Selective Harmonic Elimination via Optimal Control

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Abstract

El problema de *Selective Harmonic Elimination pulse-width modulation*(SHE) es planteado como el problema de control ptimo, con el fin de encontrar soluciones de ondas escaln sin prefijar el nmero de ngulos de conmutacin. De esta manera, la metodologa de control ptimo es capaz de encontar la forma de onda ptima y de encontrar la localizaciones de los ngulos de conmutacin, incluso sin prefijar el nmero de conmutaciones. Este es un nuevo enfoque para el problema SHE en concreto y para los sistemas de control con un conjunto finito de controles admisibles en general.

Key words: Selective Harmonic Elimination; Finite Set Control, Piecewise Linear function.

1 Introduction and motivations

Selective Harmonic Elimination (SHE) [4] is a well-known methodology allowing to generate signals in the form of a step function with a desired harmonic spectrum.

In broad terms, this means to obtain a function $u(\tau)$ defined on $[0, 2\pi)$ whose values in the whole interval can only belong to a finite set of real numbers $\mathcal{U}$ and which, in addition to that, has certain predetermined Fourier coefficients.

Due to the application in power converters, we will focus on functions with half-wave symmetry, i. e. $u(\tau+\pi) = -u(\tau)$ for all $\tau \in [0, \pi)$. In this way, the function $u(\tau)$ is determined by its values in the interval $[0, \pi)$ and its Fourier series expansion takes the form

$$u(\tau) = \sum_{\substack{i\in\mathbb{N} \\ i\ odd}} a_i \cos(i\tau) + \sum_{\substack{j\in\mathbb{N} \\ j\ odd}} b_j \sin(j\tau), \tag{1.1}$$

where the coefficients a_i and b_j are given by

$$\begin{aligned} a_i &= \frac{2}{\pi}\int_0^\pi u(\tau)\cos(i\tau)\,d\tau, \\ b_j &= \frac{2}{\pi}\int_0^\pi u(\tau)\sin(j\tau)\,d\tau. \end{aligned} \tag{1.2}$$

In view of this, the SHE problem can be formulated as:

Problem 1.1 (SHE) *Let $\mathcal{U} = \{u_k\}_{k=1}^K$ be a finite subset of the interval $[-1, 1]$ and let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two sets of odd numbers with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$, respectively. Given the target vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, we look for a function $u(\tau)$, $\tau \in [0, \pi)$, such that $u(\tau)$ can only take values within $\mathcal{U}$ and whose Fourier coefficients satisfy:*

$$\begin{aligned} a_i &= (\boldsymbol{a}_T)_i, \quad \text{for all } i \in \mathcal{E}_a, \\ b_j &= (\boldsymbol{b}_T)_j, \quad \text{for all } j \in \mathcal{E}_b. \end{aligned}$$

Roughly speaking, this problem can be understood as finding a multi-set $\mathcal{S} = \{s_k\}_{k=1}^{\widehat{K}}$ with $s_k \in \mathcal{U}$, defining the values that the function $u(\tau)$ will assume and in which order they will appear (see [3]). In this way, given the sequence $\mathcal{S}$ we can focus in finding the exact locations where the function $u(\tau)$ changes its values. Following the terminology introduced in the SHE literature ([6,3,5]),

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we will refer to these locations of the changes in the values of the waveform $\mathcal{S}$ as *switching angles*. To finding of the switching angles given the sequence S can be addressed as a minimization problem where the variables are the angles while the cost functional is the Euclidean distance between the obtained Fourier coefficients and the desired ones.

Since the cardinality of S is not known a priori, meaning that we do not know how many switches will be necessary, it appears that the only solution to the SHE problem consists in fix the number of changes and counting all the possible combinations, to later solve an optimization problem for each one of them. Taking into account that the number of possible sequences S is given by $|\mathcal{U}|^{|S|}$, it is evident that the complexity of the above approach increases rapidly with the cardinality of S. This problem has been studied in [6] where, through appropriate algebraic transformations, the authors are able to convert the SHE problem into a polynomial system whose solutions' set contains all the possible waveforms for a given set $\mathcal{U}$ and number of elements in the sequence S which, however, is predetermined.

On the other hand, we shall also mention that the SHE methodology has been developed to provide in real-time different target Fourier coefficients con with a KHz latency. This makes impossible to find real-time solutions by optimization, making then necessary to pre-determine solutions that can later be interpolated. Nevertheless, it is well-known that, fixed a sequence S, the continuity of the switching locations with respect to a continuous variation of the target Fourier coefficients may be quite cumbersome. In the majority of the cases, it is impossible to find a continuous solution in a large interval, an it is necessary to change the waveform S while moving across different solution regions ([6,7]). This makes difficult the interpolation of solutions and their finding.

In this document, we will present the SHE problem as an optimal control one, where the optimization variable is the signal $u(\tau)$ defined in the entire interval $[0, \pi)$. In particular, we will describe how the Fourier coefficients of the function $u(\tau)$ can be seen as the final state of a system controlled by $u(\tau)$. Hence, the optimization is performed among all the possible functions that satisfy $|u(\tau)| < 1$ and can control the final state at the desired Fourier coefficients. Then we will show how to design a control problem so that the solution is a step function.

The present document is organized as follows. In Section 2, we will present the classical formulations of the SHE problem os an optimization one, introducing some seminal concept and discussing its advantages and disadvantages. In Section 3, we will formulate the SHE problem as a controlled dynamical system. In Section 4, we will present the optimal control problem allowing us to obtain the solutions to the SHE problem. In Section 5, we will present our numerical experiments. Finally, in Section 6, we will equation the conclusions of our study.

2 Classical SHE approach

In the classical SHE approach, the piece-wise definition of the function $u(\tau)$ is exploited to simplify the problem. In this formulation, $u(\tau)$ can be fully characterized by the switching angles and the constant values it may assume. We define these two concepts as follows.

Definition 2.1 (Switching angles) *Given a function $u : [0, \pi] \to \mathcal{U}$, the switching locations are the values of $\tau \in [0, \pi]$ where $u(\tau)$ changes its value discontinuously. We will denote the commutation angles as $\boldsymbol{\phi} = \{\phi_0, \phi_1, \ldots, \phi_M, \phi_{M+1}\}$, where we have taken $\phi_0 = 0$ and $\phi_{M+1} = \pi$.*
Definition 2.2 (Waveform) *Given $\mathcal{U}$ a finite subset of $[-1, 1]$, we will call a waveform a finite set $S = \{s_1, s_2, s_3, \ldots, s_{M+1}\}$ of elements of $\mathcal{U}$ with repetition.*

Then a waveform S indicates the values that the function will take and in which order they will appear within the interval $[0, \pi)$, while $\boldsymbol{\phi}$ indicates the switching locations. Considering these two elements, we can rewrite the Fourier coefficients as

$$\begin{aligned} a_i(\boldsymbol{\phi}) &= \frac{2}{i\pi} \sum_{k=1}^{M+1} s_k \Big[\sin(i\phi_k) - \sin(i\phi_{k-1}) \Big] \\ b_j(\boldsymbol{\phi}) &= \frac{2}{j\pi} \sum_{k=1}^{M+1} s_k \Big[\cos(j\phi_{k-1}) - \cos(j\phi_k) \Big] \end{aligned} \tag{2.1}$$

In this way, we can reformulate Problem 1.1 as follows.

Problem 2.1 (Optimization for SHE) *Given a waveform S, we look for the switching angles $\boldsymbol{\phi}$ by means of the following minimization problem:*

$$\min_{\boldsymbol{\phi} \in [0,\pi]^M} \sum_{i \in \mathcal{E}_a} \|a_T^i - a_i(\boldsymbol{\phi})\|^2 + \sum_{j \in \mathcal{E}_b} \|b_T^j - b_j(\boldsymbol{\phi})\|^2 \tag{2.2}$$
subject to:
$$0 < \phi_1 < \phi_2 < \cdots < \phi_{M-1} < \phi_M < \pi$$

In this formulation, the SHE problem converts in a minimization problem with restrictions which can be solved by well-known techniques. Since the problem has several minimizers, we shall solve it employing global optimizers. Furthermore, since the choice of the waveform is arbitrary, we shall proceed in the same way for each possible waveform.

Tengo que enlazar estas seccin con la formulacion de control ptimo.

3 Selective Harmonic Elimination as dynamical system

Inspired by the continuous nature of the optimization variable $u(\tau)$, we propose in this document the formulation from the optimal control. In this way we avoid the choice of the waveform, so that the optimization problem chooses the most convenient one in each case. So we look for $u(\tau) \in \mathcal{U}$, $\tau \in [0, \pi)]$, that has the desired Fourier coefficients.

We will use the fundamental theorem of differential calculus to rewrite the expression of the Fourier coefficients (1.2) as the evolution of a dynamical system. That is to say, for all i and j

$$\alpha_i(\tau) = \frac{2}{\pi}\int_0^\tau u(\tau)\cos(i\tau)d\tau \Rightarrow \begin{cases} \dot{\alpha}_i(\tau) = \frac{2}{\pi}u(\tau)\cos(i\tau) \\ \alpha_i(0) = 0 \end{cases}$$

$$\beta_j(\tau) = \frac{2}{\pi}\int_0^\tau u(\tau)\sin(j\tau)d\tau \Rightarrow \begin{cases} \dot{\beta}_j(\tau) = \frac{2}{\pi}u(\tau)\sin(j\tau) \\ \beta_j(0) = 0 \end{cases}$$

The evolution of the dynamical systems $\alpha_i(\tau)$ and $\beta_j(\tau)$ from the time $\tau = 0$ to $\tau = \pi$ gives us the coefficients a_i and b_j.

We introduce notation to refer to vectors $\boldsymbol{\alpha} = \{\alpha_i\}_{i\in\mathcal{E}_a}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathcal{E}_b}$. In this way, the general SHE problem (1.1) can be formulated as a control problem for a dynamical system where $\boldsymbol{\alpha}(\tau)$ and $\boldsymbol{\beta}(\tau)$ are the states and $u(\tau)$ is the control variable, and whose objective will be to bring the states from the origin to the objective vectors $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ in time $\tau = \pi$.

In order to obtain a compact expression of the problem that simplifies our understanding of it, we will introduce notation. So if we consider a problem with sets of odd numbers:

$$\begin{aligned} \mathcal{E}_a &= \{e_a^1, e_a^2, e_a^3, \dots, e_a^{N_a}\} \\ \mathcal{E}_b &= \{e_b^1, e_b^2, e_b^3, \dots, e_b^{N_b}\} \end{aligned} \tag{3.1}$$

then we can define the vectors $\boldsymbol{\mathcal{D}}^\beta(\tau) \in \mathbb{R}^{N_a}$ y $\boldsymbol{\mathcal{D}}^\beta(\tau) \in \mathbb{R}^{N_b} \mid \forall\tau \in (0, \pi]$ such that:

$$\boldsymbol{\mathcal{D}}^\alpha(\tau) = \begin{bmatrix} \cos(e_a^1\tau) \\ \cos(e_a^2\tau) \\ \dots \\ \cos(e_a^{N_a}\tau) \end{bmatrix}, \boldsymbol{\mathcal{D}}^\beta(\tau) = \begin{bmatrix} \sin(e_b^1\tau) \\ \sin(e_b^2\tau) \\ \dots \\ \sin(e_b^{N_b}\tau) \end{bmatrix} \tag{3.2}$$

So the dynamical system can be written as:

$$\begin{cases} \dot{\boldsymbol{\alpha}}(\tau) = \left(\frac{2}{\pi}\right)\boldsymbol{\mathcal{D}}^\alpha(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{\alpha}(0) = 0 \end{cases}$$
$$\begin{cases} \dot{\boldsymbol{\beta}}(\tau) = \left(\frac{2}{\pi}\right)\boldsymbol{\mathcal{D}}^\beta(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{\beta}(0) = 0 \end{cases} \tag{3.3}$$

Compressing the notation even more we can call the total state of the system $\boldsymbol{x}(\tau)$ to the concatenation of the states $\boldsymbol{\alpha}(\tau)$ and $\boldsymbol{\beta}(\tau)$ so that:

$$\boldsymbol{x}(\tau) = \begin{bmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{bmatrix} \quad \boldsymbol{x}_0 = \begin{bmatrix} \boldsymbol{a}_T \\ \boldsymbol{b}_T \end{bmatrix} \quad \boldsymbol{\mathcal{D}}(\tau) = \begin{bmatrix} \boldsymbol{\mathcal{D}}^\alpha(\tau) \\ \boldsymbol{\mathcal{D}}^\beta(\tau) \end{bmatrix} \tag{3.4}$$

So for a pair of sets $\mathcal{E}_a$ and $\mathcal{E}_b$ we have the following associated dynamical system:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = \left(\frac{2}{\pi}\right)\boldsymbol{\mathcal{D}}(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = 0 \end{cases} \tag{3.5}$$

Then we look for a function $u(\tau)$ such that it leads the dynamical system to the point $\boldsymbol{x}_T$, that is, the final state $\boldsymbol{x}(\pi)$ is $\boldsymbol{x}_0$. Since in control theory one usually steers a dynamical susytem from a given initial datum to zero, we introduce the change of variables $\boldsymbol{x}(\tau) \mapsto \boldsymbol{x}(\tau) - \boldsymbol{x}_0$. Our dynamical system then becomes

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = -\left(\frac{2}{\pi}\right)\boldsymbol{\mathcal{D}}(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \tag{3.6}$$

In this way, if the state at time $\tau = \pi$ coincides with zero, then the Fourier coefficients of the control $u(\tau)$ which has steered it to that configuration are the ones associated with the final condition $\boldsymbol{x}_T$.

Remark 3.1 (Quarter-wave symmetry) *In the SHE literature, is usual to distinguish among the half-wave symmetry problem (addressed in the present paper) and the quarter-wave symmetry one. If we wanted to formulate the quarter-wave symmetry problem, we should simply modify system 3.6 in such a way that we eliminate the states $\alpha_j(\tau)$ and the integration is for $\tau \in (0, \pi/2]$.*

4 Optimal Control for SHE

Since the SHE problem is equivalent to controlling a dynamic system from the origin of coordinates to a specific point, we must formulate a control problem that solves this task but also complies with the restrictions on the values that the control can take. It is necessary to set a finite subset $\mathcal{U}$ of the interval $[-1, 1]$, the optimal control is such that it can only take the values allowed in the discretization. In other words, the control problem is:

Problem 4.1 (OCP for SHE) *Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ and given the target vector $\boldsymbol{x}_T \in \mathbb{R}^{N_a+N_b}$, also given a set $\mathcal{U}$ of admissible controls, we look for the function $u(\tau) \mid \tau \in [0, \pi)$ such that:*

$$\min_{u(\tau)\in\mathcal{U}} \frac{1}{2}||\boldsymbol{x}(\pi)||^2 \tag{4.1}$$

suject to:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = -\left(\frac{2}{\pi}\right)\boldsymbol{\mathcal{D}}(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \tag{4.2}$$

The solution of this control problem is complex due to the restriction on the admissible control values. In order to obtain a problem that can be solved by classical control theory we can formulate the following control problem:

Problem 4.2 (Regularized OCP for SHE) *Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ and given the target vector $\boldsymbol{x}_T \in \mathbb{R}^{N_a+N_b}$, we look for the function $|u(\tau)| < 1 \mid \tau \in [0, \pi)$ such that:*

$$\min_{|u(\tau)|<1} \left[\frac{1}{2}||\boldsymbol{x}(\pi)||^2 + \epsilon\int_0^\pi \mathcal{L}(u(\tau))d\tau\right] \tag{4.3}$$

suject to:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = -\left(\frac{2}{\pi}\right)\boldsymbol{\mathcal{D}}(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \tag{4.4}$$

Where we will choose $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ such that the optimal control u^* only takes values in the discretization $\mathcal{U}$ of the interval $[-1.1]$. Furthermore, the parameter ϵ should be small so that the solution minimizes the distance from the final state and the target. Next we will study the optimality conditions of the problem, for a general $\mathcal{L}$ function, and then specify how $\mathcal{L}$ should be so that the optimal control u^* only takes the allowed values in $\mathcal{U}$.

4.1 Optimality conditions

To write the optimility conditions of the problem we will use the principle of the Pontryagin minimum [2, Chapter 2.7]. For them it is necessary to define the Hamiltonian of the system, which in this case is:

$$H(u, \boldsymbol{p}, \tau) = \epsilon\mathcal{L}(u) - \left(\frac{2}{\pi}\right)[\boldsymbol{p}^T(\tau)\cdot\boldsymbol{\mathcal{D}}(\tau)]u(\tau) \tag{4.5}$$

Where the variable $\boldsymbol{p}(\tau)$ called adjoint state is introduced, which is associated with the restriction imposed by the system. This has the same dimension of the state vector, so that

$$\boldsymbol{x}(\tau) = \begin{bmatrix}\boldsymbol{\alpha}(\tau)\\ \boldsymbol{\beta}(\tau)\end{bmatrix} \Leftrightarrow \boldsymbol{p}(\tau) = \begin{bmatrix}\boldsymbol{p}^\alpha(\tau)\\ \boldsymbol{p}^\beta(\tau)\end{bmatrix} \tag{4.6}$$

In what follows, we will enumerate the optimality conditions arising from the Pontryagin principle.

(1) **Final condition of the adjoint**: This optimiality condition is obtained from the cost in the final time $\tau = \pi$ of the control problem in this case $\Psi(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}(\pi) - \boldsymbol{x}_T\|^2$.

$$\boldsymbol{p}(\pi) = \nabla_{\boldsymbol{x}}\Psi(\boldsymbol{x}) = (\boldsymbol{x}(\pi) - \boldsymbol{x}_T) \tag{4.7}$$

(2) **Adjoint evolution equation**:

$$\dot{\boldsymbol{p}}(\tau) = -\nabla_x H(u(\tau), \boldsymbol{p}(\tau), \tau) = 0 \tag{4.8}$$

From where it can be deduced that $\boldsymbol{p}(\tau)$ is a constant so that $\boldsymbol{p}(\tau) = \boldsymbol{x}(\pi) - \boldsymbol{x}_T \mid \forall\tau \in [0, \pi)$ so from now on we will refer to it simply as $\boldsymbol{p}$, noting that it is invariant in time.

(3) **Optimal Waveform**: We known that $u^* = \arg\min_{|u|<1} H(\tau, \boldsymbol{p}^*, u)$, so in this case we can write:

$$u^*(\tau) = \underset{|u|<1}{\arg\min}\left[\epsilon\mathcal{L}(u(\tau)) - \left(\frac{2}{\pi}\right)[\boldsymbol{p}^{*T}\cdot\boldsymbol{\mathcal{D}}(\tau)]u(\tau)\right] \tag{4.9}$$

Therefore, this optimality condition reduces to the optimization of a function in a variable within the interval $[-1, 1]$.

It is important to recall that

$$[\boldsymbol{p}^{*T}\cdot\boldsymbol{\mathcal{D}}(\tau)] = \sum_{i\in\mathcal{E}_a} p^*_\alpha\cos(i\tau) + \sum_{j\in\mathcal{E}_b} p^*_\beta\sin(j\tau) \tag{4.10}$$

we build a function $\mathcal{H}_m : [-1, 1] \to \mathbb{R}$ such that:

$$\mathcal{H}_m(u) = \epsilon\mathcal{L}(u) - mu|\forall m \in \mathbb{R} \tag{4.11}$$

where we replaced the term $(2/\pi)[\boldsymbol{p}^{*T}\cdot\boldsymbol{\mathcal{D}}(\tau)]$ with a parameter m which may assume all real values. In other words, the problem of designing an optimal control which may only take real values in $\mathcal{U}$ is reduced to construct a one-dimensional function with parameter m whose minima are the elements in $\mathcal{U}$ for all values of m.

4.2 Piecewise linear penalization

In this section, we discuss how to design the penalization term $\mathcal{L}(u)$ so that the optimal control is always contained in $\mathcal{U}$.

In more detail, we can choose the affine interpolation of a parabola $\mathcal{L} : [-1, 1] \to \mathbb{R}$ as a penalization term. That is

$$\mathcal{L}(u) = \begin{cases} [(u_{k+1} + u_k)(u - u_k) + u_k^2] & \text{if} u \in [u_k, u_{k+1}[\\ 1 & \text{if } u = u_{N_u} \end{cases} \tag{4.12}$$

$$\forall k \in \{1, \ldots, N_u - 1\}$$

Nevertheless, to compute the minimum of $\mathcal{H}_m(u)$, we shall take into account that this function is not differentiable and the optimality condition then requires to work with the subdifferential $\partial\mathcal{L}(u)$, which given by

$$\partial\mathcal{L}(u) = \begin{cases} \{u_1 + u_2\} & \text{if } u = u_1 \\ \{u_k + u_{k+1}\} & \text{if } u \in]u_k, u_{k+1}[\quad \dagger \\ [u_k + u_{k-1}, u_{k+1} + u_k] & \text{if } u = u_k \quad \ddagger \\ \{u_{N_u} + u_{N_u - 1}\} & \text{if } u = u_{N_u} \end{cases} \tag{4.13}$$

$$\dagger\ \forall k \in \{1, \ldots, N_u - 1\} \quad \ddagger\ \forall k \in \{2, \ldots, N_u - 1\}$$

Hence, we have $\partial H_m = \epsilon\partial\mathcal{L} - m$. This means that, given $m \in \mathbb{R}$, we look for $u \in [-1, 1]$ minimizing $\mathcal{H}_m(u)$. It is then necessary to determine whether zero belongs to $\partial\mathcal{H}_m(u)$.

- **Case 1:** $m \leq \epsilon(u_1 + u_2)$: since m is less than the minimum of all subdifferentials, then zero does not belong to any of the intervals we defined. Hence, the minimum is in one of the extrema

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_1 \tag{4.14}$$

- **Case 2:** $m = \epsilon(u_{k+1} + u_k)$: taking into account that $\forall k \in \{1, \ldots, N_u - 1\}$,

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = [u_k, u_{k+1}[\tag{4.15}$$

- **Case 3:** $\epsilon(u_k + u_{k-1}) < m < \epsilon(u_{k+1} + u_k)$: taking into account that $\forall k \in \{2, \ldots, N_u - 1\}$,

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_k \tag{4.16}$$

- **Case 4:** $m > \epsilon(u_{N_u} + u_{N_u - 1})$:

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_{N_u} \tag{4.17}$$

In other words, only when $m = \epsilon(u_{k+1} + u_k)$ the minima of the Hamiltonian belong to an interval. For all the other values of $m \in \mathbb{R}$, these minima are contained in $\mathcal{U}$. So that under a continuous variation of m, Case 2 can only occur pointwise. Recalling the optimal control problem $m(\tau) = [\boldsymbol{p}^T(\tau) \cdot \boldsymbol{\mathcal{D}}(\tau)]$, we can notice that Case 2 corresponds to the instants τ of change of value.

5 Numerical simulations

In this section, we will present several examples in which we solve our optimal control problem through the direct method and the non-linear constrained optimization tool `CasADi` [1].

5.1 *Smooth approximation of piece-wise linear penalization*

With the final aim of using an optimization software to solve our optimal control problem, we will approximate our piece-wise linear penalization with the help of the Heaviside function $h : \mathbb{R} \to \mathbb{R}$ and its smooth approximation defined as follows:

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \begin{cases} h^\eta(x) = (1 + \tanh(\eta x))/2 \\ \eta \to \infty \end{cases} \tag{5.1}$$

Using h, we can define the (smooth) function $\Pi^\eta_{a,b} : \mathbb{R} \to \mathbb{R}$ as:

$$\begin{aligned} \Pi^\eta_{[a,b]}(x) &= -1 + h^\eta(x - a) + h^\eta(-x + b) \\ &= \frac{\tanh[\eta(x - a)] + \tanh[\eta(b - x)]}{2}. \end{aligned}$$

In this way, we can define the smooth version of (4.12):

$$\mathcal{L}^\eta(u) = \sum_{k=1}^{N_u - 1} \left[(u_{k+1} + u_k)(u - u_k) + u_k^2\right] \Pi^\eta_{[u_k, u_{k+1}]}(u) \tag{5.2}$$

So that, when $\eta \to \infty$, then $\mathcal{L}^\eta \to \mathcal{L}$.

5.2 *Direct method for OCP-SHE*

To solve the optimal control problem (4.2), we use a direct method. If we consider a partition $\mathcal{P} = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of interval $[0, T]$, we can represent a function $\{u(\tau) \mid \tau \in [0, T]\}$ as a vector $\boldsymbol{u} \in \mathbb{R}^T$ where component $u_t = u(\tau_t)$. Then the optimal control problem (4.1) can be written as optimization problem with variable $\boldsymbol{u} \in \mathbb{R}^T$. This problem is a nonlinear programming, for this we use CasADi software to solve. Hence, given a partition of the interval $[0, \pi)$, we can formulate the problem (4.1) as the following one in discrete time
Problem 5.1 (Numerical OCP) *Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively, given the target vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$, so that $\boldsymbol{x}_0 = [\boldsymbol{a}_T, \boldsymbol{b}_T]^T$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$ and a partition $\mathcal{P}_\tau = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of the interval $[0, \pi)$, we search a*

vector $\boldsymbol{u} \in \mathbb{R}^T$ that minimizes the following function:

$$\min_{\boldsymbol{u}\in\mathbb{R}^T} \left[||\boldsymbol{x}^T||^2 + \epsilon \sum_{t=0}^{T-1} \mathcal{L}^\eta(u_t)\Delta\tau_t \right] \tag{5.3}$$

suject to:

$$\forall \tau \in \mathcal{P} \begin{cases} \boldsymbol{x}^{t+1} = \boldsymbol{x}^t - (2/\pi)\Delta\tau_t \boldsymbol{\mathcal{D}}(\tau_t) u_t \\ \boldsymbol{x}^0 = \boldsymbol{x}_0 \end{cases} \tag{5.4}$$

5.3 Resultados

All our simulations have been performed with a laptop with $8Gb$ of RAM memory and the execution time for finding the solution given a target vector is os the order of $1s$. IN what follows, we will discuss all the numerical results we have obtained

(1) **OCP with quarter-wave symmetry**: we will consider the problem with a set of odd numbers $\mathcal{E}_b = \{1, 5\}$ and a discretization of $[0, \pi/2]$ with $T = 200$. We show the solutions for the target vectors $b_T^1 = \{(-0.4, -0.3, \ldots, 0.3, 0.4)\}$ keeping $b_T^5 = 0$ for each case. We display the optimal trajectories we obtained in Figure **??**, where we can see a continuity of the solutions with respect to the target vector.

(2) **OCP with quarter-wave symmetry for an interval of** b_1: for this example we consider the following set of odd numbers: $\mathcal{E}_b = \{1, 5, 7, 11, 13\}$. Moreover, we consider the target vector $\boldsymbol{b}_T = [m_a, 0, 0, 0, 0]$, where $m_a \in [-1, 1]$ is a parameter, with three penalization terms: $\mathcal{L}(u) = -f$, $\mathcal{L}(u) = +f$ and $\mathcal{L}(u) = -f^2$ obtained by direct method with uniform partition of interval $[0, \pi/2]$ with $T = 400$ and penalization parameter $\epsilon = 10^{-5}$. For each one of the penalization terms we will employ, the distance between the Fourier coefficients is of the order 10^{-4}. Nevertheless, when the penalization term is $\mathcal{L}(u) = -f^2$, the solution does not present continuity with respect to the target vector. On the other hand, it is important to mention that the solutions for the penalization terms $\mathcal{L}(u) = -f$ y $\mathcal{L}(u) = f$ are symmetric, so that inverting those solutions with respect to the origin and inverting their sign, it can be observed that they are the same.

(3) **SHE with three levels**: we can see that in the case in which the control $u(\tau)$ can only take values in $[0, 1]$, we obtain signals which can take three levels in the interval $[0, 2\pi]$ due to the quarter-wave symmetry. If we solve the optimal control problem this time with restrictions $\{0 < u(\tau) < 1\}$. We repeat the same procedure as before, thus obtaining solutions for the same penalization terms and obtaining Figure **??**. There we show the continuity of the solutions and that they are of the order 10^{-4}.

(4) **Changes in the commutations number**: thanks to the optimal control formulation of SHE, we can vary the number of commutation angles. This is illustrated in the following example, where we considered the set of odd numbers $\mathcal{E}_b = \{1, 3, 9, 13, 17\}$. Moreover, we consider the target vector $\boldsymbol{b}_T = [m_a, 0, 0, 0, 0]$, where $m_a \in [0, 1]$ is a parameter. In this problem, we used the penalization $\mathcal{L} = f$ with a penalization parameter $\epsilon = 10^{-4}$. We can see in Figure **??** that the optimal control problem is capable of moving among different solution sets.

(5) **OCP for SHE with half-wave symmetry**: we considered the optimal control problem in half-wave with $\mathcal{E}_a = \{1, 3, 5\}$ and $\mathcal{E}_b = \{1, 3, 5, 9\}$, where $\boldsymbol{a}_T = [m_a, 0, 0]$, $\boldsymbol{b}_T = [m_a, m_a, 0, 0]$ and $m_a \in [-0.6, 0.6]$. We chose the penalization $L(u) = +f$

6 Conclusiones

We presented the SHE problem from the point of view of control theory. This methodology allows obtaining a 10^{-4}–10^{-5} precision in the distance to the target vector. Nevertheless, comparing with methodologies where the commutation number is fixed a priori, our approximation is computationally more expensive. Notwithstanding that, the optimal control provides solutions in the entire range of the modulation index, although the number of solutions or their locations change dramatically.

This methodology for the SHE problem connects control theory with harmonic elimination. In this way, the SHE problem can be solved through classical tools.

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