

# Compact MILP formulations for the $p$ -center problem

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**Abstract.** The  $p$ -center problem consists in selecting  $p$  centers among  $M$  to cover  $N$  clients, such that the maximal distance between a client and its closest selected center is minimized. For this problem we propose two new and compact integer formulations.

Our first formulation is an improvement of a previous formulation. It significantly decreases the number of constraints while preserving the optimal value of the linear relaxation. Our second formulation contains less variables and constraints but it has a weaker linear relaxation bound.

We besides introduce an algorithm which enables us to compute strong bounds and significantly reduce the size of our formulations.

Finally, the efficiency of the algorithm and the proposed formulations are compared in terms of quality of the linear relaxation and computation time over instances from OR-Library.

## 1 Introduction

We consider  $N$  clients  $\{C_1, \dots, C_N\}$  and  $M$  potential facility sites  $\{F_1, \dots, F_M\}$ . Let  $d_{ij}$  be the distance between  $C_i$  and  $F_j$ . The objective of the  $p$ -center problem is to open up to  $p$  facilities such that the maximal distance (called *radius*) between a client and its closest selected site is minimized.

This problem is very popular in combinatorial optimization and has many applications. We refer the reader to the recent survey [2]. Very recent publications include [7, 6] which provide heuristic solutions and [3] on an exact solution method.

In this paper, we will focus on mixed-integer linear programming formulations of the  $p$ -center problem.

Let  $\mathcal{M}$  and  $\mathcal{N}$  respectively be the sets  $\{1, \dots, M\}$  and  $\{1, \dots, N\}$ . The most classical formulation, denoted by  $(P_1)$ , for the  $p$ -center problem (see for example [4]) considers the following variables:

- $y_j$  is a binary variable equal to 1 if and only if  $F_j$  is open;
- $x_{ij}$  is a binary variable equal to 1 if and only if  $C_i$  is assigned to  $F_j$ ;
- $R$  is the radius.

$$(P_1) \left\{ \begin{array}{ll} \min R & (1a) \\ \text{s.t. } \sum_{j=1}^M y_j \leq p & (1b) \\ \sum_{j=1}^M x_{ij} = 1 & i \in \mathcal{N} \\ x_{ij} \leq y_j & i \in \mathcal{N}, j \in \mathcal{M} \\ \sum_{j=1}^M d_{ij} x_{ij} \leq R & i \in \mathcal{N} \\ x_{ij}, y_j \in \{0, 1\} & i \in \mathcal{N}, j \in \mathcal{M} \\ r \in \mathbb{R} & \end{array} \right.$$

Constraint (1b) ensures that no more than  $p$  facilities are opened. Each client is assigned to exactly one facility through Constraints (1c). Constraints (1d) link variables  $x_{ij}$  and  $y_j$  while (1e) ensure the coherence of the objective.

A more recent formulation, denoted by  $(P_2)$ , was proposed in [5]. Let  $D^0 < D^1 < \dots < D^K$  be the different  $d_{ij}$  values  $\forall i \in \mathcal{N} \forall j \in \mathcal{M}$ . Note that, if many distances  $d_{ij}$  have the same value,  $K$  may be significantly lower than  $M \times N$ . Let  $\mathcal{K}$  be the set  $\{1, \dots, K\}$ . Formulation  $(P_2)$  is based on the variables  $y_j$ , previously introduced, and one binary variable  $z^k$ , for each  $k \in \mathcal{K}$ , equals to 1 if and only if the optimal radius is greater than or equal to  $D^k$ :

$$(P_2) \left\{ \begin{array}{l} \min D^0 + \sum_{k=1}^K (D^k - D^{k-1}) z^k \\ \text{s.t. } 1 \leq \sum_{j=1}^M y_j \leq p \\ z^k + \sum_{j : d_{ij} < D^k} y_j \geq 1 \quad i \in \mathcal{N}, k \in \mathcal{K} \\ y_j, z^k \in \{0, 1\} \quad j \in \mathcal{M}, k \in \mathcal{K} \end{array} \right. \quad (2a)$$

$$(2b)$$

$$(2c)$$

Constraints (2c) ensure that if no facility located at less than  $D^k$  of client  $C_i$  is selected, then the radius must be greater than or equal to  $D^k$ .

This formulation has been proved to be tighter than  $(P_1)$  [5]. However, its size strongly depends on the value  $K$  (*i.e.*, the number of distinct distances  $d_{ij}$ ).

It also has recently been adapted to the  $p$ -dispersion problem which consists in selecting  $p$  facilities among  $N$  such that the minimal distance between two selected facilities is maximized [8].

A last formulation, that can be deduced from  $(P_2)$  by a change of variables, has been recently introduced [3] and named  $(P_4)$ . It contains, for all  $k \in \mathcal{K}$ , a binary variable  $u_k$  equal to 1 if and only if the optimal radius is  $D^k$  (*i.e.*,  $u_k = z^k - z^{k+1}$  and  $z^k = \sum_{q=k}^K u_q$ ):

$$(P_4) \left\{ \begin{array}{l} \min \sum_{k=1}^K D^k u_k \\ \text{s.t. } (2b) \\ \sum_{j : d_{ij} \leq D^k} y_j \geq \sum_{q=1}^k u_q \quad i \in \mathcal{N}, k \in \mathcal{K} \\ \sum_{k=1}^K u_k = 1 \quad (3c) \\ y_j, u_k \in \{0, 1\} \quad j \in \mathcal{M}, k \in \mathcal{K} \end{array} \right. \quad (3a)$$

$$(3b)$$

They also proposed a weaker version of this formulation, called  $(P_3)$ , obtained by replacing the left-hand side of constraints (3b) by  $u_k$ . They proved that  $(P_4)$  leads to the same linear relaxation bound and has the same size as  $(P_2)$ .

The rest of the paper is organized as follows. Section 2 presents our two new formulations. In Section 3 we introduce an algorithm. Finally, Section 4 describes numerical results on instances from the OR-Library.

## 2 Our new formulations

### 2.1 Formulation $(CP_1)$

In  $(P_2)$ , for all  $k \in \mathcal{K}$ , variable  $z^k$  is equal to 1 if and only if the optimal radius is greater than or equal to  $D^k$ . As a consequence, the following constraints are valid

$$z^k \geq z^{k+1} \quad k \in \{1, \dots, K-1\}. \quad (4)$$

We first show that these inequalities are redundant for  $(P_2)$ . Let  $(P'_2)$  be the formulation obtained when constraints (4) are added to  $(P_2)$  and let  $v(\overline{F})$  be the optimal value of the linear relaxation of a given formulation  $F$ . We now prove that adding constraints (4) does not improve the quality of the linear relaxation.

**Proposition 1.**  $v(\overline{P'_2}) = v(\overline{P_2})$

*Proof.* We show that an optimal solution  $(\tilde{y}, \tilde{z})$  of the relaxation of  $(P_2)$  satisfies (4). For each distance  $D^k$  there exists a client  $i(k)$  such that

$$\tilde{z}^k + \sum_{j : d_{i(k)j} < D^k} \tilde{y}_j = 1 \quad (5)$$

otherwise  $\tilde{z}^k$  can be decreased and  $(\tilde{y}, \tilde{z})$  is not optimal.

We now assume that  $\tilde{z}^{k-1} < \tilde{z}^k$  for some index  $k \in \{2, \dots, K\}$ . It follows that

$$\tilde{z}^{k-1} + \sum_{j : d_{i(k)j} < D^{k-1}} \tilde{y}_j < \tilde{z}^k + \sum_{j : d_{i(k)j} < D^k} \tilde{y}_j = 1$$

The last equality follows from (5). Therefore, constraints (2c) for  $i(k)$  and  $k - 1$  is violated.  $\square$

We now prove that a large part of constraints (2c) are redundant in  $(P'_2)$ .

Let  $N_i^k$  be the set of facilities located at less than  $D^k$  from client  $C_i$ . We can observe that  $N_i^k$  is included in  $N_i^{k+1}$ , for all  $k \in \mathcal{K}$ . Moreover,  $N_i^k$  is equal to  $N_i^{k+1}$  if and only if there is no facility at distance  $D^k$  from client  $C_i$ . Let  $S_i$  be the set of indices  $k \in \{1, \dots, K - 1\}$  such that  $N_i^k$  is different from  $N_i^{k+1}$ . Observe that  $|S_i| \leq \min(M, K)$ .

We define Formulation  $(CP_1)$  as Formulation  $(P'_2)$  where only the constraints (2c) such that  $k \in S_i$  or  $k = K$  are kept.

$$(CP_1) \left\{ \begin{array}{ll} \min D^0 + \sum_{k=1}^K (D^k - D^{k-1}) z^k & (6a) \\ \text{s.t. (2b), (4)} & \\ z^k + \sum_{j : d_{ij} < D^k} y_j \geq 1 & i \in \mathcal{N}, k \in S_i \cup \{K\} \\ y_j, z^k \in \{0, 1\} & j \in \mathcal{M}, k \in \mathcal{K} \end{array} \right. \quad (6b)$$

The number of constraints is dominated by the number of constraints (6b). This number is bounded by both  $NM$  and  $NK$ .

The following proposition proves that  $(CP_1)$  is a valid formulation.

**Proposition 2.**  $(CP_1)$  is a valid formulation of the p-center problem.

*Proof.* We show that the constraints removed from  $(P'_2)$  are dominated. If  $N_i^k = N_i^{k+1}$ , then  $\sum_{j : d_{ij} < D^k} y_j = \sum_{j : d_{ij} < D^{k+1}} y_j$ . Since  $z^k \geq z^{k+1}$ , we have:

$$z^k + \sum_{j : d_{ij} < D^k} y_j \geq z^{k+1} + \sum_{j : d_{ij} < D^{k+1}} y_j \geq 1.$$

As a consequence, the constraint (2c) associated with  $i$  and  $k$  is dominated by the one associated with  $i$  and  $k + 1$ .  $\square$

We now prove that Formulations  $(P_2)$  and  $(CP_1)$  lead to the same bound by linear relaxation.

**Proposition 3.**  $v(\overline{CP_1}) = v(\overline{P_2})$ .

*Proof.* The arguments used in the proof of Proposition 2 can be used again to show that the constraints removed from  $(P'_2)$  do not impact the value of the linear relaxation.  $\square$

To sum up,  $(CP_1)$  is a valid formulation that has the same LP bound as  $(P_2)$ . However, as detailed in Table 1, Formulation  $(CP_1)$  is much smaller since it reduces the number of constraints by a factor of up to  $N$ .

## 2.2 Formulation $(CP_2)$

We now introduce a second formulation, denoted by  $(CP_2)$ , which contains less variables and constraints than  $(CP_1)$ .

We replace the  $K$  binary variable  $z^k$  with a unique general integer variable  $r$  which represents the index of a radius:

$$(CP_2) \left\{ \begin{array}{ll} \min r \\ \text{s.t. (2b)} \\ r + k \sum_{j : d_{ij} < D^k} y_j \geq k & i \in \mathcal{N}, k \in S_i \cup \{K\} \\ y_j \in \{0, 1\} & j \in \mathcal{M} \\ r \in \{0, \dots, K\} \end{array} \right. \quad (7a)$$

Constraints (7a) play a similar role to Constraints (6b).

Formulation  $(CP_2)$  does not directly provide the value of the optimal radius  $R$  but its index  $r$  such that  $D^r = R$ . We now prove that Formulation  $(CP_2)$  is valid.

**Proposition 4.**  $(CP_2)$  is a valid formulation of the  $p$ -center problem.

*Proof.* Let  $(\tilde{y}, \tilde{z})$  be an integer solution of  $(CP_1)$ . We first show that there exists an integer solution  $(\bar{y}, \bar{r})$  of  $(CP_2)$  which provides the same radius by setting  $\bar{y} = \tilde{y}$  and  $\bar{r} = \sum_{k=1}^K \tilde{z}^k$ . We need to prove that constraints (7a) are satisfied. We know that

$$\tilde{z}^k + \sum_{j : d_{ij} < D^k} \tilde{y}_j \geq 1$$

is satisfied for any client  $C_i$  and any distance  $D^k$ .

If  $\tilde{z}^k$  is equal to 0, the corresponding Constraint (7a) is satisfied, as  $\sum_{j : d_{ij} < D^k} \tilde{y}_j \geq 1$ . Otherwise, the same result is obtained since the  $\tilde{z}^k$  variables are ordered in decreasing order which leads to  $\bar{r} \geq k$ . These two solutions provide the same radius as  $D^0 + \sum_{k=1}^K (D^k - D^{k-1}) \tilde{z}^k = D^{\sum_{k=1}^K \tilde{z}^k}$ .

We now prove that for any solution  $(\tilde{y}, \tilde{r})$  of  $(CP_2)$  there exists an equivalent solution  $(\bar{y}, \bar{z})$  of  $(CP_1)$ . We set  $\bar{y} = \tilde{y}$  and  $\bar{z}^k = 1$  if and only if  $\tilde{r} \geq k$ . Constraint

$$\tilde{r} + k \sum_{j : d_{ij} < D^k} \tilde{y}_j \geq k \quad (8)$$

is satisfied for any  $k \in \mathcal{K}$ . If  $\tilde{r}$  is lower than  $k$ , then at least one variable  $\tilde{y}_j$  from equation (8) is equal to 1 and the corresponding constraint (6b) is satisfied. Otherwise,  $\bar{z}^k$  is equal to 1 and the same conclusion is reached.  $\square$

We now prove that the linear relaxation of  $(CP_1)$  is stronger than the one of  $(CP_2)$ .

**Assumption 1.** We shall suppose  $D^0 = 0$  and  $\forall k \in \mathcal{K}, D^k - D^{k-1} = 1$ .

This assumption is not restrictive, one can transform any instance by replacing any distance  $D^k$  by its rank  $k$ . The transformed problem is equivalent as if the optimal radius is  $D^{k^*}$ , then the optimal solution of the transformed problem is  $k^*$ .

Under this assumption, problems  $(CP_1)$  and  $(CP_2)$  have the same optimal values, both of them compute the rank of the optimal radius.

**Proposition 5.** Let  $\overline{CP_1}$  and  $\overline{CP_2}$  respectively be the LP relaxation of  $(CP_1)$  and  $(CP_2)$ ,  $v(\overline{CP_1}) \geq v(\overline{CP_2})$  under Assumption 1.

*Proof.* Let  $(\tilde{y}, \tilde{z})$  be a solution of  $\overline{CP_1}$ . We build a solution  $(\bar{y}, \bar{r})$  of  $\overline{CP_2}$  with the same value. We take  $\bar{y} = \tilde{y}$  and  $\bar{r} = \sum_{k=1}^K \tilde{z}^k$ .

We need to prove that constraints (7a) are satisfied.

Since the  $z^k$  variables are ordered in decreasing order by Constraints 4, it follows that  $\bar{r} \geq k \tilde{z}^k \forall k \in \mathcal{K}$ . This and Constraints (2c) imply that Constraints (7a) are satisfied.  $\square$

Table 1 summarizes the size of the previously mentioned formulations.

Formulation	# of variables	# of constraints
$(P_1)$	$\mathcal{O}(NM)$	$\mathcal{O}(NM)$
$(P_2), (P_3), (P_4)$	$\mathcal{O}(M + K)$	$\mathcal{O}(NK)$
$(CP_1)$	$\mathcal{O}(M + K)$	$\mathcal{O}(\min(NM, NK))$
$(CP_2)$	$\mathcal{O}(M)$	$\mathcal{O}(\min(NM, NK))$

Table 1: Size of the four formulations ( $K \leq NM$ ).

### 3 A two-step resolution algorithm

We present, in this section, a two-step algorithm to solve more efficiently the  $p$ -center problem.

Let  $lb$  be a lower bound of the optimal radius. We suppose that  $lb$  is one of the distances  $D^k$  since, otherwise,  $lb$  can be set to the next distance. All the distances  $d_{ij}$  lower than  $lb$  can be replaced by  $lb$ .

Similarly, all the distances  $d_{ij}$  greater than an upper bound  $ub$  can be replaced by  $ub + 1$  in order to discard solutions of value greater than  $ub$ .

The size of Formulations  $(P_2)$  and  $(CP_1)$  strongly depends on  $K$ . This value can be reduced by identifying lower and upper bounds. Such bounds can easily be obtained, as mentioned in [5].

Our resolution algorithm, depicted in Figure 1, can be applied to any formulation  $F$  of the  $p$ -center problem including  $(P_1)$ ,  $(P_2)$ ,  $(P_3)$ ,  $(P_4)$ ,  $(CP_1)$  and  $(CP_2)$ . It is mainly based on the idea that whenever the optimal value  $\bar{v}$  of the linear relaxation of  $F$  is not equal to an existing distance, then there exists  $k \in K$  such that  $D^{k-1} < \bar{v} < D^k$ . In that case,  $D^k$  constitutes a stronger lower bound than  $\bar{v}$  and the linear relaxation can be solved again. This process is repeated until an existing distance is obtained as the optimal value of the linear relaxation. This constitutes Step 1 of the algorithm.

The bound obtained when applying this algorithm over  $(P_2)$  or  $(CP_1)$  corresponds to the one called  $LB^*$ , computed by a binary search algorithm in [5].

Step 1 can be further improved by introducing the notion of *dominated clients* and *dominated facilities* within some reduction rules. A facility  $F_a$  is dominated if there exists another facility  $F_b$  such that  $d_{ia} \geq d_{ib}$  for all clients  $i$ . Such a facility can be removed as it will always be at least as interesting to assign a client to  $F_b$  than to  $F_a$ . Similarly, a client  $C_a$  is said to be dominated if there exists another client  $C_b$  such that  $d_{aj} \leq d_{bj}$  for all facilities  $j$ . Dominated clients can also be ignored.

Instructions 3 and 4 are repeated since new dominated clients and facilities may be found when a bound is improved, and vice versa.

Step 2 of Algorithm 1 consists in solving Formulation  $F$  to optimality with the improved bounds  $lb$  and  $ub$  computed in Step 1.

#### Algorithm 1:

```

F: formulation of the  $p$ -center problem
p: maximal number of centers
d: distances
lb, ub: initial bounds
Result: The optimal radius
// Step 1
1 repeat
2   repeat
3     Remove dominated clients and facilities // Reduction rules
4      $(lb, ub) \leftarrow$  Compute bounds
5   until  $lb$  and  $ub$  are not improved and no more dominated clients or facilities have been found
6    $\bar{v} \leftarrow$  SolveLinearRelaxation( $F$ ,  $lb$ ,  $ub$ )
7    $lb \leftarrow \min_k\{D^k : \bar{v} \leq D^k\}$ 
8 until  $\bar{v} = lb$  // until  $\bar{v}$  is one of the existing distances
// Step 2
9  $r^* \leftarrow$  SolveOptimally( $F$ ,  $lb$ ,  $ub$ )
10 return  $r^*$ 

```

Figure 1: Algorithm used to solve the  $p$ -center problem through  $F$ , a  $p$ -center formulation.

		(P <sub>1</sub> )	(P <sub>2</sub> )	(CP <sub>1</sub> )	(CP <sub>2</sub> )
<b>Instance 1</b>	number of variables	10101	286	286	101
	number of constraints	12209	18602	6089	5903
	(LB <sub>0</sub> = 0)	LP bound	97,57	106,54	106,54
	(UB <sub>0</sub> = 186)	resolution time (s)	9,14	251,28	<b>3,16</b>
<b>Instance 2</b>	number of variables	10101	277	277	101
	number of constraints	12473	17702	6094	5917
	(LB <sub>0</sub> = 0)	LP bound	76,72	85,68	85,68
	(UB <sub>0</sub> = 178)	resolution time (s)	15,69	47,31	<b>2,99</b>
<b>Instance 3</b>	number of variables	10101	305	305	101
	number of constraints	11293	20502	6852	6647
	(LB <sub>0</sub> = 0)	LP bound	73,24	83,28	83,28
	(UB <sub>0</sub> = 205)	resolution time (s)	11,68	21,02	<b>2,85</b>
<b>Instance 4</b>	number of variables	10101	299	299	101
	number of constraints	12009	19902	6403	6204
	(LB <sub>0</sub> = 0)	LP bound	54,55	64,16	64,16
	(UB <sub>0</sub> = 204)	resolution time (s)	3,19	43,02	<b>1,64</b>
<b>Instance 5</b>	number of variables	10101	270	270	101
	number of constraints	11777	17002	6263	6093
	(LB <sub>0</sub> = 0)	LP bound	30,37	37,82	37,82
	(UB <sub>0</sub> = 169)	resolution time (s)	1,93	25,10	<b>1,66</b>

Table 2: Size and resolution times (1 thread) of the formulations for the five first OR-Library instances with  $lb = LB_0$  and  $ub = UB_0$ .

## 4 Numerical results

We implement Formulations  $(P_1)$ ,  $(P_2)$ ,  $(CP_1)$  and  $(CP_2)$  as well as Algorithm 1 on an Intel XEON E3-1280 with 3,5 GHz and 32Go of RAM with the Java API of CPLEX 12.7. Following several authors, we consider instances from the OR-Library [1].

### 4.1 Comparing sizes and computation times on 5 instances

Table 2 presents a comparison of the sizes of the four formulations on the five first instances of the OR-Library with  $N = M = 100$ . We use the initial lower bound  $LB_0 = \max_{i \in \mathcal{N}} \min_{j \in \mathcal{M}} d_{ij}$  and initial upper bound  $UB_0 = \min_{j \in \mathcal{M}} \max_{i \in \mathcal{N}} d_{ij}$  introduced in [5].

As expected, the number of variables in  $(CP_1)$  and  $(P_2)$  are equal and are significantly lower than in  $(P_1)$ . Formulation  $(P_2)$  has more constraints than Formulation  $(P_1)$ . Formulation  $(CP_1)$  has by far less constraints than  $(P_2)$ . All this explains why  $(CP_1)$  has the best performances in every aspect.

Formulation  $(CP_2)$  is the most compact but this does not fully compensate the poor quality of its LP bound.

### 4.2 Relaxation and computation times on the 40 OR-Library instances

In Table 3, we perform a larger comparison with stronger bounds  $lb$  and  $ub$  equal to the bounds  $LB_1$  and  $UB_1$  introduced in [5]. The resolution is then performed by CPLEX with its default parameters but with a maximal CPU time of 1 hour.

The first column is the instance number. The three following columns provide  $N$ ,  $p$  and the optimal value of the instances ( $N = M$  in these instances). Columns 5 and 6 contain the initial bounds  $LB$  and  $UB$ . For each formulation, column “b” corresponds the optimal value of the linear relaxation and column “t” to the resolution time in seconds.

We can first observe that Formulations  $(CP_1)$  and  $(P_2)$  solve all the 40 instances within 1 hour while ten instances are not solved with  $(P_1)$  and one instance is not solved with  $(CP_2)$ . We can even observe that  $(CP_1)$  solves the whole set of instances in less than 50 minutes and  $(P_2)$  in less than 85 minutes.

Formulation  $(P_2)$  outperforms  $(CP_1)$  mainly on instances 36 and 39. This is possibly due to some difficulty of the solver to find good feasible solutions.

### 4.3 Results of Algorithm 1

Table 4 presents the results of Algorithm 1 with formulations  $(CP_1)$  and  $(CP_2)$ . Columns “t1” and “t2” respectively correspond to the time of the first phase and the total time.

Formulation  $(CP_2)$  is now able to solve all the instances within 1 hour. We observe that the total time to solve the 40 instances is reduced by approximately 6 times for  $(CP_1)$  and 14 times for  $(CP_2)$  if compared to Table 3.

## 5 Conclusion

We introduced two new compact formulations of the  $p$ -center problem. We theoretically compared the quality of their LP bounds and their sizes to existing formulations. Numerical experiments confirmed these results and highlighted the fact that our new formulation  $(CP_1)$  outperforms the previously known formulations  $(P_1)$  and  $(P_2)$  at all levels. Our more compact formulation  $(CP_2)$  suffers from the poor quality of its linear relaxation. Another aspect of our work was to embed the formulations within a two-step algorithm in order to obtain better computation times.

Our future work will focus on improving our compact formulation through polyhedral studies.

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	N	p	opt	lb	ub	(P <sub>1</sub> ) b	t	(P <sub>2</sub> ) b	t	(CP <sub>1</sub> ) b	t	(CP <sub>2</sub> ) b	t
1	100	5	127	59	133	98	2,4	107	75,3	107	<b>1,0</b>	85	4,0
2	100	10	98	56	117	77	2,9	86	7,3	86	<b>0,5</b>	71	5,2
3	100	10	93	55	116	74	2,9	84	2,5	84	<b>0,2</b>	69	3,1
4	100	20	74	41	127	55	0,7	65	7,9	65	<b>0,6</b>	53	3,4
5	100	33	48	23	87	31	0,8	38	1,0	38	<b>0,1</b>	30	1,5
6	200	5	84	38	94	68	35,9	75	106,7	75	<b>2,7</b>	59	47,1
7	200	10	64	34	79	51	20,5	58	100,2	58	<b>1,8</b>	46	26,1
8	200	20	55	30	72	41	20,7	48	87,2	48	<b>1,6</b>	38	19,6
9	200	40	37	22	73	28	8,9	33	14,9	33	<b>1,4</b>	27	29,8
10	200	67	20	11	44	15	1,6	18	0,8	18	<b>0,3</b>	14	5,5
11	300	5	59	34	67	50	99,0	54	30,4	54	<b>6,2</b>	44	68,1
12	300	10	51	30	72	43	229,7	48	71,0	48	<b>7,2</b>	39	98,7
13	300	30	36	20	56	28	114,0	33	44,6	33	<b>4,7</b>	26	106,9
14	300	60	26	14	60	19	157,1	23	33,4	23	<b>12,9</b>	18	151,7
15	300	100	18	10	42	13	8,6	16	9,4	16	<b>0,9</b>	13	30,2
16	400	5	47	26	51	41	403,2	45	25,3	45	<b>3,3</b>	36	54,5
17	400	10	39	21	47	33	737,8	36	35,0	36	<b>24,9</b>	29	149,2
18	400	40	28	16	50	22	664,7	25	96,4	25	<b>22,1</b>	20	431,4
19	400	80	18	10	40	14	226,2	16	81,4	16	<b>18,5</b>	13	116,9
20	400	133	13	7	32	10	9,0	12	3,0	12	<b>0,9</b>	10	22,5
21	500	5	40	23	48	35	2581,0	37	118,3	37	<b>13,6</b>	31	194,6
22	500	10	38	21	49	31	-	35	924,4	35	<b>24,6</b>	28	507,8
23	500	50	22	13	38	17	1375,8	20	212,2	20	<b>38,4</b>	16	481,8
24	500	100	15	9	35	12	573,7	14	51,0	14	<b>29,6</b>	11	209,2
25	500	167	11	6	27	8	57,2	10	5,1	10	<b>2,0</b>	8	23,1
26	600	5	38	21	43	32	3093,6	35	106,0	35	<b>13,6</b>	28	152,4
27	600	10	32	18	39	28	3118,9	30	104,3	30	<b>48,3</b>	25	341,5
28	600	60	18	10	33	14	-	16	176,2	16	<b>103,3</b>	13	-
29	600	120	13	7	36	10	-	12	130,7	12	<b>77,8</b>	9	893,6
30	600	200	9	5	29	7	106,5	8	<b>12,4</b>	8	15,7	7	89,8
31	700	5	30	16	34	27	1793,8	28	68,8	28	<b>12,5</b>	24	139,9
32	700	10	29	16	35	25	-	27	718,7	27	<b>127,3</b>	22	944,5
33	700	70	15	9	26	13	-	14	155,1	14	<b>76,0</b>	12	890,1
34	700	140	11	6	30	9	2617,9	10	168,7	10	<b>32,8</b>	8	464,9
35	800	5	30	16	32	27	-	29	23,0	29	<b>13,0</b>	23	170,6
36	800	10	27	16	34	24	-	26	<b>130,3</b>	26	821,7	21	1056,6
37	800	80	15	8	26	12	-	14	222,5	14	<b>90,9</b>	11	1706,9
38	900	5	29	15	35	25	-	27	68,8	27	<b>19,0</b>	21	300,1
39	900	10	23	13	28	20	-	22	<b>348,4</b>	22	1190,0	18	1786,4
40	900	90	13	7	22	10	-	12	551,0	12	<b>129,5</b>	10	1059,9
				Total		57699		5129		<b>2991</b>		16390	

Table 3: Comparison of the different formulations with  $lb = LB_1$  and  $ub = UB_1$ . For each instance, the smallest time appears in bold. Symbol “-” means that the instance was not solved within 1 hour.

N	p	opt	(CP <sub>1</sub> )		(CP <sub>2</sub> )		
			t1	t2	t1	t2	
1	100	5	127	0,2	<b>0,3</b>	0,3	0,7
2	100	10	98	0,2	<b>0,2</b>	0,3	0,4
3	100	10	93	0,2	<b>0,3</b>	0,3	0,4
4	100	20	74	0,3	<b>0,4</b>	0,4	0,5
5	100	33	48	0,1	<b>0,2</b>	0,3	0,4
6	200	5	84	1,9	<b>2,7</b>	5,2	6,3
7	200	10	64	1,1	<b>1,4</b>	3,0	3,4
8	200	20	55	0,8	<b>1,0</b>	2,8	3,0
9	200	40	37	2,0	<b>2,7</b>	4,5	5,4
10	200	67	20	0,4	<b>0,6</b>	0,9	1,1
11	300	5	59	0,8	<b>0,9</b>	2,2	2,2
12	300	10	51	3,4	<b>4,6</b>	10,2	12,5
13	300	30	36	3,6	<b>4,6</b>	8,8	9,8
14	300	60	26	3,5	<b>4,5</b>	14,8	17,5
15	300	100	18	1,5	<b>2,1</b>	3,3	3,7
16	400	5	47	1,4	<b>1,4</b>	6,4	6,4
17	400	10	39	3,3	<b>4,3</b>	9,5	10,6
18	400	40	28	5,8	<b>8,3</b>	29,1	33,3
19	400	80	18	4,1	<b>6,2</b>	9,8	12,1
20	400	133	13	2,5	<b>3,0</b>	4,0	5,0
21	500	5	40	3,1	<b>4,0</b>	9,7	10,3
22	500	10	38	16,6	<b>26,5</b>	38,6	48,3
23	500	50	22	7,0	<b>9,9</b>	31,5	37,1
24	500	100	15	7,6	<b>11,4</b>	18,5	23,7
25	500	167	11	3,7	<b>4,6</b>	7,5	9,0
26	600	5	38	4,6	<b>5,3</b>	19,3	20,7
27	600	10	32	9,5	<b>12,5</b>	23,0	26,2
28	600	60	18	14,4	<b>17,5</b>	42,0	48,7
29	600	120	13	23,4	<b>32,7</b>	91,0	111,4
30	600	200	9	10,5	<b>15,1</b>	17,4	21,9
31	700	5	30	8,2	<b>9,3</b>	15,8	17,5
32	700	10	29	18,8	<b>71,8</b>	33,8	109,8
33	700	70	15	10,2	<b>14,3</b>	25,4	34,4
34	700	140	11	34,2	<b>46,4</b>	90,1	107,6
35	800	5	30	2,2	<b>2,2</b>	11,8	12,0
36	800	10	27	20,0	<b>30,3</b>	40,5	53,1
37	800	80	15	21,8	<b>27,8</b>	50,2	60,9
38	900	5	29	12,2	<b>12,7</b>	29,7	30,3
39	900	10	23	36,6	<b>49,7</b>	45,5	153,4
40	900	90	13	21,8	<b>31,2</b>	50,3	70,7
<b>Total</b>				<b>484</b>		1142	

Table 4: Results obtained with Algorithm 1 of Figure 1 with  $lb = LB_1$  and  $ub = UB_1$ .