Finite-Dimensional Vector Space

DJN_DL

Given a list of vectors $v_1, v_2, ..., v_n \in V(F)$*, and a list of scalars* $c_1, c_2, ..., c_n (i = 1, 2, ..., n) \in F$

$$c_1 v_1 + c_2 v_2 + ... + c_n v_n$$

is then called linear combination *of* $\{v_1, v_2, ..., v_n\}$*. All linear combinations of a certain list of vectors form a set called* spanning set*, notation by* $\text{span}\{v_1, v_2, ..., v_n\}$.

Span is the smallest containing subspace

Theorem 1. *The span of a list of vectors is the smallest subspace containing all the vectors of the list.*

Proof. For a list of vectors $v_1, v_2, ..., v_n \in V$

Let $a_i = 0$ for $\quad i = 1, 2, ..., n$, $a_1 v_1 + a_2 v_2 + ... + a_n v_n = 0 \in \text{span}(v_1, v_2, ..., v_n)$

$\{b_i\}$ for $i = 1, 2, ..., n$ is another list of scalars than $\{a_i\}$, $a_i \neq 0$ here

$$\underbrace{(b_1 v_1 + ... + b_n v_n)}_{\in \text{span}} + \underbrace{(a_1 v_1 + a_2 v_2 + ... + a_n v_n)}_{\in \text{span}} = (a_1 + b_1) v_1 + ... + (a_n + b_n) v_n \in \text{span}\{v_1, v_2, ..., v_n\}$$

λ is a scalar in F

$$\lambda(a_1 v_1 + ... + a_n v_n) = \lambda a_1 v_1 + ... + \lambda a_n v_n \in \text{span}\{v_1, v_2, ..., v_n\}$$

Thus $\text{span}\{v_1, v_2, ..., v_n\}$ is a subspace of V, and since any subspace containing all vectors of $\{v_i\}$ must contain their span (from the definition of vector space), any subspace other than the span is "larger" (containing more vectors) than the span.

□

If a list of vectors form a span which is equal to V, in other words, $\text{span}\{v_1, v_2, ..., v_n\} = V$, we say $v_1, v_2, ..., v_n$ span V. If a vector space can be spanned by a finite set of vectors within itself, it is a *finite dimensional vector space*, otherwise, an *infinite vecotr space.* (Not really defined yet here)

Polynomial $P(F)$

A function $P: F \to F$ is said to be a *polynomial* with coeffecients if there are $a_0, a_1, a_2, ..., a_m \in F$ s.t.

$$p(x) = a_0 + a_1 x + a_2 x^2 + ... + a_m x^m \text{ for } x \in F$$

$P(F)$ is the set of all polynomials with coefficients in F, it is a subspace of F^F. Every polynomial can be uniquely determined by its coefficients.

A polynomial is said to have a *degree* of m if there are coefficients $a_m \neq 0, a_i (i > m) = 0$ s.t.

$$p(x) = a_0 + a_1 x + ... + a_m x^m + \underbrace{(a_i x^{m+1} + ...)}_{=0}$$

Zero polynomial is defined to have a degree $-\infty$

$P_m(F)$ is defined as the set containing all polynomials of which degree is no more than m.

What if a vector in a span can be uniquely represented, i.e. only one list of scalars makes it a linear combination of a certain list of vectors. Here we define *linear independence*.

Suppose a list of vectors $v_1, v_2, ..., v_n$ in V_F, if the only choice to make $a_1 v_1 + a_2 v_2 + ... + a_n v_n$ is $a_1 = a_2 = ... = a_n = 0$, then $v_1, v_2, v_3, ..., v_n$ are *linear independent.*

The definition is just a unique of representatin of zero. Suppose there is a nonzero vector $v \in \text{span}\{v_1, v_2, ..., v_n\}$, and that it cannot be uniquely represented, that is there exist $\{a_i\}$ $\{b_i\}$ s.t.

$$v = a_1 v_1 + a_2 v_2 + \ldots + a_n v_n \tag{1}$$

$$v = b_1 v_1 + b_2 v_2 + \ldots + b_n v_n \tag{2}$$

$(1) - (2)$: $(a_1 - b_1)v_1 + \ldots + (a_n - b_n)v_n = 0$

Since $\{v_i\}$ is linearly independent, $a_i - b_i = 0\,(i = 1, 2, \ldots, n) \rightarrow a_i = b_i$, thus v must be uniquely represented by $\{v_i\}$.

Conversely, if all vectors can be uniquely represented, choose 0, all coefficients must be zero, implying linear independence (the definition of linear independence). Thus, for a list of vectors, linear independence is equivalent to unique representation. And *linear independence* is so defined that we can coefficients other than all zeros s.t. $a_1 v_1 + \ldots. + a_n v_n = 0$.