Multilevel Selective Harmonic Modulation via Optimal Control

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Abstract

We consider the *Selective Harmonic Modulation* (SHM) problem, consisting in the design of a staircase control signal with some prescribed frequency components. In this work, the SHM problem is addressed as an optimal control one in which the admissible controls are piece-wise constant functions, taking values only in a given finite set. In order to fulfill this constraint, we introduce a cost functional with piece-wise linear penalization which, by means of Pontryagin's maximum principle, makes the optimal control have the desired staircase form. An advantage of our approach, relevant in practical power electronics engineering applications, is that the number of commutations and the waveform need not be specified a priori. Indeed, our algorithm provides an admissible waveform as well as the location of the switches. Up to the best of our knowledge, this approach to the SHM problem via optimal control is new. Moreover, our methodology may be applicable to other optimal control problems with a finite-set constraint on the control. We also provide several numerical examples in which the SHM problem is solved by using our approach.

Key words: Selective Harmonic Modulation; Optimal Control Theory; Finite-Set Control; Pontryagin's maximum principle; Piece-wise linear penalization.

1 Introduction and motivations

Selective Harmonic Modulation (SHM) [Rodríguez et al., 2002] is a well-known methodology in power electronics engineering, employed to improve the performances of a converter by controlling the phase and amplitude of the harmonics in its output voltage. As a matter of fact, this technique allows to increase the power of the converter and, at the same time, to reduce its losses.

Because of the growing complexity of modern electrical networks, consequence for instance of the high penetration of renewable energy sources, the demand in power of electronic converters is day by day increasing. For this and other reasons, SHM has been a preeminent research interest in the power electronics community, and a plethora of SHM-based techniques has been developed in recent years. An incomplete bibliography includes [Duranay and Guldemir, 2017, Janabi et al., 2020, Yang et al., 2017].

In broad terms, the technique consists in generating a *control signal* with a desired harmonic spectrum by modulating or eliminating some specific lower-order Fourier coefficients. In practice, the signal is constructed as a step function with a finite number of switches, taking values only in a given finite set. Such a signal can be fully characterized by two features (see Figure 1):

1. The *waveform*, i.e. the sequence of values that the function takes in its domain.
2. The *switching angles*, i.e. the sequence of points where the signal switches from one value to following one.

Using this simple characterization of the signal, in many practical situations the SHM problem is reduced to a finite-dimensional optimization one in which, given a suitable waveform, the aim is to find the optimal loca-

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tion of the switching angles.

One of the principal difficulties in addressing the SHM problem through finite-dimensional optimization is that the choice of a suitable waveform may not always be evident. In addition to that, also determining the optimal number of switching angles may be a quite cumbersome task (see Section 3).

To overcome these difficulties, in this work we propose a new approach to SHM based on optimal control theory (see Section 4): the Fourier coefficients in the SHM problem are identified with the terminal state of a controlled dynamical system, where the control is actually the signal, solution to the SHM problem.

The main difficulty in our approach arises from the fact that we look for piece-wise constant control signals, taking values only in a given finite set. This formulation is in analogy with what is known in the control literature as *switching controls* or *switching systems* (see [Liberzon, 2003, Zuazua, 2011, Liu and Gong, 2014]), whose goal is to control the dynamics of a system by switching from an actuator to another in a systematic way so that, at each instant of time, only one actuator is active.

In the control theory literature, there exist different methods to obtain this type of control, which can be classified as follows:

1. Methods with a predetermined number of switches, whose locations have to be optimized [Xu and Antsaklis, 2002].
2. Methods without a predetermined number of switches. These methods provide a control with finite constraint set in a infinite time-horizon [Quevedo et al., 2004].

Nevertheless, as we shall see, the SHM problem we are going to consider translates in an optimal control problem with finite time horizon, for which the above methodologies are not applicable. This kind of constraint on the control prevents one from applying the classical tools in optimal control theory, such as Pontryagin's maximum principle.

In order to bypass this difficulty, we introduce a penalization term for the control, which guarantees that the optimal control will have the desired staircase structure. The optimization is then carried out over the controls in L^∞, and then, standard computational algorithms in optimal control theory can be implemented to solve the SHM problem.

The main contributions of the present paper are as follows:

1. We reformulate the SHM problem as an optimal control problem with a staircase constraint on the control.
2. We do not need to determine a priori the waveform of the solution nor the number of switching angles.
3. We introduce a penalization term for the control which implicitly induces the staircase property on the solution. Here we distinguish two cases: when the control set contains only two elements (bi-level or bang-bang control) and when it contains more than two elements (multilevel control).
4. Our approach allows one to compute solutions to the SHM problem with different waveforms.

This document is structured as follows. In Section 2, we introduce a mathematical formulation of a general SHM problem. In Section 3, we show the classical formulation in SHM literature, and we show the main problems related to this approach. In Section 4, we present our main contributions. In particular, where explain how the SHM can be cast as an optimal control one and we show the strategies to obtain a piece-wise function as optimal control. In Section 5, we show several numerical examples of concrete SHM problems solved with our methodology. In Section 6, we present the proofs of our theoretical results. Finally, in Section 7, we summarize and comment our results, and present some open problems.

2 Preliminaries

This section is devoted to the mathematical formulation of the SHM problem and to introduce the notation that will be used throughout the paper. Let

$$\mathcal{U} = \{u_1, \dots, u_L\} \tag{2.1}$$

be a given set of $L \geq 2$ real numbers satisfying

$$u_1 = -1,\ u_L = 1 \ \text{ and } \ u_k < u_{k+1} \quad \forall k \in \{1, \dots, L\}.$$

The goal is to construct a step function

$$u(t) : [0, 2\pi) \to \mathcal{U},$$

with a finite number of switches, such that some of its lower-order Fourier coefficients take specific values prescribed a priori.

Due to applications in power converters, it is typical to only consider functions with *half-wave symmetry*, i.e. satisfying

$$u(t + \pi) = -u(t) \quad \text{for all } t \in [0, \pi). \tag{2.2}$$

In this way, we only need to determine u in the interval $[0, \pi)$. Moreover, as a consequence of this symmetry, the Fourier series of u only involves the odd terms (as the even terms just vanish), i.e.

$$u(t) = \sum_{\substack{j \in \mathbb{N} \\ j\ odd}} a_j \cos(jt) + \sum_{\substack{j \in \mathbb{N} \\ j\ odd}} b_j \sin(jt),$$

with

$$\begin{aligned} a_j &= \frac{2}{\pi} \int_0^{\pi} u(\tau) \cos(j\tau)\, d\tau, \\ b_j &= \frac{2}{\pi} \int_0^{\pi} u(\tau) \sin(j\tau)\, d\tau. \end{aligned} \tag{2.3}$$

In view of the half-wave symmetry (2.2), in what follows, we will only work with the restriction $u|_{[0,\pi)}$, which, with some abuse of notation, we still denote by u.

As we anticipated, we are only considering piece-wise constant functions with a finite number of switches and taking values in $\mathcal{U}$. More precisely, we look for $u : [0, \pi) \to \mathcal{U}$ of the form

$$u(t) = \sum_{m=0}^{M} s_m \chi_{[\phi_m, \phi_{m+1})}(t), \quad M \in \mathbb{N} \tag{2.4}$$

for some $\mathcal{S} = \{s_m\}_{m=0}^{M}$ satisfying

$$s_m \in \mathcal{U} \ \text{ and } \ s_m \neq s_{m+1} \quad \forall m \in \{0, \ldots, M\}$$

and $\Phi = \{\phi_m\}_{m=1}^{M}$ such that

$$0 = \phi_0 < \phi_1 < \ldots < \phi_M < \phi_{M+1} = \pi.$$

In (2.4), $\chi_{[\phi_m, \phi_{m+1})}$ denotes the characteristic function of the interval $[\phi_m, \phi_{m+1})$. With these notations we just introduced, we can now define the waveform and the switching angles as follows.

Definition 2.1 *For a function $u : [0, \pi) \to \mathcal{U}$ of the form (2.4), we refer to $\mathcal{S}$ as the* waveform, *and we refer to Φ as the* switching angles.

Observe that any u of the form (2.4) is fully characterized by its waveform $\mathcal{S}$ and switching angles Φ. An example of such a function is displayed in Figure 1.

In the practical engineering applications that motivated our study, due to technical limitations, it is preferable to employ signals taking consecutive values in $\mathcal{U}$. In the sequel, we will refer to this property of the waveform as the *staircase property*.

We can rigorously formulate this property as follows: we say that a signal u of the form (2.4) fulfills the staircase property if its waveform $\mathcal{S}$ satisfies

$$(s_m^{min}, s_m^{max}) \cap \mathcal{U} = \emptyset, \quad \forall m \in \{0, \ldots, M-1\}, \tag{2.5}$$

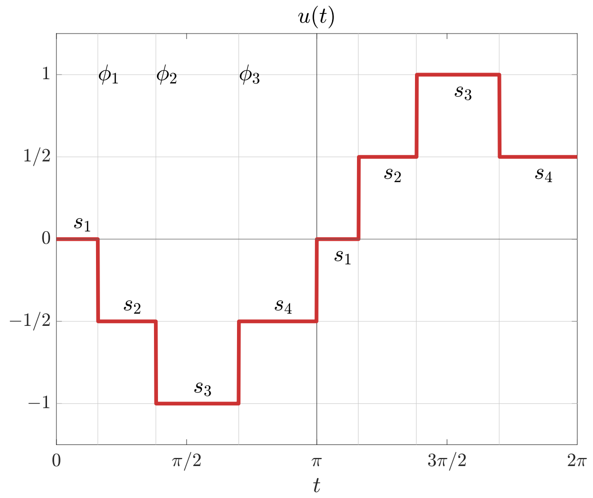

Fig. 1. A possible solution to the SHM Problem, where we considered the finite set of control $\mathcal{U} = \{-1, -1/2, 0, 1/2, 1\}$. We show the switching angles Φ and the waveform $\mathcal{S}$ (see Definition 2.1). The function $u(t)$ is displayed on the whole interval $[0, 2\pi)$ to highlight the half-wave symmetry introduced in (2.2).

where

$$\begin{aligned} s_m^{min} &= \min\{s_m, s_{m+1}\} \\ s_m^{max} &= \max\{s_m, s_{m+1}\}. \end{aligned}$$

Note that when $\mathcal{U} = \{-1, 1\}$ (the bi-level case), this property is satisfied for any u of the form (2.4).

We can now formulate the SHM problem as follows:

Problem 2.1 (SHM) *Let $\mathcal{U}$ be given as in (2.1), and let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two finite sets of odd numbers of cardinality $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively. For any two given vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, construct a function $u : [0, \pi) \to \mathcal{U}$ of the form (2.4), satisfying (2.5), such that the vectors $\boldsymbol{a} \in \mathbb{R}^{N_a}$ and $\boldsymbol{b} \in \mathbb{R}^{N_b}$, defined as*

$$\boldsymbol{a} = (a_j)_{j \in \mathcal{E}_a} \quad \text{and} \quad \boldsymbol{b} = (b_j)_{j \in \mathcal{E}_b} \tag{2.6}$$

satisfy

$$\boldsymbol{a} = \boldsymbol{a}_T \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{b}_T,$$

where the coefficients a_j and b_j in (2.6) are given by (2.3).

Remark 2.1 (SHE) *In Problem 2.1, we gave a very general formulation of SHM. This formulation contains also the so-called* Selective Harmonic Elimination (SHE) *problem (see [Sun et al., 1996]), in which the target vectors are such that*

$$\begin{aligned} (a_T)_1 \neq 0 \quad (a_T)_{i \neq 1} = 0 \quad \forall i \in \mathcal{E}_a \\ (b_T)_1 \neq 0 \quad (b_T)_{j \neq 1} = 0 \quad \forall j \in \mathcal{E}_b. \end{aligned}$$

SHE is of great relevance in the electric engineering literature. Its objective is to generate a signal with amplitude $m_1 = \sqrt{a_1^2 + b_1^2}$ *and phase* $\varphi_1 = \arctan(b_1/a_1)$*, removing the high-frequency components. In this way, SHE may be understood as a generator of clean Fourier modes through a staircase signal.*

3 SHM as a finite-dimensional optimization problem

A typical approach to address the SHM Problem 2.1 ([Yang et al., 2015, Konstantinou and Agelidis, 2010, Sun et al., 1996]) consists in looking for solutions having a specific waveform determined a priori and looking for suitable switching angles Φ allowing to reach the desired target.

Note that, for a fixed waveform $\mathcal{S}$, the Fourier coefficients of a function u of the form (2.4) can be written in terms of the switching angles Φ in the following way:

$$a_j = a_j(\Phi) = \frac{2}{j\pi}\sum_{m=0}^{M} s_m\Big[\sin(j\phi_{m+1}) - \sin(j\phi_m)\Big]$$

$$b_j = b_j(\Phi) = \frac{2}{j\pi}\sum_{m=0}^{M} s_m\Big[\cos(j\phi_m) - \cos(j\phi_{m+1})\Big]$$

Hence, for two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ as in Problem 2.1, and any fixed waveform $\mathcal{S}$, we can define the functions

$$\boldsymbol{a}_{\mathcal{S}}(\Phi) := \big(a_j(\Phi)\big)_{j\in\mathcal{E}_a} \in \mathbb{R}^{N_a}$$

and

$$\boldsymbol{b}_{\mathcal{S}}(\Phi) := \big(b_j(\Phi)\big)_{j\in\mathcal{E}_b} \in \mathbb{R}^{N_b}.$$

In this framework, SHM can then be cast as a finite-dimensional optimization process consisting in minimizing, over the choice of $\Phi = \{\phi_m\}_{m=1}^M$, the euclidean distance between the vectors $(\boldsymbol{a}, \boldsymbol{b})$ and $(\boldsymbol{a}_T, \boldsymbol{b}_T)$ as defined in Problem 2.1.

Problem 3.1 (Optimization problem for SHM) *Let* $\mathcal{E}_a$*,* $\mathcal{E}_b$ *and the targets* $\boldsymbol{a}_T$ *and* $\boldsymbol{b}_T$ *be given as in Problem 2.1. Let* $\mathcal{S} := \{s_m\}_{m=0}^M$ *be a fixed waveform satisfying (2.5). Find the switching angles* Φ *solution to the following minimization problem:*

$$\min_{\Phi\in[0,\pi]^M}\left(\|\boldsymbol{a}_{\mathcal{S}}(\Phi) - \boldsymbol{a}_T\|^2 + \|\boldsymbol{b}_{\mathcal{S}}(\Phi) - \boldsymbol{b}_T\|^2\right)$$

$$\textit{subject to: } 0 = \phi_0 < \phi_1 < \ldots < \phi_M < \phi_{M+1} = \pi.$$

At this regard, it is important to notice that the optimization Problem 3.1 solves the original Problem 2.1 only when the optimal cost is zero. This makes necessary to fully characterize the space of targets $(\boldsymbol{a}_T, \boldsymbol{b}_T)$ for which it exists a solution. With this aim, we will define the the *optimal cost function* and the *solvable set* as follows:

Definition 3.1 (optimal cost function) *We call optimal cost function* $V_{\mathcal{S}} : \mathbb{R}^{N_a} \times \mathbb{R}^{N_b} \to \mathbb{R}$*, the function that take as input variables the vector targets* $\boldsymbol{a}_T$ *and* $\boldsymbol{b}_T$ *and return the optimal cost of the Problem 3.1.*

Definition 3.2 (solvable set) *We define a solvable set* $\mathcal{R}_{\mathcal{S}}$ *as:*

$$\mathcal{R}_{\mathcal{S}} = \{(\boldsymbol{a}_T, \boldsymbol{b}_T) \in \mathbb{R}^{N_a+N_b} | V_{\mathcal{S}}(\boldsymbol{a}_T, \boldsymbol{b}_T) = 0\} \tag{3.1}$$

In practical applications, one is only interested in solutions of Problem 3.1 corresponding to targets $(\boldsymbol{a}_T, \boldsymbol{b}_T) \in \mathcal{R}_{\mathcal{S}}$, so that it would be desirable to obtain a control function containing the switching angles $\Phi^*_{\mathcal{S}}$ for the whole set $\mathcal{R}_{\mathcal{S}}$. We will refer to this function as the *policy* with waveform $\mathcal{S}$

Definition 3.3 (policy) *We will call policy a function* $\Pi_{\mathcal{S}} : \mathcal{R}_{\mathcal{S}} \to [0,\pi]^M$ *such that* $\Phi^* = \Pi_{\mathcal{S}}(\boldsymbol{a}_T, \boldsymbol{b}_T)$*, with* Φ^* *the optimal switching angles solutions of Problem 2.1 for the targets* $(\boldsymbol{a}_T, \boldsymbol{b}_T)$*.*

With the aim of reconstructing the policy $\Pi_{\mathcal{S}}$, a typical approach is to solve numerically Problem 3.1 for a limited number of points in $\mathbb{R}^{N_a+N_b}$ and check that the optimal cost is zero. Secondly, one interpolate the function $\Pi_{\mathcal{S}}$ in the convex set generated by the points previously obtained.

Let us now point out some of the main difficulties and drawback arising in this approach.

3.1 Combinatory problem

In practice, one does not dispose of a suitable waveform $\mathcal{S}$ which yields a solution to the Problem 2.1. A common approach to solve the SHM problem consists in fixing the number of switches M, and then solve Problem 3.1 for all the possible combinations of M elements of $\mathcal{U}$. However, taking into account that the number of possible M-tuples in $\mathcal{U}$ is of the order $(L-1)^M$, it is evident that the complexity of the above approach increases rapidly when $L > 1$. This problem has been studied in [Yang et al., 2015] where, through appropriate algebraic transformations, the authors are able to convert the SHM problem into a polynomial system whose solutions' set contains all the possible waveforms $\mathcal{S}$ of M elements in $\mathcal{U}$. As a drawback of this approach, the number of switches M needs to be prefixed. However, in some cases, determining the number of switches which is necessary to reach the desired Fourier coefficients is not a straightforward task.

3.2 Solvable set problem

Given a waveform $\mathcal{S}$, the corresponding solvable set $\mathcal{R}_{\mathcal{S}}$ is usually very small, yielding to policies $\Pi_{\mathcal{S}}$ which are not very effective. This issue is typically addressed by solving Problem 3.1 for a set of waveforms $\{\mathcal{S}_l\}_{l=1}^{r}$ and obtaining different policies $\{\Pi_{\mathcal{S}_l}\}_{l=1}^{r}$ and solvable sets $\{\mathcal{R}_{\mathcal{S}_l}\}_{l=1}^{r}$ for each one of them. By gathering them, on then creates a new policy which is applicable in a larger set. Nevertheless, due to the fact that in general to different waveforms may correspond disjoint or even overlapping solvable sets, this union of policies is chaotic, providing different solutions for the same target vector $(\boldsymbol{a}_T, \boldsymbol{b}_T)$, or even generating regions with no solution (see Figure 2).

3.3 Policy problem

Due to the complexity of a policy generated by the union of different waveforms, the continuity of the switching angles cannot be guaranteed. This is a well known problem ion the SHM community [Agelidis et al., 2008, Dahidah and Agelidis, 2008, Dahidah et al., 2015, Yang et al., 2017] (see Figure 2).

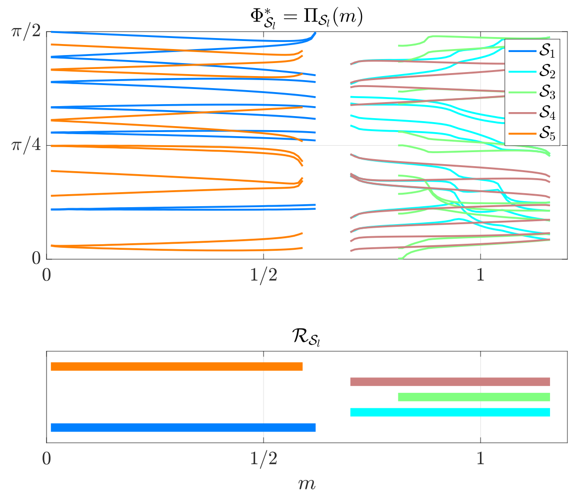

Fig. 2. In the first picture, we display the optimal switching angles $\Phi^*_{\mathcal{S}}$ associated to different waveforms $\{\mathcal{S}_l\}_{l=1}^{7}$ for a SHE problem (see Remark 2.1), considering the sets $\mathcal{E}_a = \{1\}$ and $\mathcal{E}_b = \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31\}$. We chose the target $(a_1, b_1) = (m, 0)$ $\forall m \in [0, 1.2]$. The second figure shows the solvable sets for each waveform we considered.

4 SHM problem as an Optimal Control Problem

Our main contribution in the present paper consists in formulating the SHM problem from the viewpoint of optimal control. In this formulation, the Fourier coefficients of the signal $u(t)$ are identified with the terminal state of a controlled dynamical system of $N_a + N_b$ components defined in the time-interval $[0, \pi]$. The control of the system is precisely the signal $u(t)$, defined as a function $[0, \pi] \to \mathcal{U}$, which has to steer the state from the origin to the desired values of the prescribed Fourier coefficients.

The starting point of this approach is to rewrite the Fourier coefficients of the function $u(t)$ as the final state of a dynamical system controlled by $u(t)$.

To this end, let us first note that, in view of (2.3), for all $u \in L^\infty([0, \pi]; \mathbb{R})$ any Fourier coefficient a_j satisfies

$$a_j = y(\pi),$$

with $y \in C([0, \pi]; \mathbb{R})$ defined as

$$y(t) = \frac{2}{\pi} \int_0^t u(\tau) \cos(j\,\tau) d\tau.$$

Besides, as a consequence of the fundamental theorem of calculus, $y(\cdot)$ is the unique solution to the differential equation

$$\begin{cases} \dot{y}(t) = \dfrac{2}{\pi} \cos(j\,t) u(t), & t \in [0, \pi] \\ y(0) = 0. \end{cases} \tag{4.1}$$

Analogously, we can also write the Fourier coefficients b_j, defined in (2.3), as the solution at time $t = \pi$ of a differential equation similar to (4.1).

Hence, for $\mathcal{E}_a$, $\mathcal{E}_b$, $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ given, the SHM Problem 2.1 can be reduced to finding a control $u(\cdot)$ of the form (2.4), satisfying (2.5), such that the solution $\boldsymbol{y} \in C([0, \pi]; \mathbb{R}^{N_a + N_b})$ to the dynamical system

$$\begin{cases} \dot{\boldsymbol{y}}(t) = \dfrac{2}{\pi} \boldsymbol{\mathcal{D}}(t) u(t), & t \in [0, \pi] \\ \boldsymbol{y}(0) = 0. \end{cases} \tag{4.2}$$

satisfies

$$\boldsymbol{y}(\pi) = [\boldsymbol{a}_T; \boldsymbol{b}_T]^\top,$$

where

$$\boldsymbol{\mathcal{D}}(t) = \left[\boldsymbol{\mathcal{D}}^a(t); \boldsymbol{\mathcal{D}}^b(t) \right]^\top, \tag{4.3}$$

with $\boldsymbol{\mathcal{D}}^a(t) \in \mathbb{R}^{N_a}$ and $\boldsymbol{\mathcal{D}}^b(t) \in \mathbb{R}^{N_b}$ given by

$$\boldsymbol{\mathcal{D}}^a(t) = \begin{bmatrix} \cos(e_a^1 t) \\ \cos(e_a^2 t) \\ \vdots \\ \cos(e_a^{N_a} t) \end{bmatrix}, \quad \boldsymbol{\mathcal{D}}^b(t) = \begin{bmatrix} \sin(e_b^1 t) \\ \sin(e_b^2 t) \\ \vdots \\ \sin(e_b^{N_b} t) \end{bmatrix} \tag{4.4}$$

Here, e_a^i and e_b^i denote the elements in $\mathcal{E}_a$ and $\mathcal{E}_b$, i.e.

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \dots, e_a^{N_a}\}, \quad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \dots, e_b^{N_b}\}.$$

Moreover,for notation simplicity, we reverse the time in (4.2) using the transformation $\boldsymbol{x}(t) = \boldsymbol{y}(\pi - t)$. In this way, the SHM problem converts into the following null-controllability one with initial condition $\boldsymbol{x}(0) = [\boldsymbol{a}_T; \boldsymbol{b}_T]^\top$ (see also Figure 3):

Problem 4.1 (SHM as null-controllability problem) *Let $\mathcal{U}$ be a given set as in (2.1), and let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two finite sets of odd numbers of cardinality $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively. For any two given vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, we look for a function $u : [0, \pi] \to [-1, 1]$ of the form (2.4), satisfying (2.5), such that the solution to the initial-value problem*

$$\begin{cases} \dot{\boldsymbol{x}}(t) = -\dfrac{2}{\pi}\boldsymbol{\mathcal{D}}(t)u(t), & t \in [0, \pi] \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 := [\boldsymbol{a}_T; \boldsymbol{b}_T] \end{cases} \tag{4.5}$$

satisfies $\boldsymbol{x}(\pi) = 0$, where $\boldsymbol{\mathcal{D}}$ is given by (4.3)–(4.4).

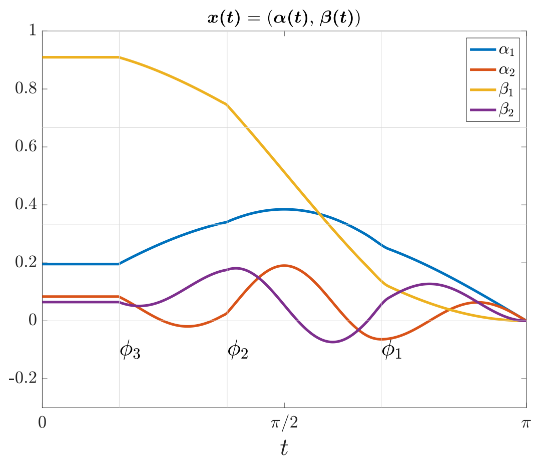

Fig. 3. Evolution of the dynamical system (4.5) with $\mathcal{E}_a = \{1, 2\}$ and $\mathcal{E}_b = \{1, 2\}$ corresponding to the control u in Figure 1. The positions of the switching angles $\boldsymbol{\phi}$ are displayed as well.

A natural approach for controllability problems such as Problem 4.1 is to formulate them as an optimal control one, where the cost functional to be minimized is the euclidean distance from the final state $\boldsymbol{x}(\pi)$ to the target, in this case, the origin. In what follows, for a given vector $\boldsymbol{v} \in \mathbb{R}^d$, we shall always denote by $\|\boldsymbol{v}\|$ the euclidean norm $\|\boldsymbol{v}\|_{\mathbb{R}^d}$.

Problem 4.2 (OCP for SHM) *Let $\mathcal{U}$ be a given set as in (2.1) and let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two finite sets of odd numbers of cardinality $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively. For any two given vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, we look for an admissible control $u \in \mathcal{U}_{ad}$ solution to the following optimal control problem:*

$$\min_{u \in \mathcal{U}_{ad}} \frac{1}{2}\|\boldsymbol{x}(\pi)\|^2 \quad \textit{subject to the dynamics } (4.5),$$

where the set of admissible controls $\mathcal{U}_{ad}$ is defined as

$$\mathcal{U}_{ad} := \{u \in L^\infty([0, \pi]; [-1, 1]) \\ \textit{of the from } (2.4) \textit{satisfying } (2.5)\}.$$

Remark 4.1 *Note that, due to the coercivity of the cost functional, Problem 4.2 admits at least one solution for any target $[\boldsymbol{a}_T; \boldsymbol{b}_T]^\top$. However, we say that the SHM problem admits a solution if and only if the minimum in Problem 4.2 is equal to zero. Otherwise, we say that the target $[\boldsymbol{a}_T; \boldsymbol{b}_T]^\top$ is unreachable. In this work, we will not discuss the reachable set for the control problem (4.2).*

A main feature of the SHM problem is that we are looking for signal functions u of the form (2.4) satisfying (2.5). In principle, this can be guaranteed by by adding directly this constraint on the set of admissible controls $\mathcal{U}_{ad}$. Notwithstanding that, the inclusion of such a constraint makes the optimal control problem extremely difficult, as it prevents us from applying standard arguments such as the Pontryagin maximum principle, and implementing all the computational techniques developed in the last decades to solve optimal control problems. As a matter of fact, the classical tools in optimal control theory seem not to be adapted to handle such constraints on the control. In order to bypass this difficulty, we propose a modification of Problem 4.2 by adding to the cost functional a penalization term for the control.

In what follows, if not stated differently, we will always denote

$$L^\infty := L^\infty([0, \pi]; [-1, 1]).$$

We can then give the following equivalent formulation of Problem 4.2.

Problem 4.3 (Penalized OCP for SHM) *Fix $\varepsilon > 0$ and a function $\mathcal{L} \in C([-1, 1]; \mathbb{R})$. Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two finite sets of odd numbers of cardinality $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively. For any two given vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, we look for a control $u \in L^\infty$ solution to the following optimal control problem:*

$$\min_{u \in L^\infty} \frac{1}{2}\|\boldsymbol{x}(\pi)\|^2 + \varepsilon \int_0^\pi \mathcal{L}(u(t))dt$$

subject to the dynamics (4.5).

Observe that, in Problem 4.3, we do not impose the constraint that the control has to be the form (2.4) and satisfy the staircase property. Nevertheless, as we shall see, these properties of u will arise naturally from a suitable choice of the Lagrangian $\mathcal{L}$.

Another important aspect that needs to be taken into account is that the penalization term for the control might prevent the optimal trajectory from reaching the target. In other words, even if there exists a control for which the optimal trajectory satisfies $\boldsymbol{x}(\pi) = 0$, the optimal control in Problem 4.3 might not do so, and therefore, the solution to Problem 4.3 would not be a solution to the SHM problem. This issue may be prevented by a proper selection of the weighting parameter ε which allows to tune the precision of the optimal control for the perturbed problem, guaranteeing that the final state of the optimal trajectory is close enough to zero. In more detail, we have the following result.

Proposition 4.1 *Assume that $[\boldsymbol{a}_T, \boldsymbol{b}_T]^\top$ is such that Problem 4.1 admits a solution, and let $u^* \in L^\infty$ be the solution to Problem 4.3. Then the associated trajectory $\boldsymbol{x}^* \in C([0,\pi];\mathbb{R})$, solution to (4.5), satisfies*

$$\|\boldsymbol{x}^*(\pi)\|^2 \leq 4\varepsilon\pi\|\mathcal{L}\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the max-norm in $C([-1,1];\mathbb{R})$.

The proof of Proposition 4.1 is postponed to Section 6. Let us now describe the construction of penalization functions $\mathcal{L}$ which guarantee that any solution to Problem 4.3 has the form (2.4) and satisfies (2.5). To this end, we will distinguish two cases, depending on the cardinality of $\mathcal{U}$.

4.1 Bilevel SHM problem via OCP (Bang-Bang Control)

In this case, the control set $\mathcal{U}$ defined in (2.1) has only two elements, i.e. $\mathcal{U} = \{-1, 1\}$. In the control theory literature, a control taking only two values is known as *bang-bang control*. In the SHM literature, this kind of solution are called *bi-level solutions*. Note that in this case, any u with the form (2.4) trivially satisfies the staircase property (2.5).

The first main result of the present paper is the following.

Theorem 4.1 *Let $\mathcal{U} = \{-1, 1\}$, and consider Problem 4.3. If the function $\mathcal{L} \in C([-1,1];\mathbb{R})$ is concave, then any optimal control u^*, solution to Problem 4.3, has a bang-bang structure, i.e. it has the form (2.4).*

The proof of Theorem 4.1 is postponed to Section 6, and follows from the optimality conditions given by the Pontryagin's maximum principle. The fact that the control u acts linearly in (4.5), together with the concavity of $\mathcal{L}$, implies that the Hamiltonian is also concave, and then, the minimum is always attained at the limits of the interval $[-1, 1]$.

We point out that, by choosing different penalization functions $\mathcal{L}$, we can obtain solutions to the SHE problem with different waveforms. Typical choices which give rise to bang-bang controls are

$$\mathcal{L}(u) = u, \quad \mathcal{L}(u) = -u, \quad \mathcal{L}(u) = -u^2.$$

See Figure 7 for an illustration.

4.2 Multilevel SHM problem via OCP

Inspired by the ideas of the previous subsection, we can address the case when $\mathcal{U}$ contains more than two elements. This is known in the power electronics literature as the *multilevel SHM problem*. Now, the goal is to construct a function $\mathcal{L}$ such that the Hamiltonian associated to Problem 4.3 always attains the minimum at points in $\mathcal{U}$. A way to construct such a function $\mathcal{L}$ is to interpolate a parabola in $[-1, 1]$ by affine functions, considering the elements in $\mathcal{U}$ as the interpolating points. Since, between any two points in $\mathcal{U}$, the function $\mathcal{L}$ is a straight line, the Hamiltonian is a concave function in these intervals, and hence, the minimum is always attained at points in $\mathcal{U}$.

Theorem 4.2 *Let $\mathcal{U}$ be a given set as in (2.1). For some $0 < a \in \mathbb{R}$ and $b \in \mathbb{R}$, set the function*

$$\mathcal{P}(u) = a(u-b)^2.$$

Consider the Problem 4.3 with

$$\mathcal{L}(u) = \begin{cases} \lambda_k(u) & \text{if } u \in [u_k, u_{k+1}[\\ \mathcal{P}(1) & \text{if } u = u_L \end{cases} \qquad \forall k \in \{1, \dots, L-1\} \tag{4.6}$$

where

$$\lambda_k(u) = \frac{(u-u_k)\mathcal{P}(u_{k+1}) + (u_{k+1}-u)\mathcal{P}(u_k)}{u_{k+1}-u_k}. \tag{4.7}$$

Then any optimal control u^*, solution to Problem 4.3, has the form (2.4) and satisfies (2.5).

Remark 4.2 (*Bang-off-bang control*) *We note that when $\mathcal{U} = \{-1, 0, 1\}$, as penalization function we can just use the L^1-norm of the control, i.e. $\mathcal{L}(u) = |u|$. This yields to the so-called* bang-off-bang *controls, that are widely studied in the literature [Nagahara et al., 2013, Ikeda and Nagahara, 2016]. By taking a different parabola $\mathcal{P}$, one can then obtain different bang-off-bang solutions to the SHM problem.*

We illustrate in Figure 8 different examples of penalization functions $\mathcal{L}$ giving rise to multilevel solutions to the

SHM problem. We point out that, by varying the values of a and b in Theorem 4.2, we can obtain solutions with different waveforms.

Remark 4.3 *For completeness, we shall mention that in Theorem 4.2 the Lagrangian $\mathcal{L}$ can actually have a more general form than (4.6), still yielding to an optimal control u^* with the form (2.4) and satisfying (2.5). In more detail, given a strictly convex function $\mathcal{P}(u)$, we can define $\mathcal{L}(u)$ as in (4.6) but with a more general selection of λ_k. For having controls in the desired form for the SHM problem, we only have to take into account the sufficient condition that, for all $k \in \{1,\dots,N_u-1\}$, the functions λ_k are concave.*

Examples of functions which are compatible with this situation are

1. *A union of concave functions (see the second plot in Figure 8):*

$$\lambda_k(u) = -4u^2 + 2(u_k + u_{k+1}) - 2u_k.$$

2. *Linear approximation of shifted parabola (see the third plot in Figure 8)*

$$\lambda_k(u) = \frac{1}{4}\Big[(u_{k+1}+u_k)(u-u_k-1)+u_k^2\Big].$$

5 Numerical simulations

In this section, we present several examples in which we solve our optimal control problem through the direct method [Rao, 2009] and the non-linear constrained optimization tool `CasADi` [Andersson et al., 2019]. All the simulations we are going to present can be found also in [Oroya, 2021]. Our Experiments were conducted on a personal MacBook Pro laptop (1,4 GHz Quad-Core Intel Core i5, 8GB RAM, Intel Iris Plus Graphics 1536 MB).

We will start by presenting a suitable numerical approximation for the penalization functions $\mathcal{L}$ introduced in Section 4. Secondly, we will briefly mention the methodology we shall employ to solve the optimal control problem. Finally, we will show three examples of SHM problem solved via the optimal control approach.

5.1 *Smooth approximation of piece-wise linear penalization*

With the final aim of using an optimization software to solve our optimal control problem, we will approximate our piece-wise linear penalization with the help of the Heaviside function $h:\mathbb{R}\to\mathbb{R}$ and its smooth approximation defined as follows:

$$h(x) = \begin{cases} 1 & \text{if } x>0 \\ 1/2 & \text{if } x=0 \\ 0 & \text{if } x<0 \end{cases}, \qquad \begin{cases} h^\theta(x) = \dfrac{1+\tanh(\theta x)}{2} \\ \lim\limits_{\theta\to+\infty} h^\theta = h. \end{cases}$$

Using h^θ, for all $k\in\{1,\dots,N_u-1\}$ we can then define the (smooth) function $\Pi^\theta_{u_k,u_{k+1}}:\mathbb{R}\to\mathbb{R}$ as:

$$\begin{aligned}\Pi^\theta_{u_k,u_{k+1}}(x) &= -1 + h^\theta(x-u_k) + h^\theta(-x+u_{k+1}) \\ &= \frac{\tanh[\theta(x-u_k)] + \tanh[\theta(u_{k+1}-x)]}{2}.\end{aligned}$$

and use it to build the smooth version of (4.6)

$$\mathcal{L}^\theta(u) = \sum_{k=1}^{N_u-1} \lambda_k \Pi^\theta_{[u_k,u_{k+1}]}(u)$$

where

$$\lambda_k = (u_{k+1}+u_k)(u-u_k) + u_k^2 \tag{5.1}$$

so that $\mathcal{L}^\theta \to \mathcal{L}$ when $\theta\to+\infty$.

In this way, we can obtain a smooth numerical approximation of our penalization function.

5.2 *Direct method for OCP-SHE*

To solve the optimal control problem (4.3), we use a direct method. If we consider a partition $\mathcal{T}=\{t_0,t_1,\dots,t_{N_t-1}\}$ of interval $[0,\pi]$, we can represent a function $\{u(\tau)\mid\tau\in[0,\pi]\}$ as a vector $\boldsymbol{u}\in\mathbb{R}^{N_t}$ where component $u_t = u(\tau_t)$. Then the optimal control problem (4.2) can be written as optimization problem with variable $\boldsymbol{u}\in\mathbb{R}^{N_t}$. This problem is a nonlinear programming, for this we use `CasADi` software to solve. Hence, given a partition of the interval $[0,\pi)$, we can formulate the problem 4.3 as the following one in discrete time.

Problem 5.1 (Numerical OCP) *Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively, given the target vectors $\boldsymbol{a}_T\in\mathbb{R}^{N_a}$, so that $\boldsymbol{x}_0 = [\boldsymbol{a}_T,\boldsymbol{b}_T]^\top$ and $\boldsymbol{b}_T\in\mathbb{R}^{N_b}$ and a partition $\mathcal{T}=\{t_k\}_{k=0}^{N_t-1}$ of the interval $[0,\pi)$, we search a vector $\boldsymbol{u}\in\mathbb{R}^{N_t}$ that solves the following minimization* *problem:*

$$\min_{\boldsymbol{u}\in\mathbb{R}^{N_t}} \left[\|\boldsymbol{x}^{N_t}\|^2 + \varepsilon \sum_{k=0}^{N_t-1} \left[\frac{\mathcal{L}^\eta(u_{t_k}) + \mathcal{L}^\eta(u_{t_{k+1}})}{2} \Delta t_k \right] \right]$$

subject to:

$$\forall \tau \in \mathcal{P} \begin{cases} \boldsymbol{x}_{t_{k+1}} = \boldsymbol{x}_{t_k} - \Delta t_k (2/\pi) \boldsymbol{\mathcal{D}}(t_k) \\ \boldsymbol{x}_{t_0} = \boldsymbol{x}_0 \end{cases}$$

where

$$\Delta t_k = t_{k+1} - t_k \quad \forall k \in \{1, \ldots, N_t - 1\}. \tag{5.2}$$

5.3 Examples

All the examples that we are going to present will share the following common parameter: $\varepsilon = 10^{-5}$, $\eta = 10^{-5}$ and $\mathcal{P}_t = \{0.0, 0.1, 0.2, \ldots, \pi\}$. Moreover, we will consider $\mathcal{E}_a = \{1, 5, 7, 11, 13, 15\}$ and $\mathcal{E}_b = \{1, 5, 7, 11, 13, 15\}$, and the target vectors $\boldsymbol{a}_T = (m, 0, 0)^T$ and $\boldsymbol{b}_T = (m, 0, 0w, 0, 0, 0)^T$ for all $m \in [-0.8, 0.8]$. We will treat all the cases specific types of controls we mentioned before, that is, *bang-bang*, *bang-off-bang* and *multilevel*.

Simulation 5.1 (Bang-Bang) *We consider Problem 5.1 with a set of admissible controls:*

$$\mathcal{U} = \{-1, 1\}$$

In accordance with Theorem 4.1, we can see in Figure 4 how for all $m \in [-0.8, 0.8]$ the optimal control only takes the values $\{-1, 1\}$ displayed in the plot in blue and red, respectively. Ademş de esto, la poltica

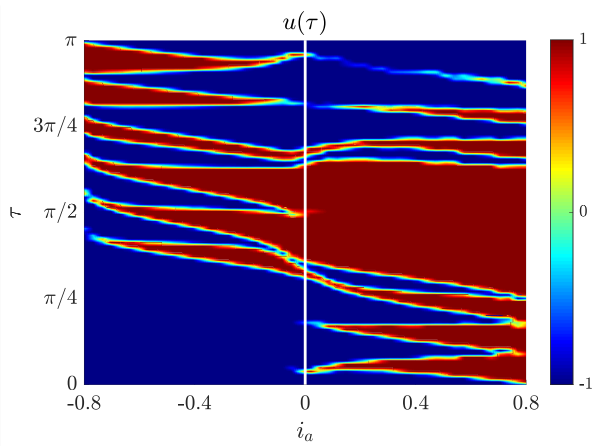

Fig. 4. Results of simulation of Simulation 5.1.

Simulation 5.2 (Bang-off-Bang) *We consider Problem 5.1 with a set of admissible controls:*

$$\mathcal{U} = \{-1, 0, 1\}$$

In the same way as before, when we consider three possible values $\mathcal{U} = \{-1, 0, 1\}$, we recover the *bang-off-bang* controls (Figure 5).

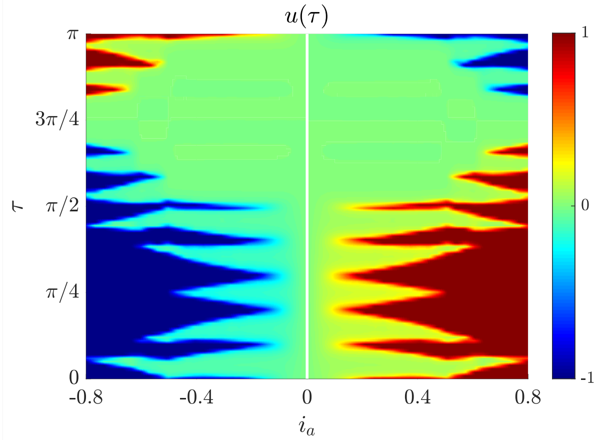

Fig. 5. Results of simulation of Simulation 5.2.

Simulation 5.3 (Multi-level) *We consider Problem 5.1 with a set of admissible controls:*

$$\mathcal{U} = \{-1, -1/2, 0, 1/2, 1\}$$

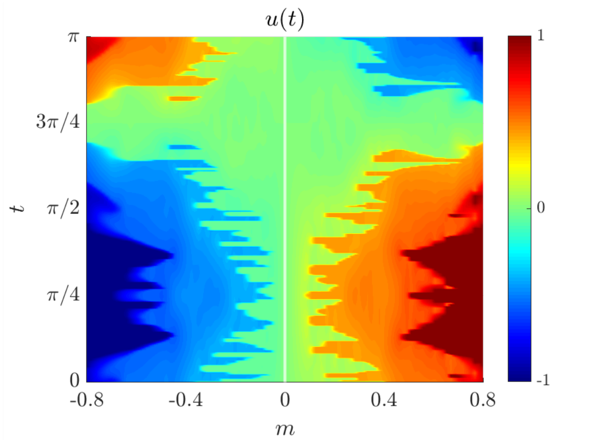

Fig. 6. Results of simulation of Simulation 5.3.

Finally, we can see in Figure 6 the same behavior when considering $\mathcal{U} = \{-1, 1\}$.

6 Proofs of results in Section 4

We give here the proofs of the results presented in Section 4. We start with the proof of Proposition 4.1, which gives an upper estimate for the error in the solution to the optimal control problem with the penalization term for the control.

Proof: **(of Proposition 4.1)** *Since we are supposing that Problem 4.1 has a solution, there exists a control* $\tilde{u} \in L^\infty$ *such that its corresponding trajectory* $\tilde{\boldsymbol{x}}$*, solution to* (4.5)*, satisfies* $\tilde{\boldsymbol{x}}(\pi) = 0$*.*

Now, let u^* *be the solution to Problem 4.3, and let* $\boldsymbol{x}^*$ *be its corresponding trajectory. By the optimality of* u^* *we have*

$$\frac{1}{2}\|\boldsymbol{x}^*(\pi)\|^2 + \varepsilon \int_0^\pi \mathcal{L}(u^*(\tau))d\tau \leq \varepsilon \int_0^\pi \mathcal{L}(\tilde{u}(\tau))d\tau,$$

and hence, we deduce that $\|\boldsymbol{x}^*(\pi)\|^2 \leq 4\varepsilon\pi\|\mathcal{L}\|_\infty$*.* □

6.1 Proofs of Theorems 4.1 and 4.2

The proofs of Theorems 4.1 and 4.1 are based on the optimality conditions for the Optimal Control Problem 4.3, which can be deduced by means of Pontryagin's maximum principle [Bryson, 1975, Chapter 2.7].

To this end, let us first introduce the Hamiltonian function associated to the Optimal Control Problem 4.3:

$$\mathcal{H}(u, \boldsymbol{p}, t) = \varepsilon\mathcal{L}(u) - \frac{2}{\pi}\big(\boldsymbol{p} \cdot \boldsymbol{\mathcal{D}}(t)\big)u(t), \tag{6.1}$$

where $\boldsymbol{p} \in \mathbb{R}^{N_a+N_b}$ is the so-called adjoint variable, which arises from the restriction imposed by the dynamical system (4.5). In view of the definition of $\boldsymbol{\mathcal{D}}(t)$ in (4.3)-(4.4), we will sometimes write the state and the adjoint variables using the following notation:

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{a}(t) \\ \boldsymbol{b}(t) \end{bmatrix} \quad \text{and} \quad \boldsymbol{p}(t) = \begin{bmatrix} \boldsymbol{p}^a(t) \\ \boldsymbol{p}^b(t) \end{bmatrix}.$$

Now, let us derive the optimality conditions arising from Pontryagin's principle.

1. **The adjoint system**: for any $u^* \in L^\infty$, solution to the Problem 4.3, there exists a unique adjoint trajectory $\boldsymbol{p}^* \in C([0, \pi]; \mathbb{R}^{N_a+N_b})$ which satisfies the following terminal-value problem

$$\begin{cases} \dot{\boldsymbol{p}}^*(t) = -\nabla_x\mathcal{H}(u(t), \boldsymbol{p}^*(t), t), & t \in [0, \pi] \\ \boldsymbol{p}^*(\pi) = \nabla_x\Psi(\boldsymbol{x}^*(\pi)) \end{cases}$$

where Ψ is the terminal cost of the Optimal Control Problem 4.3. In our case, we have $\Psi(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}\|^2$. Moreover, since the Hamiltonian does not depend on the state variable $\boldsymbol{x}$, we simply have $\dot{\boldsymbol{p}}^*(t) = 0$ for all $t \in [0, \pi]$. We therefore deduce that the adjoint trajectory is constant, and given by

$$\boldsymbol{p}^*(t) = \boldsymbol{x}^*(\pi), \qquad \forall t \in [0, \pi]. \tag{6.2}$$

2. **The Optimal Control**: now, using the optimal adjoint trajectory, we can deduce the necessary optimality condition for the control, which reads as follows:

$$u^*(t) \in \operatorname*{arg\,min}_{|u|\leq 1} \mathcal{H}(t, \boldsymbol{p}^*(t), u), \quad \forall t \in [0, \pi]. \tag{6.3}$$

As we will see, for penalization functions $\mathcal{L}$ as the ones we consider in Theorems 4.1 and 4.2, this argmin is a singleton for almost every $t \in [0, \pi]$. Hence, given the adjoint $\boldsymbol{p}^*$, condition (6.3) uniquely determines the control as a function in L^∞.

In view of (6.1) and (6.2), we can write the optimality condition (6.3) as

$$\begin{aligned} u^*(t) \in &\operatorname*{arg\,min}_{|u|\leq 1} \mathcal{J}(u, \mu^*(t)) \\ &:= \operatorname*{arg\,min}_{|u|\leq 1} [\varepsilon\mathcal{L}(u) - \mu^*(t)u] \end{aligned} \tag{6.4}$$

where

$$\begin{aligned} \mu^*(t) :=& \frac{2}{\pi}\big(\boldsymbol{x}^*(\pi) \cdot \boldsymbol{\mathcal{D}}(t)\big) \\ =& \sum_{i\in\mathcal{E}_a} a_i^*(\pi)\cos(it) + \sum_{j\in\mathcal{E}_b} b_j^*(\pi)\sin(jt). \end{aligned} \tag{6.5}$$

6.2 Proof of Theorem 4.1

We need to prove that, if $\mathcal{L}$ is a concave function, then any optimal control u^* has the form (2.4) with $\mathcal{U} = \{-1, 1\}$. It then suffices to prove that the argmin in (6.3) is the singleton $\{-1\}$ or $\{1\}$ for all $t \in [0, \pi]$ except for a finite set of t.

Since $\mathcal{L}$ is a concave function, so is $\mathcal{J}(u, \mu)$ as a function of u. Hence, in view of (6.4), we have

$$u^*(t) = -1 \quad \text{whenever} \quad \mathcal{J}(-1, \mu^*(t)) < \mathcal{J}(1, \mu^*(t))$$

and

$$u^*(t) = 1 \quad \text{whenever} \quad \mathcal{J}(1, \mu^*(t)) < \mathcal{J}(-1, \mu^*(t)).$$

Now, in view of the definition of $\mathcal{J}$ in (6.4), we see that $u^*(t)$ is uniquely determined, and belongs to $\mathcal{U}$, for all $t \in [0, \pi]$ except when

$$\mu^*(t) = \frac{\varepsilon}{2}\Big[\mathcal{L}(1) - \mathcal{L}(-1)\Big].$$

Finally, in view of the form of μ^* in (6.5), it is clear that the above equality can only hold a finite number of times in $[0,\pi]$. Indeed, the times $t \in [0,\pi]$ such that it holds are precisely the switching angles. $\square$

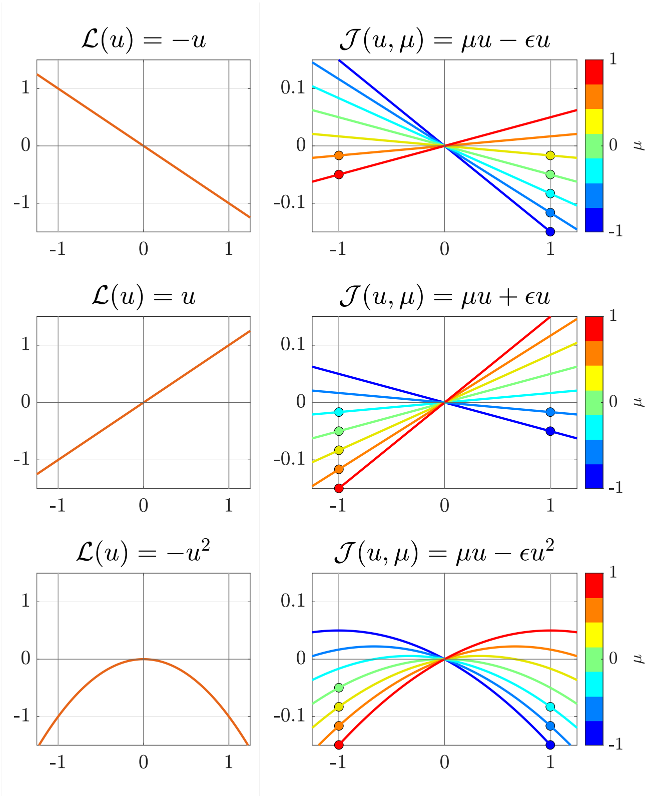

Fig. 7. Bi-level SHE: in the left column we show three types of concave penalization compatibles with Theorem 4.1. In the right columns we display the behavior of the corresponding Hamiltonian for different values of μ.

6.3 *Proof of Theorem 4.2*

In this case, we suppose that $\mathcal{U} = \{u_i\}_{i=1}^L$ is a finite set of real numbers in $[-1,1]$ satisfying

$$-1 = u_1 < u_2 < \ldots < u_L = 1, \quad \text{with } L > 2.$$

The case $L = 2$ is just the bi-level case. As in the previous proof, we only need to show that the argmin in (6.4) is a singleton and belongs to $\mathcal{U}$ for every $t \in [0,\pi]$ except for a finite number of times $t \in [0,\pi]$.

In this case, the study of the minimizers of $\mathcal{J}$ is slightly more involved since the penalization function $\mathcal{L}$ defined in (4.6)-(4.7) is not differentiable. However, $\mathcal{L}$ is an affine interpolation of a convex function, so it is Lipschitz and convex. Hence, $\mathcal{J}$ is also Lipschitz and convex as a function of u.

In view of this, we know that u is a minimizer of $\mathcal{J}(u,\mu)$ if and only if

$$0 \in \partial_u \mathcal{J}(u,\mu), \tag{6.6}$$

where ∂_u denotes the subdifferential with respect to u. Let us recall the definition of subdifferential from convex analysis:

$$\begin{aligned}\partial_u \mathcal{J}(u,\mu) = \{c \in \mathbb{R} \quad & \text{s.t.} \\ & \mathcal{J}(v,\mu) - \mathcal{J}(u,\mu) \geq c(v-u) \\ & \forall v \in [-1,1]\}.\end{aligned}$$

In the case of a convex function, one may show that the subdifferential at $u \in (-1,1)$ is the nonempty interval $[a,b]$, where a and b are the one-sided derivatives

$$a = \lim_{v \to u^-} \frac{\mathcal{J}(v,\mu) - \mathcal{J}(u,\mu)}{v-u}$$

$$b = \lim_{v \to u^+} \frac{\mathcal{J}(v,\mu) - \mathcal{J}(u,\mu)}{v-u}.$$

Moreover, the subdifferential at $u = -1$ and $u = 1$ are given by $(-\infty, b]$ and $[a, +\infty)$ respectively.

Using this characterization of the subdifferential, we can compute $\partial_u \mathcal{J}(u,\mu)$ for all $u \in [-1,1]$ in terms of μ. Let us define

$$p_k := \frac{\mathcal{P}(u_{k+1}) - \mathcal{P}(u_k)}{u_{k+1} - u_k}$$

for all $k \in \{1,\ldots,L-1\}$. Now, using the definition of $\mathcal{J}$ in (6.4) and $\mathcal{L}$ in (4.6), we can compute

$$\begin{aligned}\partial_u \mathcal{J}(-1,\mu) &= (-\infty, \varepsilon p_1 - \mu], \\ \partial_u \mathcal{J}(1,\mu) &= [\varepsilon p_{L-1} - \mu, +\infty), \\ \partial_u \mathcal{J}(u_k,\mu) &= [\varepsilon p_{k-1} - \mu,\, \varepsilon p_k - \mu],\end{aligned}$$

for all $k \in \{2,\ldots,L-1\}$, and

$$\partial_u \mathcal{J}(u,\mu) = \{\varepsilon p_k - \mu\},$$

for all $u \in (u_k, u_{k+1})$ and all $k \in \{1,\ldots,L-1\}$.

Now we observe that

$$\begin{aligned}0 \in \partial_u \mathcal{J}(-1,\mu) \quad & \text{iff} \quad \mu \leq \varepsilon p_1, \\ 0 \in \partial_u \mathcal{J}(1,\mu) \quad & \text{iff} \quad \mu \geq \varepsilon p_{L-1}, \\ 0 \in \partial_u \mathcal{J}(u_k,\mu) \quad & \text{iff} \quad \varepsilon p_{k-1} \leq \mu \leq \varepsilon p_k,\end{aligned} \tag{6.7}$$

for all $k \in \{2,\ldots,L-1\}$, and, for all $k \in \{1,\ldots,L-1\}$,

$$0 \in \partial_u \mathcal{J}(u,\mu) \quad \text{for all } u \in [u_k, u_{k+1}]$$

if and only if $\mu = \varepsilon p_k$.

Using this, along with the optimality condition (6.6), we deduce that, for almost every $\mu \in \mathbb{R}$, we have

$$\arg\min_{u \leq 1} \mathcal{J}(u, \mu) = \{u_k\}$$

for some $u_k \in \mathcal{U}$. Indeed, (6.6) does not hold if and only if $\mu = \varepsilon p_k$, for some $k \in \{1, \dots, L-1\}$.

Hence, using the optimality condition for $u^*(t)$ in (6.4), we see that $u^*(t)$ is uniquely determined, and belongs to $\mathcal{U}$, for all $t \in [0, \pi]$ except when

$$\mu^*(t) = \varepsilon \frac{\mathcal{P}(u_{k+1}) - \mathcal{P}(u_k)}{u_{k+1} - u_k}$$

for some $k \in \{1, \dots, L-1\}$. In view of the form of $\mu^*(t)$ in (6.5), it is clear that the above equality can only hold a finite number of times in the interval $[0, \pi]$. Indeed, we refer to the set of times when the above equality holds as the switching angles. Due to the continuity of $\mu^*(t)$, along with (6.7), it is clear that $u^*(t)$ does not change value between two consecutive switching angles. The staircase property of the waveform (2.5) can also be deduced from (6.4) and (6.7), along with the continuity of the function $\mu^*(t)$. $\square$

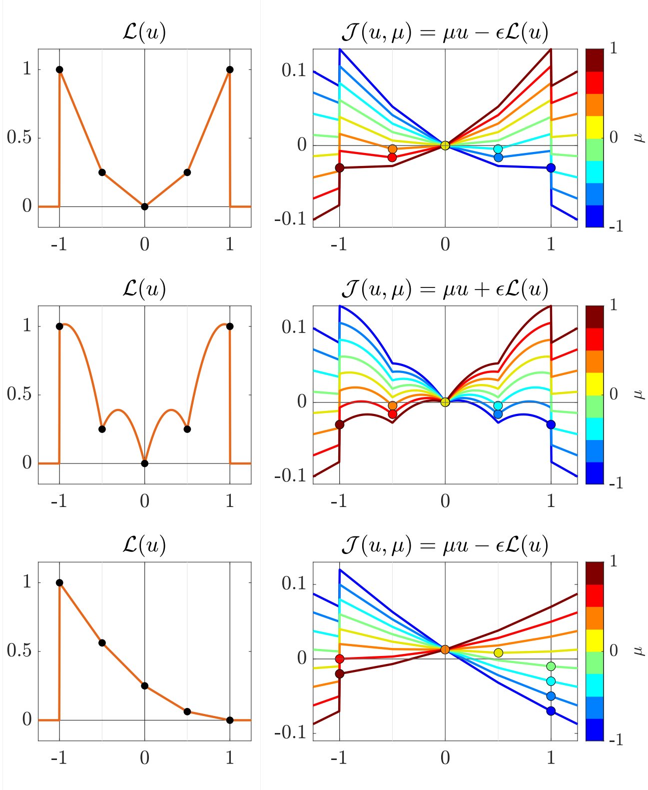

Fig. 8. Multilevel SHM: in the left column we show three types of penalizations which are compatible with our theoretical results. In the right column we show the behavior of the corresponding Hamiltonian for different values of μ.

7 Conclusions

Acknowledgements

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