3] EM Algorithm

A] $Pr(z|x) = \dfrac{Pr(x|z)\, Pr(z)}{Pr(x)}$

Using this formula,

$$\gamma(G,H) = Pr(G|H) = \frac{\frac{1}{2}p}{\frac{1}{2}p + \frac{1}{4}(1-p)}$$

$$= \frac{\frac{1}{2}p}{\frac{1}{2}\left(p + \frac{1}{2}(1-p)\right)}$$

$$= \frac{p}{\frac{2p+1-p}{2}}$$

$$\gamma(G,H) = \frac{2p}{p+1}$$

$$\gamma(G,T) = Pr(G|T) = \frac{\frac{1}{2}p}{\frac{1}{2}p + \frac{3}{4}(1-p)} = \frac{p}{p + \frac{3}{2}(1-p)}$$

$$= \frac{2p}{3-p}$$

$$\gamma(S,H) = Pr(S|H) = \frac{\frac{1}{4}(1-p)}{\frac{1}{4}(1-p) + \frac{1}{2}p} = \frac{1-p}{1-p+2p}$$

$$= \frac{1-p}{p+1}$$

$$\gamma(S|T) = Pr(S|T) = \frac{\frac{3}{4}(1-p)}{\frac{1}{2}p + \frac{3}{4}(1-p)} = \frac{1-p}{\frac{2}{3}p + 1 - p}$$

$$= \frac{1-p}{2p+3-3p}$$

$$= \frac{1-p}{3-p}$$

B]

$$L_p(p') = \sum_{i=1}^{N} \sum_{(G,S)} \gamma_p(z, x_i) \cdot \log Pr(z, x_i; g')$$

$$= \sum_{i=1}^{N} \Big(\gamma_p(G, x_i) \log Pr(G, x_i; g') + \gamma_p(S, x_i) \log Pr(S, x_i; g') \Big)$$

[$\because$ breaking Sum $\sum_{G,S}$]

Now, Total N button presses out of which some are Head out comes some are Tail outcomes. So, breaking $\sum_{i=1}^{N}$] [$N_H + N_T = N$]

And Expanding term $Pr(z, x_i, g')$

$$= \sum_{i=1}^{NH} \Big(\gamma_p(G,H) \cdot \log Pr(H|G) \cdot Pr(G) + \gamma_p(S,H) \cdot \log Pr(H|S) \cdot Pr(S) \Big)$$

$$+ \sum_{i=1}^{N_T} \Big(\gamma_p(G,T) \log Pr(T|G) \, Pr(G) + \gamma_p(S,T) \log Pr(T|S) \cdot Pr(S) \Big)$$

Now, Put values of $Pr(G) = p'$

$Pr(S) = 1 - p'$ and

then differentiate it w.r.t. g' For $\arg\max_{p'} L_p(p')$. And we can remove $\sum_{i=1}^{NH}$ and $\sum_{i=1}^{N_T}$ by multiplying simply by N_H and N_T respectively

$$= N_H \Big(\gamma(G,H) \cdot \frac{1}{\cancel{\log}\, Pr(H|G) \cdot \rho'} \cdot \cancel{\log}\, Pr(H|G) +$$

$$\gamma(\cancel{G}\, S, H) \cdot \frac{1}{Pr(\cancel{H} H|S) \, (1-\rho')} \cdot Pr(H|S) \Big)$$

$$+$$

$$N_T \Big(\gamma(G,T) \cdot \frac{1}{Pr(\cancel{H} T|G) \, \rho'} \cdot Pr(T|G) +$$

$$\gamma(S,T) \cdot \frac{1}{Pr(T|S) \cdot \cancel{\rho'} \, (1-\rho')} \, Pr(T|S) \Big)$$

$$= N_H \left(\frac{2p}{p+1} \cdot \frac{1}{1/2} \cdot \frac{1}{\rho'} + \frac{1-p}{p+1} \cdot \frac{1}{1/4} \cdot \frac{1}{1-\rho'} \right)$$

$$+$$

$$N_T \left(\frac{2p}{3-p} \cdot \frac{1}{\frac{1}{2}} \cdot \frac{1}{\rho'} + \frac{1-p}{3-p} \, \frac{1}{\frac{3}{4}} \cdot \frac{1}{1-\rho'} \right)$$

$$= N_H \left(\frac{4p}{\rho' (p+1)} + \frac{4(1-p)}{(p+1)(1-\rho')} \right) +$$

$$N_T \left(\frac{4p}{(3-p) \, \rho'} + \frac{4(1-p)}{(3-p)\,(1-\rho')} \right)$$

$$= \frac{N_H \cdot (4p)}{\rho'} \left(\frac{1}{1+p} + \frac{1}{3-p} \right) + \frac{N_T \cdot 4 \cdot (1-p)}{1-\rho'} \left[\frac{1}{p+1} + \frac{1}{3-p} \right]$$

$$= \frac{4 \cdot N_H \cdot p}{p'}\left[\frac{3-p+1+p}{(3-p)(1+p)}\right] + \frac{4 \cdot N_T(1-p)}{1-p'}\left[\frac{3-p+p+1}{(p+1)(3-p)}\right]$$

$$= \cancel{4} \frac{8 N_H \cdot p}{p'(3-p)(1+p)} + \frac{8 N_T(1-p)}{(1-p')(3-p)(1+p)}$$