Forest Sci., Vol. 31, No. 4, 1985, pp. 975–990

The Use of Bayes/Empirical Bayes Estimation in Individual Tree Volume Equation Development

EDWIN J. GREEN
WILLIAM E. STRAWDERMAN

ABSTRACT. Two empirical Bayes estimators were presented for the parameters in the combined variable volume equation. For the slope coefficient, an informative prior distribution was used, while noninformative priors were specified for the error term and the intercept. The empirical Bayes estimators were compared to weighted least squares estimators for loblolly pine, white oak, black cherry, and red maple through simulation studies. The empirical Bayes estimators were superior in terms of predictive ability for white oak and black cherry. There was little difference between empirical Bayes and least squares for loblolly pine, and least squares was superior for red maple. These results were attributed to the quality of the prior information for each species. Finally, there was an indication that in cases where good prior information exists, empirical Bayes estimation may be used to reduce the amount of data necessary to construct volume equations. FOREST SCI. 31:975–990.

ADDITIONAL KEY WORDS. Weighted least squares, combined variable equation, loblolly pine, eastern hardwoods.

IN THIS PAPER we present the results of a study performed to investigate the possibility of using empirical Bayes estimation to construct volume equations with greater predictive ability than that obtained using conventional volume equation development techniques. We also investigated the feasibility of using empirical Bayes estimation to reduce the sample size necessary to achieve a given level of predictive ability.

Empirical Bayes (EB) estimation of unknown parameters has become a popular, albeit controversial, topic in the last few decades.[1] Essentially the procedure involves combining existing knowledge about the probable value of a parameter

[1] The term "empirical Bayes" has been applied to two different types of estimators in the past. In one type, the prior distribution is based on past data. In the other, the prior is not explicitly recognized, and the parameters are estimated with a Stein-rule technique. In this paper, we shall use the term empirical Bayes in the former context.

The authors are Assistant Professor of Forestry, Department of Horticulture and Forestry, Cook College, Rutgers University, and Professor of Statistics, Department of Statistics, Rutgers College, Rutgers University, P.O. Box 231, New Brunswick, NJ 08903. This work was partially funded by NJAES Project Number D-12386-31-85, supported by the N.J. Agricultural Experiment Station and U.S. McIntire-Stennis Act funds. Paper of the Journal Series, N.J. Agric. Exp. Stn., Cook College, Rutgers Univ., New Brunswick, N.J. The authors thank H. E. Burkhart, Thomas M. Brooks Professor of Forest Biometrics, School of Forestry and Wildlife Resources, VPI & SU, for contribution of the loblolly pine data, and A. J. Martin, Assistant Professor and Extension Forester, Univ. of Wisconsin–Madison and the U.S. Forest Service for the eastern hardwood data used in the study. Manuscript received 29 June 1984.

with sample evidence to determine an estimate of the parameter or its distribution. Swindel (1972) has discussed Bayesian inference in a forestry context. Related work on Bayesian estimation in forest inventory has been reported by Ek and Issos (1978) and Burk and Ek (1982).

EMPIRICAL BAYES ESTIMATION

EB estimation is based on Bayes theorem, which is not controversial. However, its extension into inference and estimation theory has been a subject of lively debate for years (Swindel 1972). In the EB procedure previous knowledge about an unknown parameter $\underline{\theta}$ is characterized by a prior probability distribution, $p(\underline{\theta})$. The sample evidence $\underline{y}$ is a vector of n observations whose distribution $p(\underline{y}|\underline{\theta})$ depends on the unknown $\underline{\theta}$. Then according to Bayes theorem:

$$p(\underline{\theta}|\underline{y}) = p(\underline{y}|\underline{\theta})p(\underline{\theta})/p(\underline{y}). \tag{1}$$

Fisher (1922) explained that once the data have been observed, $p(\underline{y}|\underline{\theta})$ may be regarded as a function of $\underline{\theta}$, not $\underline{y}$. When so viewed, $p(\underline{y}|\underline{\theta})$ may be interpreted as the likelihood function for $\underline{\theta}$ given $\underline{y}$. This concept is the basis for the familiar maximum likelihood estimation procedure (e.g., see Bickel and Doksum 1977). As $p(\underline{y})$ is independent of $\underline{\theta}$, it is of no interest in the estimation problem. Thus we may write:

$$p(\underline{\theta}|\underline{y}) = cL(\underline{\theta}|\underline{y})\, p(\underline{\theta}) \tag{2}$$

where

$L(\underline{\theta}|\underline{y})$ = likelihood for $\underline{\theta}$ given $\underline{y}$
c = constant to be determined such that $p(\underline{\theta}|\underline{y})$ integrates to unity.

Equation (2) is the basis for Bayesian inference. In the parlance of Bayesian statistics, $p(\underline{\theta})$ is referred to as the prior distribution of $\underline{\theta}$ and is assumed to represent the investigator's knowledge about $\underline{\theta}$ *before* observing data. Then $p(\underline{\theta}|\underline{y})$ is the posterior distribution of $\underline{\theta}$, given $\underline{y}$, and represents his or her beliefs about $\underline{\theta}$ *after* viewing data.

To derive Empirical Bayes (EB) estimators, one must make an assumption about the proper likelihood and prior distribution to use in (2). This is similar to choosing a likelihood function for maximum likelihood estimation. Following this, solution of (2) is a problem of calculus. Once the posterior distribution of $\underline{\theta}$ $[p(\underline{\theta}|\underline{y})]$ is known, the logical estimator for $\underline{\theta}$ is the mean of $p(\underline{\theta}|\underline{y})$, which may be shown to be the minimum mean square error estimator given the distribution $p(\underline{\theta}|\underline{y})$ (e.g., Box and Tiao 1973).

DATA

The data used in this study consisted of a loblolly pine data set and an eastern hardwoods data set. The loblolly data included observations on inside bark cubic volume, dbh (D), and total height (H) for 427 plantation grown trees in the Piedmont and Coastal Plain regions of Virginia and the Coastal Plain regions of Delaware, Maryland, and North Carolina. Details of this data set have been published by Burkhart and others (1972a) and Cao and others (1980). The eastern hardwoods data contained observations on outside bark cubic volume, D, and H for 18 eastern species, with an average of 65 trees per species. The data was collected in natural stands in western Virginia and West Virginia. Details of this data set have been reported by Martin (1981).

As described in a subsequent section, volume equations were derived for loblolly pine (*Pinus taeda* L.) and the three hardwood species for which we had the most observations. These species were white oak, (*Quercus alba* L.) (84 trees), black cherry (*Prunus serotina* Ehrh.) (78 trees), and red maple (*Acer rubrum* L.) (76 trees).

Derivation of Estimators for Volume Equation Coefficients

A common problem in forestry is the development of an equation to predict the cubic volume content of a tree given D and H. As reported by Spurr (1952), foresters have been examining this problem since at least the early nineteenth century. It is logical then to expect that good prior information on the values of volume equation coefficients would exist. To simplify the problem, we chose to investigate only one equation, the common combined variable volume equation:

$$V = \beta_0 + \beta_1 D^2 H. \tag{3}$$

This equation has been widely used, and a brief search of the literature would suffice to demonstrate that there is a great deal of uniformity in reported $\hat{\beta}_1$ values. For inside bark cubic foot volumes where D is measured in inches and H in feet, $\hat{\beta}_1$ tends to be close to 0.002, while for outside bark volumes it tends to be slightly greater (e.g., see Gevorkiantz and Olsen 1955; Smith and Breadon 1964; Bailey and Clutter 1970; Burkhart and others 1972a, 1972b).[2] There does not appear to be any corresponding unity among $\hat{\beta}_0$ values or mean square error (MSE) values (when reported). This realization led to the hypothesis that it would be possible to achieve greater precision for a given sample size or achieve equivalent precision with a smaller sample by using an EB estimator for β_1 instead of a conventional estimator. It is well known that the usual least squares assumption of homogeneous variance of the dependent variable is inappropriate in the case of volume equation development, and that weighted least squares (WLS) is preferable to ordinary least squares for estimating volume equation coefficients (e.g., Cunia 1964, Smalley and Bower 1968, Clutter and others 1983). Thus we considered WLS to be the conventional procedure and the standard against which an EB estimator should be judged.

In order to develop WLS estimators, it was necessary to determine the appropriate weights. For this purpose, the data for each species were sorted by D^2H values and grouped into discrete classes of equal size in ascending order. Ten groups were established for loblolly pine, while six groups were established for each of the hardwood species. Equation (3) was fitted to the data in each group by ordinary least squares. The resulting mean squared error (MSE) values were then related to the group mean D^2H values via the following equation:

$$\ln(\text{MSE}) = a + b \ln(D^2H) \tag{4}$$

The estimate of b found by ordinary least squares applied to equation (4) was 2.17, 2.08, 2.44, and 2.21 for loblolly pine, white oak, black cherry, and red maple, respectively. Thus it was assumed that the variance of stem volume was proportional to the square of D^2H for each of the four species. The appropriate weighting factor to transform equation (3) into an equation in which the homogeneity of variance assumption was more nearly met was $(1/D^2H)$, and the transformed equation was

$$V/D^2H = \beta_0/D^2H + \beta_1 \tag{5}$$

Ordinary least squares fitting of equation (5) to the data for each species provided the WLS estimates of the regression coefficients. For loblolly pine, white oak, red

---

[2] When volume is in cubic meters, D in centimeters and H in meters, $\hat{\beta}_1$ tends to be close to 0.000029.

maple, and black cherry, the WLS estimates were then modified by the prior distributions according to Bayes theorem to produce EB estimates.

Specifically, for EB estimation we assumed:

1. $\underline{Y} = X\underline{\beta} + \underline{\epsilon}$ (6)

where:

$$\underline{Y} = (\mathrm{V/D^2H_1}, \mathrm{V/D^2H_2}, \ldots, \mathrm{V/D^2H}_n)'$$
$$\underline{\epsilon} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_n)'$$
$$\underline{\beta} = (\beta_0, \beta_1)'$$
$$X = \begin{bmatrix} 1/\mathrm{D^2H_1} & 1 \\ 1/\mathrm{D^2H_2} & 1 \\ \vdots & \vdots \\ 1/\mathrm{D^2H}_n & 1 \end{bmatrix}$$

2. $\epsilon \sim N(\underline{0}, \sigma^2 I)$.
3. $\underline{\beta}$ and σ^2 independent, i.e., $p(\underline{\beta}, \sigma^2) = p(\underline{\beta})p(\sigma^2)$.
4. Noninformative prior for σ^2, i.e., $p(\sigma^2) \propto \sigma^{-2}$.
5. Noninformative prior for β_0, i.e., $p(\beta_0) \propto k$, where k is a constant.
6. β_0 and β_1 independent, i.e., $p(\underline{\beta}) = p(\beta_0)p(\beta_1)$.
7. Prior distribution on β_1 is $N(\theta_1, \sigma_\theta^2)$, i.e., $p(\beta_1) \propto \exp[-(\beta_1 - \theta_1)^2/2\sigma_\theta^2]$.

Noninformative prior distributions are useful when one wishes to rely on the sample evidence for the estimation of a particular parameter. The noninformative priors specified in assumptions 4 and 5 have been shown to be appropriate for linear regression parameters (e.g., see Box and Tiao 1973). According to Zellner (1971), assumptions 3 and 7 are reasonable when one has prior information about the regression parameter but has little or no good prior knowledge about the distribution of σ^2. It was felt that this was appropriate in the present case. Although we had good prior information on β_1 and could estimate a prior variance for β_1 from the sample variance of reported values, we did not feel we had adequate prior information to specify an informative prior distribution for σ^2.

Assumptions 1 and 2 imply the following likelihood for $\underline{\beta}$ and σ^2:

$$L(\underline{\beta}, \sigma^2) \propto \exp\left[-\frac{1}{2\sigma^2}\{\nu s^2 + (\underline{\beta} - \underline{\hat{\beta}})'X'X(\underline{\beta} - \underline{\hat{\beta}})\}\right] \tag{7}$$

where

$$\nu = n - 2$$
$$\underline{\hat{\beta}} = (X'X)^{-1}X'\underline{Y}$$
$$s^2 = (\underline{Y} - \underline{\hat{Y}})'(\underline{Y} - \underline{\hat{Y}})(1/\nu).$$

Equation (7) and assumptions 3 through 7, when applied to equation (2), lead to

$$p(\underline{\beta}, \sigma^2 \mid \underline{Y}) = k'\sigma^{-2}\left\{\exp\left[-\frac{1}{2\sigma_\theta^2}(\beta_1 - \theta_1)^2\right]\right\}\sigma^{-n} \cdot\exp\left\{-\frac{1}{2\sigma^2}[\nu s^2 + (\underline{\beta} - \underline{\hat{\beta}})'X'X(\underline{\beta} - \underline{\hat{\beta}})]\right\} \tag{8}$$

where k' = constant determined such that

$$\int_{-\infty}^{\infty}\int_0^{\infty} p(\underline{\beta}, \sigma^2 \mid \underline{Y})\, d\sigma^2\, d\underline{\beta} = 1.$$

As we are interested in an EB estimator for β_1 only, we need to determine the marginal posterior distribution of β_1:

$$p(\underline{\beta}\,|\,\underline{Y}) = \int_0^\infty p(\underline{\beta}, \sigma^2\,|\,\underline{Y})\, d\sigma^2$$

$$\propto \exp\left[-\frac{1}{2\sigma_\theta^2}(\beta_1 - \theta_1)^2\right]$$

$$\cdot \int_0^\infty \sigma^{-(n+2)}\exp\left\{-\frac{1}{2\sigma^2}[\nu s^2 + (\underline{\beta} - \hat{\underline{\beta}})'X'X(\underline{\beta} - \hat{\underline{\beta}})]\right\} d\sigma^2$$

$$\propto \exp\left[-\frac{1}{2\sigma_\theta^2}(\beta_1 - \theta_1)^2\right]\left\{-\frac{1}{2}[\nu s^2 + (\underline{\beta} - \hat{\underline{\beta}})'X'X(\underline{\beta} - \hat{\underline{\beta}})]\right\}^{-(n/2)} \Gamma(n/2) \quad (9)$$

where $\Gamma(n/2)$ = gamma function with parameter $n/2$. And,

$$p(\beta_1\,|\,\underline{Y}) = \int_{-\infty}^\infty p(\underline{\beta}, \underline{Y})\, d\beta_0$$

$$\propto \exp\left[-\frac{1}{2\sigma_\theta^2}(\beta_1 - \theta_1)^2\right]$$

$$\cdot \int_{-\infty}^\infty \left\{-\frac{1}{2}[\nu s^2 + (\underline{\beta} - \hat{\underline{\beta}})'X'X(\underline{\beta} - \hat{\underline{\beta}})]\right\}^{-(n/2)} d\beta_0$$

$$\propto \exp\left[-\frac{1}{2\sigma_\theta^2}(\beta_1 - \theta_1)^2\right]\left[1 - \frac{(\beta_1 - \hat{\beta}_1)^2/c_{22}}{\nu s^2}\right]^{-\left(\frac{n-1}{2}\right)} \quad (10)$$

where c_{22} = element in 2nd row, 2nd column of $(X'X)^{-1}$.

An appropriate EB estimator for β_1, $\tilde{\beta}_1$, would be the mean of $p(\beta_1\,|\,\underline{Y})$:

$$\tilde{\beta}_1 = \int_{-\infty}^\infty \beta_1 k''\exp\left[-\frac{1}{2\sigma_\theta^2}(\beta_1 - \theta_1)^2\right]\left[1 - \frac{(\beta_1 - \hat{\beta}_1)^2/c_{22}}{\nu s^2}\right]^{-\left(\frac{n-1}{2}\right)} d\beta_1 \quad (11)$$

where k'' = constant determined such that

$$\int_{-\infty}^\infty p(\beta_1\,|\,\underline{Y})\, d\beta_1 = 1.$$

However, this integral is mathematically intractable. Instead, one may estimate $\tilde{\beta}_1$ by recognizing that $p(\beta_1\,|\,\underline{Y})$ is a product of a normal kernel, $\exp[-(\beta_1 - \theta_1)^2/2\sigma_\theta^2]$, and a student's τ kernel, $[1-\{(\beta_1 - \hat{\beta}_1)^2/c_{22}\nu s^2\}]^{-\left(\frac{n-1}{2}\right)}$. As the normal and τ distributions tend to be very similar in shape, a logical approximation would be to substitute a normal kernel with the same mean and variance as the τ kernel for the τ kernel in (10). This yields

$$p(\beta_1\,|\,\underline{Y}) \cong k''\exp\left[-\frac{1}{2\sigma_\theta^2}(\beta_1 - \theta_1)^2\right]\exp\left[-\frac{1}{2q^2}(\beta_1 - \hat{\beta}_1)^2\right] \quad (12)$$

where

$$q^2 = (n - 2)s^2c_{22}/(n - 4). \quad (13)$$

From Appendix A1.1 in Box and Tiao (1973), we find

$$(\beta_1 | \underline{Y}) \stackrel{\cdot}{\sim} N[\bar{\beta}_1, (\sigma_\theta^{-2} + q^{-2})^{-1}]$$

where

$$\bar{\beta}_1 = (\sigma_\theta^{-2} + q^{-2})^{-1}(\sigma_\theta^{-2}\theta_1 + q^{-2}\hat{\beta}_1) \tag{14}$$

and $\stackrel{\cdot}{\sim}$ implies approximately distributed. Thus a reasonable estimator for β_1 is $\bar{\beta}_1$. Zellner (1971) proposed a different solution to the problem of the product of the normal and τ kernels. He expanded the τ kernel through a Taylor series and then dropped all but the leading term in the expansion (Zellner 1971, p. 240). The resulting estimator for β_1 (i.e., the mean of the posterior for β_1) was

$$\bar{\bar{\beta}}_1 = [\sigma_\theta^{-2} + (c_{22}s^2)^{-1}]^{-1}[\sigma_\theta^{-2}\theta_1 + (c_{22}s^2)^{-1}\hat{\beta}_1]. \tag{15}$$

Note the similarity between $\bar{\beta}_1$ and $\bar{\bar{\beta}}_1$. As both estimators are reasonable, we chose to investigate each of them. The estimator given in equation (14) shall be referred to as the GS estimator, and that in equation (15) as the ZE estimator. The two differ by the ratio $(n - 2)/(n - 4)$ which is applied to the variance of $\hat{\beta}_1$ in the GS estimator and not in the ZE estimator. This should tend to put more weight on the prior information in the GS estimator and make the ZE estimator more conservative.

The EB estimator for β_0, given the noninformative prior distribution used for this parameter, is equivalent to the WLS estimator of β_0.[3]

Prior Distributions

The prior distributions for β_1 were determined in two manners. For loblolly pine, six reported $\hat{\beta}_1$ values were gathered from studies in which equation (3) was fitted to total cubic foot volume inside bark data for loblolly. These six studies and the corresponding $\hat{\beta}_1$ values are displayed in Table 1. A plot of the six values on normal probability paper following the procedures given by Draper and Smith (1981) revealed no obvious trends toward non-normality.[4] Thus, the prior for β_1 was assumed to be normal with mean and variance equal to the sample mean and sample variance of the six values.

The procedure for establishing the prior distribution for the hardwoods was slightly different. EB estimators were developed for the three hardwood species with the most observations (white oak, red maple, and black cherry). Rather than basing the prior on reported $\hat{\beta}_1$ values, equation (3) was fitted by WLS to the fifteen remaining hardwood species. The resulting $\hat{\beta}_1$ values are shown in Table 2. Again, a normal plot of these values revealed no obvious trends toward non-normality. For each of the three hardwood species tested, the prior distribution for β_1 was assumed to be normal with mean and variance equal to the sample mean and sample variance of the fifteen values in Table 2.

In some cases it may be advisable to use a data-based empirical distribution function (EDF) as the prior rather than a parametric distribution such as the normal. However in this case, with only 6–15 points, we believe an EDF prior would be unstable, and thus we used the normal as a more stable procedure.

---

[3] We also examined the use of informative prior distributions for both β_1 and β_0. However this yielded results no better than those obtained using prior information on β_1 and a noninformative prior for β_0.

[4] This test has low power for such a small sample size. However we used the test to determine if there were any gross departures from normality in the reported coefficients. It is a widely accepted practice to employ normal distributions when there is a lack of contrary evidence.

TABLE 1. Reported values of slope coefficient ($\hat{\beta}_1$) from studies in which combined variable total inside bark cubic foot volume equation fitted to loblolly pine data, by author.

Author(s)	$\hat{\beta}_1$	Author(s)	$\hat{\beta}_1$
Bailey and Clutter (1970)	0.00193	Burkhart and others (1972b)	0.00205
Matney and Sullivan (1982)	.00195	Shipman[a]	.00206
Hasness and Lenhart (1972)	.00196	Smalley and Bower (1968)	.00207

[a] Shipman, R. D. Volume tables for loblolly pine plantations in the South Carolina Piedmont. Dep of For, SC Agric Exp Stn, Clemson, SC; SC State Comm of For; and USDA Soil Conserv Serv Forest Res Ser No 3. No date.

Testing Procedure

The objective of the study was to determine if EB procedures could be used in volume table construction to achieve better predictive ability than standard WLS techniques for a given sample size, or achieve equivalent predictive ability with a reduced sample size. To achieve this objective, EB estimation on reduced sample sizes was compared to WLS estimation on equal and greater sample sizes.

For each of the four species considered, 25 percent of the observations were randomly withdrawn and placed into a validation set for evaluating the predictive ability of the fitted equations. Equation (3) was then fitted to the remaining 75 percent of the data (the fitting set) using WLS. The resulting estimates shall be referred to as 100 percent WLS estimates, as they were obtained using 100 percent of the fitting data. Next, prespecified percentages of the observations in the fitting set were randomly selected, and equation (3) was fitted to the data in each subsample using WLS and the two EB estimators. For loblolly pine, the percentages were 3, 6, 15, 33, 66, and 85 percent. There were too few observations in the hardwood data sets for removals of 3 or 6 percent to be meaningful, and thus only 15, 33, 66, and 85 percent of the observations were removed for each of these species. The observations were randomly divided into fitting and validation sets 50 times. The subsampling procedure (randomly selecting prespecified percentages of the observations in the fitting set) was performed 10 times for each fitting set. A record of the observations in the validation set corresponding to each fitting set was maintained. Thus for each species there were 50 estimates of β_1 based on WLS on the complete fitting data, and 500 estimates of β_1 based on WLS and the two EB estimators for each level of subsampling from the fitting data. For each method of estimation we examined the following criteria:

TABLE 2. Values of slope coefficient ($\hat{\beta}_1$) in combined variable total outside bark cubic foot volume equation used to form prior distribution for slope for hardwood species, by species.

Species	$\hat{\beta}_1$	Species	$\hat{\beta}_1$
Yellow birch	0.00242	Black locust	0.00257
Sugar maple	.00244	Cucumbertree	.00258
White ash	.00248	Hickory sp.	.00258
Black oak	.00248	Scarlet oak	.00259
Red oak	.00250	Yellow-poplar	.00263
American beech	.00251	Black tupelo	.00268
Sweet birch	.00251	American basswood	.00275
Chestnut oak	.00252		

1. Mean difference ($\overline{D}$), mean absolute difference ($|\overline{D}|$), and mean squared difference ($\overline{D^2}$) between observed volume and predicted volume per tree on the validation data.
2. Mean difference, mean absolute difference, and mean squared difference between the sum of observed volumes and the sum of predicted volumes on the validation data.
3. Mean difference, mean absolute difference, and mean squared difference between estimated β_1 and the "true" β_1, obtained from WLS on all data (the fitting *and* validation sets) for each species.

Criterion 1 provides a test of an equation's predictive ability on a per-tree basis. While this is important, it is also important to determine how well an equation performs when predicted volumes are summed, as is normally done to estimate the total cubic volume of a stand. Criterion 2 addresses the latter question. Furthermore, although predictive ability is of primary importance in this case, it is also interesting to examine how well the EB estimators estimate the parameter in question, β_1, as opposed to WLS estimators. For this purpose, the WLS estimate of β_1 for each species over all the data (fitting and validation) for that species was determined. This value was assumed to represent the "true" value of β_1. The ability of each estimator to estimate the "true" value was then studied using criterion 3 above.

Under criteria 1 and 2 above, $\overline{D}$ measures the bias in the predictions of a fitted equation, while $|\overline{D}|$ and $\overline{D^2}$ account for the bias *and* precision in the predictions of the equation. Under criterion 3, $\overline{D}$ measures the bias in an estimator of β_1 while $|\overline{D}|$ and $\overline{D^2}$ account for the bias *and* precision of the estimator. As $|\overline{D}|$ and $\overline{D^2}$ measure bias *and* precision while $\overline{D}$ measures only bias, $|\overline{D}|$ and $\overline{D^2}$ may be regarded as better indicators of a fitted model's predictive ability than $\overline{D}$. In this study we shall consider $\overline{D}$ to indicate a model's bias and $|\overline{D}|$ and $\overline{D^2}$ to indicate the model's predictive ability.

It is important to recognize that the WLS and EB estimators based on 85, 66, 33, 15, 6, and 3 percent samples involve measuring fewer trees, resulting in a cost savings. It is not necessary that the estimators based on smaller samples actually yield $\overline{D}$, $|\overline{D}|$, or $\overline{D^2}$ values less than those obtained with more data. An estimator which yields $\overline{D}$, $|\overline{D}|$, or $\overline{D^2}$ values slightly greater than another may be preferable from a cost-plus-loss viewpoint if it involves measuring fewer trees.

RESULTS[5]

The mean differences between observed and predicted volumes per tree and summed total volumes are presented in Tables 3 and 4. The mean differences between the estimated and "true" value of β_1 are shown in Table 5. The EB estimators are biased and thus would be expected to perform more poorly under this statistic than the unbiased WLS estimators. However, as shown in Tables 3 and 4, the EB estimators were generally less biased in terms of predicted volumes for two species (white oak and black cherry) than the WLS estimators. The cause of this result is easy to determine. The WLS estimates of β_1 for white oak and black cherry based on all the data were 0.00256 and 0.00252, which are close to 0.00255, the mean of the values in Table 2, and thus the EB estimators were shrunk toward the correct value. The WLS estimate of β_1 based on all the data

[5] Although this study is concerned with only three estimators (WLS, GS, and ZE), for convenience the estimators from differing subsampling levels will be referred to as separate estimators. For instance, we shall discuss 100 percent WLS estimators and 85 percent WLS estimators, although these are the same estimator based on different data.

TABLE 3. Mean difference, mean absolute difference, and mean squared difference between observed and predicted individual tree cubic foot volumes, by estimator, sampling scheme, and species.

Species and percent sample	Mean difference			Mean absolute difference			Mean squared difference		
	WLS	ZE	GS	WLS	ZE	GS	WLS	ZE	GS
White oak									
$N = 84$, $n_f = 63$, $n_v = 21$[a]									
100%	−0.217			2.339			16.307		
85%	−.209	−0.247	−0.237	2.339	2.322	2.322	16.347	16.077	16.069
66%	−.210	−.183	−.182	2.350	2.323	2.322	16.525	16.098	16.085
33%	−.186	−.183	−.182	2.426	2.337	2.334	17.410	16.129	16.083
15%	−.308	−.188	−.188	2.585	2.380	2.383	19.445	16.083	16.029
Red maple									
$N = 76$, $n_f = 57$, $n_v = 19$[a]									
100%	−0.413			3.123			31.651		
85%	−.384	−0.757	−0.770	3.133	3.104	3.103	31.651	32.245	32.275
66%	−.411	−.862	−.881	3.143	3.110	3.110	31.866	32.608	32.658
33%	−.401	−1.170	−1.225	3.244	3.171	3.174	33.250	33.927	34.082
15%	−.314	−1.555	−1.696	3.346	3.271	3.298	34.265	35.343	36.003
Black cherry									
$N = 78$, $n_f = 59$, $n_v = 19$[a]									
100%	0.374			2.912			23.591		
85%	.362	0.267	0.264	2.915	2.888	2.887	23.638	23.140	23.124
66%	.366	.253	.248	2.923	2.886	2.885	23.871	23.162	23.133
33%	.280	.117	.104	2.945	2.868	2.864	24.546	22.947	22.848
15%	.305	.082	.054	3.082	2.892	2.882	26.829	22.673	22.372
Loblolly pine									
$N = 427$, $n_f = 320$, $n_v = 107$[a]									
100%	−0.005			0.267			0.211		
85%	−.004	−0.011	−0.011	.265	0.265	0.265	.210	0.210	0.210
66%	−.006	−.014	−.014	.266	.266	.266	.210	.211	.211
33%	−.009	−.024	−.024	.267	.267	.267	.213	.214	.214
15%	−.003	−.034	−.035	.270	.269	.269	.216	.217	.217
6%	−.010	−.059	−.061	.281	.276	.276	.234	.226	.226
3%	0[b]	−.072	−.077	.293	.283	.285	.252	.231	.232

[a] N = total sample size for species.
n_f = no. of trees in fitting data for species.
n_v = no. of trees in validation data for species.
[b] Zero to four decimal places, i.e., 0.0000.

for red maple was 0.00243, less than all but one of the values in Table 2. Similarly, the WLS estimate using all the data for loblolly pine was 0.00194, less than all but one of the values in Table 1. Thus for these two species the EB estimators were shrunk to a value greater than the appropriate one, as indicated by the full data set for each species. The results presented in Table 5 indicate that the EB estimators are more biased than the WLS estimators for estimating the "true" value of β_1.

The $|\overline{D}|$ and $\overline{D^2}$ values on a per-tree basis are presented in Table 3. For black cherry, under both $|\overline{D}|$ and $\overline{D^2}$, all the EB estimators were superior to all the WLS estimators, including the 100 percent WLS estimator. For white oak, the EB estimators were always superior under $|\overline{D}|$ and $\overline{D^2}$ to the WLS estimator from

TABLE 4. Mean difference, mean absolute difference, and mean squared difference between sum of observed cubic foot volumes and sum of predicted cubic foot volumes, by estimator, sampling scheme, and species.

Species and percent sample	Mean difference			Mean absolute difference			Mean squared difference		
	WLS	ZE	GS	WLS	ZE	GS	WLS	ZE	GS
White oak									
$N = 84$, $n_f = 63$, $n_v = 21$[a]									
100%	−4.567			14.816			346.646		
85%	−4.393	−3.957	−3.944	15.111	13.699	13.660	356.573	292.523	290.843
66%	−4.412	−3.835	−3.813	15.757	13.655	13.588	394.478	294.325	291.389
33%	−3.902	−3.844	−3.820	19.468	13.735	13.511	574.340	288.725	279.773
15%	−6.463	−5.184	−4.985	24.384	14.260	14.481	984.481	326.679	333.076
Red maple									
$N = 76$, $n_f = 57$, $n_v = 19$[a]									
100%	−7.844			22.498			883.125		
85%	−7.300	−14.388	−14.634	22.690	23.466	23.534	884.235	970.459	975.473
66%	−7.820	−16.394	−16.761	23.364	24.550	24.670	924.959	1,037.213	1,045.810
33%	−7.617	−22.247	−23.282	26.773	27.854	28.296	1,221.589	1,303.396	1,332.754
15%	−5.978	−29.561	−32.215	28.332	32.133	34.005	1,358.570	1,657.202	1,801.960
Black cherry									
$N = 78$, $n_f = 59$, $n_v = 19$[a]									
100%	7.106			20.563			668.855		
85%	6.861	5.069	5.006	20.426	19.350	19.312	681.268	593.885	591.008
66%	6.949	4.802	4.709	21.233	19.516	19.446	725.586	598.531	593.500
33%	5.302	2.202	1.972	23.336	19.312	19.048	860.038	570.401	553.397
15%	5.788	1.541	1.014	27.503	18.284	17.550	1,248.162	522.940	480.183

TABLE 4. Continued.

Species and percent sample	Mean difference			Mean absolute difference			Mean squared difference		
	WLS	ZE	GS	WLS	ZE	GS	WLS	ZE	GS
Loblolly pine									
$N = 427$, $n_f = 320$, $n_v = 107$[a]									
100%	−0.557			3.676			20.196		
85%	−.488	−1.196	−1.201	3.718	3.706	3.706	21.384	21.695	21.701
66%	−.616	−1.503	−1.511	3.819	3.843	3.843	22.723	23.306	23.319
33%	−.988	−2.582	−2.608	4.495	4.488	4.492	32.656	32.875	32.931
15%	−.344	−3.658	−3.761	5.660	5.412	5.442	51.850	44.044	44.335
6%	−1.036	−6.284	−6.586	8.986	7.427	7.575	125.987	78.883	81.271
3%	.002	−7.664	−8.328	11.466	9.164	9.632	206.248	122.595	134.416

[a] N = total sample size for species.
n_f = no. of trees in fitting data for species.
n_v = no. of trees in validation data for species.

TABLE 5. Mean difference, mean absolute difference, and mean squared difference between estimated and "true" value of β_1, by estimator, sampling scheme, and species.

Species and percent sample	Mean difference			Mean absolute difference			Mean squared difference		
	WLS	ZE	GS	WLS	ZE	GS	WLS	ZE	GS
White oak									
$N = 84$, $n_f = 63$[a]									
100%	0[b]			1.8×10^{-5}			5.14×10^{-10}		
85%	0.1×10^{-5}	0.3×10^{-5}	0.3×10^{-5}	2.4×10^{-5}	1.8×10^{-5}	1.8×10^{-5}	9.27×10^{-10}	4.95×10^{-10}	4.85×10^{-10}
66%	0[b]	$.2 \times 10^{-5}$	$.2 \times 10^{-5}$	3.6×10^{-5}	2.4×10^{-5}	2.4×10^{-5}	20.06×10^{-10}	9.24×10^{-10}	8.94×10^{-10}
33%	$.5 \times 10^{-5}$	$.5 \times 10^{-5}$	$.5 \times 10^{-5}$	6.8×10^{-5}	3.4×10^{-5}	3.2×10^{-5}	69.01×10^{-10}	17.24×10^{-10}	15.46×10^{-10}
15%	$.2 \times 10^{-5}$	$.5 \times 10^{-5}$	$.5 \times 10^{-5}$	11.4×10^{-5}	3.5×10^{-5}	2.8×10^{-5}	215.14×10^{-10}	20.29×10^{-10}	13.35×10^{-10}
Red maple									
$N = 76$, $n_f = 57$[a]									
100%	-0.1×10^{-5}			2.0×10^{-5}			6.57×10^{-10}		
85%	$.1 \times 10^{-5}$	-2.4×10^{-5}	-2.5×10^{-5}	2.5×10^{-5}	2.7×10^{-5}	2.8×10^{-5}	9.26×10^{-10}	11.27×10^{-10}	11.59×10^{-10}
66%	0[b]	-3.1×10^{-5}	-3.2×10^{-5}	3.2×10^{-5}	3.5×10^{-5}	3.5×10^{-5}	16.06×10^{-10}	18.01×10^{-10}	18.58×10^{-10}
33%	$.4 \times 10^{-5}$	-4.9×10^{-5}	-5.2×10^{-5}	6.1×10^{-5}	5.5×10^{-5}	5.7×10^{-5}	58.07×10^{-10}	42.67×10^{-10}	44.53×10^{-10}
15%	1.5×10^{-5}	-6.9×10^{-5}	-7.8×10^{-5}	9.3×10^{-5}	7.3×10^{-5}	8.0×10^{-5}	140.90×10^{-10}	68.81×10^{-10}	75.94×10^{-10}
Black cherry									
$N = 78$, $n_f = 59$[a]									
100%	0[b]			2.0×10^{-5}			6.40×10^{-10}		
85%	-0.1×10^{-5}	-0.7×10^{-5}	-0.7×10^{-5}	2.5×10^{-5}	2.1×10^{-5}	2.1×10^{-5}	10.10×10^{-10}	7.41×10^{-10}	7.33×10^{-10}
66%	$-.2 \times 10^{-5}$	$-.9 \times 10^{-5}$	$-.9 \times 10^{-5}$	3.3×10^{-5}	2.7×10^{-5}	2.7×10^{-5}	17.44×10^{-10}	11.64×10^{-10}	11.44×10^{-10}
33%	$-.9 \times 10^{-5}$	-1.8×10^{-5}	-1.9×10^{-5}	5.4×10^{-5}	3.8×10^{-5}	3.7×10^{-5}	46.25×10^{-10}	22.67×10^{-10}	21.35×10^{-10}
15%	-1.1×10^{-5}	-2.4×10^{-5}	-2.6×10^{-5}	8.9×10^{-5}	4.5×10^{-5}	4.0×10^{-5}	122.45×10^{-10}	31.49×10^{-10}	24.91×10^{-10}

TABLE 5. *Continued.*

Species and percent sample	Mean difference			Mean absolute difference			Mean squared difference		
	WLS	ZE	GS	WLS	ZE	GS	WLS	ZE	GS
Loblolly pine									
$N = 427$, $n_f = 320$[a]									
100%	-0.1×10^{-5}			0.6×10^{-5}			0.65×10^{-10}		
85%	$-.1 \times 10^{-5}$	-0.4×10^{-5}	-0.4×10^{-5}	$.8 \times 10^{-5}$	0.8×10^{-5}	0.8×10^{-5}	1.10×10^{-10}	1.18×10^{-10}	1.18×10^{-10}
66%	$-.2 \times 10^{-5}$	$-.7 \times 10^{-5}$	$-.7 \times 10^{-5}$	1.1×10^{-5}	1.2×10^{-5}	1.2×10^{-5}	2.00×10^{-10}	2.13×10^{-10}	2.14×10^{-10}
33%	$-.5 \times 10^{-5}$	-1.3×10^{-5}	-1.4×10^{-5}	2.0×10^{-5}	2.0×10^{-5}	2.0×10^{-5}	6.30×10^{-10}	6.23×10^{-10}	6.24×10^{-10}
15%	0[b]	-1.8×10^{-5}	-1.9×10^{-5}	3.3×10^{-5}	2.8×10^{-5}	2.8×10^{-5}	17.46×10^{-10}	12.07×10^{-10}	12.06×10^{-10}
6%	$-.7 \times 10^{-5}$	-3.5×10^{-5}	-3.6×10^{-5}	6.0×10^{-5}	4.1×10^{-5}	4.2×10^{-5}	57.62×10^{-10}	24.66×10^{-10}	24.34×10^{-10}
3%	$.1 \times 10^{-5}$	-3.9×10^{-5}	-4.3×10^{-5}	8.4×10^{-5}	4.6×10^{-5}	4.7×10^{-5}	113.42×10^{-10}	28.55×10^{-10}	27.79×10^{-10}

[a] N = total sample size for species.
n_f = no. of trees in fitting data for species.
[b] Zero to six decimal places, i.e., 0.000000.

the same subsampling scheme and all the EB estimators except those at the 15 percent level were superior to the 100 percent WLS estimator under $|\overline{D}|$. Again, all the EB estimators were superior to all the WLS estimators under $\overline{D^2}$. In the case of red maple, under $|\overline{D}|$, the 66 and 85 percent EB estimators were superior to all the WLS estimators, while the 15 and 33 percent EB estimators were superior to the WLS estimator from the same subsampling level. Under $\overline{D^2}$, the EB estimators for each subsampling level were close, but never superior, to the WLS estimators based on the same subsampling level. For loblolly pine, there was very little difference between the WLS and EB estimators under $|\overline{D}|$ or $\overline{D^2}$ except at the low subsampling levels (3 and 6 percent). At these levels, the EB estimators were superior to the WLS estimators.

The mean absolute and mean squared differences for summed total volumes are presented in Table 4 and show the same general pattern as the per-tree differences. The EB estimators performed better than WLS estimation for white oak and black cherry, and there was little difference except at the lower of subsampling levels for loblolly pine. The only case in which the trends for summed total volumes (Table 4) differed from individual tree volumes (Table 3) was that of red maple. For this species the results in Table 4 indicate that WLS estimation was somewhat better than EB estimation in terms of total volumes. As noted above, this pattern was not clearly evident in Table 3.

The mean absolute and mean squared differences between the estimated and "true" values of β_1 show that for all species, the EB estimators are superior to the WLS estimators at the low subsampling levels. At higher subsampling levels, EB tends to be better than WLS for white oak and black cherry, there is little difference between the estimators for loblolly pine, and WLS tends to be better for red maple.

A further result is that as previously mentioned, the ZE estimators do seem to be more conservative than the GS estimators. When the prior information is good (white oak and black cherry) the GS estimators perform better under $|\overline{D}|$ and $\overline{D^2}$ than the ZE estimators. When this is not the case, the ZE estimators tend to be superior. However, the differences are usually small, and the two share an identical relationship with the WLS estimators.

Conclusions

One reason for the slow acceptance of empirical Bayes methods by foresters has been the fact that such procedures are biased. However, as demonstrated here, if the prior mean is close to the actual mean, the bias is very small. Thus when good prior information is available, as it was for white oak and black cherry in this case, the bias in the EB estimators is negligible. Furthermore, it is not clear that unbiasedness is a necessary condition for an optimal estimator. Most practitioners would probably feel more comfortable with a biased estimator which was always close to the true value of the parameter in question than with an unbiased estimator which usually tended to be farther away. For these reasons, we feel that predictive ability is usually a more important property than unbiasedness. In this study, because of the nature of the problem, we believe that it is the mean absolute and mean squared differences of the predicted individual tree and summed total volumes that are of primary interest. In light of this, the above results have three implications. First, it appears that it would be advisable to use EB estimators for volume equation coefficients when good prior information is available. This is supported by the complete dominance of the EB estimators over the WLS estimators from the same subsampling schemes for white oak and black cherry. Our prior information was not as good for red maple and loblolly pine. However even for these species, EB estimation was competitive with WLS.

In practice it may not be prudent to combine data from different species or from different regions for the same species to determine a prior distribution as done here. For example, for future studies on red maple we recommend using the value found from WLS on all the red maple data (0.00243) as the prior mean. For loblolly pine, one should use a value determined from past studies in the region of interest. This should lead to gains for these two species similar to those demonstrated here for black cherry and white oak. At this point we cannot recommend a procedure other than the one followed here for determining prior variances. More studies are needed to determine the true variability of the slope parameter for specific populations.

The second implication is that, of the two EB estimators presented (GS and ZE), the one to be employed depends upon the investigator's willingness to gamble. As most investigators are somewhat conservative, we suspect that most would prefer the more conservative ZE estimator.

Finally, there is a strong indication that through the use of EB estimation sample sizes may be reduced and predictive ability nearly as good or better as that from WLS estimation obtained. Our results indicate that with good prior information one may be able to collect only 15–33 percent as much information and still develop estimators of comparable, or perhaps even superior, predictive ability.

In summary, it appears that empirical Bayes estimation is a promising technique for developing volume equations. More research is needed to determine whether or not it is possible to specify conditions under which EB estimation will dominate conventional estimation procedures.

Literature Cited

Bailey, R. L., and J. L. Clutter. 1970. Volume tables for old-field loblolly pine in the Georgia Piedmont. Ga Forest Res Counc Rep 22, Ser 2, 4 p.

Bickel, P. J., and K. A. Doksum. 1977. Mathematical statistics: basic ideas and selected topics. Holden-Day, San Francisco, CA. 493 p.

Box, G. E. P., and G. C. Tiao. 1973. Bayesian inference in statistical analysis. Addison-Wesley Publ Co, Reading, MA. 588 p.

Burk, T. E., and A. R. Ek. 1982. Application of empirical Bayes/James-Stein procedures to simultaneous estimation problems in forest inventory. Forest Sci 28:753–771.

Burkhart, H. E., R. C. Parker, M. R. Strub, and R. G. Oderwald. 1972a. Yields of old-field loblolly pine plantations. Va Polytech Inst and State Univ, Div For and Wildl Resour, FWS-3-72, 51 p.

Burkhart, H. E., R. C. Parker, and R. G. Oderwald. 1972b. Yields for natural stands of loblolly pine. Va Polytech Inst and State Univ, Div For and Wildl Resour, FWS-2-72, 63 p.

Cao, Q. V., H. E. Burkhart, and T. A. Max. 1980. Evaluation of two methods for cubic-volume prediction of loblolly pine to any merchantable limit. Forest Sci 26:71–80.

Clutter, J. L., J. C. Fortson, L. V. Pienaar, G. H. Brister, and R. L. Bailey. 1983. Timber management: a quantitative approach. John Wiley and Sons, New York. 333 p.

Cunia, T. 1964. Weighted least squares method and construction of volume tables. Forest Sci 10: 180–191.

Draper, N. R., and H. Smith. 1981. Applied regression analysis. Second ed. John Wiley and Sons, New York. 709 p.

Ek, A. R., and J. N. Issos. 1978. Bayesian theory and multi-resource inventory. *In* Integrated inventories of renewable natural resources: proceedings of the workshop, p 291–298. USDA Forest Serv Gen Tech Rep RM-55, 482 p.

Fisher, R. A. 1922. On the mathematical foundations of theoretical statistics. Reprinted in Contributions to mathematical statistics (by R. A. Fisher, 1950). John Wiley and Sons, New York.

Hasness, J. R., and J. D. Lenhart. 1972. Cubic-foot volumes for loblolly pine trees in old-field plantations in the Interior West Gulf Coastal Plain. Stephen F. Austin State Univ, Texas For Pap 12, 7 p.

Gevorkiantz, S. R., and L. P. Olsen. 1955. Composite volume tables for timber and their application in the Lake States. USDA Tech Bull 1104, 51 p.

MARTIN, A. J. 1981. Taper and volume equations for selected Appalachian hardwood species. USDA Forest Serv Res Pap NE-490, 22 p.
MATNEY, T. G., and A. D. SULLIVAN. 1982. Compatible stand and stock tables for thinned and unthinned loblolly pine stands. Forest Sci 28:161–171.
SMALLEY, G. W., and D. R. BOWER. 1968. Volume tables and point-sampling factors for loblolly pines in plantations on abandoned fields in Tennessee, Alabama, and Georgia highlands. USDA Forest Serv Res Pap SO-32, 13 p.
SMITH, J. H. G., and R. E. BREADON. 1964. Combined variable equations and volume-basal area ratios for total cubic foot volumes of the commercial trees of B.C. For Chron 40:258–261.
SPURR, S. H. 1952. Forest inventory. John Wiley and Sons, New York. 472 p.
SWINDEL, B. F. 1972. The Bayesian controversy. USDA Forest Serv Res Pap SE-95, 12 p.
ZELLNER, A. 1971. An introduction to Bayesian inference in econometrics. John Wiley and Sons, New York. 431 p.

Forest Sci., Vol. 31, No. 4, 1985, pp. 990–994

Structural Instability Analysis; The Case of Newsprint Consumption in the United States

Anders Baudin and Anders Westlund

ABSTRACT. The objective of this study is to analyse the problem of parameter instability which may be inherent in a newsprint consumption model. This instability is analysed by various test statistics, and the consumption model is recursively estimated. FOREST SCI. 31:990–994.

ADDITIONAL KEY WORDS. Parameter instability, recursive estimation.

THE CONCERN of this note is the problem of structural instabilities in the newsprint consumption system in the United States. There are several phenomena which have caused or may cause structural instabilities within the system. Some of those are discussed, e.g., in Baudin and Westlund (1984), which also gives the identification and estimation of the following consumption model for newsprint in the US (t-values within parentheses):

$$\begin{aligned} \ln C_t = 2.278 &+ \underset{(3.93)}{0.603} \ln C_{t-1} - \underset{(-2.25)}{0.339} \ln C_{t-2} \\ &+ \underset{(3.35)}{0.760} \ln \mathrm{PCN}_{t-1} - \underset{(-1.80)}{0.260} \ln P_{t-3} \\ (R^2 = 0.64,& \quad s_e = 0.056), \end{aligned} \tag{1}$$

where C = consumption of newsprint in the US (thousands of metric tons/quarter); PCN = private consumption of "nondurable goods" (deflated); and P = price index for newsprint (1975 = 100, deflated).

The time lags $t - 1$, $t - 2$, $t - 3$ in (1) are determined by empirical means, applying a backward elimination regression procedure. Insignificant factors have been eliminated one by one, and careful examination of the structure and residual analysis has been performed after each step.

Comparisons with other similar structural analyses show that this model (estimated with data from 72:I to 83:I) is in line with results given, e.g., by Buongiorno (1978).

An analysis of structural instability presupposes some formal interpretation of the structure concept. The present analysis concentrates on the structural parameters of (1). The idea that economic and social structures are always characterized by a certain structural instability is probably commonly accepted now (see, e.g., Johnson 1977, 1980), but it is

---

The authors are Associate Professors in Statistics, at Umeå University and at the Research Council for Humanities and Social Sciences in Sweden, respectively. They wish to thank anonymous reviewers and a board member for very helpful reviews and suggestions. Manuscript received 6 December 1983.