RECURRENCE RELATIONS

Many of the counting problems

basic

counting

usin
g

(1)

cannot be solved

counting techniques and hence requires solution by finding relationships, called relations.

recurrence

Note that, sequences can be defined recursively.

So, each

recurrence relation is

particular sequence.

connected to a

[Definition: A recurrence relation for a sequence {ant

an equation that

expresses

is

an

in terms of

no

one or more of the previous terms of the sequence

namely, ao, as an-1 for all integers n' no,

where

[Note

no

an

is

a

non

A sequence

recurrence

relation.

negative integer.

is called a solution of a

relation if its

terms satisfy the recurrence

an-2

for n=2

Example

an

= An-1

with

ao

3

and an

5

Each

of

the

relations

11

the

VID the

or formula,

recurrence

has C the

expression

restriction or range

initial conditions.

Using the initial conditions and

determine

the rest

of

can

92

= a 1

ал

ао

3

5

八

2

and

So

a

3

and

the formula, we

the sequence. the terms of

=

=

a 2

2

=-3

on.

Therefore, $\{a_n\} = \{3, 5, 2,$

$-3,$

a_n

$- 5$

$\}$

2

[Example Determine whether the sequence $\{a_n\}$, where @ for every nonnegative integer n, is a solution recurrence relation

of

$a_n = 3n$

the

Answer the same

consider the

572

Now, for

$2a_{n-1}$

$-a_{n-2}$

=

when

a_{n-2}, n

$-2,5x2.$

$a_n = 2a_{n-1}$

$a_n = 2n$

$a_n = 2a_{n-1} - a_{n-2}.$

we

see that,

recurrence relation

and

$a_n = 3n$

$2[3(n-1)] - 3(n-2) = 3n = a_n.$

Therefore, $\{a_n\}$, where

the

recurrence

Suppose that,

$a_n = 3n$, is

relation.

a solution of

for $n \geq 0$. Then $a_0 = 1, a_1 = 2$

$a_2 = 4$. Because, 201

гал

relation

$a_n =$

2n

and

and

we

see

that

the

considering

the

$\{a_n = 2^n\}$ is

relation.

[Note] The

$a_0 = 2.2-1=3792$

a_n

$= 2a_{n-1} - a_{n-2}$,

not a

solution of

recurrence

recurrence

relation and the initial

a sequence.

Conditions uniquely determine

Consider the recurrence relation,

$a_n = 2a_{n-1}$

a_{n-}

2

check

He

can

are

solutions

▪ { 0, 3, 6, 9,

1 £9, 9, 9, 9,

Modellin

g

Recurrence

2572

that the following sequences of the

recurrence relation.

} $= \{a_n = 3n\}$

with

Recurrence

relations

can

wide range of

interest, counting of

Hanoi problem

{ an

be

=

تم و

Relations

used to model a

as compound

such

problems

bit strings

strings, solving

Tower

etc.

Let,

an

the

hour

i...,

Example The number of bacteria in

a colony triples

every hour. If a colony begins with five bacteria,

how

be present in

many

Since

will

number

of

number

an

of

n hours?

bacteria

$3a_{n-1}$

of

nhours

triples in every

and $a_0 = 5$

bacteria at the end

n = 1

aл

=3a0 зао

2

a2

= зал

= 3

3290

93=

23 00

33

and som.

ao

Hence,

an

зп

3nao

ao

of

of

n

hours,

we

the

At the end

9 bacteria

in

Example Suppose

that in a savings account

year with interest

will

be

in

Let

the

=3".5

colony.

at a

have (35) number

a person deposits Rs. 25,000 bank

yielding 12%. per compounded

annually. How much account after

20 years?

the account after n

years.

I_n : amount in

Since, the

ire.

I_n

amount after

n years.

amount after $(n-1)$ years + interest of th

I_{n-1}

$+0.12 . I_{n-1}$ with

$1.12 \, I_{n-1}$

1. $I_1 = 1.12 \, I_0$

I_2

$= 1 \cdot 12 \, I_1$

year.

$L_0 = 25{,}000$

$1.12 \times 25\,000$

$(1+12) \times (1+12).$

$I_0 = (1+12)^2 \, I,$

Hence,

In

$= (1 \cdot 12)$ " Io $(1+12)$

x 25000

=

120

=

$(1+12)$ 20 x

25000

a

[Example Find

recurrence relation and give

initial conditions for the number of bit strings of

length that do not have two

bit strings

Let,

n

are

n

consecutive *Os*. How many such

there of length ten?

number of bit

strings of

an : number

that do not

have

two

lengt

h

n

consecutive

os.

bit

strings of

length

One can note that, the number of

that do not have two consecutive os equals

such bit strings starting with 1,

plus number of such bit strings starting with

0.

the number

the

oF

For the particular problem,

5

that

the

The

do

we

bit strings of length n

not

assume that n7,3,

bit string has at least three

bits.

of length n, starting with ↑

that

Os are precisely the

two consecutive

have two

consecutive

bit strings of length (n-1) with no

1

OS

with

a

there

are

an-1

added at the beginning.

Consequently

bit strings.

such

of length n starting with a 0,

that do

strings of length

n

Bit Strings

have two consecutive Os

bit; otherwise they would

n

ot 2nd

a

must have 1 as the would beco end with

bair of Os. It follows that the bit strings of length in starting with

consecutive

Os

)

that have no two

are precisely the bit strings of length (n-2) with

added

An-2

no

two

consecutive

Os with 10

at the beginning. Consequently, there are

bit strings

such

1

does not contain two consecutive os

n

(n-1) length bit string

length bit string

I does not contain two consecutive Os

n-1 length

in length

may contain two

consecutive Os

as the bit can be o

does not contain Fibo consecutive o

n-2 length

01

n length

ire.,

start with 1:

any bit string of

length(n-1)

with no two consecutive OS

start with 0:12hd bit | Jany bit

string of length (n-2)

pn

d

イ

with no two consecutive Os

an =

So, we

conclude that,

are

The initial conditions

both

the

two

have

As the valid

an-1

No.

5

bit

strings of

length n with no

two consecut

OS

an−1

an-2

Total": an=an-tan-2

+an-2

573

aя

a1 = 2

= 2, because

1

do not

are

01,10 and

1-length bit strings o and

consecutive

Os,

bit strings of length

2

11, we have, a1 =3!

So,

aз

93=

3+2

5

-

94 =5+3=8

and

So

on .

= 89 +55 144.

Therefore, a 10 = ag + ag

Solving linear recurrence relations (homogeneous) with constant coefficients

A

degree 1

b

order

орко

recurrence

and Ghear homogeneous recurrence relation

of

constant coefficients is of the

form

к

L

with

relation.

where

An

C1, C2

Cля

$$= CA\ a_{n-1} + C_2\ a_{n-2}$$

+

a

$+C_k a_{n-k}$

A

, .Ck are real numbers and Cx70.

The degree is the highest power of the variable in the

Examples

2

an

$= 2 a_{n-1}$

$+ a_{n-2}$

"

an

$+ a_{n-1}$

$a_n = 3 n^2 a_{n-1}$

IV

an

recurrence relation.

Non-Linear

Non-homogeneous Non-constant coefficients

degree 1 and

$3a_{n-1}$

$+5a_{n-2}$

order 2

$\sqrt{a_n} = a_n = 1^2$

The

R. H.S

$+a_{n-3}$

recurrence relation

is

a

Å

6

Non-linear, homogeneous recurrence relation with Constant coefficients, **and** of degre ree 2 and of order 3

is linear because the

sum of previous terms of the

sequence,

each having power 1.

(A) is

are

homogeneous because

not multiples

nonzero

A

is

of the

occur, that

does not contain

no

terms

ajs

non-sequential terms.

ork bag any

a recurrence relation with all the constant

coefficients,

rather

order

than functions that depend

because

on n.

an is expressed in

k- ferms of the sequence.

To solve (A)

is

K

The a

ferms

of

the

previous

The degree

of

'A'

i's one.

Note

we

need

k no. of

initial

conditions.

of

of recurrence

relation

A

Characteristic equation

recurrence relation

$+ C_2 a_{n-2} +$

$+ C_k a_{n-k}$

real numbers and C_K £0.

basic approach for solving

Consider

the

$a_n =$

$C_1 a_{n-1}$

9, C_2,

c_k

are

The

Solutions

Constant.

Note

an =rn is

rn

=

of the form

a

Now dividing by

介

ran

an

Σ

a solution of

+ C2rn−2 +

(A

is to look for

where

ris a

A if and only if

+ Ckrn-k

obtain,

Cz

п-к

не

Jon-1

=

a

рп-к

bon-k

+ C2 jan-2

+.

pn-k

+CK=

n-k за

rn-k

rk

a = Grk-1

K-2

+

C2

за

+

+CK

characteristic equation of the recurrence relation

an is

an

number of

the

distinct roots

The characteristic equation is a polynomial of degree K.

K

[Theorem 1 Let rk-Grk -

1_c2rk-2

Characteristic equation with constant coefficients and

K

59,82,

-CK=0 is the

rk. Then

solution

to therecurrence relation

=

C1 an-1

+ c 2 an-2 +

TCK aп-K

, if and only if

an

for

n = 0, 1 1, 2,

$= 211n + 2282n + .$

and

$+ \ a\ krm$

K

n

da,X2,..,KK are constants.

Example Find the solution to the recurrence relation

an =

6an-1

Replace

Then

Comp

11 an-2 +6 an-3,90=2, 94 = 5, 92 =15.

ak =rk in the given

recurrence relation

the characteristic

equation,

rk = 6rk-1 -118 K-2

Dividing by

r

-3

rk-3

we

precurre

nce

+6rk-3

obtain,

$= 6r2-11+6$

$\Rightarrow \sqrt{3}-682^{2} + 118-6=0$

[Characteristic

29"]

82

2, 83=3

Call roots are distinct)

$\rightarrow (8-1) \quad (8-2)(r-3)=0$

Then

the

an

solution

Can

be written as,

<1(ra)" + d2 (82)

"+dz (rz)"

Now, for 90 =
2,

=2, we

obtain,

<1 (r1) ° + d2 (82)°+ α3
(83) ° = 2

Xq + x2 + x 3 = 2

Applying the remaining initial conditions

a1 =5

and

92 = 15
az

we
,

obtain,

21+2×2+323 = 5

21 + 422 + 923 = 15

Solving the system of linear eqpes

<1 = 1
Да

Th

erefore

Example

$\not\propto 2$

$= -1$

the

we

have,

$\underline{2}23=$

solution, $a_n = 1 - (2)^n + 2.(3)^n$

Solve the

recurrence

$a_n = a_{n-1} + 2a_{n-2}$

relation

$a_0 = 2, \quad a_1 = 7$

given recurrence relation

Characteristic eqn of the given

Hence,

$r^2=8+2$ चे

ね

js $2-p-2 = 0$

$= (r-2)(r+1)=0$

$81=2,\ 52$

$\sqrt{2} = -1$

$an = xq(2)" + x2$

$(-1)".$

Applying the initial conditions,

$n = 0, 1, 2,$

$21\ 20 + \alpha 2 (-1)^{\circ}$

$= 2$

ख

$21 + 22$

$= 2$

$\times 1\ 21 + \alpha 2 (-1)\ 1$

$= 7$

$\wedge(-1)1$

$221-22 = 7$

1

20

we

obtain

$3.(2)" - 1.\ (-1)".$

ao

90 = 2

⇒

a1 = 7 ал

>7

Solving D

and

an =

Theorem 2

Let

C1, C2,

Cк and

roots

) r

Such

rk-ark-1-c2r"

ľ'

11

the solution,

K-2

·CK = 0 be

repeated

the characteristic eq" with constant coefficients

mt,

Then

ck 70

• have

t

with multiplicity m1, m2...

that mix 1 and

a sequence {an} is

recurrence

if and

relation

and only

only if

an

an = (α1+α2n+x3n2+

2

to

+ (Bat B2n +

+ . .

2

a

ma+m2t . . +ME=k.

solution

of

the

= Gan-n+Gan-2+. . FCK 9n-w

+

Bm2-1

ท

тл

^) (r

) "

nm2-1) (^2)^2 +

...

ท

Mt-

. + C & 1 + 2 + -

+ m2 - n"=-1)

(rz)"

for n=0,1,2,

and Li, Bi,-ği are

Constants

Example Solve

Replacin

g

an =

9 0 = 1, a

an by pn=-3xn-1

Dividing by

son on

23-3

2n-3

rn

3an-1-3an-2-an-3

2 a

A 2

=-1

we obtain,

-3∞n-2

3

r

382

-ph-3

we have,

rn-3

35-1

+382 +31 + 1 = 0

203

3

10

= 1,-1,-1

Solution!

3

д

n-2

ron-3
ねり
ph-3 rn-3

(x+1)3=0

an = (21 +n22+
n2αz) (r)"
= (α1+ na2+n2α3)
(-1)"

Applying I.

C.S

Hence, the solution is

Example If roots of a
relation
are

× 1 = 1, α2 = 3, dz=-2
22

an= (1+3n-2n2)

(−1)"

for n=0, 1, 2,

(9

linear homogeneous recurrence 2,2,

3, 5, 5 and 9 then the form

of the general solution

is

an = (x1+α2in) (2)" +B1 (3)" +

(81 + 82. n) (5) + S1 (9)"

') (3)" +

(81+82.n) +8,

where, n = 0, 1, 2,

бл

Solvi
ng

a re

1

and 21, 22, B1, 81, 8, and

∝

,

2

constants, (can be determined using I.C.S)

recurrence relations using substitution methods

Forward substitution Method

Consider the

n =

1

for n for n = 2

recurrence

an

3

2an-1

ал

92

=

200

= 291

relation

,571, 90 = 3

B

= 2.29 o

222.3

,

2.3

Cusing the

recurrence

relation)

for n=3, az

= 292 =

= 23.00 = 23.3

A 3

an

n

27.90 = 2", 3

Hence, {an = 2.37

is

recurrence

Backward

relation.

Method

a solution of the given

previous recurrence relation given

inR

Consider

the

$a_n = 2a_{n-1}$

$= 2a_n -$

$= 2.2\ 2\ A_{n-2} = 2\ a_{n-2}^{2}$

Cusing

recuror

relation)

1=

$2^3 a_{n-3}$

n

2.8

2

a o

a solution

Hence, $\{a_n =$

$2, 3\}$ is

Nobel Solution of

a

of

(B)

recurrence relation using

forward substitution method is to initiate the method the initial conditions (ao) and obtain

in terms of

with

the value of

for the

an

the I.C.S.

backward substitution method, the method

requires to express the value of an in terms of previous terms

obtain the expression

of an

and proceed to

in terms of

Forward Method

Co the

I.C.S.

Backward Method

ao

→ an

Using

Forward

Backward

root

method,

we

an

contain

in

an

→ 90

and the characteristic

obtain the expression of

closed form formula that does not

any previous sequence term.

Generat

ing

Function

Generating

functions can be relations

by translating

the terms of

a

(↑

used to solve recurrence

relation for

a recurrence

n

a sequence into an equation involving generating function. This equation can then be solved to find a closed form for the generating fr From the closed form, the coefficients of the power series for the generating fr. can be found, solving the original recurrence relation.

[Definition The generating for for the sequence aosu...

ак,

series

The called

is the infinite

of

real numbers

+axxx+... = Zax xx

$G\ (x) = ao$+agat.. takx

K20

ак x

present form of the generating for often

ordinary generating for. of {ak}.

the

3}

Example C

{ax = 3 } The

generating fr

$G(x) = \Sigma a_k \cdot$

$K=0$

=

$\Sigma \varepsilon \; 3, \chi$

K20

Note

We

can

k

$\{ a_x = 2k \}$ The generating

fo $G(z) = [a_k . x^k$

N

2

K20

Z2k2k

K20

+

@

fak=kneh

The

generating fr

ال

2

K20

$$=\Sigma(k+1).x$$

2

K20

Consider the finite sequence {

3,2,1,5,6}

set the finite sequ

by setting a 5 = 96 =

=

a2

Then

1, 93

the

5,

a4

= 6.

to infinite seq

=0 and 90 = 3, a = 2,

corresponding generating

for is,

$G(x) =$

=

I ak sike

R=0

2

Nα

such that a5 =α6=-=C

95296

$3 + 2x + x2 + 5x3+6x4$

2

Then for the finite sequence {ak}={90, 91,

and 12 for fixed n, the

• corresponding infinite

sea?+ represented as the

generating f?.

can

be

$G(x) =$

a o + an x +

+aqx+

an +1

= an+2

n.

=

+anan, such that

= 0

is

a

polynomial

of

$\{1,1,1,1,1,1$

$\}$

of degree

Example find the

generating f

$G(x)$

Example! Find

Find

oF

1 +1.2 + 1. x2

+

1 + x + x2 +

z xn

=

n=0

1. x 3 + 1. x 4 + 1.

x5

3

x 3 + x 1 +25

+24+25

x

6

-1

x 1

ぬの

the generating f" in

closed form the sequence

{0,1,0,0,1,0,0,1,

Zi ax x K

G(x)

Example

M

14

K=0

aK

*

}

K [90=0, 91 = 1, 92 = 0,

a3 = 0, 94 = 1,

*

,

*

0 + 1. x + 0. x2 +0.x2 + 1.x2 +0.25 +0. x2

+ *1. x*

9 +0. x8 +0 . x

4

$= x + x$ ̄ ̄

=4

$G(x)$

$+ 1,$

1.x

7

10

$+ x$

$$x(1 + x^3 + x^6 + x^9$$

x

$$x \, 1-x^3$$

find

+

+

10+

$1 \; x < 1$

$>$

7

the generating for of

$\{1, 2, 3, 4, ...\}$ [90 = 1,

$a1 = 2$, $a2 = 3$, $93 = 4, ...$]

cκ

Σ ak xk

κ 20

aκ x

2

$1 + 2x + 3x2 +$

1

$(1-x)2$

2

g

Consider

the

generating

fr

sequence

{an)

G(x) = Σak.xx

Solving recurrence relations using generating functions

aκ

(13)

and

the

correspondin

Now, x. G (x)

44

x2. G (x)

N

·K20

ακ

Zak x. Zi ak.

K 20

2

x2.

Σ axixk

K=0

$\Sigma\alpha$ =

Σaxxx+

K20

Z x

aK

K=0

K+1

K+2

and

So

on.

Example

Solve

the

зак-1

of

we

aκ

recurrence relation κ > 1,

ao

A0 = 2

Consider the corresponding generating the seq" {an} as, G(x) = 21 ak xk

by substituting

K=0

= ao + a1 x + α 2 x2+ . . = ao + Σi ak

K21

aκ

x

K

Now, R AK = 3AK-1 in the

κ

3ak-1 KM, in the expression

expression of G(2)

aк

obtain,

a o

$$G(x) = a_0 + \sum_{K=1}^{\infty} (3\alpha k-1), ak$$

$\longrightarrow G(x)-a0$

= 3

AK-1

K x

K21

$\Rightarrow G(x)-2$

– 3 Cao xa x tan 234

N

JV

HM

$$3x\,(\alpha 0 + \alpha 1\, x + a2\, x2^{2} +$$

3x . 3x

a

Дакак

K20

3x, G(x)

G(x) - 3xG(x) = 2

2

ت

=>

G (x) =

1-3x

2

= 2(1-3x)=

= 2 (1 + 3x

=1

N

1

2 I 3K. x

K20

Comparing

with

we

obtain

Example

Solve

the

aκ

The

K20

aκ

(2.35). a

+

K

∞

(3x)2 + (3x)

3+

ЯК

x

$G(x) = \Sigma\ axx$

aк

recurrence

K=0

= 2.3

aк

K

N

∞

14

Z(2.3K).

**,

K20

relation using

generating f

K 712

A0 = 3,91 = 1

+ зак-2 ak = 2ax-1 corresponding

generating for for the

sequence {an) is

defined

G(x)

=

We consider the

⇒

aк

ने

aк

Since

ак

ак

K> 2

а так

ак

K = 2

as,

к

Дак

x

K=0

relation

2 ак-1

2ak-1

comp

2ак-1

+3 ак-2

= 0

зак−2

xk zo

2

зак-2

X

к

[Multiplying by xk J summing

for K = 2, 3,

2 Σ ak-1

x

K=2

Solving individually

1st term

x

j

K

Дак як

K=2

ما

к - 3 Σ

= 92

2

K = 2

x2 + A3 аз

тал

2,3,.

ак-2

3+

x

we

get,

ак

20

x+a2x2 + ...) -

(aotani

(α0+91x)

=(a

o

Σακακ

=

=

Kzo

$G(x) = (ao+aq\ x)$

2nd term! 2Z ak-1

K = 2

3rd

term : 3Σak-2.

K = 2

K

x

$=294x2+2a2x2+$

2

$= 2x\ (aqxt\ azxt$

2

15

.)

$\bullet(90+ax+92x2+ \ ..) -$

$2x\alpha 0$

$= 2\ x$

$=2x,\ G(x)$

aK

=

=

2 ,

2x 90

зaox '+ зax

3

+

$3x2$ (aotaqx $+92x2+$

$3x2$ G (x)

Substitutin
g

11

and

M

in C

we obtain,

$3x26\,(x)=0$

podemos do

$G(x) - a_0 - ax - 2x\,G(x) + 2x\,a_0$

$3x^2 G(x)$

$\rightarrow G(x) - 2x\,G(x) - 3x^2\,G(x) = a_0 + a_1 x - 2x\,a_0$

$G(x)\,[1 - 2x - 3x^2]$

$=$

$3 + x + 6x$

$6x$

$\Rightarrow$

$G(x) =$

$3-5x$

$1-2x$

$=$

$=$

$3x^2$

$3-5x$

$(1-3x)(1+x)$

A

be determined.

B

$(1-3x$

3x)

1+x)

where

A

and

B

are to

Noro,

$G\ (x)$

Σ

Then

equating the

$A(1+x) - B\ (1-3x)$

$(1-3x)\ (1+x)$

$A(1+x) - B(1-3x) = 3-5x$

3-5x

$(1-3x)$

$(1+x)$

numerators, we

obtain,

3-5x.

A +3B=-5 .

$$\Rightarrow (4-B) + (4 +3B) x =$$

#ence,

solving

A–B

the

}}

3

eqns, we

A = 1, B = 2

and

obtain

Therefore,

G(a)

1

1-3 x

+

2

1+x.

一フ

(^-3x)=1+2(1+x

) = 1

(1 + 3 x + (3 x)2 +

(3x)

= + .

11

NJ

=

+2(1 - x + x2 - x3

+ +2(1-x+x

Ž

x

K

Σ 3K, 2K + Σ 2.

(−1) xk

K=0

3KX

K

K=0

K x

Σ [3*

+2.(-15] xx

K=0

:)

$> \mid ax\ xx > Z\ (3^* + 2.$

$(-1)\ ^*)\ ^{**}$

Now,

G(x) =

Σ

K20

aк

ל

K=0

i.e., {ax = 3x + 2.

(−1) *}, "

aк

K = 0, 1, 2,

(16)