

RECURRENCE RELATIONS

Many of the counting problems

basic
counting

using

(1)

cannot be solved

counting techniques and hence
requires solution by finding relationships, called
relations.

recurrence

Note that, sequences can be defined
recursively.

So, each

recurrence relation is
particular sequence.

connected to a

[Definition: A recurrence relation for a sequence $\{a_n\}$

is an equation that expresses

a_n

in terms of

a_0, a_1, \dots, a_{n-1}

for

one or more of the previous terms of the sequence
namely, $a_n = f(n, a_0, a_1, \dots, a_{n-1})$ for all integers $n \geq n_0$,
where

[Note

n_0

a_n

is

a

non

A sequence

recurrence

relation.

negative integer.

is called a solution of a
 relation if its
 terms satisfy the recurrence

Example

$$a_n = a_{n-2} + a_{n-1} \quad \text{for } n \geq 2$$

with

$$a_0 = 0, \quad a_1 = 1$$

$$a_2 = 1$$

$$a_3 = 2$$

and $a_4 = 3$

$$a_5 = 5$$

Each

of

the

relations

$$a_6 = 8$$

the

VID the
 or formula,

recurrence

has C the

expression

restriction or range

initial conditions .

Using the initial conditions and
determine

the rest

of

can

92

= a_1
 a_n

a_0
3

5

八

2

and

So

a

3

and

the formula, we

the **sequence**. the terms of

=

=

a_2

2

$$= -3$$

on.

Therefore, $\{a_n\} = \{3, 5, 2, -3,$

$$a_{n+1}$$

$$- 5$$

}

2

[Example Determine whether the sequence $\{a_n\}$, where a_n for every nonnegative integer n , is a solution recurrence relation

of

$$a_n = 3^n$$

the

Answer the same

consider the

$$572$$

Now, for

$$2a_{n-1}$$

$$-a_{n-2}$$

=

when

$$a_{n-2,n} = -2, 5 \times 2.$$

$$a_n = 2a_{n-1}$$

$$a_n = 2^n$$

$$a_n = 2a_{n-1} - a_{n-2}.$$

we

see that,

recurrence relation

and

$$a_n = 3^n$$

$$2[3^{(n-1)}] - 3^{(n-2)} = 3^n = a_n.$$

Therefore, $\{a_n\}$, where

the

recurrence

Suppose that,

$$a_n = 3^n, \text{ is}$$

relation.

a solution of

for $n \geq 0$. Then $90 = 1, 99 = 2$

$A_2 = 4$. Because, 201

gal

relation

$$a_n =$$

$$2^n$$

and

and

we

see

that

the

considering

the

$\{a_n = 2^n\}$ is
relation.

[Note] The

$$a_0 = 2 \cdot 2^{-1} = 3792$$

a_n

$$= 2a_{n-1} - a_{n-2},$$

not a

solution of

recurrence

recurrence

relation and the initial

a sequence.

Conditions uniquely determine

Consider the recurrence relation,

$$a_n = 2a_{n-1}$$

a_{n-2}

2

check

He

can

are

solutions

▪ { 0, 3, 6, 9, 12, 15, 18, 21, 24, 27, 30, 33, 36, 39, 42, 45, 48, 51, 54, 57, 60, 63, 66, 69, 72, 75, 78, 81, 84, 87, 90, 93, 96, 99, 102, 105, 108, 111, 114, 117, 120, 123, 126, 129, 132, 135, 138, 141, 144, 147, 150, 153, 156, 159, 162, 165, 168, 171, 174, 177, 180, 183, 186, 189, 192, 195, 198, 201, 204, 207, 210, 213, 216, 219, 222, 225, 228, 231, 234, 237, 240, 243, 246, 249, 252, 255, 258, 261, 264, 267, 270, 273, 276, 279, 282, 285, 288, 291, 294, 297, 300, 303, 306, 309, 312, 315, 318, 321, 324, 327, 330, 333, 336, 339, 342, 345, 348, 351, 354, 357, 360, 363, 366, 369, 372, 375, 378, 381, 384, 387, 390, 393, 396, 399, 402, 405, 408, 411, 414, 417, 420, 423, 426, 429, 432, 435, 438, 441, 444, 447, 450, 453, 456, 459, 462, 465, 468, 471, 474, 477, 480, 483, 486, 489, 492, 495, 498, 501, 504, 507, 510, 513, 516, 519, 522, 525, 528, 531, 534, 537, 540, 543, 546, 549, 552, 555, 558, 561, 564, 567, 570, 573, 576, 579, 582, 585, 588, 591, 594, 597, 600, 603, 606, 609, 612, 615, 618, 621, 624, 627, 630, 633, 636, 639, 642, 645, 648, 651, 654, 657, 660, 663, 666, 669, 672, 675, 678, 681, 684, 687, 690, 693, 696, 699, 702, 705, 708, 711, 714, 717, 720, 723, 726, 729, 732, 735, 738, 741, 744, 747, 750, 753, 756, 759, 762, 765, 768, 771, 774, 777, 780, 783, 786, 789, 792, 795, 798, 801, 804, 807, 810, 813, 816, 819, 822, 825, 828, 831, 834, 837, 840, 843, 846, 849, 852, 855, 858, 861, 864, 867, 870, 873, 876, 879, 882, 885, 888, 891, 894, 897, 900, 903, 906, 909, 912, 915, 918, 921, 924, 927, 930, 933, 936, 939, 942, 945, 948, 951, 954, 957, 960, 963, 966, 969, 972, 975, 978, 981, 984, 987, 990, 993, 996, 999, 1002, 1005, 1008, 1011, 1014, 1017, 1020, 1023, 1026, 1029, 1032, 1035, 1038, 1041, 1044, 1047, 1050, 1053, 1056, 1059, 1062, 1065, 1068, 1071, 1074, 1077, 1080, 1083, 1086, 1089, 1092, 1095, 1098, 1101, 1104, 1107, 1110, 1113, 1116, 1119, 1122, 1125, 1128, 1131, 1134, 1137, 1140, 1143, 1146, 1149, 1152, 1155, 1158, 1161, 1164, 1167, 1170, 1173, 1176, 1179, 1182, 1185, 1188, 1191, 1194, 1197, 1200, 1203, 1206, 1209, 1212, 1215, 1218, 1221, 1224, 1227, 1230, 1233, 1236, 1239, 1242, 1245, 1248, 1251, 1254, 1257, 1260, 1263, 1266, 1269, 1272, 1275, 1278, 1281, 1284, 1287, 1290, 1293, 1296, 1299, 1302, 1305, 1308, 1311, 1314, 1317, 1320, 1323, 1326, 1329, 1332, 1335, 1338, 1341, 1344, 1347, 1350, 1353, 1356, 1359, 1362, 1365, 1368, 1371, 1374, 1377, 1380, 1383, 1386, 1389, 1392, 1395, 1398, 1401, 1404, 1407, 1410, 1413, 1416, 1419, 1422, 1425, 1428, 1431, 1434, 1437, 1440, 1443, 1446, 1449, 1452, 1455, 1458, 1461, 1464, 1467, 1470, 1473, 1476, 1479, 1482, 1485, 1488, 1491, 1494, 1497, 1500, 1503, 1506, 1509, 1512, 1515, 1518, 1521, 1524, 1527, 1530, 1533, 1536, 1539, 1542, 1545, 1548, 1551, 1554, 1557, 1560, 1563, 1566, 1569, 1572, 1575, 1578, 1581, 1584, 1587, 1590, 1593, 1596, 1599, 1602, 1605, 1608, 1611, 1614, 1617, 1620, 1623, 1626, 1629, 1632, 1635, 1638, 1641, 1644, 1647, 1650, 1653, 1656, 1659, 1662, 1665, 1668, 1671, 1674, 1677, 1680, 1683, 1686, 1689, 1692, 1695, 1698, 1701, 1704, 1707, 1710, 1713, 1716, 1719, 1722, 1725, 1728, 1731, 1734, 1737, 1740, 1743, 1746, 1749, 1752, 1755, 1758, 1761, 1764, 1767, 1770, 1773, 1776, 1779, 1782, 1785, 1788, 1791, 1794, 1797, 1800, 1803, 1806, 1809, 1812, 1815, 1818, 1821, 1824, 1827, 1830, 1833, 1836, 1839, 1842, 1845, 1848, 1851, 1854, 1857, 1860, 1863, 1866, 1869, 1872, 1875, 1878, 1881, 1884, 1887, 1890, 1893, 1896, 1899, 1902, 1905, 1908, 1911, 1914, 1917, 1920, 1923, 1926, 1929, 1932, 1935, 1938, 1941, 1944, 1947, 1950, 1953, 1956, 1959, 1962, 1965, 1968, 1971, 1974, 1977, 1980, 1983, 1986, 1989, 1992, 1995, 1998, 2001, 2004, 2007, 2010, 2013, 2016, 2019, 2022, 2025, 2028, 2031, 2034, 2037, 2040, 2043, 2046, 2049, 2052, 2055, 2058, 2061, 2064, 2067, 2070, 2073, 2076, 2079, 2082, 2085, 2088, 2091, 2094, 2097, 2100, 2103, 2106, 2109, 2112, 2115, 2118, 2121, 2124, 2127, 2130, 2133, 2136, 2139, 2142, 2145, 2148, 2151, 2154, 2157, 2160, 2163, 2166, 2169, 2172, 2175, 2178, 2181, 2184, 2187, 2190, 2193, 2196, 2199, 2202, 2205, 2208, 2211, 2214, 2217, 2220, 2223, 2226, 2229, 2232, 2235, 2238, 2241, 2244, 2247, 2250, 2253, 2256, 2259, 2262, 2265, 2268, 2271, 2274, 2277, 2280, 2283, 2286, 2289, 2292, 2295, 2298, 2301, 2304, 2307, 2310, 2313, 2316, 2319, 2322, 2325, 2328, 2331, 2334, 2337, 2340, 2343, 2346, 2349, 2352, 2355, 2358, 2361, 2364, 2367, 2370, 2373, 2376, 2379, 2382, 2385, 2388, 2391, 2394, 2397, 2400, 2403, 2406, 2409, 2412, 2415, 2418, 2421, 2424, 2427, 2430, 2433, 2436, 2439, 2442, 2445, 2448, 2451, 2454, 2457, 2460, 2463, 2466, 2469, 2472, 2475, 2478, 2481, 2484, 2487, 2490, 2493, 2496, 2499, 2502, 2505, 2508, 2511, 2514, 2517, 2520, 2523, 2526, 2529, 2532, 2535, 2538, 2541, 2544, 2547, 2550, 2553, 2556, 2559, 2562, 2565, 2568, 2571, 2574, 2577, 2580, 2583, 2586, 2589, 2592, 2595, 2598, 2601, 2604, 2607, 2610, 2613, 2616, 2619, 2622, 2625, 2628, 2631, 2634, 2637, 2640, 2643, 2646, 2649, 2652, 2655, 2658, 2661, 2664, 2667, 2670, 2673, 2676, 2679, 2682, 2685, 2688, 2691, 2694, 2697, 2700, 2703, 2706, 2709, 2712, 2715, 2718, 2721, 2724, 2727, 2730, 2733, 2736, 2739, 2742, 2745, 2748, 2751, 2754, 2757, 2760, 2763, 2766, 2769, 2772, 2775, 2778, 2781, 2784, 2787, 2790, 2793, 2796, 2799, 2802, 2805, 2808, 2811, 2814, 2817, 2820, 2823, 2826, 2829, 2832, 2835, 2838, 2841, 2844, 2847, 2850, 2853, 2856, 2859, 2862, 2865, 2868, 2871, 2874, 2877, 2880, 2883, 2886, 2889, 2892, 2895, 2898, 2901, 2904, 2907, 2910, 2913, 2916, 2919, 2922, 2925, 2928, 2931, 2934, 2937, 2940, 2943, 2946, 2949, 2952, 2955, 2958, 2961, 2964, 2967, 2970, 2973, 2976, 2979, 2982, 2985, 2988, 2991, 2994, 2997, 3000, 3003, 3006, 3009, 3012, 3015, 3018, 3021, 3024, 3027, 3030, 3033, 3036, 3039, 3042, 3045, 3048, 3051, 3054, 3057, 3060, 3063, 3066, 3069, 3072, 3075, 3078, 3081, 3084, 3087, 3090, 3093, 3096, 3099, 3102, 3105, 3108, 3111, 3114, 3117, 3120, 3123, 3126, 3129, 3132, 3135, 3138, 3141, 3144, 3147, 3150, 3153, 3156, 3159, 3162, 3165, 3168, 3171, 3174, 3177, 3180, 3183, 3186, 3189, 3192, 3195, 3198, 3201, 3204, 3207, 3210, 3213, 3216, 3219, 3222, 3225, 3228, 3231, 3234, 3237, 3240, 3243, 3246, 3249, 3252, 3255, 3258, 3261, 3264, 3267, 3270, 3273, 3276, 3279, 3282, 3285, 3288, 3291, 3294, 3297, 3300, 3303, 3306, 3309, 3312, 3315, 3318, 3321, 3324, 3327, 3330, 3333, 3336, 3339, 3342, 3345, 3348, 3351, 3354, 3357, 3360, 3363, 3366, 3369, 3372, 3375, 3378, 3381, 3384, 3387, 3390, 3393, 3396, 3399, 3402, 3405, 3408, 3411, 3414, 3417, 3420, 3423, 3426, 3429, 3432, 3435, 3438, 3441, 3444, 3447, 3450, 3453, 3456, 3459, 3462, 3465, 3468, 3471, 3474, 3477, 3480, 3483, 3486, 3489, 3492, 3495, 3498, 3501, 3504, 3507, 3510, 3513, 3516, 3519, 3522, 3525, 3528, 3531, 3534, 3537, 3540, 3543, 3546, 3549, 3552, 3555, 3558, 3561, 3564, 3567, 3570, 3573, 3576, 3579, 3582, 3585, 3588, 3591, 3594, 3597, 3600, 3603, 3606, 3609, 3612, 3615, 3618, 3621, 3624, 3627, 3630, 3633, 3636, 3639, 3642, 3645, 3648, 3651, 3654, 3657, 3660, 3663, 3666, 3669, 3672, 3675, 3678, 3681, 3684, 3687, 3690, 3693, 3696, 3699, 3702, 3705, 3708, 3711, 3714, 3717, 3720, 3723, 3726, 3729, 3732, 3735, 3738, 3741, 3744, 3747, 3750, 3753, 3756, 3759, 3762, 3765, 3768, 3771, 3774, 3777, 3780, 3783, 3786, 3789, 3792, 3795, 3798, 3801, 3804, 3807, 3810, 3813, 3816, 3819, 3822, 3825, 3828, 3831, 3834, 3837, 3840, 3843, 3846, 3849, 3852, 3855, 3858, 3861, 3864, 3867, 3870, 3873, 3876, 3879, 3882, 3885, 3888, 3891, 3894, 3897, 3900, 3903, 3906, 3909, 3912, 3915, 3918, 3921, 3924, 3927, 3930, 3933, 3936, 3939, 3942, 3945, 3948, 3951, 3954, 3957, 3960, 3963, 3966, 3969, 3972, 3975, 3978, 3981, 3984, 3987, 3990, 3993, 3996, 4000, 4003, 4006, 4009, 4012, 4015, 4018, 4021, 4024, 4027, 4030, 4033, 4036, 4039, 4042, 4045, 4048, 4051, 4054, 4057, 4060, 4063, 4066, 4069, 4072, 4075, 4078, 4081, 4084, 4087, 4090, 4093, 4096, 4099, 4102, 4105, 4108, 4111, 4114, 4117, 4120, 4123, 4126, 4129, 4132, 4135, 4138, 4141, 4144, 4147, 4150, 4153, 4156, 4159, 4162, 4165, 4168, 4171, 4174, 4177, 4180, 4183, 4186, 4189, 4192, 4195, 4198, 4201, 4204, 4207, 4210, 4213, 4216, 4219, 4222, 4225, 4228, 4231, 4234, 4237, 4240, 4243, 4246, 4249, 4252, 4255, 4258, 4261, 4264, 4267, 4270, 4273, 4276, 4279, 4282, 4285, 4288, 4291, 4294, 4297, 4300, 4303, 4306, 4309, 4312, 4315, 4318, 4321, 4324, 4327, 4330, 4333, 4336, 4339, 4342, 4345, 4348, 4351, 4354, 4357, 4360, 4363, 4366, 4369, 4372, 4375, 4378, 4381, 4384, 4387, 4390, 4393, 4396, 4399, 4402, 4405, 4408, 4411, 4414, 4417, 4420, 4423, 4426, 4429, 4432, 4435, 4438, 4441, 4444, 4447, 4450, 4453, 4456, 4459, 4462, 4465, 4468, 4471, 4474, 4477, 4480, 4483, 4486, 4489, 4492, 4495, 4498, 4501, 4504, 4507, 4510, 4513, 4516, 4519, 4522, 4525, 4528, 4531, 4534, 4537, 4540, 4543, 4546, 4549, 4552, 4555, 4558, 4561, 4564, 4567, 4570, 4573, 4576, 4579, 4582, 4585, 4588, 4591, 4594, 4597, 4600, 4603, 4606, 4609, 4612, 4615, 4618, 4621, 4624, 4627, 4630, 4633, 4636, 4639, 4642, 4645, 4648, 4651, 4654, 4657, 4660, 4663, 4666, 4669, 4672, 4675, 4678, 4681, 4684, 4687, 4690, 4693, 4696, 4699, 4702, 4705, 4708, 4711, 4714, 4717, 4720, 4723, 4726, 4729, 4732, 4735, 4738, 4741, 4744, 4747, 4750, 4753, 4756, 4759, 4762, 4765, 4768, 4771, 4774, 4777, 4780, 4783, 4786, 4789, 4792, 4795, 4798, 4801, 4804, 4807, 4810, 4813, 4816, 4819, 4822, 4825, 4828, 4831, 4834, 4837, 4840, 4843, 4846, 4849, 4852, 4855, 4858, 4861, 4864, 4867, 4870, 4873, 4876, 4879, 4882, 4885, 4888, 4891, 4894, 4897, 4900, 4903, 4906, 4909, 4912, 4915, 4918, 4921, 4924, 4927, 4930, 4933, 4936, 4939, 4942, 4945, 4948, 4951, 4954, 4957, 4960, 4963, 4966, 4969, 4972, 4975, 4978, 4981, 4984, 4987, 4990, 4993, 4996, 5000, 5003, 5006, 5009, 5012, 5015, 5018, 5021, 5024, 5027, 5030, 5033, 5036, 5039, 5042, 5045, 5048, 5051, 5054, 5057, 5060, 5063, 5066, 5069, 5072, 5075, 5078, 5081, 5084, 5087, 5090, 5093, 5096, 5099, 5102, 5105, 5108, 5111, 5114, 5117, 5120, 5123, 5126, 5129, 5132, 5135, 5138, 5141, 5144, 5147, 5150, 5153, 5156, 5159, 5162, 5165, 5168, 5171, 5174, 5177, 5180, 5183, 5186, 5189, 5192, 5195, 5198, 5201, 5204, 5207, 5210, 5213, 5216, 5219, 5222, 5225, 5228, 5231, 5234, 5237, 5240, 5243, 5246, 5249, 5252, 5255, 5258, 5261, 5264, 5267, 5270, 5273, 5276, 5279, 5282, 5285, 5288, 5291, 5294, 5297, 5300, 5303, 5306, 5309, 5312, 5315, 5318, 5321, 5324, 5327, 5330, 5333, 5336, 5339, 5342, 5345, 5348, 5351, 5354, 5357, 5360, 5363, 5366, 5369, 5372, 5375, 5378, 5381, 5384, 5387, 5390, 5393, 5396, 5399, 5402, 5405, 5408, 5411, 5414, 5417, 5420, 5423, 5426, 5429, 5432, 5435, 5438, 5441, 5444, 5447, 5450, 5453, 5456, 5459, 5462, 5465, 5468, 5471, 5474, 5477, 5480, 5483, 5486, 5489, 5492, 5495, 5498, 5501, 5504, 5507, 5510, 5513, 5516, 5519, 5522, 5525, 5528, 5531, 5534, 5537, 5540, 5543, 5546, 5549, 5552, 5555, 5558, 5561, 5564, 5567, 5570, 5573, 5576, 5579, 5582, 5585, 5588, 5591, 5594, 5597, 5600, 5603, 5606, 5609, 5612, 5615, 5618, 5621, 5624, 5627, 5630, 5633, 5636, 5639, 5642, 5645, 5648, 5651, 5654, 5657, 5660, 5663, 5666, 5669, 5672, 5675, 5678, 5681, 5684, 5687, 5690, 5693, 5696, 5699, 5702, 5705, 5708, 5711, 5714, 5717, 5720, 5723, 5726, 5729, 5732, 5735, 5738, 5741, 5744, 5747, 5750, 5753, 5756, 5759, 5762, 5765, 5768, 5771, 5774, 5777, 5780, 5783, 5786, 5789, 5792, 5795, 5798, 5801, 5804, 5807, 5810, 5813, 5816, 5819, 5822, 5825, 5828, 5831, 5834, 5837, 5840, 5843, 5846, 5849, 5852, 5855, 5858, 5861, 5864, 5867, 5870, 5873, 5876, 5879, 5882, 5885, 5888, 5891, 5894, 5897, 5900, 5903, 5906, 5909, 5912, 5915, 5918, 5921, 5924, 5927, 5930, 5933, 5936, 5939, 5942, 5945, 5948, 5951, 5954, 5957, 5960, 5963, 5966, 5969, 5972, 5975, 5978, 5981, 5984, 5987, 5990, 5993, 5996, 6000, 6003, 6006, 6009, 6012, 6015, 6018, 6021, 6024, 6027, 6030, 6033, 6036, 6039, 6042, 6045, 6048, 6051, 6054, 6057, 6060, 6063, 6066, 6069, 6072, 6075, 6078, 6081, 6084, 6087, 6090, 6093, 6096, 6099, 6102, 6105, 6108, 6111, 6114, 6117, 6120, 6123, 6126, 6129, 6132, 6135, 6138, 6141, 6144, 6147, 6150, 6153, 6156, 6159, 6162, 6165, 6168, 6171, 6174, 6177, 6180, 6183, 6186, 6189, 6192, 6195, 6198, 6201, 6

wide range of
interest, counting of
Hanoi problem

{ an

be

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Relations

used to model a

as compound

such

problems

bit strings

strings, solving
Tower

etc.

Let,

an

the

hour

i....,

Example The number of bacteria in

a colony triples

every hour. If a colony begins with five bacteria,

how

be present in

many

Since

will

number

of

number

an

of

n hours ?

bacteria

3^{n-1}

of

n hours

triples in every

and $a_0=5$

bacteria at the end

$$n = 1$$

ал

$$= 3a_0 \text{ } 3a_0$$

2

a2

$$= 3a_1$$

$$= 3$$

3290

$$93 =$$

$$23 \text{ } 00$$

$$33$$

and som.

ao

Hence,

an

3π

3nao

ao

of

of

n

hours,

we

the

At the end

9 bacteria

in

Example Suppose

that in a savings account

year with interest
will

be

in

Let

the

$= 3\% .5$

colony.

at a

have (35) number

a person deposits Rs. 25,000 bank

yielding 12%. per compounded

annually. How much account after

20 years?

the account after n

years.

I_n : amount in

Since, the

=

ire.

I_n

amount after

n years.

amount after $(n-1)$ years + interest
of th

I_{n-1}

+ $0.12 \cdot I_{n-1}$ with

$1.12 I_{n-1}$

$$1. I_1 = 1.12 I_0$$

I_2

$$= 1.12 I_1$$

=

year.

$$I_0 = 25,000$$

$$1.12 \times 25,000$$

$$(1.12) \times (1.12)$$

$$I_0 = (1.12)^2 I,$$

Hence,

In

$$= (1 \cdot 12) \cdot 10^{12} \times 25000$$

=

120

=

$$(1+12) \cdot 10^{12} \times 25000$$

a

[Example Find

recurrence relation and give

initial conditions for the number of bit strings of

length that do not have two

bit strings

Let,

n

are

n

consecutive 0s. How many such
there of length ten?
number of bit
strings of

an : number

that do not

have

two

length
n

consecutive

0s.

bit

strings of
length

One can note that, the number of
that do not have two consecutive 0s equals
such bit strings starting with 1,
plus number of such bit strings starting with
0.

the number

the

of

For the particular problem,

5

that

the

The

do

we

bit strings of length n

not

assume that $n \geq 3$,

bit string has at least three bits.

of length n , starting with \uparrow
that

0s are precisely the
two consecutive

have two

consecutive

bit strings of length $(n-1)$ with no

1

0s

with

a

there

are

a_{n-1}

added at the beginning.

Consequently

bit strings.

such

of length n starting with a 0,

that do

strings of length

n

Bit Strings

have two consecutive 0s

bit; otherwise they would

n

at 2nd

a

must have 1 as the would beco end with

pair of 0s. It follows that the bit strings of length n starting with consecutive

0s that have no two 0s are precisely the bit strings of length $(n-2)$ with added

A_{n-2} no two consecutive 0s with 10 at the beginning. Consequently, there are

bit strings such

1 does not contain two consecutive 0s

n $(n-1)$ length bit string

length bit string

It does not contain two consecutive 0s

$n-1$ length

in length

may contain two consecutive 0s

as the bit can be 0

does not contain two consecutive 0s

$n-2$ length

01

n length

ire.,

start with 1 :

any bit string of
length $(n-1)$

with no two consecutive 0s

start with 0: 1st bit | Any bit
string of length $(n-2)$

p_n

d

1

with no two consecutive Os

$a_n =$

So, we

conclude that,

are

The initial conditions

both

the

two

have

As the valid

a_{n-1}

No.

5

bit

strings of

length n with no
two consecut

OS

a_{n-1}

a_{n-2}

Total": $a_n = a_{n-1} + a_{n-2}$

$+a_{n-2}$

573

ая

$a_1 = 2$

= 2, because

1

do not

are

01, 10 and

1-length bit strings of 0 and 1
consecutive

0s,

bit strings of length

2

11, we have, $a_1 = 3!$

So,

a_3

93=

$3+2$

=

5

-

$$94 = 5 + 3 = 8$$

and

So

on .

$$= 89 + 55 = 144 .$$

Therefore, $a_{10} = a_9 + a_8$

Solving linear recurrence relations (homogeneous) with constant coefficients

A

degree 1

b

order

of

recurrence

and the homogeneous recurrence relation

of

constant coefficients is of the

form

K

L

with

relation.

where

$$A_n = C_1 a^{n-1} + C_2 a^{n-2} + \dots + C_k a^{n-k} + A$$

, C_k are real numbers and $C \neq 0$.

The degree is the highest power of the variable in the

Examples

$$a_n = 2a_{n-1} + a_{n-2}$$

"

$$a_n = 3n^2 a_{n-1}$$

IV

a_n

recurrence relation.

Non-Linear

Non-homogeneous

Non-constant
coefficients

degree 1 and

$$a_n = a_{n-1} + 3a_{n-2} + 5a_{n-3}$$

order 2

The

R. H.S

$$+a_{n-3}$$

recurrence relation

is

a

A

Non-linear, homogeneous recurrence relation with Constant
coefficients, and of degree

degree 2 and of order 3

is linear because the

sum of previous terms of the
sequence,

each having power 1.

(A) is

are

homogeneous because

not multiples

nonzero

A

is

of the

occur, that

does not contain

no

terms

are

non-sequential terms.

or any

a recurrence relation with all the constant

coefficients,

rather

order

than functions that depend

because

on n .

an is expressed in

k - terms of the sequence.

To solve (A)

is

K

The a

terms

of

the

previous

The degree

of

'A'

is one .

Note

we

need

k no. of

initial

conditions .

of
of recurrence
relation

A

Characteristic equation

recurrence relation

$$+ C_2 a_{n-2} +$$

$$+ C_k a_{n-k}$$

real numbers and $C_k \neq 0$.

basic approach for solving

Consider

the

$$a_n =$$

$$C_1 a_{n-1}$$

$$C_2,$$

$$C_k$$

are

The

Solutions

Constant.

Note

$a_n = r^n$ is

r^n

of the form

a

Now dividing by

r^n

r^n

a_n

Σ

a solution of

$+ C_2 r^{n-2} +$

(A

is to look for

where

r is a

A if and only if

$+ C_k r^{n-k}$

obtain,

distinct roots

The characteristic equation is a polynomial of degree K .

[Theorem 1 Let r_k - $G r_k$ - $1_{-c_2 r_k - 2$

Characteristic equation with constant coefficients and K

59,82,

$-CK=0$ is the

r_k . Then

solution

to the recurrence relation

=

$C_1 a_{n-1}$

$+ c_2 a_{n-2} +$

$TCK a_{n-K}$

, if and

only if

a_n

for

$n = 0, 1, 2,$

$$= 211n + 2282n + \dots$$

and

$$+ a_{k,n}$$

a, X_2, \dots, X_K are constants.

Example Find the solution to the recurrence relation

$$a_n =$$

$$6a_{n-1}$$

Replace

Then

Copy

$$11 a_{n-2} + 6 a_{n-3}, 90=2, 94 = 5, 92 =15.$$

$a_k = r_k$ in the given
recurrence relation

the characteristic

equation,

$$r_k = 6r_{k-1} - 11r_{k-2}$$

Dividing by

$$r_{k-3}$$

$$-3$$

$$r_{k-3}$$

we

precurre

nce

$$+6rk-3$$

obtain,

$$= 6r^2-11 + 6$$

$$\Rightarrow \sqrt[3]{3-682} + 118-6=0$$

[Characteristic
29"]

$$82$$

$$2, \quad 83=3$$

Call roots are distinct)

$$\rightarrow (8-1) \quad (8-2) \quad (r-3) = 0$$

Then

the

an

=1

solution

Can

be written as,

$$<1(r a) " + d2 (82)$$

$$+dz(rz)$$

Now, for $\theta = 2$,

$\theta = 2$, we

obtain,

$$x_1^2 + x_2^2 + x_3^2$$

$$= 2$$

$$x_1 + x_2 + x_3 = 2$$

Applying the remaining initial conditions

$$x_1 = 5$$

and

$$x_2 = 15$$

at

we

j

obtain,

$$2^2 + 2 \times 2 + 3 \times 2 = 5$$

$$2^2 + 4 \times 2 + 9 \times 2 = 15$$

Solving the system of linear eqpes

$$x_1 = 1$$

Da

Th

Therefore

Example

$$r^2 + 2r - 2 = -1$$

the

we

have,

$$r_1 = 2, r_2 = -3$$

solution, $a_n = 1 \cdot (2)^n + 2 \cdot (-3)^n$

Solve the

recurrence

$$a_n = a_{n-1} + 2a_{n-2}$$

relation

$$a_0 = 2, a_1 = 7$$

given recurrence
relation

Characteristic eqn of the given

Hence,

$$r^2 - 8r + 2 = 0$$

we

$$r^2 - 8r + 2 = 0$$

$$= (r-2)(r+1)=0$$

$$r_1=2, \quad r_2$$

$$\sqrt{2} = -1$$

$$a_n = x^n (2)^n + x^{2n} (-1)^n.$$

Applying the initial conditions,

$$n = 0, 1, 2,$$

$$2^1 - 2^0 + \alpha 2^0 (-1)^0 = 2$$

ख

$$2^1 + 2^2$$

$$= 2$$

$$\times 1 - 2^1 + \alpha 2^1 (-1)^1 = 7$$

$$^1 (-1)^1$$

$$2^2 - 2^1 = 7$$

1

$$2^0$$

we

obtain

$$3 \cdot (2)^n - 1 \cdot (-1)^n.$$

$$a_0 = 2$$

\Rightarrow

$$a_1 = 7, a_n > 7$$

Solving D

and

$$a_n =$$

Theorem 2

Let

$C_1, C_2,$

C_k and

roots

r

Such

$$r^k - 1 - C_2 r^n$$

"

11

the solution,

$K-2$

$\cdot C_k = 0$ be

repeated
the characteristic eq" with constant coefficients

mt,

Then

CK 70

• have

t

with multiplicity m1, m2...

that mix 1 and

a sequence $\{a_n\}$ is

recurrence

if and

relation

and only

only if

a_n

$$a_n = (\alpha_1 + \alpha_2 n + \alpha_3 n^2 +$$

2

to

$$+ (B_1 + B_2 n +$$

+ . .

2

a

$$ma+m2t...+ME=k.$$

solution

of

the

$$= G_{an-n}+G_{an-2}+... FCK \ 9n-w$$

+

$$B_{m2-1}$$

n

π

$$^{\wedge}) \ (r$$

$$)^{"$$

$$nm2-1) (^2)^{\wedge}2 \ +$$

...

n

Mt-

$$. \ + \ C \ \& \ 1 \ + \ 2 \ + \ -$$

$$+ m^2 - n'' = -1)$$

$$(rz)''$$

for $n=0,1,2,$

and Li, Bi,-gi are

Constants

Example Solve

Replacin

g

$$a_n =$$

$$90 = 1, a$$

$$a_n \text{ by } p_n = -3x_n - 1$$

Dividing by

son on

$$23-3$$

$$2n-3$$

m

$$3a_{n-1} - 3a_{n-2} - a_{n-3}$$

$$= \frac{2}{3} a_{n-2} + \frac{1}{3} a_{n-3} - a_{n-2} = -\frac{1}{3} a_{n-2} + \frac{1}{3} a_{n-3}$$

we obtain,

$$-3a_{n-2} + a_{n-3}$$

$$3r^3 - 3r^2 + r$$

$$3r^2 - 3r + 1$$

$$-r^3 - 3r^2 + 3r - 1$$

we have,

$$r^n - 3r^{n-1} + 3r^{n-2} - r^{n-3}$$

$$35 - 1$$

$$+382 + 31 + 1 = 0$$

$$203$$

$$3$$

$$10$$

$$= 1, -1, -1$$

Solution !

$$3$$

$$a_{n-2}$$

ron-3
ねり

ph-3 m-3

$$(x+1)^3=0$$

$$\begin{aligned} a_n &= (21 + n^2 + n^2 \alpha z) (r)'' \\ &= (\alpha 1 + n a^2 + n^2 \alpha 3) \\ &\quad (-1)'' \end{aligned}$$

Applying I .

C . S

Hence, the solution is

Example If **roots** of a
relation

are

$$\times 1 = 1, \quad \alpha_2 = 3, \quad dz = -2$$

22

$$a_n = (1 + 3n - 2n^2)$$

$$(-1)^n$$

for $n=0, 1, 2,$

(9

linear homogeneous recurrence $2, 2,$
 $3, 5, 5$ and 9 then the form
of the general solution
is

$$a_n = (x_1 + \alpha_2 i)^n (2)'' + B_1 (3)'' + (81 + 82. n) (5) + S_1 (9)'' \\ ') (3)'' + (81 + 82. n) + 8,$$

where, $n = 0, 1, 2,$
 ∞

Solving

are

α_1
 and $\alpha_2, B_1, 81, 8,$ and
 α_2
 α_1
 α_2

constants, (can be determined using I.C.S)

recurrence relations using substitution methods

Forward substitution Method

Consider the

$$\begin{aligned}
 & a_n = 1 \\
 & \text{for } n = 2 \\
 & \text{recurrence} \\
 & a_n = 2a_{n-1} \\
 & a_1 = 92 \\
 & = 200 \\
 & = 291
 \end{aligned}$$

relation

$$a_n = 2a_{n-1} + 90$$

B

$$= 2.29 \times 10^3$$

$$222.3$$

,

2.3

Cusing the
recurrence
relation)

for $n=3$, a_3

$$= 292 =$$

$$= 23.00 = 23.3$$

A_3

a_n

$$27.90 = 2.37$$

Hence, $\{a_n = 2.37$

is

recurrence

Backward

relation.

Method

a solution of the given

previous recurrence relation given
in R

Consider

the

$$a_n = 2a_{n-1} - 1$$

$$= 2(2a_{n-2} - 1) - 1$$

$$= 2^2 a_{n-2} - 2 - 1$$

Cusing
recuror
relation)

$$= 2^3 a_{n-3} - 2^2 - 2 - 1$$

$$= 2^n a_0 - 2^{n-1} - 2^{n-2} - \dots - 2$$

a solution

Hence, $\{a_n = 2^n a_0 - 2^{n-1} - 2^{n-2} - \dots - 2\}$ is

Nobel Solution of

a

of

(B)

recurrence relation using

forward substitution method is to initiate the method

the initial conditions (a_0) and obtain
in terms of

with

the value of

for the

an

the I.C.S.

backward substitution method, the method

requires to express the value of an in terms
of previous terms

obtain the expression

of an

and proceed to

in terms of

Forward Method

Co the

I.C.S.

Backward Method

a_0

$\rightarrow a_n$

Using

Forward

Backward

root

method,

we

an

contain

in

an

→ 90

and the characteristic

obtain the expression of
closed form formula that does not
any previous sequence term.

Generat
ing

Function

Generating

functions can be relations
by translating

the terms of

a

(↑

used to solve recurrence

relation for

a recurrence

n

a sequence into an equation involving generating function. This equation can then be solved to find a closed form for the generating function. From the closed form, the coefficients of the power series for the generating function can be found, solving

the

original

recurrence relation.

[Definition The generating function for the sequence

a_0, a_1, a_2, \dots

a_k ,
series

The called

is the infinite

of

real numbers

$$a_0 + a_1x + a_2x^2 + \dots = \sum_{k=0}^{\infty} a_k x^k$$

$$G(x) = a_0 + a_1x + a_2x^2 + \dots$$

$k \geq 0$

$a_k x^k$

present form of the generating for often
ordinary generating for. of $\{a_k\}$.

the

$\{a_k\}$

Example C

$\{a_k = 3^k\}$ The
generating fr

$$G(x) = \sum_{k=0}^{\infty} a_k x^k$$

$$= \sum_{k=0}^{\infty} \frac{a_k}{2^k} x^{2k}$$

Note

We can

$\{a_k \mid a_k = 0 \text{ if } k \text{ is odd}\}$
 The generating function for $G(z) = \sum_{k=0}^{\infty} a_k z^k$

$$\sum_{k=0}^{\infty} \frac{a_k}{2^k} z^{2k} = \sum_{k=0}^{\infty} \frac{a_k}{2^k} (z^2)^k = G(z^2)$$

@fak=kneh
 The generating function for

$$a_k = \sum_{j=0}^{k-1} (j+1) \cdot x^j$$

Consider the finite sequence {
3,2,1,5,6}

set the finite sequ

by setting $a_5 = 96$

$a_2 =$

Then

$a_1 = 93$

the

$a_3 = 5$

a_4

$= 6$

to infinite seq

$a_0 = 0$ and $a_5 = 96$, $a_6 = 2$,

corresponding generating

for is,

$$G(x) =$$

=

$\sum_{k=0}^{\infty} a_k x^k$

$R=0$

2

Na

such that $a_5 = a_6 = \dots = C$

95296

$$3 + 2x + x^2 + 5x^3 + 6x^4$$

Then for the finite sequence $\{a_k\} = \{90, 91, \dots, 12\}$ for fixed n , the

• corresponding infinite sequence can be represented as the generating function.

can

be

$$G(x) =$$

$$a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + a_n x^n + \dots$$

a_{n+1}

$$= a_{n+2}$$

$n.$

$=$

$+a_n a_n,$ such that

$$= 0$$

is

a

polynomial

of

$$\{1, 1, 1, 1, 1, 1\}$$

of degree

Example find the
generating f

$$G(x)$$

Example ! Find

Find

oF

$$1 + 1.2 + 1. x^2 + \dots + 1. x^n + \dots = \frac{1. x^3 + 1. x^4 + 1. x^5 + \dots + 1. x^{25} + 1. x^{26} + \dots}{x^6 - 1}$$

the generating f" in closed form the sequence

$$\{0, 1, 0, 0, 1, 0, 0, 1, \dots\}$$

$$G(\mathbf{x})$$

Example

M

14

K=0

a_K

*

}

K [90=0, 91 = 1, 92 = 0,

a3 = 0, 94 = 1,

*

*

0 + 1. x + 0. x² + 0. x² + 1. x² + 0.25 + 0. x²
+ 1. x

9 + 0. x⁸ + 0. x

$$=x+x^{-5},$$

$$=4$$

$$G(x)$$

$$+1, \\ 1.x$$

$$7$$

$$10$$

$$+x$$

$$x\left(1+x^3+x^6+x^9\right. \\ \left.x\right)$$

$$x(1-x^3)$$

find

$$+$$

$$+$$

$$10+$$

$$1-x<1$$

$$>$$

the generating for of

$$\{1, 2, 3, 4, \dots\} \quad [90 = 1,$$

$$a_1 = 2, a_2 = 3, a_3 = 4, \dots]$$

CK

$$\sum a_k x^k$$

K 20

ak x

$$1 + 2x + 3x^2 +$$

1

$$(1-x)^2$$

2

g

Consider

the

generating

fr

sequence

$\{a_n\}$

$$G(x) = \sum a_k x^k$$

Solving recurrence relations using generating functions

a_k

(13)

and

the
correspondin

Now, $x \cdot G(x)$

44

$$x^2 \cdot G(x)$$

N

$\cdot K20$

αK

$$\sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_{k+1} x^{k+1}$$

$K20$

$$x^2.$$

$$\sum_{k=0}^{\infty} a_k x^k$$

$$\sum \alpha =$$

$$\sum_{k=0}^{\infty} a_k x^k +$$

$$k=0$$

$$\sum_{k=0}^{\infty} x^k$$

$$a_k$$

$$k=0$$

$$k+1$$

$$k+2$$

and

So

on.

Example

Solve

the

$$\sum_{k=0}^{\infty} a_k x^k$$

of

we

a_k

recurrence relation $k > 1$,

a_0

$$a_0 = 2$$

Consider the corresponding generating the seq" $\{a_n\}$ as, $G(x)$
 $= \sum_{k=0}^{\infty} a_k x^k$

by substituting

$$= a_0 + a_1 x + a_2 x^2 + \dots = a_0 + \sum_{k=1}^{\infty} a_k x^k$$

Now, $R A_k = 3A_{k-1}$ in the
 k

$3\alpha_{k-1}$ KM, in the expression
expression of $G(2)$

a_k
obtain,

$$G(x) = a_0 + \sum_{k=1}^{\infty} (3\alpha_{k-1}) a_k$$

$$\rightarrow G(x) - a_0$$

$$= 3 \sum_{k=1}^{\infty} \alpha_{k-1} x^{k-1}$$

Kx

$$\Rightarrow G(x) - a_0 = 3 \sum_{k=1}^{\infty} \alpha_{k-1} x^{k-1}$$

N

$$= 3 \alpha_0 x + 3 \alpha_1 x^2 + 3 \alpha_2 x^3 + \dots$$

JV

HM

$$3x (\alpha_0 + \alpha_1 x + \alpha_2 x^2 + \dots)$$

$$3x \cdot 3x$$

a

Дакак

K20

$$3x, G(x)$$

$$G(x) - 3xG(x) = 2$$

2

ت

=>

$$G(x) =$$

$$\frac{1-3x}{2}$$

$$= 2(1-3x)=$$

$$= 2(1 + 3x$$

=1

N

1

$$2 \int 3K. x$$

K20

Comparing

with

we

obtain

Example

Solve

the

ak

The

k20

ak

(2.35). a

+

k

∞

(3x)2 + (3x)

3+

як

х

$$G(x) = \sum a_{xk}$$

аk

recurrence

k=0

$$= 2.3$$

аk

k

n

∞

$$Z(2.3k).$$

**

,

relation using
generating f

K 712

$$A_0 = 3,91 = 1$$

$$+ 3a_{k-2} \quad a_k = 2a_{k-1} \text{ corresponding}$$

generating for for the
sequence $\{a_n\}$ is
defined

$$G(x)$$

=

We consider the

\Rightarrow

a_k

ने

a_k

Since

a_k

1

a_k

$k > 2$

а так

a_k

$k = 2$

$a_s,$

k

Дак

x

$k=0$

relation

$2 a_{k-1}$

$2 a_{k-1}$

отпр
1

1

$2 a_{k-1}$

$+ 3 a_{k-2}$

$= 0$

$$3a_{k-2}$$

$$x^k z^0$$

$$3a_{k-2}$$

$$x^k$$

[Multiplying by x^k & summing
for $k = 2, 3,$

$$2 \sum_{k=2}^{\infty} a_{k-1} x^k$$

$$k=2$$

Solving individually

1st term

$$\sum_{k=2}^{\infty} a_{k-1} x^k$$

Дак як

$$k=2$$

$$k - 3 \sum_{k=2}^{\infty} a_k$$

$$= 92$$

$$2 \sum_{k=2}^{\infty} a_k$$

$$x^2 + A^3_{az}$$

тал

$$2,3,.$$

$$a_{k-2}$$

$$3+$$

x

we

get,

a_k

20

$$x+a^2x^2 + \ldots) -$$

$$(aotani$$

$$(\alpha_0+91x)$$

$$=(a$$

0

$$\sum a_k a_k$$

=

=

Kzo

$$G(x) = (a_0 + a_1 x)$$

2nd term! $2 \sum_{k=2}^{\infty} a_{k-1}$

3rd

term: $3 \sum_{k=2}^{\infty} a_{k-2}$

$\sum_{k=0}^{\infty}$

$$= 2a_1 x + 2a_2 x^2 + \dots$$

2

$$= 2x (a_1 + a_2 x + \dots)$$

2

15

.)

$$\bullet (a_0 + a_1 x + a_2 x^2 + \dots)$$

$$= 2x a_0$$

$$= 2x$$

$$= 2x, G(x)$$

a_k

-

$$G(x) - \alpha_0 - ax - 2x G(x) + 2x90$$

$$3x^2 G(2)$$

$$\rightarrow G(x) - 2x G(x) - 3x^2 G(a) = 90 + 91x^2 + 90$$

$$G(x) [1 - 2x - 3x^2]$$

=

$$3 + x + 6x$$

$$6x$$

\Rightarrow

$$G(x) =$$

$$3 - 5x$$

$$1 - 2x$$

=

=

$$3x^2$$

$$3 - 5x$$

$$(1 - 3x)(1 + x)$$

A

be determined.

B

$$(1 - 3x$$

$$3x)$$

$$1+x)$$

where

A

and

B

are to

Now,

$$G(x) = \frac{3-5x}{(1-3x)(1+x)}$$

Then

equating the

$$\frac{A(1+x) + B(1-3x)}{(1-3x)(1+x)} = \frac{3-5x}{(1-3x)(1+x)}$$

$$A(1+x) + B(1-3x) = 3-5x$$

$$\frac{3-5x}{(1-3x)(1+x)}$$

numerators, we

obtain,

$$3-5x.$$

$$A + 3B = -5.$$

$$\Rightarrow (4-B) + (4+3B)x =$$

Hence,

solving

$$A-B$$

the

$$3$$

eqns, we

$$A = 1, B = 2$$

and

obtain

Therefore,

$G(x)$

$$\frac{1}{1-3x}$$

+

$$\frac{2}{1+x}.$$

→

$$(\frac{1}{1-3x}) = 1 + 2(\frac{1}{1+x})$$

$$)=1$$

$$(1+3x+(3x)^2+(3x)^3)$$

$$=+.$$

11
NJ

$$+2(1-x+x^2-x^3$$

$$++2(1-x+x$$

$$\check{Z}$$

$$x$$

$$K$$

$$\sum 3K, \quad 2K + \sum 2.$$

$$(-1)^{xk}$$

$$K=0$$

$$3KX$$

$$K$$

$$K=0$$

$$K_x$$

$$\sum [3^*$$

$$+2.(-15]_{xx}$$

$$K=0$$

:)

$$\langle 1 | a x | x \rangle = \int (3^* + 2. (-1)^*)^{**}$$

Now,

$$G(x) =$$

$$\sum_{k=0}^{\infty} a_k x^k$$

$$\text{i.e., } \{ a_k = 3x + 2. \}$$

$$(-1)^* \}, "$$

$$k = 0, 1, 2, \dots \quad (16)$$