# ESCI654 Project 1

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## Problem

### Setting and Motivation

The Portsmouth Brewery is a popular restaurant in Portsmouth, New Hampshire. Frequently, soups must be made in large batches to prepare for the coming days of business. After a soup is prepared it must be cooled to a temperature of at most 41 °F within 6 hours to prevent unsafe bacteria growth[1]. In order to cool soups before storing in the walk-in fridge, after cooking the soup is transferred to a cylindrical 5-gallon bucket made of food-grade high density polyethylene plastic. The bucket is then immersed in an ice-bath in one of the sinks. The subject of interest in this case is the temperature of a prepped soup, which must be reduced to 41 °F or lower within 6 hours or else the soup must be thrown away.

### Justification

The sought quantity in this case is what the minimum temperature the ice bath needs to be in order to cool the soup to 41 °F within 6 hours. This is important because if the ice bath is not cold enough then the food will spoil and end up costing the Portsmouth Brewery time and money. There is also risk of food poisoning if the soup is carelessly stored prior to proper cooling.

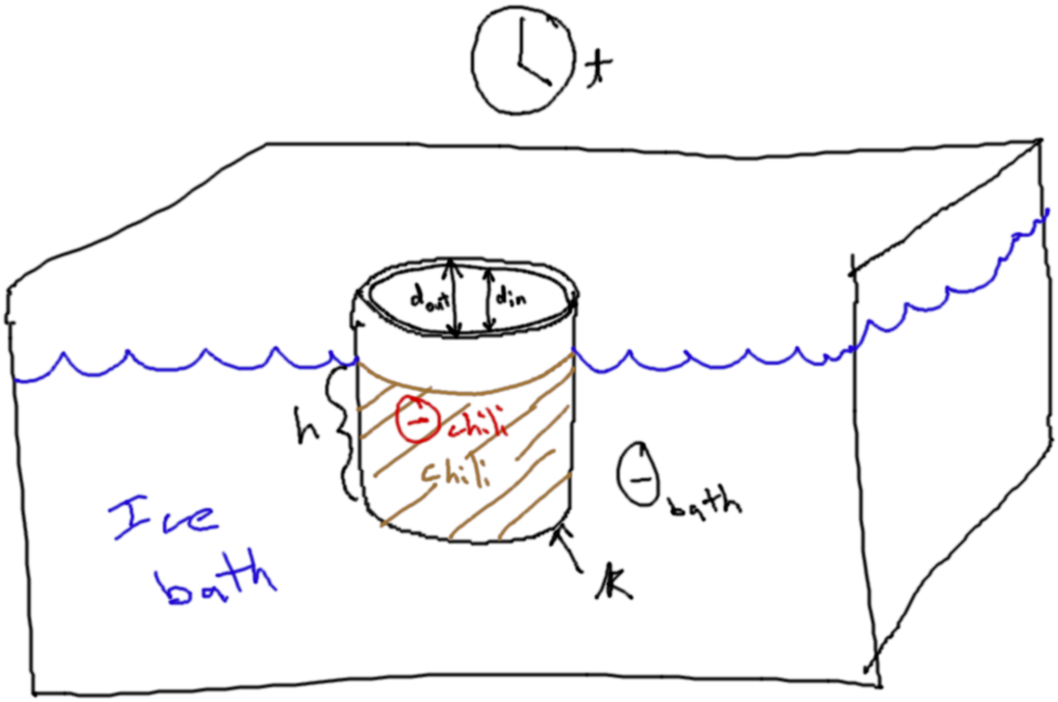
### Problem Statement

A 5-gallon bucket is filled with freshly prepared chili with an initial temperature of $91.0 \pm 0.1\,°C$. The dimensions of the bucket, as measured by a tape measure, are inner diameter $d_{in} = 28.60 \pm 0.05\,cm$, outer diameter $d_{out} = 30.05 \pm 0.05\,cm$, and the height of the chili in the bucket is $h = 28.00 \pm 0.05\,cm$. The plastic wall of the bucket has a thermal conductivity[2] of $k = 0.49\,\frac{W}{mK}$. The bucket is immersed in a large sink with a saltwater icebath with a temperature of $-5.0 \pm 0.1\,°C$. Will the chili cool in time to be safely stored for the next day (the chili must be cooled to 41 °F or lower)?

[1]Source: Centers of Disease Control and Prevention, "Best Food Cooling Practices for Restaurants", 17 June 2019, https://www.cdc.gov/nceh/ehs/ehsnet/plain_language/food-cooling-practices.htm

[2]Source: INEOS Olefins & Polymers USA, "Typical Engineering Properties of High Density Polyethylene", https://www.ineos.com/globalassets/ineos-group/businesses/ineos-olefins-and-polymers-usa/products/technical-information--patents/ineos-typical-engineering-properties-of-hdpe.pdf

## Solution




We can use an integral form of the Law of Conservation of Energy to solve this problem. For this problem we will write it in terms power.

$$\frac{dQ}{dt} = \sum P_{qin} - \sum P_{qout} + \sum P_{rin} - \sum P_{rout}$$

Where $\frac{dQ}{dt}$ is the change in energy over time, $\sum P_{qin}$ is the power due to heat flux into the bucket of chili, $\sum P_{qout}$ is the power due to heat flux out of the bucket of chili, $\sum P_{rin}$ is the power output due to chemical reactions in the chili, and $\sum P_{rout}$ is the power consumed due to chemical reactions in the chili. Since there are no exothermic or endothermic reactions occuring in the chili $\sum P_{rin} = \sum P_{rout} = 0$. We can also assume that there is no power going into the chili once it has been removed from the stove. Therefore $\sum P_{qin} = 0$. That leaves us with

$$\frac{dQ}{dt} = -\sum P_{qout}$$

This is a conduction problem, so we know that power is equal to heat flux times the surface area of conduction. That means that the power leaving the chili equals the surface area of the chili times the heat flux leaving the chili; $P_{qout} = A\dot{q}_{out}$

Since there is only one power source leaving the chili in the form of heat, we are left with

$$\frac{dQ}{dt} = -A\dot{q}_{out}$$

Fourier's Law states that

$$\dot{q}_{out} = k\frac{\Theta_{chili} - \Theta_{bath}}{l}$$

Where $k$ is the thermal conductivity of the bucket, $\Theta_{chili}$ is the temperature of the chili, $\Theta_{bath}$ is the temperature of the ice bath, and $l$ is the thickness of the bucket. $l$ is also the outer diameter of the bucket minus the inner diameter of the bucket, so the equation can be rewritten as

$$\dot{q}_{out} = k\frac{\Theta_{chili} - \Theta_{bath}}{d_{out} - d_{in}}$$

The specific heat of the chili is also needed, which is given by

$$Q = VC_V(\Theta_{chili} - \Theta_{bath})$$

Where $V$ is the volume of the chili and $C_V$ is the volumetric specific heat capacity of the chili. After substitution into our conservation of power equation we get

$$\frac{d}{dt}(VC_V(\Theta_{chili} - \Theta_{bath})) = -Ak\frac{\Theta_{chili} - \Theta_{bath}}{d_{out} - d_{in}}$$

We can assume $V$ and $C_V$ are constants, so we can move them outside of the derivative.

$$VC_V\frac{d}{dt}(\Theta_{chili} - \Theta_{bath}) = -Ak\frac{\Theta_{chili} - \Theta_{bath}}{d_{out} - d_{in}}$$

Next we will define $\Theta_{chili} - \Theta_{bath}$ as an excess temperature, or the amount by which the chili exceeds the bath temperature:

$$\Theta^* = \Theta_{chili} - \Theta_{bath}$$

And then substitute that into our equation

$$VC_V\frac{d}{dt}(\Theta^*) = -Ak\frac{\Theta^*}{d_{out} - d_{in}}$$

Next we rearrange in order to integrate using separation of variables.

$$\frac{1}{\Theta^*}d\Theta^* = \frac{-Ak}{VC_V(d_{out} - d_{in})}dt$$

$$\int \frac{1}{\Theta^*}d\Theta^* = \int \frac{-Ak}{VC_V(d_{out} - d_{in})}dt$$

$$\ln \Theta^* = \frac{-Ak}{VC_V(d_{out} - d_{in})}t + D$$

Where $D$ is an integration constant. To evaluate $D$ we can use what we know about the initial conditions. At time $t = 0$, we know that the excess temperature of the chili is equal to the initial excess temperature of the chili, given by the initial temperature of the chili minus the temperature of the ice bath: $\Theta_o^* = \Theta_o - \Theta_{bath}$ Therefore

$$\ln \Theta_o^* = \frac{-Ak}{VC_V(d_{out} - d_{in})} * 0 + D = D$$

$$\ln \Theta^* = \frac{-Ak}{VC_V(d_{out} - d_{in})}t + \ln \Theta_o^*$$

$$\ln \frac{\Theta^*}{\Theta_o^*} = \frac{-Ak}{VC_V(d_{out} - d_{in})}t$$

$$\frac{VC_V(d_{out} - d_{in})}{-Ak} \ln \frac{\Theta^*}{\Theta_o^*} = t$$

Additionally, the volume $V$ of the chili is given by the measured dimensions of the chili in the bucket

$$V = \pi(\frac{d_{in}}{2})^2 h$$

The surface area in contact with the ice bath can be simplified as the average of the inner and outer circumference of the bucket times the height of the chili.

$$A = 2\pi(\frac{\frac{d_{in}+d_{out}}{2}}{2})h = 2\pi(\frac{d_{in}+d_{out}}{4})h = \frac{d_{in}+d_{out}}{2}h\pi$$

Substituting into our equation gives

$$\frac{\pi(\frac{d_{in}}{2})^2 hC_V(d_{out}-d_{in})}{-\frac{d_{in}+d_{out}}{2}h\pi k}\ln\frac{\Theta^*}{\Theta^*_o} = t$$

And after simplification we end up with

$$\frac{(\frac{d_{in}}{2})^2 C_V(d_{out}-d_{in})}{-\frac{d_{in}+d_{out}}{2}k}\ln\frac{\Theta^*}{\Theta^*_o} = t$$

Finally, resubstitution of our definitions of excess temperature and initial excess temperature

$$\frac{(\frac{d_{in}}{2})^2 C_V(d_{out}-d_{in})}{-\frac{d_{in}+d_{out}}{2}k}\ln\frac{\Theta_{chili}-\Theta_{bath}}{\Theta_o-\Theta_{bath}} = t$$

We are almost ready to solve for $t$. However the last thing we need is the volumetric specific heat of the chili. This is not a quantity that is readily-available in literature; chili composition is also variable. Therefore I took some chili home in order to figure out the volumetric specific heat capacity myself. The procedure is located in the appendix. The result was $C_V = 6.4 * 10^6 \frac{J}{m^3 {}^\circ C} \pm 10\%$. Now we can solve for t by plugging in all known variables.

$$\frac{(\frac{(28.60\pm0.05\,cm)(\frac{1m}{100cm})}{2})^2(6.4*10^6\frac{J}{m^3 {}^\circ C}\pm10\%)((30.05\pm0.05\,cm)(\frac{1m}{100cm})-(28.60\pm0.05\,cm)(\frac{1m}{100cm}))}{-\frac{(30.05\pm0.05\,cm)(\frac{1m}{100cm})+(28.60\pm0.05\,cm)(\frac{1m}{100cm})}{2}(0.49\,\frac{W}{mK})}$$

$$*\ln\frac{((41.00^\circ F-32^\circ F)(\frac{5}{9}\frac{1}{^\circ F})+273.15\,K)-(-5.0+273.15\pm0.1\,K)}{(91.0+273.15\pm0.1\,K)-(-5.0+273.15\pm0.1\,K)} = t$$

We'll solve all of the addition and subtraction first to simplify the equation. We will need to propagate uncertainty. For addition and subtraction uncertainty is added in quadrature for random and uncorrelated uncertainty. The diameter measurements *are* correlated because I used the same measuring tape to measure both, so those uncertainties are added directly. Likewise, temperature measurements are correlated because the same thermometer was used for each measurement. Chili temperature does not have uncertainty because it is a target temperature of 41 °F, thus it is treated as a known quantity.

$$\frac{(\frac{(0.2860\pm0.005\,m)}{2})^2(6.4*10^6\frac{J}{m^3 {}^\circ C}\pm10\%)(0.0145\pm0.0010m)}{-\frac{0.5865\pm0.0010m}{2}(0.49\,\frac{W}{mK})}\ln\frac{(10.00\pm0.1\,K)}{96.0\pm0.2\,K} = t$$

Next is multiplication and division. Measurements for diameter are correlated; measurements for temperature are also correlated with each other, so their relative uncertainties are added directly. First convert all of the absolute uncertainties to relative uncertainties

$$\frac{(\frac{(0.2860\,m\pm1.748\%)}{2})^2(6.4*10^6\frac{J}{m^3 {}^\circ C}\pm10\%)(0.0145\,m\pm6.8966\%)}{-\frac{0.5865\,m\pm0.1705\%}{2}(0.49\,\frac{W}{mK})}\ln\frac{(10.00\,K\pm1\%)}{96.0\,K\pm0.2083\%} = t$$

Then we multiply and divide the correlated values first.

$$\frac{(0.0002965\,m^3\pm10.3926\%)(6.4*10^6\frac{J}{m^3 {}^\circ C}\pm10\%)}{-0.1436925\,\frac{W}{K}\pm0.1705\%}\ln(0.10417\pm1.2083\%) = t$$

Finally multiply the uncorrelated values. Uncertainties are added in quadrature.

$$\frac{1897.6672\frac{J}{^\circ C}\pm\sqrt{0.103926^2+0.10^2}}{-0.1436925\,\frac{W}{K}\pm0.1705\%}\ln(0.10417\pm1.2083\%) = t$$

Note that $1\,°C = 1\,K$, and $1\,W = 1\,\frac{J}{s}$, therefore

$$\frac{1897.6672\frac{J}{K} \pm 14.422\%}{-0.1436925\,\frac{J}{sK} \pm 0.1705\%} \ln\left(0.10417 \pm 1.2083\%\right) = t$$

$$(-13206.44571\,s \pm \sqrt{0.14422^2 + 0.001705^2}) \ln\left(0.10417 \pm 1.2083\%\right) = t$$

$$(-13206.44571\,s \pm 14.423\%) \ln\left(0.10417 \pm 1.2083\%\right) = t$$

The natural log of a relative uncertainty is the same relative uncertainty.

$$(-13206.44571\,s \pm 14.423\%)(-2.2617 \pm 1.2083\%) = t$$

$$29869.01826\,s \pm \sqrt{0.14423^2 + 0.012083^2} = t$$

$$8.2969 \pm (8.2969)(14.48\%)\,hr = t$$

$$8.2969 \pm 1.2014\,hr = t$$

$$t \approx 8.3 \pm 1.2\,hr$$

The chili will not cool in time even in the best-case scenario if the ice bath is only $-5\,°C$. Either the temperature of the bath should be lowered or the container should be made of a more thermally-conductive material.

# Model Assumptions

## Chili and ice baths are fully thermally mixed at all times

In a real situation a prep cook such as myself would ideally stir the chili mixture every ten minutes to ensure it cools evenly. Even if the average temperature of the chili reaches $41\,°F$, if the center of the chili is still at a higher temperature then bacteria will still grow to unsafe levels.

My calculations depend on the assumption that the temperature of the chili is constant throughout; it is because of that that I am able to use just the thermal conductivity of the bucket. If the chili were not fully thermally mixed then I would have to take its thermal conductivity into account. By stirring the chili I ensure that the thermal conductivity of the chili is negligible.

## Chili and ice baths have no exothermic or endothermic chemical reactions

It is safe to assume that there are no exothermic or endothermic chemical reactions in the ice bath. With regards to the chili, chemical reactions *do* take place during the cooking process. However, even if there are exothermic or endothermic reactions taking place within the chili, their effect on the temperature of the chili is most likely negligible. If they were *not* negligible, then the conservation of power equation would require additional terms to account for them. If there were significant net exothermic reactions taking place then the chili would take even longer to cool. If there were significant net endothermic reactions taking place then the chili would not take as long to cool.

## The temperature of the ice bath is constant

The ice bath is large enough such that the temperature should be constant. Additionally, the temperature of ice is in fact constant during phase change; additionally, the ice in the bath should ideally be replenished every few hours by employees of Portsmouth Brewery to keep the temperature constant. Therefore it makes sense to treat the ice bath temperature as constant. If the temperature were not constant (for example, if the ice all melted and prep cooks ignored it) then we would have to take the change in bath temperature into account.

# Appendix

## Determining volumetric specific heat capacity of chili

### Materials and Procecure

A variable power supply, a resistor, $100. \pm 5\,mL$ of chili, a digital thermometer, a stopwatch, and a small wooden spoon were used.

The ambient (initial) chili temperature was measured. The power supply was connected to the resistor and set to $10.0 \pm 0.1\,V$ with a current measuring $0.29 \pm 0.01\,A$, then inserted into the chili; chili was stirred with a small wooden spoon. Final temperature reading taken $10.00 \pm 0.25\,min$ later.

### Results

Initial temperature $\Theta_o = 22.3 \pm 0.1\,°C$. Final temperature $\Theta_f = 25.0 \pm 0.1\,°C$. Input voltage $E = 10.0 \pm 0.1\,V$ and input current $I = 0.29 \pm 0.01\,A$.

### Calculating volumetric specific heat capacity

$$Q = C_V V \Delta\Theta$$

Where $Q$ is the energy absorbed or released by the chili, $C_V$ is the volumetric specific heat capacity of the chili, $V$ is the volume of the chili, and $\Delta\Theta$ is the change in temperature of the chili. Dividing both sides by time gives us

$$\frac{Q}{t} = \frac{1}{t} C_V V \Delta\Theta$$

Since power $P = \frac{Q}{t}$

$$P = \frac{1}{t} C_V V \Delta\Theta$$

Electrical power (for a direct current source) is represented as $P = EI$; also the change in temperature $\Delta\Theta = \Theta_f - \Theta_o$, therefore

$$EI = \frac{1}{t} C_V V (\Theta_f - \Theta_o)$$

$$C_V = \frac{EIt}{V(\Theta_f - \Theta_o)}$$

This final equation assumes that all of the electrical power is converted to heat by the resistor. Now we can substitute all of our measured values. The same procedure as the main problem is used to propagate uncertainty. Uncertainties for temperature are added directly because they are correlated (same thermometer used for both measurements).

$$C_V = \frac{(10.0 \pm 0.1\,V)(0.29 \pm 0.01\,A)(10.00 \pm 0.25\,min)(60\frac{s}{min})}{(100. \pm 5\,mL)(\frac{1m^3}{1{,}000{,}000 mL})((25.0 \pm 0.1\,°C) - (22.3 \pm 0.1\,°C))}$$

$$C_V = \frac{(2.9\,W \pm 3.59\%)(10.00 \pm 0.25\,min)(60\frac{s}{min})}{(0.000100 \pm 0.000005\,m^3)(2.7 \pm 0.2\,°C)}$$

$$C_V = \frac{(2.9\,W \pm 3.59\%)(10.00 \pm 0.25\,min)(60\frac{s}{min})}{2.7 * 10^{-4}\,m^3 °C \pm \sqrt{(\frac{0.000005}{0.000100})^2 + (\frac{0.2}{2.7})^2}}$$

$$C_V = \frac{(2.9\,W \pm 3.59\%)(10.00 \pm 0.25\,min)(60\frac{s}{min})}{2.7 * 10^{-4}\,m^3 °C \pm 8.926\%}$$

$$C_V = 6444444 \frac{J}{m^3 °C} \pm \sqrt{0.00359^2 + 0.008936^2 + (\frac{0.25}{10.00})^2}$$

$$C_V \approx 6.4 * 10^6 \frac{J}{m^3 °C} \pm 10\%$$