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ECON 3313
4/8/2021

Homework 5

Q1. Rewrite without Lag Operator

1. $(L^{\tau-1})y_t = \varepsilon_t$

$$\boxed{y_{t-1+\tau} = \varepsilon_t}$$

2. $y_t = \left(\frac{2+5L^2+.8L^4}{L-.6L^5}\right)\varepsilon_t$ ~~(L-.6L5)~~

$(L-.6L^5)y_t = (2+5L^2+.8L^4)\varepsilon_t$

$Ly_t - .6L^5 y_t = 2\varepsilon_t + 5L^2\varepsilon_t + .8L^4\varepsilon_t \rightarrow \boxed{y_{t-1} - .6y_{t-5} = 2\varepsilon_t + 5\varepsilon_{t-2} + .8\varepsilon_{t-4}}$

3. $y_t = 2(1+L^2+\frac{L^4}{L})\varepsilon_t$

$y_t = 2(1+L^2+L^3)\varepsilon_t$

$$\boxed{y_t = 2(\varepsilon_t + \varepsilon_{t-2} + \varepsilon_{t-3})}$$

Rewrite with Lag Operator

1. $y_t + y_{t-1} + \ldots + y_{t-N} = \alpha + \varepsilon_t + \varepsilon_{t-1} + \ldots + \varepsilon_{t-N}$, where α is constant

$1 + L + \ldots + L^N = \alpha_L + 1 + L + \ldots + L^N$, where α_L is constant

2. $y_t = 3\varepsilon_{t-2} + 2\varepsilon_{t-1} + \varepsilon_t$

$y_t = \varepsilon_t \sum_{l=0}^{3} b_l L$

Q2.

1. $\gamma(t,\tau) = \alpha$

- covariance structure is <u>not</u> stable overtime. To quantify the stability between y_t and $y_{t-\tau}$, displacement has to be symmetric over time
- α is not symmetric to $\tau \rightarrow \gamma(t,\tau) \not\rightarrow \gamma(\tau) \leftarrow$ not α

2. $\gamma(t,\tau) = e^{-a\tau}$

- consistent w/ covariance stationary. τ is symmetric on both sides of autocovariance function
- $\gamma(t,\tau) = e^{-a\tau} \rightarrow \gamma(t,\tau) = \gamma(-\tau)$

3. $\gamma(t,T) = \alpha \log(T)$

- autocovariance function ~~not~~ consistent w/ covariance. Log in function violates autocovariance stationary rules.
- $\gamma(t,T) \neq \alpha \log(T) \rightarrow$ not symmetric on both sides of function

4. $\gamma(t,T) = \frac{\alpha}{T}$

- autocovariance function is consistent w/ covariance stationary. function is symmetric and positive on both sides of the equation.
- $\gamma(t,T) = \frac{\alpha}{T} \rightarrow$ positive + constant on both sides

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Q3.

1. $y_t = 3000 + y_{t-1} + .9\sqrt{x_{t-1}} + \varepsilon_t,\ \varepsilon_t \sim N(0,500)$

$y_t = 3000 + 8000 + .9\sqrt{3300} = 11{,}052.01$

var: $11{,}052.01 - 8000 = \boxed{3340}$ (conditional)

2. $y_t = 3000 + 16000 + .9\sqrt{8500} = 19082.98$

var: $19{,}082.98 - 16{,}000 = \boxed{3{,}082.98}$ (unconditional)

3. It is conditional upon the sales this season. Next years expected revenue is higher @ the old firm. However, unconditionally expected revenue is higher at new firm. It is relevant for longer run. Decision depend on rate of time preference.

long run expected rev: (unconditional)

$y_t = 3000 + 18{,}000 + .9\sqrt{9000} = 21{,}085.28$

$\text{Var} = 21{,}085.38 - 18\,000 = 3{,}085.38$

expected sales based on last year (conditional)

$y_t = 3000 + 6{,}800 + .9\sqrt{2650} = 9{,}846.33$

var: $9{,}846.33 - 6800 = 3{,}046.33$

Q4.

1. $y_t = \mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}$
where $\varepsilon_t \sim$ white noise

New Condition:

Unconditional mean: $\mu_t = E(y_t) = \mu + \varepsilon(\varepsilon_t) + \theta_1 E(\varepsilon_{t-1}) + \theta_2 E(\varepsilon_{t-2})$
$= \mu \; [\varepsilon(\varepsilon_t) = 0 \;\forall t]$

Conditional mean:

$$\varepsilon(y_t/\varepsilon_t) = \varepsilon(\mu + \varepsilon_t + \theta_1 \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2} \mid \varepsilon_t)$$
$$= \mu + \varepsilon(\varepsilon_t \mid \varepsilon_t) + \theta_1 \varepsilon(\varepsilon_{t-1} \mid \varepsilon_t) + \theta_2 \varepsilon(\varepsilon_{t-2} / \varepsilon_t)$$
$$= \mu + \varepsilon_t$$
$$\approx (\varepsilon(\varepsilon_{t-1} \mid \varepsilon_t) = \varepsilon(\varepsilon_t) = 0)$$
white noise

$$\operatorname{cov}(y_t, y_{t-1}) = \operatorname{cov}(\mu + \varepsilon_t + \theta \varepsilon_{t-1} + \theta_2 \varepsilon_{t-2}, \mu + \varepsilon_{t-1} + \theta_1 \varepsilon_{t-2} + \theta \varepsilon_{t-3})$$
$$= \theta_1 \operatorname{var}(\varepsilon_{t-1}) + \theta_1 \theta_2 \operatorname{var}(\varepsilon_{t-1})$$
$$= \theta_1 \theta^2 + \theta_1 \theta_2 \sigma^2 = \theta_1 \sigma^2 (1 + \theta_2)$$

Similarly,

$\operatorname{cov}(-y_t, y_{t-2}) = \theta_2 \operatorname{var}(\varepsilon_{t-2}) = \theta_2 \sigma^2$

and $\operatorname{cov}(Y_t, Y_{t-R}) = 0 \;\forall A \geq 3$

$\operatorname{var}(y_t) = \operatorname{var}(\varepsilon_t) + \theta^2 \operatorname{var}(\varepsilon_{t-1}) + \theta^2 \operatorname{var}(\varepsilon_{t-2})$
$= \sigma^2 (1 + \theta_1^2 + \theta_2^2) = 0 \longrightarrow$ unconditional variance

where $r(k) = \operatorname{cov}(y_t, y_{t-k}) = \gamma(-k)$

$\operatorname{cov}(y_t, y_{t-k})$ is not a function of (A, k) only for k.

so, auto cov. is given by

$$y(k) = \begin{cases} \sigma^2(1 + \theta_1^2 y + \theta_2^2) & \text{if } k = 0 \\ \theta \sigma^2 (1 + \theta_2) & \text{if } |k| = 1 \\ \theta_2 \sigma^2 & \text{if } |k| = 2 \\ 0 & \text{if } |k| \geq 3 \end{cases}$$

auto correlation:

$$p(k) = \begin{cases} 1 & \text{if } k = 0 \\ \theta_1(1 + \theta_2) / (1 + \theta_1^2 + \theta_2^2) & \text{if } |k| = 1 \\ \theta_2 / (1 + \theta_1^2 + \theta_2^2) & \text{if } |k| = 2 \\ 0 & \text{if } |k| \geq 3 \end{cases}$$

2. $y_t = \phi y_{t-1} + \varepsilon_t + \theta \varepsilon_{t-1}$ [assuming $|\phi| < 1$]

$y_t(1-\phi B) = \varepsilon_t + \theta \varepsilon_{t-1} = (1+\theta B)\varepsilon_t$ where $\varepsilon_t = \varepsilon_t - \mu$

$$y_t = (1-\phi B)^{-1} \cdot (1+\theta B)\varepsilon_t$$
$$= (1 + \phi B + \phi^2 B^2 + \dots)(1+\theta B)\varepsilon_t$$
$$= [1 + \phi B + \phi^2 B^2 + \dots + \theta B + \theta\phi B^2 + \theta\phi^2 B^3 + \dots]\varepsilon_t$$
$$= \varepsilon_t + (\theta+\phi)\sum_{j=1}^{\infty} \phi^{j-1}\varepsilon_{t-j}$$

y_t is only a function of past observations, ε_t, not on future observations

$$\mathrm{cov}(y_t, y_{t-k}) = \gamma(k) = \mathrm{cov}(y_{t-k}, \phi y_{t-1} + \varepsilon_t + \theta\varepsilon_{t-1})$$
$$= \phi\,\mathrm{cov}(y_{t-k}, y_{t-1}) + \mathrm{cov}(y_{t-k}, \varepsilon_t) + \theta\,\mathrm{cov}(y_{t-k}, \varepsilon_{t-1})$$
$$= \phi\gamma(k-1) + 0 + 0$$
$$= \phi^{|k|}\gamma(0)$$

so: $\rho(k) = \dfrac{\gamma(k)}{\gamma(0)} = \phi^{|k|}$ if $k \in \mathbb{N}$, $\rho(0) = 1$

$$\gamma(0) = \mathrm{var}(y_t) = \sigma^2 + (\theta+\phi)^2 \sum_{j=1}^{\infty} \phi^{2j-2}\sigma^2$$
$$= \sigma^2\left[1 + \frac{(\theta+\phi)^2}{1-\phi^2}\right] = \frac{\sigma^2(1+2\theta\phi+\sigma^2)}{(1-\phi^2)}$$

$$E(y_t) = \phi E(y_{t-1})$$
$$\mu = \phi\mu \Rightarrow \mu = 0 \text{ on, } \phi \neq 1$$

Q5.

1. Yes, the series (CAEMP) is serially correlated. This is because the data is presented and formatted in a time-series. This allows for the data to show a continuous data point throughout X amount of time.

2. I would choose the ARMA model. I would choose this model because of its digressions found in auto and partial auto correlation models. For the auto correlation model, it showed a

gradual decline. When running the partial autocorrelation, the data exhibited a steep decay after 0.5, as the model is built for it to do.

3. The ARMA (2,2) model has the best forecast performance. When running it against the actual CAEMP data, it is the most on-par with that graph. Because of this, I would agree that the ARMA (2,2) model is the best performing one.

4. The forecast performance for the ARMA (2,2) model is excellent. Residual errors are random and show no pattern. Moving averages also tend to follow the data very closely. The residual plot confirms the ARMA (2,2) model works best in this problem.