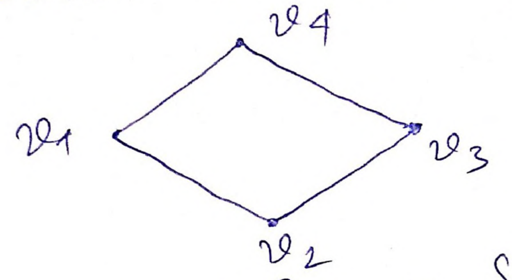
GRAPH THEORY

Definition: A graph $G=(V,E)$ or $\langle V,E\rangle$ consists of V, a non-empty set of vertices (or nodes) and E, a set of edges. Each edge has either one or two edges associated with it, called the end points.

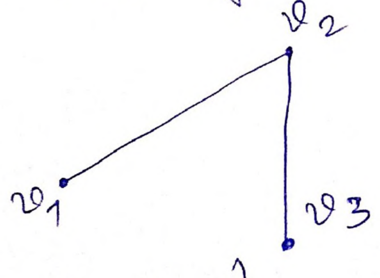


$V=\{v_1, v_2, v_3, v_4\}$, $E=\{(v_1,v_2),(v_2,v_3),(v_3,v_4),(v_4,v_1)\}$

- Consider a graph $G=(V,E)$, where V is a non-empty set.
  1. If $|V|<\infty$ or V is of finite order, then the graph G is a finite graph.
  2. If $|V|=\infty$ or V is of infinite order, then the graph G is an infinite graph.

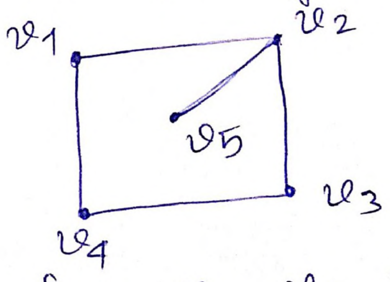
- **Simple Graph:** Consider a graph in which each edge connected two different vertices and no two edge connect the same pair of vertices. Then the graph is a simple graph.

- Ex



$V=\{v_1,v_2,v_3\}$

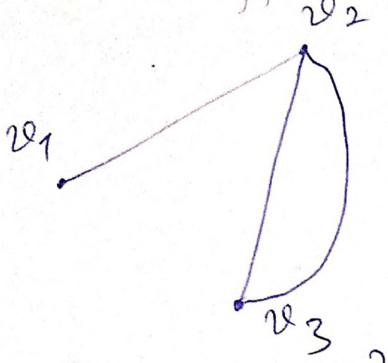
$E=\{(v_1,v_2),(v_2,v_3)\}$

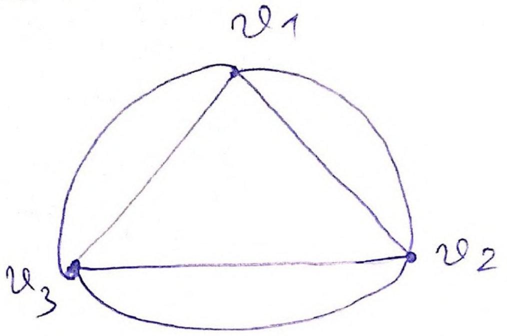


$V=\{v_1,v_2,v_3,v_4,v_5\}$

$E=\{(v_1,v_2),(v_2,v_3),(v_3,v_4),(v_4,v_1),(v_2,v_5)\}$

Multigraph: Multigraph is a graph that has multiple edges connecting the same pair of vertices, but each ~~...~~ edge connects two different vertices.





$V = \{v_1, v_2, v_3\}$

$E = \{(v_1, v_2), (v_2, v_3)\}$

The edge (v_2, v_3) has multiplicity 2.

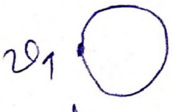
$V = \{v_1, v_2, v_3\}$

$E = \{(v_1, v_2), (v_2, v_3), (v_3, v_1)\}$

The edges (v_1, v_2), (v_2, v_3) and (v_3, v_1) have multiplicity 2.

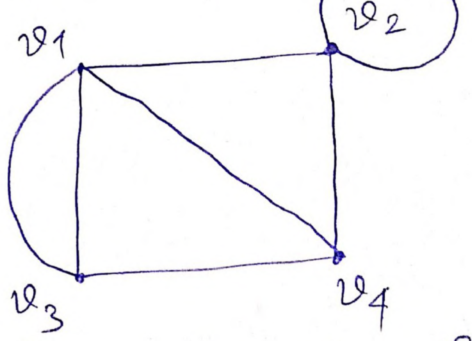
Loops: An edge that connects a vertex to itself is called a loop.

Pseudograph: A graph that may include loops and possibly contains multiple edges between the same pair of vertices is called a pseudograph.



$V = \{v_1\}$

$E = \{(v_1, v_1)\}$



$V = \{v_1, v_2, v_3, v_4\}$

$E = \{(v_1, v_2), (v_2, v_2), (v_2, v_4), (v_4, v_3), (v_3, v_1), (v_1, v_4)\}$

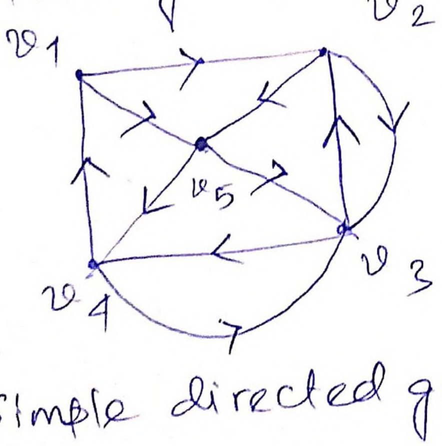
The edge (v_3, v_1) has multiplicity 2.

■ **Undirected Graph:** An undricted graph is a graph with undirected edges that are associated to an unordered pair of vertices that are end points. ③

■ **Directed Graph:** A directed graph (or digraph) (V, E) consists of a non-empty set V and a set of directed edges (or arcs) E. Each directed edge is associated with an ordered pair of vertices (u, v) that is read as start from u and end at v.

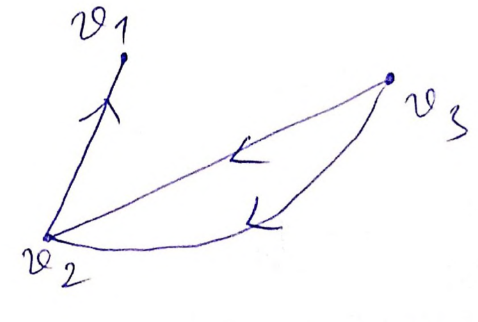
■ **simple directed graph:** Directed graphs with no loops and no multiple edges in the same direction between the same pair of vertices are called simple directed graphs.

■ **Directed Multigraph:** Directed graphs with no loops and have multiple edges in the same direction between the same pair of vertices, are called Directed Multigraph.



Simple directed graph.

No two vertices are connected by multiple edges in the same direction.



Not a simple directed graph.

The edge starts at v_3 and ends at v_2 (v_3, v_2) has multiplicity 2. So it is a directed multigraph

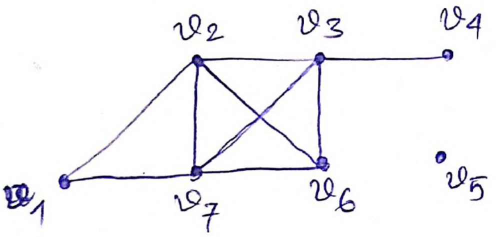
- In an undirected graph $G=(V,E)$, two vertices $u, v \in V$ are adjacent in G if u and v are end points of an edge of G.

The edge e is associated with $\{u, v\}$. Then the edge is called incident with vertices u and v.

- Degree of a vertex in an undirected graph:

Consider an undirected graph $G=(V,E)$ and $v \in V$. Then the degree of v is denoted by $\deg(v)$ and is the number of edges that are incident with it except that a loop at a vertex contributes twice to the degree.

Ex:



$\deg(v_1)=2$ $\deg(v_5)=0$
$\deg(v_2)=4$ $\deg(v_6)=3$
$\deg(v_3)=4$ $\deg(v_7)=4$
$\deg(v_4)=1$

- A vertex of deg 0 is called an isolated vertex.
- A vertex is pendant iff it has deg 1.

In the above example, the vertex v_4 is a pendant and v_5 is isolated.

Theorem | The Handshaking Lemma

Consider an undirected graph $G=(V,E)$ with e number of edges.

Then, $$2e = \sum_{v \in V} \deg(v)$$

[True even if multiple edges and loops are present]

- Each edge contributes 2 to the sum of degrees.

Ex: How many edges are there in a graph with 10 vertices each having deg 6.

→ We know that, $\sum_{v \in V} deg(v) = 10 \times 6 = 60$

$\Rightarrow 2e = 60$ [From Hand shaking Lemma].

$\Rightarrow e = 30$.

i.e., the graph has 30 edges in total.

$\boxed{\text{Theorem 2}}$ An undirected graph has even number of vertices of odd degrees.

→ Consider a graph $G = (V, E)$.

Then,
V_1 is the set of vertices of even degree
V_2 is " " " " odd degree

So, $2e = \sum_{v \in V} deg(v)$

$$\Rightarrow \underbrace{2e}_{\text{even}} = \underbrace{\sum_{v \in V_1} deg(v)}_{\text{even}} + \underbrace{\sum_{w \in V_2} deg(w)}_{\text{must be even}}$$

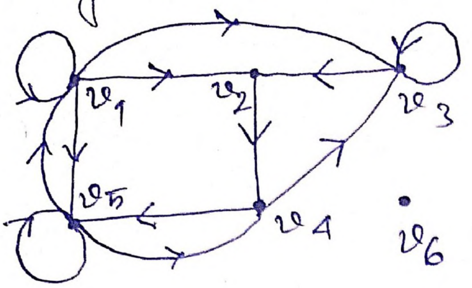
Degree of a vertex in a directed graph:

Consider a directed graph $G = (V, E)$ and an edge $(u, v) \in E$. Then u and v are defined as initial vertex and terminal vertex respectively.

For a loop. initial vertex = terminal vertex.

- In-degree ($deg^-(v)$): The in-degree of a vertex in a directed graph is the no. of edges with v as their terminal vertex.
- Out-degree ($deg^+(v)$): The out-degree of a vertex in a directed graph is the no. of edges with v as their initial vertex.

Note: A loop at a vertex contributes 1 to both in-deg and out-deg of the vertex.



$\deg^-(v_1) = 2$ $\deg^-(v_4) = 2$
$\deg^+(v_1) = 4$ $\deg^+(v_4) = 2$
$\deg^-(v_2) = 2$ $\deg^-(v_5) = 3$
$\deg^+(v_2) = 1$ $\deg^+(v_5) = 3$
$\deg^-(v_3) = 3$ $\deg^-(v_6) = 0$
$\deg^+(v_3) = 2$ $\deg^+(v_6) = 0$

$\boxed{\text{Theorem:}}$ Consider a directed graph $G = (V, E)$. Then

$$\sum_{v \in V} \deg^-(v) = \sum_{w \in V} \deg^+(w) = |E|.$$

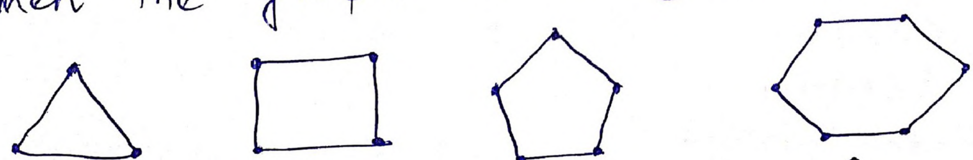
Each edge has initial vertex and termal vertex and hence contributes 1 to the summation of the in-degrees of all the vertices and 1 to the summation of the out-degrees of all the vertices.

SPECIAL GRAPHS

A. Complete Graph: Consider a simple undirected graph that contains exactly one edge between each pair of distinct vertices and no vertex is connected to itself. Then it is called a complete Graph and is denoted by K_n, where n is the no. of vertices.

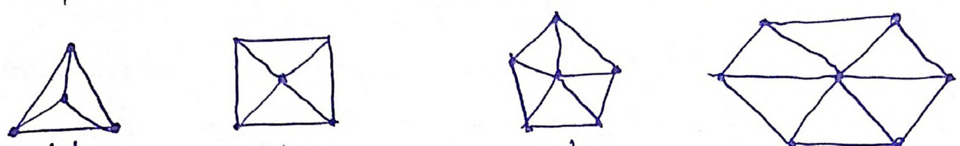
K_1 K_2 K_3 K_4 K_5

B. Cycles: Consider a graph $G = (V, E)$ with the vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E = \{(v_1, v_2), (v_2, v_3), (v_3, v_4) \ldots, (v_{n-1}, v_n), (v_n, v_1)\}$. Then the graph is a cycle and is denoted by C_n.



C_3 C_4 C_5 C_6

c. Wheels: Consider a cycle C_n with n vertices and add an additional vertex to the cycle such that the new vertex is connected to each of the n vertices in the cycle by new edges. The new graph is a ~~weel~~ wheel and is denoted by W_n.

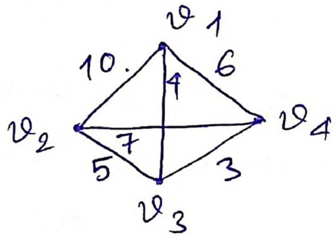


W_3 W_4 W_5 W_6

Some other graphs

Mixed graph: A graph with both directed and undirected graph is called a mixed graph.

Weighted graph: Consider a graph in which each of the edges is assigned with some weight or value. The graph is called a weighted graph.



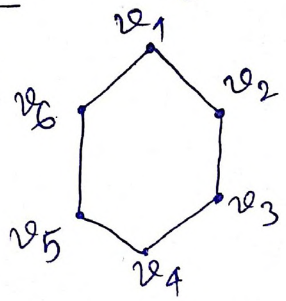
wt. is nothing but the cost of reaching from source to the destination.

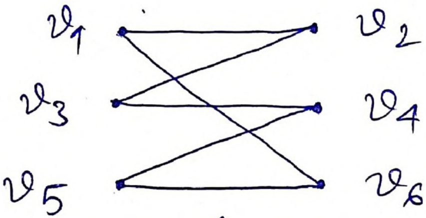
Bipartite Graph: A simple graph $G=(V,E)$ is called bipartite if the vertex set V can be partitioned into two disjoint sets V_1 & V_2 ($V=V_1 \cup V_2$), such that every edge in the ~~vertex~~ graph connects a vertex in V_1 and a vertex in V_2. furthermore no edge in G connects either two vertices in V_1 or two vertices in V_2.

(V_1, V_2) is the bipartition of the vertex set V.

Ex: C_6 is bipartite but K_3 is not.

C_6

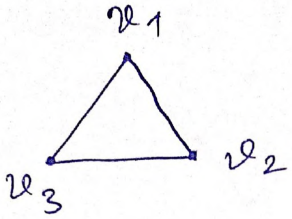




$V_1 = \{v_1, v_3, v_5\}$ $V_2 = \{v_2, v_4, v_6\}$

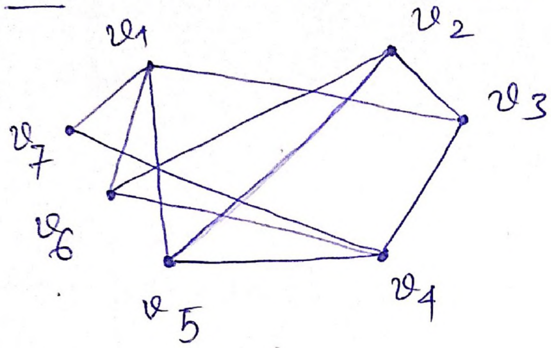
Note that, each vertex in C_6 connects a vertex in V_1 and a vertex in V_2. Also, there are no edges that connect two vertices in V_1 or two vertices in V_2.

$\underline{K_3}$

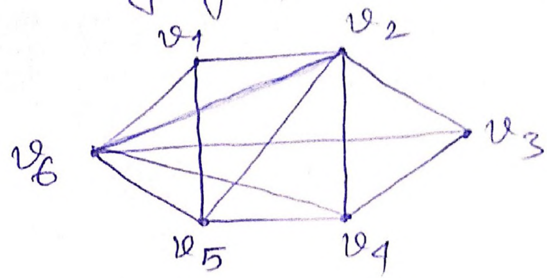


Since there are 3 vertices one of the set V_1 or V_2 will contain two vertices that will be connected by an edge. So, K_3 is not a bipartite graph.

Ex: Check whether the following graph are bipartite.



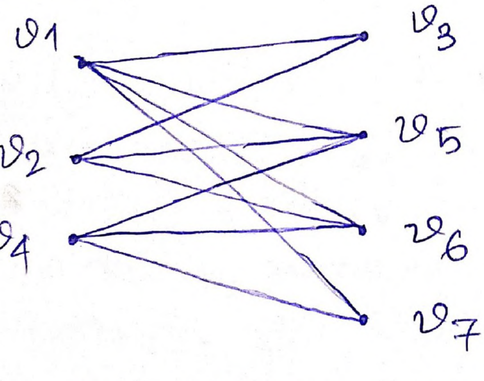
G



H

H is not a bipartite graph since it is not possible to find two disjoint set of vertices in which no edge will connect the vertices in the same set.

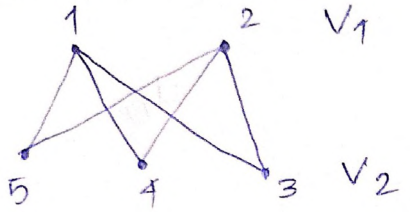
Solution:



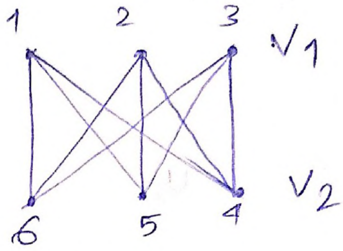
Bipartite graph

Theorem: A simple graph is bipartite iff it is possible to assign one of the two different colours to each vertex such that no two adjacent vertices are assigned the same colour.

Complete Bipartite graph: The complete bipartite graph $K_{m,n}$ is the graph that has its vertex set partitioned into two subsets of m and n vertices, respectively. There is an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.



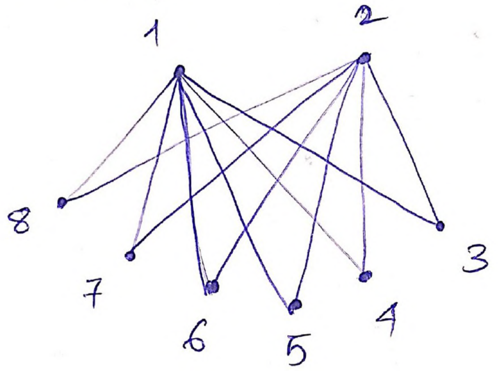
$K_{2,3}$



$K_{3,3}$

$V_1 = \{1, 2\}$, $V_2 = \{3, 4, 5\}$

$V_1 \cup V_2 = V$, $V_1 \cap V_2 = \phi$

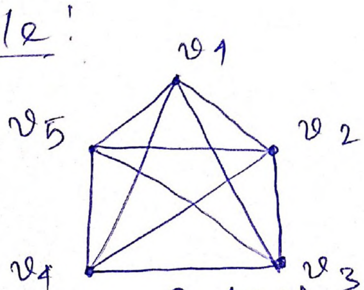


$K_{2,6}$

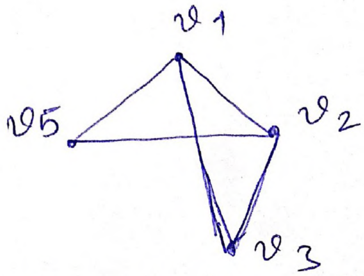
NEW GRAPHS FROM OLD

Subgraph of a graph: A subgraph of a graph $G=(V,E)$ is a graph $H=(W,F)$, where $W \subseteq V$ and $F \subseteq E$. A subgraph H of G is a proper subgraph of G if $H \neq G$.

Example:



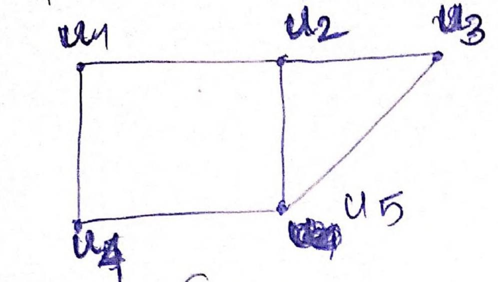
$G=(V,E)$



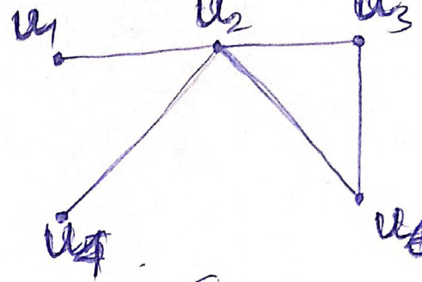
H is a subgraph of G $H = (V', E')$

$V' = \{v_1, v_2, v_3, v_5\} \subseteq V$, $E' = \{(v_1, v_2), (v_2, v_3), (v_3, v_1), (v_2, v_5)\} \subseteq E$

Union of two graph: The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and the edge set $E_1 \cup E_2$. The union is denoted by $G_1 \cup G_2$.



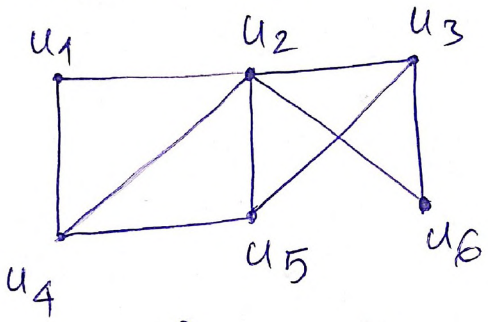
G_1



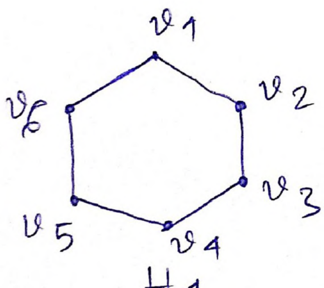
G_2

Now, $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2) = (V, E)$

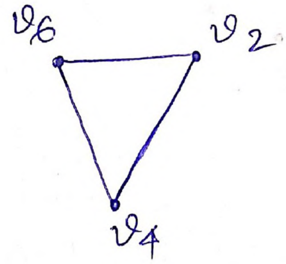
$V = \{u_1, u_2, u_3, u_4, u_5, u_6\}$



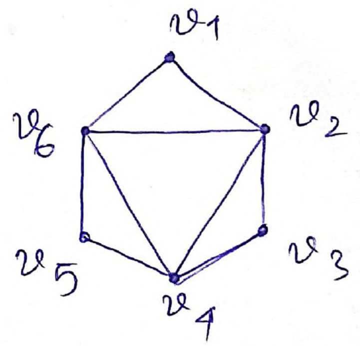
$G_1 \cup G_2$



H_1



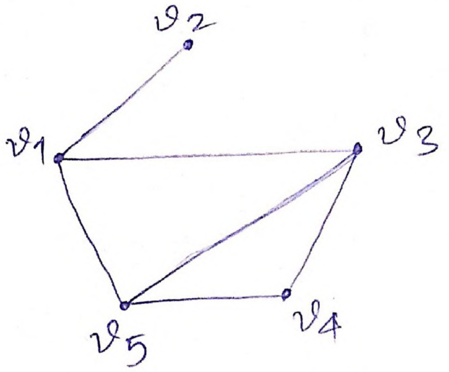
H_2



REPRESENTING GRAPHS

One way to represent a graph without multiple edges is to list all the edges of this graph. Another way is to represent a graph with no multiple edges is to use adjacency lists, that specify the vertices that are adjacent to each vertex of the graph.

Incidence Matrix: Let G = (V, E) be an undirected

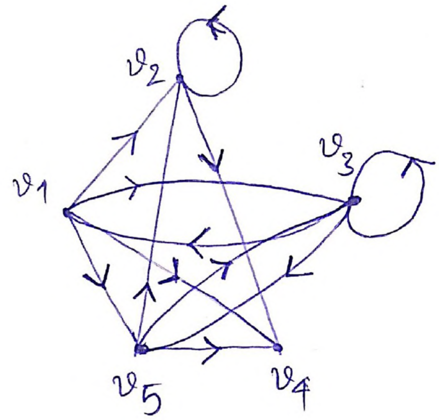


simple graph

Adjacency List

Vertex	Adjacent vertices
v_1	v_2, v_3, v_5
v_2	v_1
v_3	v_1, v_4, v_5
v_4	v_3, v_5
v_5	v_1, v_3, v_4

For directed graphs, adjacency list can be formed by listing all the vertices that are terminal vertices of edges starting at each vertex of the graph.



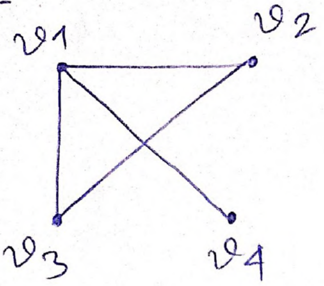
Adjacency List

Initial Vertex	Terminal Vertices
v_1	v_2, v_3, v_4, v_5
v_2	v_2, v_4
v_3	v_1, v_3, v_5
v_4	-
v_5	v_2, v_3, v_4

Adjacency Matrices: Suppose that $G = (V, E)$ is a simple graph where $|V| = n$. Suppose that the vertices of G are listed arbitrarily as $v_1, v_2, \ldots, v_n$. The adjacency matrix A of G is a binary matrix with 1 as its $(i,j)^{th}$ entry when v_i and v_j are adjacent and 0 as its $(i,j)^{th}$ entry when v_i and v_j are not adjacent. So if $A = [a_{ij}]$, then

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \text{ is an edge in } G \\ 0 & \text{otherwise.} \end{cases}$$

Ex:

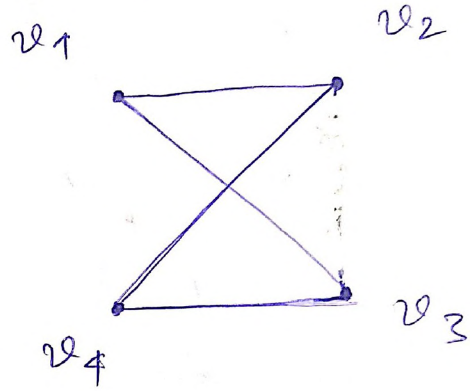


$V = \{v_1, v_2, v_3, v_4\}$

$$A = \begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

Ex: Let $A = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix}$. find the graph G with respect to the ordering of vertices as v_1, v_2, v_3 and v_4.

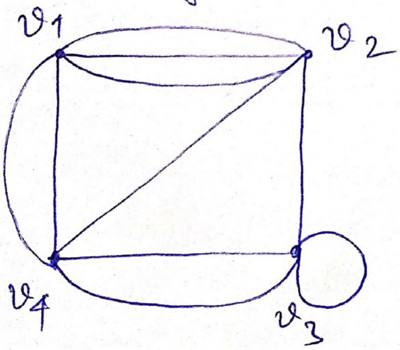
$$A = \begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$



$a_{11} = 0$ means there is no edge between v_1 and v_1, i.e, no loop at v_1.

$a_{34} = 1$ means there is an edge between v_3 and v_4.

Ex: Find the adjacency matrix to represent the following pseudograph.



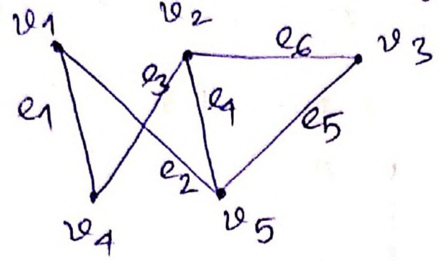
$$A = \begin{array}{cc} & \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \end{array} \\ \begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array} & \begin{bmatrix} 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \end{array}$$

2

Incidence Matrix: Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $M = [m_{ij}]$, where

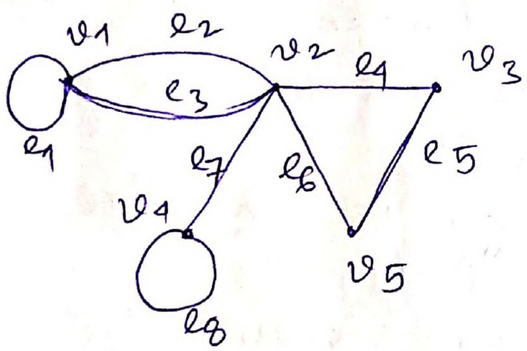
$$m_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i \\ 0 & \text{otherwise.} \end{cases}$$

Ex:



$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Ex.



$$M = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{cccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 & e_8 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \end{bmatrix} \end{array}$$

- **Mixed Graph:** A graph with both directed and undirected edges is called a Mixed Graph.

- Graph terminology:

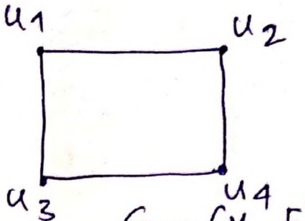
Type	Edges	Multiple edges allowed?	Loops allowed?
i) Simple graph	Undirected	No	No
ii) Multigraph	"	Yes	No
iii) Pseudograph	"	"	Yes
iv) Simple directed graph	Directed	No	No
v) Directed multigraph	"	Yes	Yes
vi) Mixed graph	both directed and undirected	Yes	Yes

Isomorphism of Graphs

We often need to know whether it is possible to draw two graphs in the same way.

Definition: The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are isomorphic if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b are adjacent in G_1 if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a, b in V_1. Such a function f is called an isomorphism.

Ex: Show that the following graphs are isomorphic.



$G = (V, E)$

$H = (W, F)$

Consider the function f,

$f(u_1) = v_1$, $f(u_2) = v_4$, $f(u_3) = v_3$, $f(u_4) = v_2$.

Then f is an one-to-one correspondence between V and W.

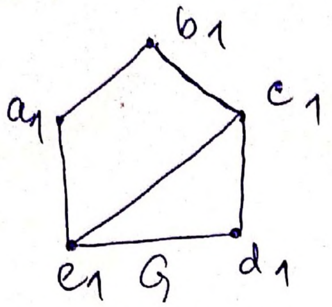
Now we consider the adjacency in both the graphs.

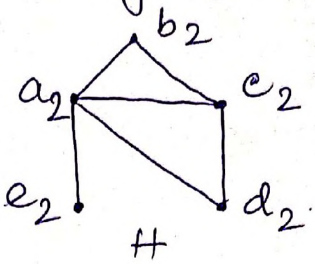
In G	In H
(u_1, u_2) is an edge	(v_1, v_4) is an edge, $v_1 = f(u_1)$, $v_4 = f(u_2)$
(u_1, u_3) " " "	(v_1, v_3) is an edge
(u_2, u_4) " " "	(v_4, v_2) is " "
(u_4, u_3) " " "	(v_2, v_3) " " "

Hence, the graphs G and H are isomorphic and is denoted by $G \cong H$.

The isomorphism $\cong$ is an equivalence relation.

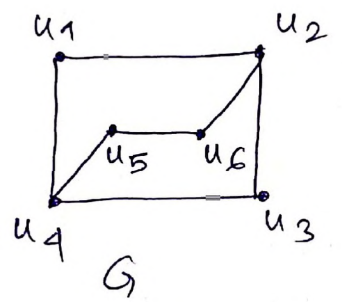
Ex : Consider the following graphs.

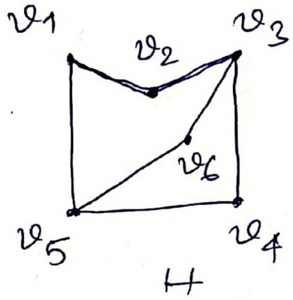




Note that the order of the sets of vertices and sets of edges are 5 and 6 in both the graphs.
However, the graph H contains a vertex e_2 with degree 1 ($\deg(e_2) = 1$), whereas G has no vertex of deg 1. Therefore, $G \not\cong H$ or G and H are not isomorphic.

Ex :





$|V| = 6$, $|E| = 7$ for both G and H.
The deg of the vertices are also matching in both G and H.
So, f may exist.
Consider the following function.

$f(u_1) = v_6$, $f(u_2) = v_3$, $f(u_3) = v_4$
$f(u_4) = v_5$, $f(u_5) = v_1$, $f(u_6) = v_2$

Now ~~by~~ comparing the adjacency matrices of both the graphs w.r.t the assignment f, we can determine that f is one-one, ~~and~~ onto and the adjacency matches in both the graphs accordingly. Hence, $G \cong H$.