Signal Denoising via Learning of Non-Linear Manifolds

Alex Mihlin

Abstract

A signal denoising method based on non-linear manifold learning was implemented. This method is applicable for a wide range of noise types. It improves with training and may be optimized for different signal types. Experiments were performed on images with Gaussian noise (Fig. 1.1) and with a superimposed image (Fig. 1.2).

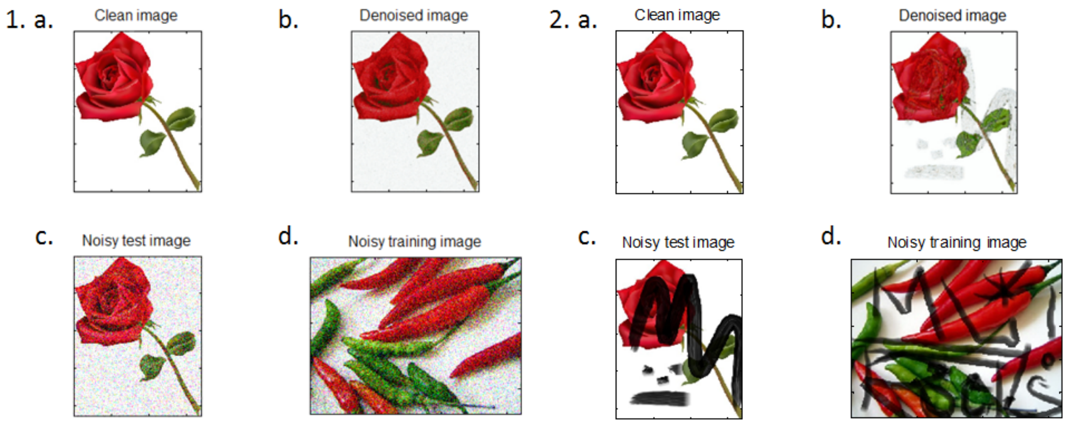

Figure 1: Denoised images. (1) Gaussian noise and (2) superimposed image. The figures illustrate: (a) the original image, (b) denoised image, (c) noisy test image and (d) noisy training image (another, clean, version of the training image was used).

Introduction

The method at hand assumes that the noise preserves the local geometry of the signal in feature space. Thus, denoising a signal amounts to inverting the global transformation induced by the noise. This inverse transformation is learned from a set of clean and noisy training signals. The learning process requires the embedding of one feature space into another. Such embedding may be done via dimensionality reduction methods.

Two canonical dimensionality reduction methods are principal component analysis (PCA) [1] and multidimensional scaling (MDS) [2]. These methods are appealing since their optimization is well understood and since they are not prone to local minima. However, the PCA and MDS methods are unable to embed the feature space into non-linear manifolds. Fig. 2.1 illustrates this shortcoming: if the feature manifold is non-linear, far away points in feature space may be embedded into close locations.

A more powerful, non-linear, dimensionality reduction method, called Locally Linear Embedding (LLE) was recently proposed [3]. This method attempts to find a low dimensional embedding, which preserves the local geometry of the feature manifold. Fig. 2.2 illustrates the advantage of this method over PCA and MDS.

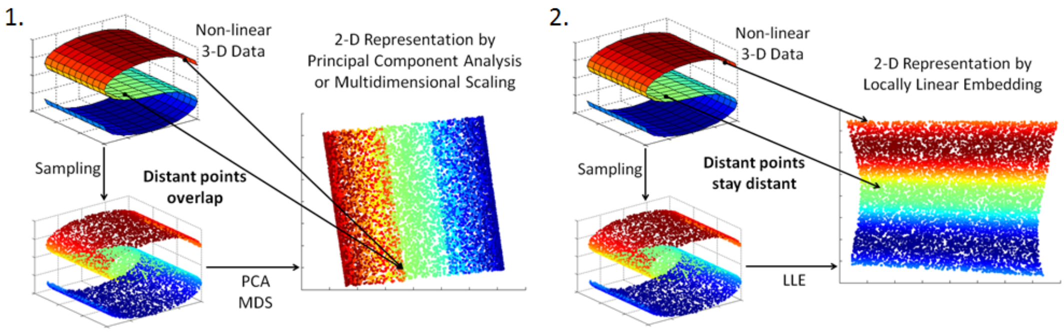

Figure 2: Reduction from 3-D into 2-D feature space by (1) PCA and MDS methods and (2) LLE method. The left figure illustrates a shortcoming of the PCA and MDS methods: distant points are embedded into close locations. The right figure demonstrates that LLE overcomes this shortcoming.

The denoising method at hand uses locally linear embedding in order to embed the feature manifold of the noisy test image into the feature manifold of the noisy training image (c.f. Fig. 3). The denoised image is then obtained by embedding this manifold into the feature manifold of the clean training image. This procedure is described in the next section.

Method description

The method at hand is based on a recent work by R. Shi, I-F. Shen and W. Chen [4]. As an example, Fig. 3 illustrates 3-dimensional training and test feature spaces. A patch (solid square) corresponding to a specific data point, I_i, is defined by the K nearest neighbours, $\{I_j\}_{j=1,\ldots,K}$, of I_i, so that

$$I_i = \sum_j W_{ij} I_j$$

where $\sum_j W_{ij} = 1$. The denoised test point, D_i, corresponding to each noisy test point, I_i, is determined by the following three steps (correspond-

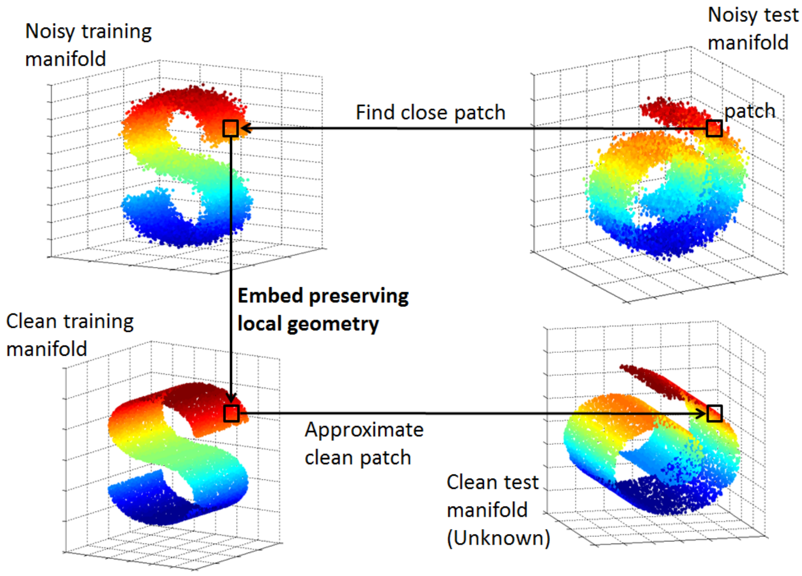

Figure 3: The denoising method. The denoising step (left arrow) assumes that the noise preserves the local geometry in feature space.

ing to the three arrows in Fig. 3):

Step 1: The test patch corresponding to data point I_i is embedded into the noisy training manifold (top arrow). This is done by locating the appropriate K nearest noisy training neighbours, $\{T_j\}$. To this end, the training and test noise types should be similar. A direct implementation of the nearest neighbours search has a large complexity of

$$\bigcirc\left[N_{train} \cdot N_{\text{nearest neighbours}}\right] \tag{1}$$

where N_{train} and $N_{\text{nearest neighbours}}$ are the numbers of training data points and nearest neighbours respectively. In order to address this issue, the training data points were arranged into a tree structure (Fig. 4). The tree brunches corresponded to points nearest to appropriate central points (cyan circles). These central points were found via k-means unsupervised learning. This method yielded an improved nearest neighbour search complexity of

$$\bigcirc\left[\log_{N_{\text{bins}}}\left(N_{\text{train}}\right) \cdot N_{\text{nearest neighbors}}\right] \tag{2}$$

The nearest neighbours of a specific data point were thus found by (i) locating the closest central point and (ii) searching for nearest neighbours only within the corresponding brunch. In order to account for points near brunch boundaries, neighbouring brunches were appended with overlapping regions (c.f. bottom of Fig. 4).

Step 2: Calculate the weights, W_{ij}, for the reconstruction of point I_i from its nearest neighbours, $\{T_j\}$. Namely, solve the following optimization prob-

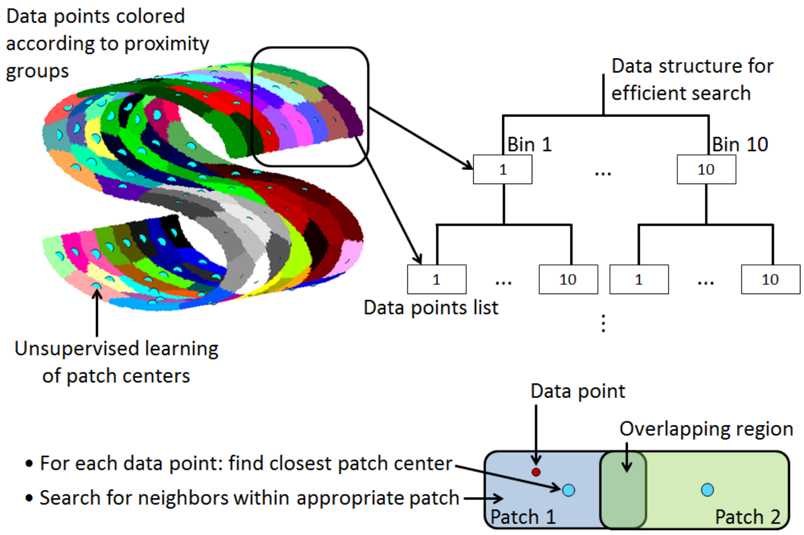

Figure 4: A tree structure for efficient nearest neighbours search.

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$$
\begin{aligned}
\min_{W_{ij}} \quad & \|I_i - \sum_{j=1}^{N_{train}} W_{ij} T_j\| \\
s.t. \quad & W_{ij} = 0 \ \text{ if } I_i,\, T_j \text{ are not nearest neighbours} \\
& \sum_{j=1}^{N_{train}} W_{ij} = 1
\end{aligned} \tag{3}
$$

This optimization problem is equivalent to the following system of linear equations:

$$
\sum_l G^i_{jl} W_{il} = 1 \tag{4}
$$

where G^i_{jl}=0 if T_j, T_l are not nearest neighbours of I_i and G^i is the local Gram matrix,

$$
G^i_{jl} = (I_i - T_j)^T (I_i - T_l)
$$

The resultant weights are normalized so that $\sum_j W_{ij} = 1$. If the number of nearest neighbours is larger than the feature space dimension, the rank of G^i is smaller than the number of nearest neighbours and equation (4) is ill defined. This issue was addressed via L_2 regularization, which resulted in an addition of a small constant to the diagonal elements of G^i. Optimization problem (3) may be made convex, by requiring that $W_{ij} \geq 0$. This requirement forces each data point to lie within a convex hull of its nearest neighbours.

Step 3: Assuming that the noise preserves the local geometry, the appropriate denoised patch is defined by the K clean training data points, $\{C_j\}$, corresponding to the K noisy training nearest neighbours, $\{T_j\}$ (left arrow in Fig. 3). The denoised data point, D_i, is thus given (bottom arrow in Fig. 3) by:

$$D_i = \sum_j W_{ij} C_j$$

where the weights, W_{ij}, were calculated in step 2.

This method was tested on images with two types of noise: (i) Gaussian noise with amplitude of 20% of the maximal feature value (Fig. 1.1) and (ii) superimposed image (Fig. 1.2). The images were divided into patches. Each N pixel patch represented a $3N$-dimensional point, corresponding to the red, green and blue colour values of each pixel. The patches were chosen with some overlap, in order to insure the smoothness of the denoised image. The images contained about 150,000, 27-dimensional (3x3 pixel patches) data points.

To account for the incompleteness of the training set, an iterative method was proposed. This method required at least two pairs of training signals. First, one of the training signals was denoised using the other. This denoised signal was then used as a training signal to further denoise an already denoised test signal. This procedure could be repeated several times, continuously refining the denoised image. The denoised image from Fig. 1.2 was obtained using two such iterations.

Conclusion

A signal denoising method (Fig. 3) was implemented based on the Locally Linear Embedding manifold learning method (Fig. 2). An efficient nearest neighbour search method was implemented via k-means unsupervised learning (Fig. 4). The use of a large number of nearest neighbours was enabled via L_2 regularization. An improved iterative denoising method was proposed and implemented. The method was successfully tested by denoising an image of Gaussian noise and of a superimposed image (Fig. 1).

References

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