$$\begin{cases} \frac{d}{dt} b(\omega, t) = -\frac{\alpha}{\pi} |D(t)\cdot\omega| \\ b(\omega,\pi) = 0 \end{cases} \longrightarrow b(\omega; t) = \frac{\alpha}{\pi}\int_t^{\pi} |D(\xi)\cdot\omega|\, d\xi$$

Notation:

$$\mathcal{E}_a = \{e_a^1; \dots; e_a^{N_a}\} \qquad \omega = (\omega_1; \dots, \omega_N) \text{ with } N = N_a + N_b$$
$$\mathcal{E}_b = \{e_b^1; \dots e_b^{N_b}\}$$

$$g(\omega; t) = D(t)\cdot\omega = \sum_{i=1}^{N_a} \cos(e_a^i t)\omega_i + \sum_{J=1}^{N_b} \sin(e_b^J t)\omega_{N_a+J}$$

$$G(\omega, t) = \int g(\omega, t)\, dt = \sum_{i=1}^{N_a} \frac{\sin(e_a^i t)}{e_a^i}\omega_i - \sum_{J=1}^{N_b} \frac{\cos(e_b^J t)}{e_b^J}\omega_{N_a+J}$$

$$b(\omega; t) = \frac{\alpha}{\pi}\int_t^{\pi} |g(\omega, \xi)|\, d\xi = \frac{\alpha}{\pi} G(\omega; \xi)\operatorname{sgn}(g(\omega; \xi))\Big|_{\xi=t}^{\xi=\pi}$$
$$= \frac{\alpha}{\pi} G(\omega, \pi)\operatorname{sgn}(g(\omega, \pi)) - \frac{\alpha}{\pi} G(\omega, t)\operatorname{sgn}(g(\omega, t))$$

Def: $b^{\theta}(\omega; t) = \frac{\alpha}{\pi} G(\omega, \pi)\tanh(\theta g(\omega, \pi)) - \frac{\alpha}{\pi} G(\omega, t)\tanh(\theta g(\omega; t))$

$\theta > 0$

Then we have $b(\omega; t) = \lim_{\theta \to +\infty} b^{\theta}(\omega, t)$

$$G(\omega, \pi) = -\sum_{J=1}^{N_b} (-1)^{e_b^J} \frac{\omega_{N_a+J}}{e_b^J} = \sum_{J=1}^{N_b} \frac{\omega_{N_a+J}}{e_b^J} =: C_G$$

$$g(\omega, \pi) = \sum_{i=1}^{N_a} (-1)^{e_a^i}\omega_i = -\sum_{i=1}^{N_a}\omega_i =: C_g$$

Hence: $b^{\theta}(\omega, t) = \frac{\alpha C_G}{\pi}\tanh(\theta C_g) - \frac{\alpha}{\pi} G(\omega; t)\tanh(\theta g(\omega; t))$

Reachable set:

$$\lambda^\theta(\omega) := -\frac{b^\theta(\omega, t)}{v \cdot \omega} \quad \text{for all} \quad t \in [0, \pi) \text{ and } v \in \mathbb{R}^N$$

$$\lambda^\theta_{opt} = \min_{\substack{\omega \in \mathbb{R}^N \\ |\omega| = 1}} \lambda^\theta(\omega)$$

$$\lambda^\theta(\omega) = \lambda^\theta(\omega_1, \dots, \omega_N) = -\left(\sum_{J=1}^N v_J \omega_J \right)^{-1} b^\theta(\omega_1, \dots, \omega_N, t)$$

$$\frac{\partial}{\partial \omega_\ell} \lambda^\theta = \left(\sum_{J=1}^N v_J \omega_J \right)^{-2} v_\ell b^\theta - \left(\sum_{J=1}^N v_J \omega_J \right)^{-1} \frac{\partial}{\partial \omega_\ell} b^\theta$$

$$\frac{\partial}{\partial \omega_\ell} b^\theta = -\frac{\alpha}{\pi} \tanh(\theta g) \frac{\partial}{\partial \omega_\ell} G - \frac{\alpha}{\pi} G \frac{\partial}{\partial \omega_\ell} \tanh(\theta g)$$

$$= -\frac{\alpha}{\pi} \tanh(\theta g) \frac{\partial}{\partial \omega_\ell} G - \frac{\alpha}{\pi} G \frac{1}{\cosh^2(\theta g)} \theta \frac{\partial}{\partial \omega_\ell} g$$

$$= -\frac{\alpha}{\pi} \tanh(\theta g) \frac{\partial}{\partial \omega_\ell} G - \frac{\alpha}{\pi} \frac{\theta G}{\cosh^2(\theta g)} \frac{\partial}{\partial \omega_\ell} g$$

$$\frac{\partial}{\partial \omega_\ell} G = \begin{cases} \dfrac{\sin(e_a^\ell t)}{e_a^\ell} & \ell = 1, \dots, N_a \\[2ex] -\dfrac{\cos(e_b^\ell t)}{e_b^\ell} & \ell = N_a+1, \dots, N_b \end{cases}$$

$$\frac{\partial}{\partial \omega_\ell} g = \begin{cases} \cos(e_a^\ell t) & \ell = 1, \dots, N_a \\ \sin(e_b^\ell t) & \ell = N_a+1, \dots, N_b \end{cases}$$

Therefore. $l = 1, \dots, N_a$

$$\frac{\partial}{\partial w_l} b^\theta = -\frac{2}{\pi e_a^l} \sin(e_a^l t) \tanh(\theta g) - \frac{2\theta}{\pi} \cos(e_a^l t) \frac{G}{\cosh^2(\theta g)}$$

$l = N_a+1, \dots, N_b$

$$\frac{\partial}{\partial w_l} b^\theta = \frac{2}{\pi e_b^l} \cos(e_b^l t) \tanh(\theta g) - \frac{2\theta}{\pi} \sin(e_b^l t) \frac{G}{\cosh^2(\theta g)}$$

Finally:

$$l = 1, \dots, N_a \longrightarrow \frac{\partial}{\partial w_l} \lambda^\theta = \frac{V_l b^\theta}{(V \cdot \omega)^2} + \frac{1}{V \cdot \omega} \left(\frac{2}{\pi e_a^l} \sin(e_a^l t) \tanh(\theta g) + \frac{2\theta}{\pi} \cos(e_a^l t) \frac{G}{\cosh^2(\theta g)} \right)$$

$$l = N_a+1, \dots, N_b \longrightarrow \frac{\partial}{\partial w_l} \lambda^\theta = \frac{V_l b^\theta}{(V \cdot \omega)^2} - \frac{1}{V \cdot \omega} \left(\frac{2}{\pi e_b^l} \cos(e_b^l t) \tanh(\theta g) - \frac{2\theta}{\pi} \sin(e_b^l t) \frac{G}{\cosh^2(\theta g)} \right)$$