École Polytechnique Fédérale de Lausanne

Convertible Bonds

Derivatives

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Chapter 1

Documentation

Convertible securities

Convertible securities are hybrid financial instruments, having characteristics of both fixed income securities and equity,Nyborg 1996a. As bonds, convertibles are able to provide income, managing the risk on potential declining markets. As equity they offer the possibility to benefit from the appreciation in the price of the company's stock, Calamos 1998.

These products can present a variety of combinations of risk/reward characteristics that were developed in the years with the increasing demand of customization of issuers and investors.

Typically they consist in a bond that can be converted into a number of stock specified at issuance. With the following introduction we aim at reviewing some of these possible combinations starting from the plain vanilla convertible bond, and then going through some of its most popular modifications as the callable convertible bonds, mandatory convertibles, contingent convertibles, exercise priorities.

Convertible bonds

Convertible bonds are debt instruments as regular bond with an additional feature that is the possibility of converting the principal into shares of common stocks.

When a company looks for new founding it could do so by increasing the number of share outstanding. But this choice might generate some negative sentiments in the investors that were already shareholders of the company, since it results in a share dilution. The share dilution makes the share less valuable since the each existing share represents a smaller percentage of ownership, thus justifies a bad sentiment of the investors .

Another option for the company is to raise founding by borrowing in the form of a bank loan or bond sale. The problem with this two options is that they are an expensive source of founding and increase default risk. These arguments: investors sentiments, cost of financing and risk of default justify the existence of convertible bonds, CHEN 2020. This is to say that, choosing convertible bonds as source of founding can moderate between these problems.

On the other hand the investors' interest is such a product can be justified by the fact that in the event of default of the company the convertible bond guarantees some securities on the principal. At the same time the in the case of an upside of the events for the company the investor can participate in this success by exercising her/his option and converting their bond holding into shares,Rebecca Baldridge 2021.

To recapitulate, the company advantage in issuing a convertible comes from the possibility of raising capital without immediately diluting their shares and paying a lower interest rate on the debt with respect to tradition forms of borrowing.

Summing up from the investor point of view convertible bonds benefits consist in receiving fixed-rate interest payments with the option to convert to stock and benefit from stock price appreciation. In addition investors get some default risk security since bondholders are paid before common stockholders. But as a consequence of the option that is left to the investor the coupon of the convertible will be lower.

A fundamental characteristic of this type of product, to which the investor must pay attention to when evaluating a possible investment is the conversion ratio.

The conversion ratio is usually defined as the number of common shares which can be received for each security at the conversion time and it is computed as the par value of the convertible security divided by the equity's conversion price. A higher conversion ratio means that more common shares can be received for each convertible security. This ratio is set when the convertible security is issued and it has an effect on the price of the security, HAYES 2021.

When the convertible security is issued, its right to convert into a predetermined number of of shares of common stock is computed. The conversion ratio is thus the specific number of shares that each issue is convertible into, which is the prospectus of the issue, T. Dinsmore, O'Keeffe and J. Dinsmore 2016.

As a consequence, the conversion value, or the intrinsic value in option terms, is the value of the convertible if it was converted into common stock at the current price of the stock. For example, if we have a bond which can be converted into 10 shares of a certain stock which has a current price of $30, them its conversion value would be 10 multiplied by $30 that would give a conversion value of $300.

Variations of convertible bonds allows to obtain other financial products with specific characteristics such as, but not limited to: callable convertible bonds, mandatory convertibles, contingent convertibles.

According to T. Dinsmore, O'Keeffe and J. Dinsmore 2016, there are three main types of convertible securities, namely convertible bonds, convertible preferred shares and mandatory convertible preferred shares.

MANDATORY CONVERTIBLES

Mandatory convertibles are a type of convertible bond where the option feature, that was given to the holder, is canceled and substituted with an obligation of conversion of the lending into shares of the company. This obligation is set to be satisfied either before or at the contractual date of conversion.

This kind of financial product can be thought as an hybrid that allows the bondholder do benefit form a guaranteed return up to the maturity date. At which point the obligation of the conversion steps in, then the holder will face a possibly positive or negative return depending on how the stock has performed.

This behavior is different from that of a standard convertible bond where there is no obligation but instead the option of exercising or not the conversion of the bond into shares of the issuing company. It is then possible to notice how the holder of a mandatory convertible bond would be in a less privileged position than the holder of a standard convertible one. To compensate for this difference investors are guaranteed an higher yield than that offered by its ordinary counterpart, CHEN 2021a.

From this it possible to retrieve the principal advantages for the two parts engaged in the contract. Investors are interested in such a product as the yield is higher than that of a standard convertible, and it allows for participation in the eventual upside of the stock move. Although as a disadvantage it also forces investor to take part in eventual downside of the of the company.

The issuer on the other hand benefits form this kind of security as, later in time, they do not pose a credit risk for the company, since there is the certainty that the bonds will be converted into shares. This is viewed positively by the rating agencies as it will ease the liabilities of the company and reduce its default risk, (T. Dinsmore, O'Keeffe and J. Dinsmore 2016).

CALLABLE CONVERTIBLE BONDS

Callable convertible bonds are convertible bond with the additional characteristic of having of potentially terminating the contract in advance upon the decision of the issuer. The company that issues the bond, in fact reserves the right to interrupt the coupon payments and repay the principal before the the occurrence of the terminal date. This structure is quite functional for companies, as when interest rates drop under the current rate that is being payed to the bond holders, the issuer can choose to terminate the contract and refinances its projects with cheaper debt, (Nyborg 1996b).

To compensate for this advantage detained by the company the holder will be compensated by a more generous coupon rate than the one that he would find for non callable bonds, (CHEN 2021b).

CONTINGENT CONVERTIBLES

Contingent convertibles ('CoCos') are another class of hybrid debt which were introduced at the end of 2009 with principal idea of offering to bank's debt holders the possibility to convert their bonds into shares, De Spiegeleer and Schoutens 2012.

In particular, these innovative instruments are characterized by an automatic conversion into equity, otherwise by a reduction of their nominal value when the bank that has issued them may go through a period of a probable incapability of growth or development. In such a situation the future prospective of the bank is uncertain for the depositors and for the bondholders, (De Spiegeleer and Schoutens 2012).

The conversion of the debt or the reduction of its nominal value is triggered by a specific event with the purpose of quantifying the situation of instability of in which is the bank during its life, (De Spiegeleer and Schoutens 2012).

The conversion of the debt which occurs automatically makes this type of product particularly attractive also from a regulatory perspective. Indeed, in order to reinforce its balance situation the bank does not need to seek other new investors because of the automatic conversion of the debt, without any shareholder's meeting requirement.

Therefore, the contingent convertibles particularly fit into a context of more stable banking system, (De Spiegeleer and Schoutens 2012).

As far as the trigger is concerned, it has been stated above that it specifies the situations in which the contingent convertible is converted into shares. In particular, the trigger event also specifies the scenario in which the bank encountering difficulties. Thus, after the trigger event the bank should obtain a stronger capital structure.

When a contingent convertible is structured there are several conditions that need to be satisfied by the trigger event:

- The trigger event needs to satisfy the clarity condition: meaning that it should bring the same message independently of the issuer's jurisdiction. Indeed, if for instance an accounting ratio is utilized to define the scenario in which the conversion of the bond occurs, than this particular accounting number cannot be calculated with different standards for each jurisdiction of the bank. If it is, then the trigger event does not satisfy the clarity requisite.
- Another requisite that must be satisfied is that the the process of conversion into share in which the life of the contingent convertible ends, should be known and well documented when the bond is issued.
- It is not possible to change the trigger event over the life of the contingent convertible, thus it should be fixed.
- The trigger event in conjunction with the data relative to a possible conversion of the contingent convertible should be publicly available, (De Spiegeleer and Schoutens 2012).

Even though there is still a debate concerning which were the main causes of the crisis that affected financial markets over the period 2007/2009, it seems that there is a wide consensus concerning the fact that one of the most relevant issues was due to the high levels of leverage ratios of the banks, which put them into serious recapitalization problems when occurred an abrupt worsening of the market conditions, (Koziol and Lawrenz 2012).

In this setting the contingent convertible bonds issued by the banks seem to be conceived as a proposal which aim is to alleviate the excessively high levels of financial leverage ratios. As it has been previously described the main characteristic of this hybrid securities is the payment of coupons just like normal bonds, but as the equity ratio drops below a determined threshold they are automatically converted into shares.

Those who support contingent convertibles tend to consider these securities as a cheaper and more efficient solution for banks that are facing a difficult financial period; and this is because such securities allow to increase the equity ratio in situations where the banks' asset value suffers high losses.

Hence, this feature is particularly useful to reduce the default risk or the intervention by supervision institutions, (Koziol and Lawrenz 2012).

Thus, the banks can benefit the effects of debt financing in good times and given the latter described feature, banks can also benefit of an alleviation of default risk.

Nevertheless, it seems that a large share of literature conveys that debt is an optimal financing arrangement for the reason that it offers coupon payments in favorable states, whereas in bad situations it establishes ownership transfer (or control rights) and this threatens firms' decision makers.

On the other hand, given the fact that contingent convertibles defer the control rights transfer, it may be that they have a distorting effect on incentives of decision makers, (Koziol and Lawrenz 2012).

As far as the investors benefits and drawbacks are concerned, it is important to mention that their incentive to invest in such a class of securities is surely the higher periodic payment that they receive if compared to most of other bonds. Indeed, investor buy such type of products with the hope that the bank will buy back the debt and, until this does not happen, they will receive above average returns associated with above average risk though. Indeed, one of the disadvantages is that investors bear this risk and have limited control when stocks are converted into stock. Moreover, the investor may hold the contingent convertible for several years; indeed, it is not granted that the convertible contingent will be converted into equity or completely redeemed by the bank. In addition, if regulators do not allow the sale, investor may encounter difficulties when they want to sell this product, (CHEN 2021c).

CONVERTIBLE SECURITIES MARKET DATA

In this paragraph we report some statistics about convertible securities. The global convertible market was valued at $525.2 billion USD (as 30-06-2021). The interest of investors in this kind of product is steadily increasing, this is part thanks to the companies that are racing to the market to ensure refinancing, and so are providing convertible bonds to the market. 2021 has seen a record volume of $170 billion of new convertibles issued. This number is expected to be around $165 billion for 2022.

From a geographical point of view the diffusion of this products is mostly located in North America and then in the EMEA (Europe, Middle East, and Africa) area. Respectively accounting for 65.8% and 20.5% of the global market as reported in the graph.

To describe the performance of this kind of asset we choose to report the performance of the iShares Convertible Bond ETF managed by Black Rock. The net asset of the found is over $1.5 billion USD and and can be considered representative enough for the USA convertible bond market. In the table are reported to have average return for one, three and five years, compared to the values updated up to the 31-03-2022.

It is also valuable to notice how this kind of products, as described in the above sections, are able to mitigate the downside of the market but still allow the investor to participate into its upside. This is nicely exposed in the reported graph.

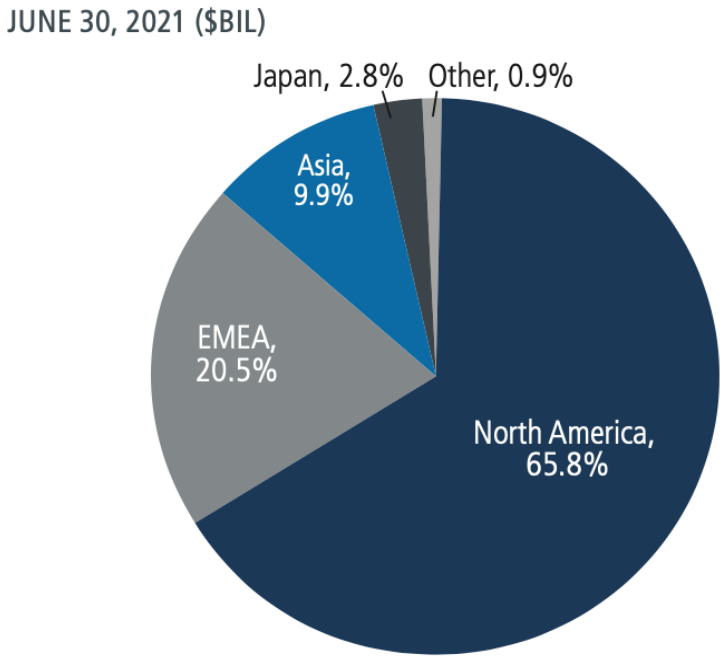


FIGURE 1.1
Global Convertible Market by Region, source: Calamos 1998

	1y	3y	5y
Total Return (%)	-8.46	18.26	14.91

FIGURE 1.2
iShares Convertible Bond ETF average return, source: Rock 2022

ANNUALIZED RETURNS

	GREAT 90s BULL MARKET 10/1/90-3/31/00	INTERNET BUBBLE CRASH 4/1/00-3/31/03	RECOVERY 4/1/03-10/31/07	FINANCIAL CRISIS 11/1/07-2/28/09	RECOVERY 3/1/09-12/31/19	COVID-19 DRAWDOWN 1/1/20-3/31/20	COVID-19 REFLATION 4/1/20-6/31/21
Equities	21.03%	-16.09%	16.13%	-41.39%	17.07%	-19.60%	52.76%
Convertible Bonds	19.00	-9.19	12.23	-32.24	13.64	-13.62	60.81

FIGURE 1.3
Convertibles Offered Equity-Like Returns with Less Downside: Calamos 1998

CHAPTER 2

PRELIMINARY ANALYSIS

2.1 Describe the position of the holder of the *mCB* as a portfolio that includes a coupon bond and a combination of European stock options.

We can recreate the position of the holder of the *mCB* as portfolio composed by:

- A long position on a coupon bond with face value equal to F and maturity equal to T_N and coupon payments of $F \cdot c$ at dates $(T_j)_{j=1}^N$.
- A long position on β European call options with strike equal to F/β.
- A short position on α European put options with strike equal to F/α.

In fact, at date $T = T_N$ the payoff will assume three possible values:

- If $S_T < F/\alpha$, the holder will receive: $F - \alpha(F/\alpha - S_T) = \alpha \cdot S_T$
- If $S_T \geq F/\alpha$ and $S_T \leq F/\beta$, the holder will receive: $F = F/S_T \cdot S_T$
- If $S_T > F/\beta$, the holder will receive: $F + \beta(S_T - F/\beta) = \beta \cdot S_T$

2.2 Describe the (pre-conversion) position of the holder of the *CB* as a portfolio that includes a standard bond and an American derivative to be specified

We can recreate the pre-conversion position of the holder of the *CB* as portfolio composed by:

- A long position on a standard bond with face value equal to F and maturity equal to T_N and coupon payments of $F \cdot c$ at dates $(T_j)_{j=1}^N$.
- A long position on γ American options with payoff at date T_j given by G_{T_j}, where

$$G_{T_j} = S_{T_j} - \frac{F}{\gamma}\left[\mathbb{E}_{T_j}^{\mathbb{Q}}\left(\frac{S_{0,T_j}}{S_{0,T_N}}\right) + \sum_{i=j+1}^{N} c \cdot \mathbb{E}_{T_j}^{\mathbb{Q}}\left(\frac{S_{0,T_j}}{S_{0,T_i}}\right)\right]$$

and $\mathbb{Q}$ is the Equivalent Martingale Measure. If we assume deterministic and constant risk free rate we can see that the payoff of the American derivative can be written as:

$$G_{T_j} = S_{T_j} - \frac{F}{\gamma}[e^{-r(T_N - T_j)}) + \sum_{i=j+1}^{N} c \cdot e^{-r(T_j - T_i)}]$$

and therefore if we decide to exercise at time T_j the American derivative we obtain the value:

$$X_{T_j} = CB_{T_j} + \gamma * S_{T_j} - \gamma \cdot \frac{F}{\gamma}[e^{-r(T_N - T_j)}) + \sum_{i=j+1}^{N} c \cdot e^{-r(T_j - T_i)}]$$

that can be written as:

$$XT_j = CB_{T_j} + \gamma * S_{T_j} - \gamma \cdot CB_{T_j}$$

2.3 Assume that markets are arbitrage-free and complete. Provide a formula for the initial prices of the mCB and the CB in terms of expectations under the EMM and the value function of an optimal stopping problem to be specified.

Let's start from the case of the *mCB*.
The initial price of the mandatory convertible bond can be computed calculating the initial price of the portfolio that replicate the payoff of the security considered. Therefore the initial price will be the sum of the initial price of the coupon bond, and of the two European option written above.

$$mCB_0 = B_0 + \beta \cdot c_0(T_n, F/\beta) - \alpha \cdot p_0(T_n, F/\alpha)$$

with:

$$B_0 = F\left[\mathbb{E}^{\mathbb{Q}}\left(\frac{S_{0,0}}{S_{0,T_N}}\right) + \sum_{i=1}^{N} c \cdot \mathbb{E}^{\mathbb{Q}}\left(\frac{S_{0,0}}{S_{0,T_i}}\right)\right] = F\left[\mathbb{E}^{\mathbb{Q}}\left(\frac{1}{S_{0,T_N}}\right) + \sum_{i=1}^{N} c \cdot \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{S_{0,T_i}}\right)\right]$$

The price of the European call and put can be calculated recursively through following formulas

$$c_t(T_n, F/\beta) = e^{-r_{t+1}\Delta}\mathbb{E}_t^{\mathbb{Q}}[\, c_{t+1}(T_n, F/\beta)\,],$$
$$p_t(T_n, F/\alpha) = e^{-r_{t+1}\Delta}\mathbb{E}_t^{\mathbb{Q}}[\, p_{t+1}(T_n, F/\beta)\,]$$

Let's analyse the case of the *CB* security.
The initial price of the *CB*, as shown at the previous point can be computed considering the American derivative and a standard coupon bond. The initial price of the coupon bond is given as before by the following

$$B_0 = F\left[\mathbb{E}^{\mathbb{Q}}\left(\frac{S_{0,0}}{S_{0,T_N}}\right) + \sum_{i=1}^{N} c \cdot \mathbb{E}^{\mathbb{Q}}\left(\frac{S_{0,0}}{S_{0,T_i}}\right)\right] = F\left[\mathbb{E}^{\mathbb{Q}}\left(\frac{1}{S_{0,T_N}}\right) + \sum_{i=1}^{N} c \cdot \mathbb{E}^{\mathbb{Q}}\left(\frac{1}{S_{0,T_i}}\right)\right]$$

The price of this American derivative is defined using American recursion, where

$$P_{T_n} = G^{+}_{T_n}, \qquad P_t = \max(G_t, C_t),$$

$$\text{where} \qquad C_t = \mathbb{E}_t^{\mathbb{Q}}\left[\frac{P_{t+1}\, S_{0,t}}{S_{0,t+1}}\right], \qquad t \in \{T_1, \dots, T_{n-1}\}$$

C_t is called continuation value of the American derivative, and the optimal rule prescribes to exercise the derivative in case of $P_t = G_t$ with $t \in \{T_1, \dots, T_{n-1}\}$.

Moreover, we can provide the definition of the Snell envelope of a discounted payoff as *discounted* process $(\hat{Z}_t)_{t=0n}^{T}$:

$$\hat{Z}_t = \max_{\theta\, \in\, \mathcal{S}_t} \mathbb{E}_t^{\mathbb{Q}}\left[\mathbf{1}_{\{\theta \le T_n\}}\hat{G}_\theta\right]$$

that we saw in class, with $\mathcal{S}_t$ which is the set of stopping times $\theta \leq T_n$. As $\mathcal{S}_{T_n} = \{T_n, \infty\}$, it implies that the Snell envelope is given by $\hat{Z}_{T_n} = \hat{G}^+_{T_n} = \max\{\hat{G}_{T_n}, 0\}$.

In particular, the Snell envelope satisfies

$$\hat{Z}_t = \mathbf{1}_{\{t=T_n\}}\hat{G}^+_t + \mathbf{1}_{\{t<T_n\}} \max\left\{\hat{G}_t, \mathbb{E}^{\mathbb{Q}}_t\left[\hat{Z}_{t+1}\right]\right\}$$

with the stopping time

$$\tau^*_t = \inf\left\{s \geq t : Z_s = G_s\right\} \in \mathcal{S}_t$$

which attains the maximum in the Snell envelope. In particular $Z_t = P_t$ holds almost surely at every date.

2.4 Show that the issuer strictly prefers to call the bond at date t if and only if $C_t >$ max$(\psi S_t, F_c)$ for some constant $\psi \geq 0$ to be determined

Denoting by B_t the t the value of the *cCB* at date t after the payment of the coupon (if any) but before any call or conversion decision, and by C_t the continuation value defined by the following

$$C_t = e^{r_{t+1}}\mathbb{E}^{\mathbb{Q}}_t[B_{t+1} + \mathbf{1}_{\{t+1\in\mathscr{C}\}}]$$

We can first analyse what the holder would do if the issuer exercises the option at a given date in order to define what is the optimal decision for the issuer at any time. The issuer knows that, once he calls the bond, the holder will choose to convert the option into γ shares only if γS_t is larger than F_c, otherwise he will prefer to receive the call price F_c .
In fact the holder has the possibility to take this choice because he retains the option to immediately convert the option once the issuer call it. Therefore the issuer will decide to call the option if and only if the continuation value $C_t > \max(F_c, \gamma S_t)$. We can interpret this optimal choice as follow: the issuer will decide to buy back the bond if the bond value is expected to be larger in the future (this value is represented by the continuation value) than the maximum that is going to pay to the holder in case he decides to buy back the option. Therefore we can identify the constant $\psi = \gamma$. At the terminal time, the issuer will prefer to call the bond if and only if $F > \max(\gamma S_T, F_c)$.

2.5 Show that if $t \in \mathscr{C}$ then $B_t =$ max$(\psi S_t, F_c)$ if $C_t >$ max$(\psi S_t, F_c)$, otherwise we have $C_t =$ max$(\psi S_t, C_t)$, where ψ is the same constant defined in the previous question. What is the corresponding formula when $t \notin \mathscr{C}$ is not a coupon date?

If t belongs to the coupon dates, we know from the previous point that the issuer strictly prefers to call the bond at date t if and only if $C_t > \max(\psi S_t, F_t)$. When this happens the holder would choose the maximum value between the Call Price F_c and $\psi \cdot S_t$. Whereas if the issuer does not exercise his option, the holder will choose the better value between $\psi \cdot S_t$ and C_t. We can then write the price B_t as :

$$B_t = \begin{cases} \max(\psi S_t, F_c) & \text{if } C_t > \max(\psi S_t, F_c) \\ \max(\psi S_t, C_t), & \text{otherwise.} \end{cases} \tag{2.1}$$

If t is not a Coupon date, we have simpler than the value of the cCB at date B_t is equal to the continuation value C_t:

$$C_t = e^{r_{t+1}}\mathbb{E}^{\mathbb{Q}}_t\left[B_{t+1} + \mathbf{1}_{\{t+1\in\mathscr{C}\}}\right].$$

Finally, the value of the option at t = T_N will be:

$$B_T = \begin{cases} \max(\gamma \mathrm{S}_{T_N}, \mathrm{F}_c) & \text{if } F > \max(\gamma S_{T_N}, F_c) \\ & \text{otherwise.} \end{cases} \tag{2.2}$$

CHAPTER 4

MODEL CALIBRATION

4.1

To show that the present value of the dividends paid by the stock over the next n years is $S_0 \cdot (1-\eta^n)$ we use an argument similar to the one we saw in Lecture 1 adapted to discrete time. We define as:

- X_t is the portfolio value at date t;
- Q_t is the initial quantity of stock held at time t;
- S_t is the stock value at date t;
- $\delta {S_t}^C$ is the share of stock value (*cum-dividend*) distributed as dividend at time t;
- $C_t = \sum_{i=1}^{t} \delta {S_i}^C$ is the cumulative cash flow of dividends up to date t.

Therefore, we consider the increment of the quantity of stock held between time t and t+1:

$$\begin{gathered}
Q_{t+1} - Q_t = Q_t + Q_t \frac{\Delta C_{t+1}}{S_{t+1}} = Q_t \delta \frac{S_{t+1}^C}{S_{t+1}^{EX}} \\
Q_{t+1} - Q_t = \frac{(C_{t+1} - C_t)Q_t}{S_{t+1}^{EX}} = Q_t \delta \frac{S_{t+1}^C}{S_{t+1}^{EX}} = \frac{\delta}{1-\delta} Q_t \\
\Rightarrow Q_{t+1} = Q_t \left(1 + \frac{\delta}{1-\delta}\right) = Q_t \left(\frac{1}{1-\delta}\right), \qquad \text{if } q = Q_t \\
Q_s = q \left(\frac{1}{1-\delta}\right)^{(s-t)}, \qquad \text{with } X_s = S_s Q_s, \qquad \forall\, s \geq t.
\end{gathered} \tag{4.1}$$

If we choose $q = (1-\delta)^{(s-t)}$ we get $X_t = q \cdot S_s = (1-\delta)^{(s-t)} \cdot S_s$ by using the law of one price. Therefore we can obtain the present value at time t of the dividends up to time T.

$$D_t(T) = S_t - P_t(T) = S_t \cdot \left[1 - (1-\delta)^{(T-t)}\right]$$

The value of future dividends up to time T is therefore:

$$D_0(T) = \left[1 - (1-\delta)^T\right] \cdot S_0$$

Therefore we can conclude that $\eta = 1 - \delta$.

4.2

To estimate the values of r and δ we start from the put-call parity equation:

$$c_0(S_0; k; T) - p_0(S_0; k; T) = S_0 - e^{-rT} \cdot k - D_0(T)$$

that we can rearrange as:

$$c_0 - p_0 - S_0 = -e^{-rT} \cdot k - D_0(T)$$

To calibrate our model we implement the following regression:

$$y_i = \alpha_i + \beta_i k + \epsilon_i$$

where we have:

$$y_i = c_0(S_0; k; T) - p_0(S_0; k; T) - S_0$$

and

$$\alpha_i = D_0(T) = \left[1 - (1-\delta)^5\right] \cdot S_0, \quad \beta_i = -e^{-rT}$$

where we have used for $D_0(T)$ the expression that we find in the point before with n=T=5.

Therefore, rearranging the terms we can obtain the estimated values of δ and r using the estimated α and β as follows:

$$\begin{cases} \hat{\delta} = -\frac{\hat{\alpha}}{S_0} = 0.0295545 \\ \hat{r} = -ln(-\hat{\beta}) = 0.05 \end{cases}$$

4.3

To calibrate a Binomial model with $T = 60$ periods of length one month ($\Delta = 1/12$) and with Equivalent Martingale Measure $q = 1/2$, first of all we use the formula of the EMM probability over one period that we derived in Lecture 4 to compute U as function of D, provided that $q = 1/2$, as we show below. We do this computation in order to reduce our optimization problem to the only one unknown variable D:

$$q = \frac{e^{r\Delta} - D}{U - D} = \frac{1}{2} \quad \Rightarrow \quad (U - D) = 2\left(e^{r\Delta} - D\right)$$
$$U = \left(e^{r\Delta} - D\right) \cdot 2 + D = 2e^{r\Delta} - D$$

Thus, we utilized this function in our model, computing the European call price as function of D and $U(D)$ and of the strikes that were provided in Table 1. Therefore we calculated the squared errors subtracting the observed call prices of Table 1 from the estimated call prices from our model. Therefore, we chose the values of U and D that minimize the squared error. In order to avoid local minimum, we focused on the interval $D \in [0.7, 1.004]$, and we obtain the following values for the up and down factor: U = 1.0544 and D = 0.9540.

Chapter 5

Analysis

5.1

Using the developed code and the calibrated model, we have estimated that, considering the following characteristics:

- Maturity 5 years
- Face value $F = 1000$
- Coupons every 6 months
- Coupon rate $c = 0.02$
- Conversion thresholds for the mCB: $\alpha = 1000$ and $\beta = 4$
- Conversion factor $\gamma = 4$
- Call price for the cCB $F_c = 1.05 \cdot F = 1050$

The price of the vanilla convertible bond CB, the price of the mandatory convertible bond mCB and the price of the callable convertible bond cCB are:

Security	*Price*
CB	995.5168
mCB	993.6688
cCB	981.5582

5.2

The Vanilla convertible bond holder at the first and second coupon dates (respectively t = 6 and t = 12) decides to hold the bond and does not exercise his option in any node of the tree.At the third coupon date (t = 18), the optimal strategy policy for the holder is to exercise at the two upper nodes of the tree: 18U - 0D and 17U - 1D. The decision comes from the fact that in these two nodes the payoff that the holder could get is larger than the continuation value.
We illustrate the optimal strategy in the following table

Time in months	*Node*	*Decision*
6	-	Not Exercise
12	-	Not Exercise
18	18U - 0D	Convert into shares
18	17U - 1D	Convert into shares

5.3
The callable Convertible Bond issuer optimal exercise policy is to not exercise his option in the first coupon period (t=6), to exercise at the second coupon date (t=12) at the first three upper nodes (12U - 0D, 11U - 1D, 10U - 2D) and to exercise at the third coupon rate at node 13 U - 5D. The optimal decision policy of the holder is to do not exercise at time 6; to exercise at time 12 at the first three nodes of the tree (12U - 0D, 11U - 1D, 10U - 2D), choosing to take respectively the stocks, the stocks and the call price; and to exercise at the third coupon rate (t=18) at the first six upper nodes (18U - 0D, 17U - 1D, 16U -2D, 15U - 3D, 14U - 4D, 13U - 5D), converting always into shares.
We illustrate the optimal strategy in the following table

Time in months	*Node*	*Issuer Decision*	*Holder Decision*
6	-	Not Buy back	Not Exercise
12	12U - 0D	Buy back	Convert into shares
12	11U - 1D	Buy back	Convert into shares
12	10U - 2D	Buy back	Take the call price
18	18U - 0D	Not Buy back	Convert into shares
18	17U - 1D	Not Buy back	Convert into shares
18	16U - 2D	Not Buy back	Convert into shares
18	15U - 3D	Not Buy back	Convert into shares
18	14U - 4D	Not Buy back	Convert into shares
18	13U - 5D	Buy back	Convert into shares

For all the other nodes that are not reported in the above reported table nor the issuer or the holder of the option are going to buy back the bond or to exercise it.
Comparing with the result of the previous point it is possible to see that nothing changes regarding the first coupon date (t = 6). At the second coupon dates we observe a difference between these two options: the issuer of the bond in three nodes of the tree will decide to call and therefore to buy back the option from the holder. This is because in these three nodes the continuation value is larger than the maximum between the call price F_c and the value of γ shares of the stock and therefore the issuer is willing to buy back the option while the holder would prefer to continue holding the option. After the call of the issuer the holder decides to immediately exercise the bond only in the first two nodes of the tree because the value of the shares that he is going to receive is larger than the call price F_c. Another difference that is possible to observe is that the holder of the option in the case of the callable option decides to exercise it also in three more nodes this is because the continuation value of the callable bond is smaller than the continuation value of the vanilla bond. In the end we observe that in one node the issuer exercise his option for the same reason explained in the second coupon date case, and the holder decides to take the shares.

5.4

We report the graphs that illustrate how the the prices of the three bonds depend on the coupon c, on the conversion factor $\gamma = \beta$ and on the call price F_c.

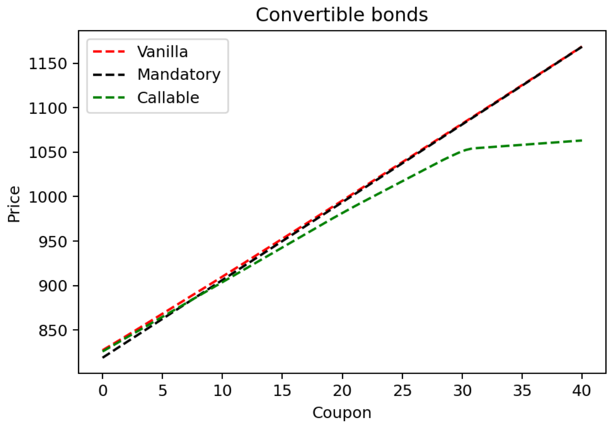


FIGURE 5.1

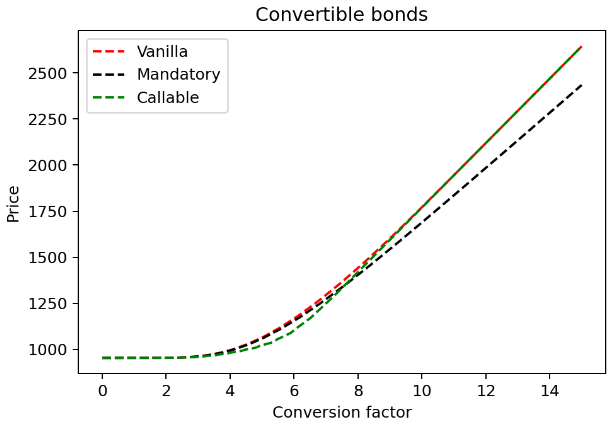


FIGURE 5.2

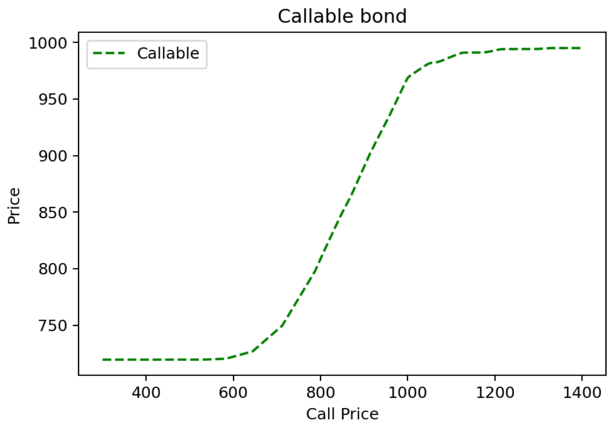


FIGURE 5.3

5.5
First we report our obtained results in tables

TABLE 5.1
Vanilla bond

γ	*Coupon*
1	25.3151
2	25.2692
3	24.3619
4	20.5207
5	12.0211
6	0

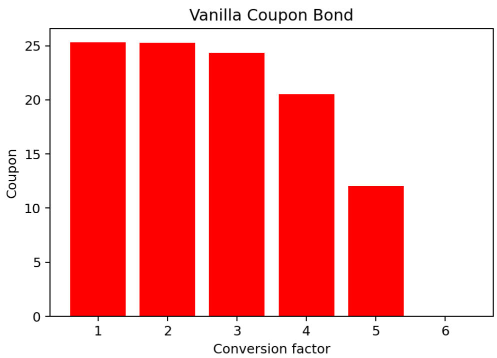
TABLE 5.2
Mandatory bond

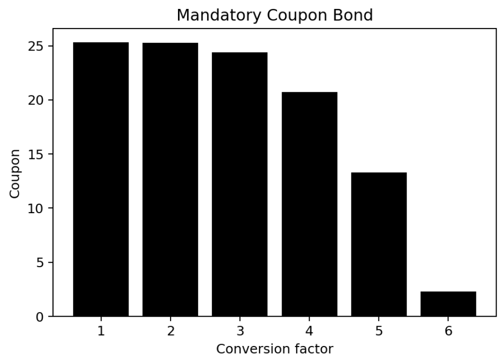
γ	*Coupon*
1	25.3151
2	25.2699
3	24.3875
4	20.7246
5	13.2904
6	2.2778

TABLE 5.3
CCB bond

γ	*Coupon*
1	25.3151
2	25.2798
3	24.7227
4	22.5587
5	15.0393
6	0

We can visualize the results that we summarized in the tables above using bar charts.





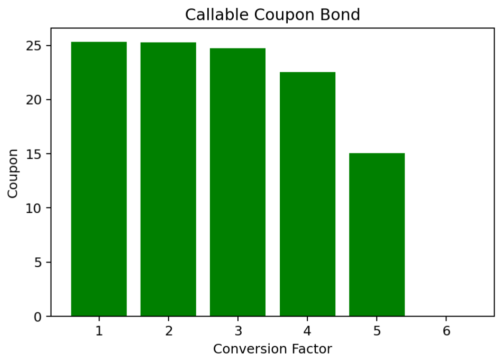


FIGURE 5.4

We calculated that the par coupon of a standard coupon bond with five year maturity and semiannual coupons is $c = 25.3151$. This value is equal to the ones that we obtain for all the three coupon types when gamma is equal to one: this is explained from the fact that the initial value of the stock S_0 is far from the face value F. We can observe that in all the three cases the par coupon rate of the three bonds decreases with the increasing of the conversion factor, Intuitively, this happens because increasing the conversion factor gives to the buyer the opportunity to buy more stocks in the case he decides to exercise his option. For the mandatory Convertible Bond, increasing β we are increasing the minimum amount of stock that we can receive, therefore the buyer will receive smaller coupon. When the conversion factor is 6, to maintain the price constant we obtain a negative coupon rate in the case of the Vanilla coupon bond and of the Callable Coupon bond, that we report as 0 in the bar chart.

5.6

We call γ_{CB} and γ_{cCB} the value of the conversion factor for the CB bond and for the cCB bond. From our calculations we have found the following values

$$\gamma_{CB} = 4.0936 \qquad \gamma_{cCB} = 4.5217$$

Given the same price F, the value of the conversion for the cCB is higher, meaning that the holder would be able to get more shares of the stock when he chooses to exercise. This is due to the fact that in the cCB case the continuation value in each of the nodes of the three is smaller or equal to the corresponding one of in the Vanilla CB one, because the seller can choose to buy back the Bond in the coupon dates.

We report in the following graph how the par conversion factors depend on the coupon c and on the call price in the case of the cCB. We can observe that both in the Vanilla Coupon Bond and in the Callable CB, when the value of the coupon increases the value of the conversion factor decreases: this is necessary to maintain constant the value F of the price. In fact if the holder receives a higher value of the coupon, at the same time he will obtain fewer shares of the stock if he decides to exercise. For the callable Coupon Bond, we can observe from the graph as when the value of the call price F_c increases, the par conversion factor decreases quickly. This happens for a similar reason as the one explained above: an higher call price gives more power to the holder, since if the issuer wants to buy back the option he has to pay a higher price.

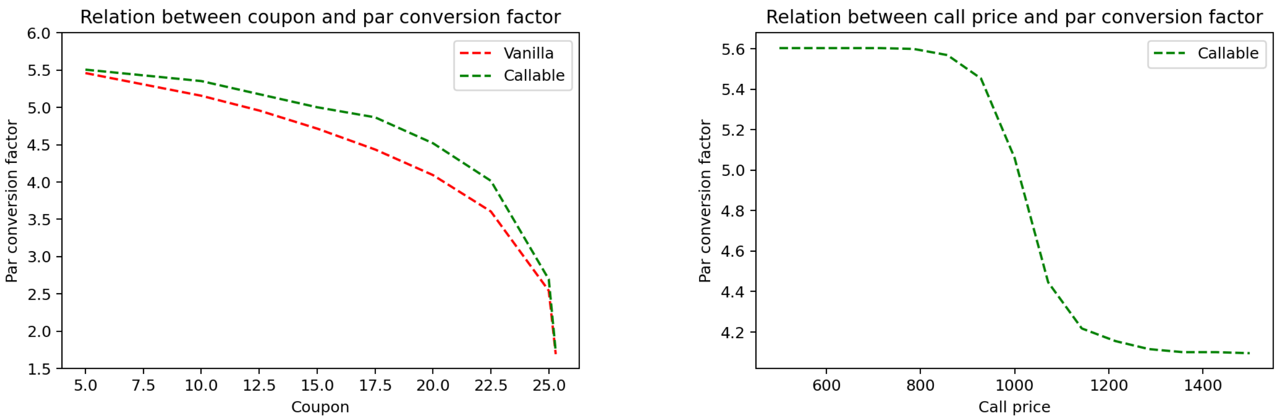


FIGURE 5.5

5.7

The following document is a Term Sheet that aims at informing J&J of available solution for the financing of its debt. For each of the products are reported the key characteristics and are highlighted the benefits and drawbacks for the issuer J&J and for its future potential investors.

Johnson & Johnson Term Sheet

Issuer Face Value Reference share price Issue date	Johnson & Johnson $ 1,000 $ 175 01.06.2022
Vanilla Convertibe Bond: CB_O1_22	
Maturity Date Issue Price Coupon Conversion Factor	01.06.2027 $ 1,000 $ 24.36 3
Vanilla Convertible Bond: CB_O2_22	
Maturity Date Issue Price Coupon paid semiannually Conversion Factor	01.06.2027 $ 1,000 $ 12.02 5
The holder retains the right, not the obligation, to convert the bond into a number of shares given by the conversion factor at each coupon date. This product allows the issuer to reduce the risk of a dilution of its shares. The investor faces unlimited upside potential and limited downside potential. The risk for the holder is lower in CB_O1_22 and higher in CB_O2_22. The difference lies in the coupon rate that is higher in the first and as a consequence will grant a lower conversion factor.	
Callable Convertible Bond: cCB_01_22	
Maturity Date Issue Price Coupon Conversion Factor Call price	01.06.2027 $ 1,000 $ 20 4.26 $ 1100
Callable Convertible Bond: cCB_O2_22	
Maturity Date Issue Price Coupon Conversion Factor Call price	01.06.2027 $ 1,000 $ 20 5.60 $ 1200
The holder retains the right (not the obligation) to convert the bond into a number of shares given by the conversion factor at each coupon date. The issuer retains the ability to exit the contact by paying a fee. This would allow the issuer to refinance its debt at a lower cost in the event of a fall in interest rates. The holder is compensated by gaining a higher coupon. The risk exposure of the holder, once again increases with the higher conversion factor which corresponds to lower coupon. Thus the cCB_O1_22 will be closer to a bond like product, while cCB_O2_22 will be closer to an equity like one	
Mandatory Convertible Bond: mCB_01_22	
Maturity Date Issue Price Coupon Conversion Threshold	01.06.2027 $ 1,000 13.29 $\alpha = 1000$ and $\beta = 5$
The holder detains the obligation to convert the bond into a number of shares given by the conversion factor at the maturity date. The issuer is favored by limiting its credit risk later in time. The holder will have the possibility to gain from unlimited upside potential but might also face unlimited downside potential. The higher risk is remunerated by a more generous coupon rate.	

Bibliography

Nyborg, Kjell Gustav (1996a). 'The use and pricing of convertible bonds'. In: *Applied Mathematical Finance* 3.3, pp. 167–190.

Calamos, John P (1998). *Convertible securities*. McGraw-Hill Professional Publishing.

CHEN, JAMES (2020). *Convertible Bond*. Investopedia. URL: https://www.investopedia.com/terms/c/convertiblebond.asp (visited on 2020).

Rebecca Baldridge, Benjamin Curry (2021). *Convertible Bonds Give You The Best Of Both Stocks And Bonds*. Forbs. URL: https://www.forbes.com/advisor/investing/convertible-bonds/ (visited on 2021).

HAYES, ADAM (2021). *Conversion Ratio Definition*. Investopedia. URL: https://www.investopedia.com/ (visited on 2020).

Dinsmore, Thomas, Jane O'Keeffe and James Dinsmore (2016). 'A review of the convertible securities market'. In: *Gabelli Funds, gabelli. com*.

CHEN, JAMES (2021a). *Mandatory Convertible*. Investopedia. URL: https://www.investopedia.com/terms/m/mandatoryconvertible.asp (visited on 2021).

Nyborg, Kjell Gustav (1996b). 'The use and pricing of convertible bonds'. In: *Applied Mathematical Finance* 3.3, pp. 167–190.

CHEN, JAMES (2021b). *Callable Bond*. Investopedia. URL: https://www.investopedia.com/terms/c/callablebond.asp (visited on 2021).

De Spiegeleer, Jan and Wim Schoutens (2012). 'Pricing contingent convertibles: A derivatives approach'. In: *The Journal of Derivatives* 20.2, pp. 27–36.

Koziol, Christian and Jochen Lawrenz (2012). 'Contingent convertibles. Solving or seeding the next banking crisis?' In: *Journal of Banking & Finance* 36.1, pp. 90–104.

CHEN, JAMES (2021c). *Contingent Convertibles (CoCos)*. Investopedia. URL: https://www.investopedia.com/terms/c/contingentconvertible.asp (visited on 2021).

Rock, Black (2022). *Callable Bond*. Black Rock. URL: https://www.blackrock.com/us/individual/products/272819/ishares-convertible-bond-etf (visited on 2022).