcing term will have no effect until $t = \tau$, so we can characterize the original PDE as $u(x, t) = 0$ for $t < \tau$. There is nothing to perturb the state until the "pulse" at $t = \tau$.

For the solution u to be integrable, u must be continuous at $t = \tau$, as well as at $x = \xi$. There can be a "jump" in the derivative at this point, which would result in the second derivative being the δ function, as seen in the original PDE. Since the solution u is continuous at $t = \tau$, $u(x, \tau) = 0$, since $u(x, t) = 0$ for $t < \tau$, as shown above.

Considering $t > \tau$, the forcing function $\delta(x - \xi)\delta(t - \tau)$ is zero, by definition of the δ function. So, considering the partial time domain, $t > \tau$, the value of the PDE would be zero, since the $\delta(t - \tau)$ term of the forcing function is zero by definition.

$$\left(\frac{\partial^2}{\partial t^2} - c^2 \frac{\partial^2}{\partial x^2}\right) u = 0,\ t > \tau$$

The only thing that must be resolved now is the rest of the "initial condition" for this partial time domain. $\frac{\partial}{\partial t} u(x, \tau) = \delta(x - \xi)$, localizing the pulse to $x = \xi$ at $t = \tau$. We have "broken" the continuity of t by splitting the time domain into these two segments. So, to capture the perturbation at $t = \tau$, we use the δ function at $x = \xi$.

(b) Solution to (a)

Solve the problem defined by (a) either by Fourier transform or by D'Alembert's method.

By D'Alembert,

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) = 0$$

This is solvable as $L(x - ct)$ and $R(x + ct)$. This leads to a general solution,

$$u(x,t) = L(x + ct) + R(x - ct)$$

So,

$$\begin{aligned}
L(x) + R(x) &= 0 \\
L(x) - R(x) &= \frac{1}{c}\int^{x} \delta(x' - \xi)dx' + K \\
&= \frac{1}{c} \\
2L(x) &= \frac{1}{c} \\
L(x) &= \frac{1}{2c} \\
2R(x) &= -\frac{1}{c} \\
R(x) &= -\frac{1}{2c}
\end{aligned}$$

Assume a solution:

$$u(x, t_\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(\omega, t_\tau)e^{-i\omega x}d\omega, \text{ where } t_\tau = t - \tau$$

Then, the PDE in (a) becomes:

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \left(U_{tt}(\omega, t_\tau) + c^2\omega^2 U(\omega, t_\tau)\right) e^{-i\omega x}d\omega = 0$$

So,

$$U_{tt} + c^2\omega^2 U = 0$$

This has the general solution:

$$U(\omega, t) = A(\omega)\sin(c\omega t) + B(\omega)\cos(c\omega t)$$

Applying the "initial conditions", $u(x,\tau) = 0$ and $\frac{\partial}{\partial t}u(x,\tau) = \delta(x - \xi)$, we get $U_t(\omega,\tau) = 1$ and $U(\omega,\tau) = 0$.

Since $U(\omega,\tau) = 0$, $B(\omega) = 0$. $U_t(\omega,\tau) = c\omega A(\omega)\cos(c\omega\tau) = 1$. So, $A(\omega) = \frac{1}{c\omega}$.

(c) Application to Nonhomogeneous

Use the result in (b) to solve:

$$\begin{aligned}
&\text{PDE}: \frac{\partial^2}{\partial t^2}u - c^2\frac{\partial^2}{\partial x^2}u = Q(x,t),\ -\infty < x < \infty,\ t > 0\\
&\text{BC}: u(x,t) \to 0 \text{ as } x \to \pm\infty,\ t > 0\\
&\text{IC}: u(x,0) = 0,\ \frac{\partial}{\partial t}u(x,0) = 0,\ -\infty < x < \infty,
\end{aligned}$$

where the forcing term Q is given by:

$$Q(x,t) = \begin{cases} 1 & \text{for } -10 < x < 10,\ t > 0 \\ 0 & \text{otherwise} \end{cases}$$