RECURRENCE RELATIONS

Many of the counting problems cannot be solved using basic counting techniques and hence requires solution by finding relationships, called recurrence relations.

* Note that, sequences can be defined recursively. So, each recurrence relation is connected to a particular sequence.

Definition: A recurrence relation for a sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers $n \geq n_0$, where n_0 is a non-negative integer.

Note A sequence is called a solution of a recurrence relation if its terms satisfy the recurrence relation.

Example

$$a_n = a_{n-1} - a_{n-2} \quad \text{for } n \geq 2$$

with $a_0 = 3$ and $a_1 = 5$.

Each of the recurrence relations has (i) the expression or formula, (ii) the restriction or range and (iii) the initial conditions.

Using the initial conditions and the formula, we can determine the rest of the terms of the sequence.

$$a_2 = a_1 - a_0 = 5 - 3 = 2, \qquad a_3 = a_2 - a_1 = 2 - 5 = -3$$

and so on . . .

Therefore, $\{a_n\} = \{3, 5, 2, -3, \ldots\}$.

Example Determine whether the sequence $\{a_n\}$, where $a_n = 3n$ for every nonnegative integer n, is a solution of the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$, $n \geq 2$. Answer the same when $a_n = 2^n$.

$\rightarrow$ Consider the recurrence relation $a_n = 2a_{n-1} - a_{n-2}$.

Now, for $n \geq 2$ and $a_n = 3n$, we see that,

$2a_{n-1} - a_{n-2} = 2[3(n-1)] - 3(n-2) = 3n = a_n$.

Therefore, $\{a_n\}$, where $a_n = 3n$, is a solution of the recurrence relation.

Suppose that, $a_n = 2^n$, for $n \geq 0$. Then $a_0 = 1, a_1 = 2$ and $a_2 = 4$. Because, $2a_1 - a_0 = 2.2 - 1 = 3 \neq a_2$ and considering the relation $a_n = 2a_{n-1} - a_{n-2}$, we see that $\{a_n = 2^n\}$ is not a solution of the recurrence relation.

Note The recurrence relation and the initial conditions uniquely determine a sequence.

Consider the recurrence relation,

$$a_n = 2a_{n-1} - a_{n-2} \quad , n \geq 2$$

We can check that the following sequences are solutions of the recurrence relation.

(i) $\{0, 3, 6, 9, \ldots\} = \{a_n = 3n\}$

(ii) $\{9, 9, 9, 9, \ldots\} = \{a_n = 9\}$

Modelling with Recurrence Relations

Recurrence relations can be used to model a wide range of problems such as compound interest, counting bit strings, solving Tower of Hanoi problem etc.

$\boxed{\text{Example}}$ The number of bacteria in a colony triples every hour. If a colony begins with five bacteria, how many will be present in n hours?

→ Let, a_n : number of bacteria at the end of n hours

Since the number of bacteria triples in every hour, i.e., $a_n = 3a_{n-1}$, $n \geq 1$ and $a_0 = 5$

$a_1 = 3a_0$; $a_2 = 3a_1 = 3^2 a_0$, $a_3 = 3^3 a_0$ and so on.

Hence, $a_n = 3^n . a_0 = 3^n . 5$

At the end of n hours, we have $(3^n . 5)$ number of bacteria in the colony.

$\boxed{\text{Example}}$ Suppose that a person deposits Rs. 25,000 in a savings account at a bank yielding 12% per year with interest compounded annually. How much will be in the account after 20 years?

→ Let I_n : amount in the account after n years.

Since, the amount after n years.

= amount after (n−1) years + interest of n^{th} year.

i.e., $I_n = I_{n-1} + 0.12 . I_{n-1}$ with $I_0 = 25,000$

$= 1.12\, I_{n-1}$

$\therefore$ $I_1 = 1.12\, I_0 = 1.12 \times 25000$

$I_2 = 1.12\, I_1 = (1.12) \times (1.12) . I_0 = (1.12)^2 I_0$

$\vdots$

$I_n = (1.12)^n I_0 = (1.12)^n \times 25000$

Hence, $I_{20} = (1.12)^{20} \times 25000$.

$\boxed{\text{Example}}$ Find a recurrence relation and give initial conditions for the number of bit strings of length n that do not have two consecutive 0s. How many such bit strings are there of length ten?

→ Let, a_n : number of bit strings of length n that do not have two consecutive 0s.

One can note that, the number of bit strings of length n that do not have two consecutive 0s equals the number of such bit strings starting with 1, plus the number of such bit strings starting with 0.

For the particular problem, we assume that $n \geq 3$, so that the bit string has at least three bits.

The bit strings of length n, starting with 1 that do not have two consecutive 0s are precisely the bit strings of length $(n-1)$ with no two consecutive 0s with a 1 added at the beginning. Consequently there are a_{n-1} such bit strings.

Bit strings of length n starting with a 0, that do not have two consecutive 0s must have 1 as the 2nd bit; otherwise they would end with a pair of 0s. It follows that the bit strings of length n, starting with 0, that have no two consecutive 0s are precisely the bit strings of length $(n-2)$ with no two consecutive 0s with 10 added at the beginning. Consequently, there are a_{n-2} such bit strings.

$$\underbrace{1\ \underbrace{\overbrace{\underline{\qquad\qquad\qquad\qquad}}^{\text{does not contain two consecutive 0s}}}_{(n-1)\text{ length bit string}}}_{n\text{ length bit string}}$$

$$\underbrace{0\ \underbrace{\overbrace{\underline{\qquad\qquad\qquad\qquad}}^{\text{does not contain two consecutive 0s}}}_{n-1\text{ length}}}_{n\text{ length}}$$

may contain two consecutive 0s as the 2nd bit can be 0.

$\underline{0}\ \underline{1}\ \underbrace{\overline{\quad\text{does not contain two consecutive 0s}\quad}}_{n-2 \text{ length}}$

$\underbrace{\qquad\qquad\qquad\qquad\qquad\qquad}_{n \text{ length}}$

		No. of bit strings of length n with no two consecutive 0s
i.e., start with 1 :	any bit string of length(n-1) with no two consecutive 0s	a_{n-1}
start with 0 : 2nd bit 1	any bit string of length(n-2) with no two consecutive 0s	a_{n-2}

Total: $a_n = a_{n-1} + a_{n-2}$

So, we conclude that, $a_n = a_{n-1} + a_{n-2}$, $n \geq 3$

The initial conditions are $a_1 = 2$, because both the 1-length bit strings 0 and 1 do not have two consecutive 0s.
As the valid bit strings of length 2 are 01, 10 and 11, we have, $a_2 = 3$.

So, $a_3 = 3+2 = 5$, $a_4 = 5+3 = 8$ and so on.

Therefore, $a_{10} = a_9 + a_8 = 89 + 55 = 144$.

Solving linear recurrence relations (homogeneous) with constant coefficients

A linear ~~...~~ homogeneous recurrence relation of order k and degree 1 with constant coefficients is a recurrence relation of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \quad \text{—— (A)}$$

where $c_1, c_2, \dots, c_k$ are real numbers and $c_k \neq 0$.

The degree is the highest power of the variable in the recurrence relation.

Examples

(i) $a_n = 2a_{n-1} + a_{n-2}^2$ — Non-linear

(ii) $a_n + a_{n-1} = 3^n$ — Non-homogeneous

(iii) $a_n = 3n^2 a_{n-1}$ — Non-constant coefficients

(iv) $a_n = 3a_{n-1} + 5a_{n-2}$ — degree 1 and order 2

(v) $a_n = a_{n-1}^2 + a_{n-3}$ Non-linear, homogeneous recurrence relation with constant coefficients, of degree 2 and of order 3

The recurrence relation (A) is linear because the R.H.S is a sum of previous terms of the sequence, each having power 1.

(A) is homogeneous because no terms occur that are not multiples of the a_js or any nonzero non-sequential terms.

(A) is a recurrence relation with all the constant coefficients, rather than functions that depend on n.

The order is k because a_n is expressed in terms of the previous k-terms of the sequence.

The degree of (A) is one.

$\boxed{\text{Note}}$ To solve (A), we need k no. of initial conditions.

$\boxed{\text{Characteristic equation of recurrence relation}}$

Consider the recurrence relation (A)

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$$

$c_1, c_2, \dots, c_k$ are real numbers and $c_k \neq 0$.

The basic approach for solving (A) is to look for solutions of the form $a_n = r^n$, where r is a constant.

$\boxed{\text{Note}}$ $a_n = r^n$ is a solution of (A) if and only if

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}$$

Now dividing by r^{n-k}, we obtain,

$$\frac{r^n}{r^{n-k}} = c_1 \frac{r^{n-1}}{r^{n-k}} + c_2 \frac{r^{n-2}}{r^{n-k}} + \dots + c_k \frac{r^{n-k}}{r^{n-k}}$$

$$\Rightarrow \boxed{r^k = c_1 r^{k-1} + c_2 r^{k-2} + \dots + c_k}$$

characteristic equation of the recurrence relation

▪ The characteristic equation is a polynomial of degree k.

$\boxed{\text{Theorem 1}}$ Let $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_k = 0$ is the characteristic equation with constant coefficients and k number of distinct roots $r_1, r_2, \dots, r_k$. Then a_n is the solution to the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k}$, if and only if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \dots + \alpha_k r_k^n$ for $n = 0, 1, 2, \dots$ and $\alpha_1, \alpha_2, \dots, \alpha_k$ are constants.

$\boxed{\text{Example}}$ Find the solution to the recurrence relation $a_n = 6a_{n-1} - 11a_{n-2} + 6a_{n-3}$, $a_0 = 2$, $a_1 = 5$, $a_2 = 15$.

→ Replace $a_k = r^k$ in the given ~~eq~~ recurrence relation. Then the characteristic equation,

$$r^k = 6r^{k-1} - 11r^{k-2} + 6r^{k-3}$$

Dividing by r^{k-3}, we obtain,

$$r^3 = 6r^2 - 11r + 6$$

$$\Rightarrow r^3 - 6r^2 + 11r - 6 = 0 \quad [\text{Characteristic eq}^n]$$

$$\Rightarrow (r-1)(r-2)(r-3) = 0$$

$$\therefore r_1 = 1,\ r_2 = 2,\ r_3 = 3 \quad (\text{all roots are distinct})$$

Then, the solution can be written as,

$$a_n = \alpha_1 (r_1)^n + \alpha_2 (r_2)^n + \alpha_3 (r_3)^n$$

Now, for $a_0 = 2$, we obtain,

$$\alpha_1 (r_1)^0 + \alpha_2 (r_2)^0 + \alpha_3 (r_3)^0 = 2$$

$$\Rightarrow \alpha_1 + \alpha_2 + \alpha_3 = 2$$

Applying the remaining initial conditions $a_1 = 5$ and $a_2 = 15$, we obtain,

$$\alpha_1 + 2\alpha_2 + 3\alpha_3 = 5$$

$$\alpha_1 + 4\alpha_2 + 9\alpha_3 = 15$$

Solving the system of linear eq^{n}s we have,
$\alpha_1 = 1$, $\alpha_2 = -1$, $\alpha_3 = 2$
Therefore, the solution, $a_n = 1 - (2)^n + 2.(3)^n$.

Example Solve the recurrence relation
$$a_n = a_{n-1} + 2a_{n-2} \quad , \quad a_0 = 2, a_1 = 7$$

→ Characteristic eq^n of the given recurrence relation
$$r^2 = r + 2 \Rightarrow r^2 - r - 2 = 0$$
$$\Rightarrow (r-2)(r+1) = 0$$
$$\therefore r_1 = 2, r_2 = -1$$
Hence, $a_n = \alpha_1 (2)^n + \alpha_2 (-1)^n \quad , n = 0, 1, 2, \ldots$

Applying the initial conditions,
$$a_0 = 2 \Rightarrow \alpha_1 2^0 + \alpha_2 (-1)^0 = 2$$
$$\Rightarrow \alpha_1 + \alpha_2 = 2 \quad \text{——— (I)}$$
$$a_1 = 7 \Rightarrow \alpha_1 2^1 + \alpha_2 (-1)^1 = 7$$
$$\Rightarrow 2\alpha_1 - \alpha_2 = 7 \quad \text{——— (II)}$$
Solving (I) and (II) we obtain the solution,
$$a_n = 3.(2)^n - 1.(-1)^n.$$

Theorem 2 Let $r^k - c_1 r^{k-1} - c_2 r^{k-2} - \ldots - c_k = 0$ be the characteristic eq^n with constant coefficients $c_1, c_2, \ldots, c_k$ and $c_k \neq 0$, have t repeated roots $r_1, r_2, \ldots, r_t$ with multiplicity $m_1, m_2, \ldots m_t$, such that $m_i \geq 1$ and $m_1 + m_2 + \ldots + m_t = k$. Then a sequence $\{a_n\}$ is a solution of the recurrence relation $a_n = c_1 a_{n-1} + c_2 a_{n-2} + \ldots + c_k a_{n-k}$ if and only if
$$a_n = (\alpha_1 + \alpha_2 n + \alpha_3 n^2 + \ldots + \alpha_{m_1 - 1} n^{m_1 - 1})(r_1)^n$$
$$+ (\beta_1 + \beta_2 n + \ldots + \beta_{m_2 - 1} n^{m_2 - 1})(r_2)^n + \ldots$$
$$+ \ldots + (\xi_1 + \xi_2 n + \ldots + \xi_{m_t - 1} n^{m_t - 1})(r_t)^n$$
for $n = 0, 1, 2, \ldots$ and $\alpha_i, \beta_i, \ldots \xi_i$ are constants.

Example Solve $a_n = -3a_{n-1} - 3a_{n-2} - a_{n-3}$

$a_0 = 1, \; a_1 = -2, \; a_2 = -1$

→ Replacing a_n by r^n we obtain,

$$r^n = -3r^{n-1} - 3r^{n-2} - r^{n-3}$$

Dividing by r^{n-3}, we have,

$$\frac{r^n}{r^{n-3}} = -3\frac{r^{n-1}}{r^{n-3}} - 3\frac{r^{n-2}}{r^{n-3}} - \frac{r^{n-3}}{r^{n-3}}$$

$$\Rightarrow r^3 = -3r^2 - 3r - 1$$

$$\Rightarrow r^3 + 3r^2 + 3r + 1 = 0 \Rightarrow (r+1)^3 = 0$$

$$\therefore r = -1, -1, -1$$

Solution: $a_n = (\alpha_1 + n\alpha_2 + n^2\alpha_3)(r)^n$

$= (\alpha_1 + n\alpha_2 + n^2\alpha_3)(-1)^n$

Applying I.C.s, $\alpha_1 = 1, \alpha_2 = 3, \alpha_3 = -2$.

Hence, the solution is, $a_n = (1 + 3n - 2n^2)(-1)^n$ for $n = 0, 1, 2, \ldots$

Example If roots of a linear homogeneous recurrence relation are 2, 2, 3, 5, 5 and 9 then the form of the general solution is

$$a_n = (\alpha_1 + \alpha_2 . n)(2)^n + \beta_1(3)^n + (\gamma_1 + \gamma_2 . n)(5)^n + \delta_1(9)^n.$$

where, $n = 0, 1, 2, \ldots$ and $\alpha_1, \alpha_2, \beta_1, \gamma_1, \gamma_2$ and δ_1 are constants. (can be determined using I.C.s)

Solving recurrence relations using substitution methods

Forward Substitution Method

Consider the recurrence relation

$$a_n = 2a_{n-1} \quad , n \geq 1, \; a_0 = 3 \quad \text{— (B)}$$

for $n = 1$, $a_1 = 2a_0 = 2.3$

for $n = 2$, $a_2 = 2a_1$ (using the recurrence relation)

$= 2.2a_0$

$= 2^2.3$

for $n=3$, $a_3 = 2a_2 = 2^3 . a_0 = 2^3 . 3$

$\vdots$

$a_n = 2^n . a_0 = 2^n . 3$

Hence, $\{a_n = 2^n . 3\}$ is a solution of the given recurrence relation.

$\boxed{\text{Backward Method}}$

Consider the previous recurrence relation given in (B).

$$\begin{aligned} a_n &= 2a_{n-1} \\ &= 2 . 2\, a_{n-2} = 2^2 a_{n-2} \quad \text{(using recurrence relation)} \\ &= 2^3 a_{n-3} \\ &\vdots \\ &= 2^n a_0 = 2^n . 3 \end{aligned}$$

Hence, $\{a_n = 2^n . 3\}$ is a solution of (B).

$\boxed{\text{Note}}$ Solution of a recurrence relation using forward substitution method is to initiate the method with the initial conditions (a_0) and obtain the value of a_n in terms of the I.C.s.

for the backward substitution method, the method requires to express the value of a_n in terms of previous terms and proceed to obtain the expression of a_n in terms of the I.C.s.

Forward Method	Backward Method
$a_0 \longrightarrow a_n$	$a_n \longrightarrow a_0$

- Using Forward, Backward and the characteristic root method, we obtain the expression of a_n in closed form formula, that does not contain any previous sequence term.

Generating Function

Generating functions can be used to solve recurrence relations by translating a recurrence relation for the terms of a sequence into an equation involving a generating function. This equation can then be solved to find a closed form for the generating $f^{\underline{n}}$. From the closed form, the coefficients of the power series for the generating $f^{\underline{n}}$ can be found, solving the original recurrence relation.

Definition The generating $f^{\underline{n}}$ for the sequence $a_0, a_1, \ldots a_k, \ldots$ of real numbers is the infinite series $G(x) = a_0 + a_1 x + \ldots + a_k x^k + \ldots = \sum_{k=0}^{\infty} a_k x^k$

* The present form of the generating $f^{\underline{n}}$ often called the ordinary generating $f^{\underline{n}}$ of $\{a_k\}$.

Example

(i) $\{a_k = 3\}$

The generating $f^{\underline{n}}$

$$G(x) = \sum_{k=0}^{\infty} a_k \cdot x^k = \sum_{k=0}^{\infty} 3 \cdot x^k$$

(ii) $\{a_k = 2^k\}$

The generating $f^{\underline{n}}$

$$G(x) = \sum_{k=0}^{\infty} a_k \cdot x^k = \sum_{k=0}^{\infty} 2^k \cdot x^k$$

(iii) $\{a_k = k+1\}$

The generating $f^{\underline{n}}$

$$G(x) = \sum_{k=0}^{\infty} a_k \cdot x^k = \sum_{k=0}^{\infty} (k+1) \cdot x^k$$

Note Consider the finite sequence $\{3, 2, 1, 5, 6\}$. We can set the finite $seq^{\underline{n}}$ to infinite $seq^{\underline{n}}$ by setting $a_5 = a_6 = \ldots = 0$ and $a_0 = 3$, $a_1 = 2$, $a_2 = 1$, $a_3 = 5$, $a_4 = 6$.

Then the corresponding generating $f^{\underline{n}}$ is,

$$G(x) = \sum_{k=0}^{\infty} a_k x^k \quad \text{such that } a_5 = a_6 = \ldots = 0$$

$$= 3 + 2x + x^2 + 5x^3 + 6x^4$$

Then for the finite sequence $\{a_k\} = \{a_0, a_1, \dots a_n\}$ for fixed n, the corresponding infinite $seq^{\underline{n}}$ can be represented as the generating $f^{\underline{n}}$

$$G(x) = a_0 + a_1 x + \dots + a_n x^n$$, such that $a_{n+1} = a_{n+2} = \dots = 0$, is a polynomial of degree n.

$\boxed{\text{Example}}$ Find the generating $f^{\underline{n}}$ of $\{1,1,1,1,1,1\}$

$$\longrightarrow G(x) = 1 + 1.x + 1.x^2 + 1.x^3 + 1.x^4 + 1.x^5$$
$$= 1 + x + x^2 + x^3 + x^4 + x^5$$
$$= \sum_{n=0}^{5} x^n = \frac{x^6 - 1}{x - 1}$$

$\boxed{\text{Example}}$ Find the generating $f^{\underline{n}}$ in closed form of the sequence $\{0,1,0,0,1,0,0,1, \dots\}$

$$\longrightarrow G(x) = \sum_{k=0}^{\infty} a_k x^k \quad [a_0 = 0, a_1 = 1, a_2 = 0, a_3 = 0, a_4 = 1, \dots]$$
$$= 0 + 1.x + 0.x^2 + 0.x^3 + 1.x^4 + 0.x^5 + 0.x^6 + 1.x^7 + 0.x^8 + 0.x^9 + 1.x^{10} + \dots$$
$$= x + x^4 + x^7 + x^{10} + \dots$$
$$= x(1 + x^3 + x^6 + x^9 + \dots)$$
$$= \frac{x}{1 - x^3}, \quad |x| < 1$$

$\boxed{\text{Example}}$ Find the generating $f^{\underline{n}}$ of $\{1,2,3,4, \dots\}$

$$\longrightarrow G(x) = \sum_{k=0}^{\infty} a_k x^k \quad [a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4, \dots]$$
$$= 1 + 2x + 3x^2 + \dots$$
$$= \frac{1}{(1-x)^2}$$

Solving recurrence relations using generating functions

Consider the sequence $\{a_K\}$ and the corresponding generating f^{n}. $G(x) = \sum_{K=0}^{\infty} a_K . x^K$.

Now, $x . G(x) = x . \sum_{K=0}^{\infty} a_K . x^K = \sum_{K=0}^{\infty} a_K x^{K+1}$

$x^2 . G(x) = x^2 . \sum_{K=0}^{\infty} a_K . x^K = \sum_{K=0}^{\infty} a_K x^{K+2}$

⋮

and so on.

Example Solve the recurrence relation

$$a_K = 3a_{K-1}, \quad K \geq 1, \quad a_0 = 2$$

→ Consider the corresponding generating f^{n} of the seq^{n} $\{a_n\}$ as, $G(x) = \sum_{K=0}^{\infty} a_K x^K$.

$$= a_0 + a_1 x + a_2 x^2 + \dots$$

$$= a_0 + \sum_{K=1}^{\infty} a_K x^K$$

Now, by substituting $a_K = 3a_{K-1}, K \geq 1$, in the expression of $G(x)$ we obtain,

$$G(x) = a_0 + \sum_{K=1}^{\infty} (3a_{K-1}) . x^K$$

$$\Rightarrow G(x) - a_0 = 3 \sum_{K=1}^{\infty} a_{K-1} x^K$$

$$\Rightarrow G(x) - 2 = 3(a_0 x + a_1 x^2 + a_2 x^3 + \dots)$$

$$= 3x(a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= 3x . \sum_{K=0}^{\infty} a_K x^K$$

$$= 3x . G(x)$$

$$\Rightarrow G(x) - 3x G(x) = 2$$

$$\Rightarrow G(x) = \frac{2}{1-3x}$$

$$= 2(1-3x)^{-1}$$
$$= 2(1 + 3x + (3x)^2 + (3x)^3 + \dots)$$
$$= 2\sum_{K=0}^{\infty} 3^K . x^K$$
$$= \sum_{K=0}^{\infty} (2.3^K). x^K$$

Comparing with $G(x) = \sum_{K=0}^{\infty} a_K x^K = \sum_{K=0}^{\infty} (2.3^K). x^K$,

we obtain, $\boxed{a_K = 2.3^K}$

$\boxed{\text{Example}}$

Solve the recurrence relation using generating f^n.

$$a_K = 2a_{K-1} + 3a_{K-2}, \quad K \geq 2, \; a_0 = 3, a_1 = 1$$

→ The corresponding generating f^n for the sequence $\{a_n\}$ is defined as,

$$G(x) = \sum_{K=0}^{\infty} a_K x^K$$

We consider the relation

$$a_K = 2a_{K-1} + 3a_{K-2}$$
$$\Rightarrow a_K - 2a_{K-1} - 3a_{K-2} = 0$$
$$\Rightarrow a_K x^K - 2a_{K-1} x^K - 3a_{K-2} x^K = 0$$

[Multiplying by x^K].

Since $K \geq 2$, summing for $K = 2, 3, \dots$, we get,

$$\sum_{K=2}^{\infty} a_K x^K - 2\sum_{K=2}^{\infty} a_{K-1} x^K - 3\sum_{K=2}^{\infty} a_{K-2} x^K = 0 \quad \text{—— (A)}$$

Solving individually,

1st term : $\sum_{K=2}^{\infty} a_K x^K = a_2x^2 + a_3x^3 + \dots$

$= (a_0 + a_1x + a_2x^2 + \dots) - (a_0 + a_1x)$

$= \sum_{K=0}^{\infty} a_K x^K - (a_0 + a_1 x)$

$= G(x) - (a_0 + a_1 x)$ —— (1)

2nd term: $2\sum_{k=2}^{\infty} a_{k-1} x^k = 2a_1x^2 + 2a_2x^3 + \ldots$

$= 2x(a_1x + a_2x^2 + \ldots)$

$= 2x(a_0 + a_1x + a_2x^2 + \ldots) - 2xa_0$

$= 2x \cdot G(x) - 2xa_0$ ——— (ii)

3rd term: $3\sum_{k=2}^{\infty} a_{k-2} x^k = 3a_0x^2 + 3a_1x^3 + \ldots$

$= 3x^2(a_0 + a_1x + a_2x^2 + \ldots)$

$= 3x^2 G(x)$ ——— (iii)

Substituting (i), (ii) and (iii), in (c), we obtain,

$$G(x) - a_0 - a_1x - 2xG(x) + 2xa_0 - 3x^2G(x) = 0$$

$$\Rightarrow G(x) - 2xG(x) - 3x^2G(x) = a_0 + a_1x - 2xa_0$$

$$\Rightarrow G(x)[1 - 2x - 3x^2] = 3 + x - 6x$$

$$\Rightarrow G(x) = \frac{3-5x}{1-2x-3x^2}$$

$$= \frac{3-5x}{(1-3x)(1+x)}$$

$$= \frac{A}{(1-3x)} - \frac{B}{(1+x)}$$

where A and B are to be determined.

Now, $G(x) = \dfrac{A(1+x) - B(1-3x)}{(1-3x)(1+x)} = \dfrac{3-5x}{(1-3x)(1+x)}$

Then equating the numerators, we obtain,

$$A(1+x) - B(1-3x) = 3-5x$$

$$\Rightarrow (A-B) + (A+3B)x = 3-5x.$$

Hence, $A - B = 3$ and $A + 3B = -5$.

solving the eq$^{\underline{n}}$s, we obtain

$$A = 1, \quad B = 2$$

Therefore, $G(x) = \frac{1}{1-3x} + \frac{2}{1+x}$.

$$= (1-3x)^{-1} + 2(1+x)^{-1}$$

$$= (1 + 3x + (3x)^2 + (3x)^3 + \dots\dots)$$

$$+ 2(1 - x + x^2 - x^3 + \dots\dots)$$

$$= \sum_{k=0}^{\infty} 3^k . x^k + \sum_{k=0}^{\infty} 2.(-1)^k x^k$$

$$= \sum_{k=0}^{\infty} [3^k + 2.(-1)^k] x^k$$

Now, $G(x) = \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} (3^k + 2.(-1)^k) x^k$

i.e., $\{a_k = 3^k + 2.(-1)^k\}$, $k = 0, 1, 2, \dots$