Multilevel Selective Harmonic Modulation via Optimal Control

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***Abstract*— We consider the *Selective Harmonic Modulation* (SHM) problem, consisting in the design of a staircase control signal with some prescribed frequency components. In this work, we propose a novel methodology to address the SHM problem: the admissible controls are obtained via an optimal control problem whose solutions are piece-wise constant functions, taking values only in a given finite set. In order to fulfill this constraint, we introduce a cost functional with piece-wise linear penalization which, by means of Pontryagin's maximum principle, makes the optimal control have the desired staircase form. Up to the best of our knowledge, this approach to the SHM problem via optimal control is new. Moreover, it has the advantage of automatically determine the optimal form of the control signal, without need of specifying it a priori. This is a big advance in the SHM literature, very relevant in practical power electronics engineering applications. Moreover, our methodology may be applicable to other optimal control problems with a finite-set constraint on the control. We also provide several numerical examples in which the SHM problem is solved by using our approach.**

***Index Terms*—Selective Harmonic Modulation; Optimal Control Theory; Finite-Set Control; Pontryagin's maximum principle; Piece-wise linear penalization.**

I. Introduction and motivations

Selective Harmonic Modulation (SHM) [1, 2] is a well-known methodology in power electronics engineering, employed to improve the performances of a converter by controlling the phase and amplitude of the harmonics in its output voltage. As a matter of fact, this technique allows to increase the power of the converter and, at the same time, to reduce its losses.

Because of the growing complexity of modern electrical networks, consequence for instance of the high penetration of renewable energy sources, the demand in power of electronic converters is day by day increasing. For this and other reasons, SHM has been a preeminent research interest in the power electronics community, and a plethora of SHM-based techniques has been developed in recent years.

In broad terms, SHM consists in generating a *control signal* with a desired harmonic spectrum by modulating some specific lower-order Fourier coefficients. In practice, the signal is constructed as a step function with a finite number of switches, taking values only in a given finite set. Such a signal can be fully characterized by two features (see Fig. 1):

1. The *waveform*, i.e. the sequence of values that the function takes in its domain.
2. The *switching angles*, i.e. the sequence of points where the signal switches from one value to following one.

Using this simple characterization of the signal, in many practical situations, the SHM problem is reduced to a finite-dimensional optimization one in which, for a given suitable waveform, the aim is to find the optimal location of the switching angles. However, this approach has the difficulty of choosing a suitable waveform, which may be quite cumbersome in some situations. In fact, even determining the number of switching angles is not in general straightforward.

To overcome these difficulties, in this work we propose a new approach to SHM based on control theory: the Fourier coefficients of the signal are identified with the terminal state of a controlled dynamical system, where the control is actually the signal, solution to the SHM problem. We then look for piecewise constant controls, taking values only in a given finite set, and satisfying the prescribed terminal condition (see Section IV for more details).

One of the main difficulties in our approach is that the constraints on the admissible controls, which must have staircase form, prevent us from applying the classical tools in optimal control theory. Specifically, one of the most popular methodology is the combination of automatic differentiation with non-linear convex optimization, achieving an algorithm good performance. However, for the use of these optimizers it is necessary to consider a convex admissible solution space, not being directly applicable in our problem.

In order to bypass this obstruction, we consider a different optimal control problem, without these type of constraints, in which the desired staircase structure is obtained by introducing a suitable penalization term for the control.

The main contributions of the present paper are the following ones:

1. We reformulate the SHM problem as an optimal control one, with a staircase-form constraint on the control. An advantage of this formulation is that neither the waveform of the solution nor the number of switching angles need to be a priori determined.
2. We introduce a penalization term for the control which implicitly induces the desired staircase property on the solution to the optimal control problem. Different choices of the penalization term can give rise to solutions to the SHM problem with different waveform.
3. For each choice of the penalization term, we prove uniqueness and continuity of the solution with respect to the target frequencies. We point out that this continuity is a highly desirable property in real applications of SHM, and sometimes, difficult to achieve.
4. We also provide some numerical examples, where we solve the SHM problem through our approach. These numerical examples confirm that the solution provided by our methodology is, effectively, continuous with respect to the target frequencies.

This document is structured as follows. In Section II, we introduce the mathematical formulation of the general SHM problem. In Section III, we recall the classical methodology casting the SHM problem through finite-dimensional optimization and we show the main criticalities related to this approach. In Section IV, we present the new approach to SHM as an optimal control problem, and state our main

Manuscript submitted March xx, 2021. This project has received funding from the European Research Council (ERC) under the European Union's Horizon 2020 research and innovation programme (grant agreement NO: 694126-DyCon). The work of U.B. and D.O. is partially supported by the Elkartek grant KK-2020/00091 CONVADP of the Basque government. The work of U.B. is partially supported by the Air Force Office of Scientific Research (AFOSR) under Award NO: FA9550-18-1-0242 and by the Grant MTM2017-92996-C2-1-R COSNET of MINECO (Spain).

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results concerning the uniqueness and stability of the solution. Section V is devoted to some numerical examples of concrete SHM problems that we have solved by means of our methodology. In Section VI, we give the proofs of the theoretical results presented in Section IV. Finally, in Section VII, we summarize and comment the conclusions of our work.

II. Preliminaries

This section is devoted to the mathematical formulation of the SHM problem and to introduce the notation that will be used throughout the paper. Let

$$\mathcal{U} = \{u_1, \dots, u_L\} \tag{II.1}$$

be a given set of $L \geq 2$ real numbers satisfying

$$u_1 = -1,\ u_L = 1 \ \text{ and } \ u_k < u_{k+1} \quad \forall k \in \{1, \dots, L\}.$$

The goal is to construct a step function

$$u(t) : [0, 2\pi) \to \mathcal{U},$$

with a finite number of switches, such that some of its lower-order Fourier coefficients take specific values prescribed a priori.

Due to applications in power converters, it is typical to only consider functions with *half-wave symmetry*, i.e. satisfying

$$u(t+\pi) = -u(t) \quad \text{for all } t \in [0, \pi). \tag{II.2}$$

In view of (II.2), in what follows, we will only work with the restriction $u|_{[0,\pi)}$, which, with some abuse of notation, we still denote by u. Moreover, as a consequence of this symmetry, the Fourier series of u only involves the odd terms (as the even terms just vanish), i.e.

$$u(t) = \sum_{\substack{j\in\mathbb{N} \\ j \text{ odd}}} a_j \cos(jt) + \sum_{\substack{j\in\mathbb{N} \\ j \text{ odd}}} b_j \sin(jt),$$

with

$$\begin{aligned} a_j &= \frac{2}{\pi}\int_0^\pi u(\tau)\cos(j\tau)\,d\tau, \\ b_j &= \frac{2}{\pi}\int_0^\pi u(\tau)\sin(j\tau)\,d\tau. \end{aligned} \tag{II.3}$$

As we anticipated, we are only considering piece-wise constant functions with a finite number of switches, taking values only in $\mathcal{U}$. In other words, we look for functions $u : [0, \pi) \to \mathcal{U}$ of the form

$$u(t) = \sum_{m=0}^{M} s_m \chi_{[\phi_m, \phi_{m+1})}(t), \quad M \in \mathbb{N} \tag{II.4}$$

for some $\mathcal{S} = \{s_m\}_{m=0}^M$ satisfying

$$s_m \in \mathcal{U} \ \text{ and } \ s_m \neq s_{m+1} \quad \forall m \in \{0, \dots, M\}$$

and $\Phi = \{\phi_m\}_{m=1}^M$ such that

$$0 = \phi_0 < \phi_1 < \ldots < \phi_M < \phi_{M+1} = \pi.$$

In (II.4), $\chi_{[\phi_m, \phi_{m+1})}$ denotes the characteristic function of the interval $[\phi_m, \phi_{m+1})$. With these notations, we can define the waveform and the switching angles as follows:

Definition II.1 *For a function $u : [0, \pi) \to \mathcal{U}$ of the form (II.4), we refer to $\mathcal{S}$ as the* waveform *and to Φ as the* switching angles.

Observe that any function u of the form (II.4) is fully characterized by its waveform $\mathcal{S}$ and switching angles Φ. An example of such a function is displayed in Fig. 1.

In the practical engineering applications that motivated our study, due to technical limitations, it is preferable to employ signals taking

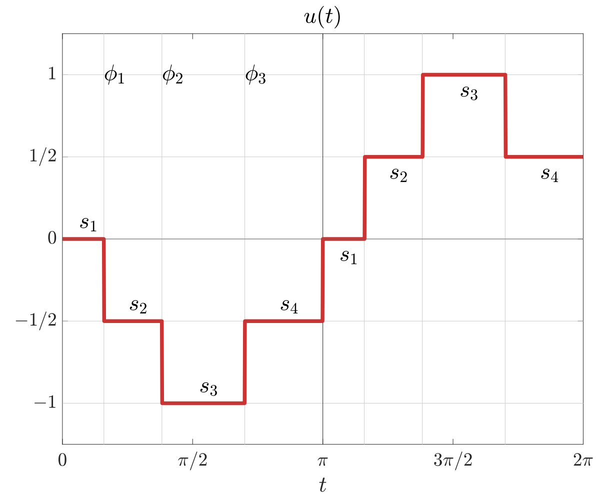

Fig. 1. A possible solution to the SHM Problem, where we considered the control-set as $\mathcal{U} = \{-1, -1/2, 0, 1/2, 1\}$. We show the switching angles Φ and the waveform $\mathcal{S}$ (see Definition II.1). The function $u(t)$ is displayed on the whole interval $[0, 2\pi)$ to highlight the half-wave symmetry introduced in (II.2).

consecutive values in $\mathcal{U}$. In the sequel, we will refer to this property of the waveform as the *staircase property*. We can rigorously formulate this property as follows:

Definition II.2 *We say that a signal u of the form (II.4) fulfills the* staircase property *if its waveform $\mathcal{S}$ satisfies*

$$(s_m^{min}, s_m^{max}) \cap \mathcal{U} = \emptyset, \quad \forall m \in \{0, \dots, M-1\}, \tag{II.5}$$

where $s_m^{min} = s_m \wedge s_{m+1}$ and $s_m^{max} = s_m \vee s_{m+1}$.

Note that when $\mathcal{U} = \{-1, 1\}$ (which is known in the SHM literature as the bi-level problem), this property is satisfied for any u of the form (II.4).

We can now formulate the SHM problem as follows:

Problem II.1 (SHM) *Let $\mathcal{U}$ be given as in (II.1), and let $\mathcal{E}_a$ and $\mathcal{E}_b$ be finite sets of odd numbers of cardinality $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively. For any two given vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, construct a function $u : [0, \pi) \to \mathcal{U}$ of the form (II.4), satisfying (II.5), such that the vectors $\boldsymbol{a} \in \mathbb{R}^{N_a}$ and $\boldsymbol{b} \in \mathbb{R}^{N_b}$, defined as*

$$\boldsymbol{a} = \left(a_j\right)_{j\in\mathcal{E}_a} \quad \text{and} \quad \boldsymbol{b} = \left(b_j\right)_{j\in\mathcal{E}_b} \tag{II.6}$$

satisfy

$$\boldsymbol{a} = \boldsymbol{a}_T \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{b}_T,$$

where the coefficients a_j and b_j in (II.6) are given by (II.3).

Remark II.1 (SHE) *In Problem II.1, we gave a very general formulation of SHM. This formulation contains also the so-called* Selective Harmonic Elimination (SHE) *problem (see [2]), in which the target vectors are such that*

$$\begin{aligned} (a_T)_1 \neq 0 \quad (a_T)_{i\neq1} = 0 \quad &\text{for all } i \in \mathcal{E}_a \\ (b_T)_1 \neq 0 \quad (b_T)_{j\neq1} = 0 \quad &\text{for all } j \in \mathcal{E}_b. \end{aligned}$$

SHE is of great relevance in the electric engineering literature. Its objective is to generate a signal with amplitude $m_1 = \sqrt{a_1^2 + b_1^2}$ and phase $\varphi_1 = \arctan(b_1/a_1)$, removing some specific high-frequency components. In this way, SHE may be understood as a generator of clean Fourier modes through a staircase signal.

III. SHM AS A FINITE-DIMENSIONAL OPTIMIZATION PROBLEM

A typical approach to the SHM Problem II.1 ([2, 3, 4]) looks for solutions u with a specific waveform $\mathcal{S}$ a priori determined, optimizing only over the location of the switching angles Φ.

Note that, for a fixed waveform $\mathcal{S}$, the Fourier coefficients of a function u of the form (II.4) can be written in terms of the switching angles Φ in the following way:

$$a_j = a_j(\Phi) = \frac{2}{j\pi}\sum_{m=0}^{M} s_m \Big[\sin(j\phi_{m+1}) - \sin(j\phi_m)\Big]$$

$$b_j = b_j(\Phi) = \frac{2}{j\pi}\sum_{m=0}^{M} s_m \Big[\cos(j\phi_m) - \cos(j\phi_{m+1})\Big]$$

Hence, for two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ as in Problem II.1, and any fixed waveform $\mathcal{S}$, we can define the functions

$$\begin{aligned} \boldsymbol{a}_{\mathcal{S}}(\Phi) &:= \big(a_j(\Phi)\big)_{j\in\mathcal{E}_a} \in \mathbb{R}^{N_a} \\ \boldsymbol{b}_{\mathcal{S}}(\Phi) &:= \big(b_j(\Phi)\big)_{j\in\mathcal{E}_b} \in \mathbb{R}^{N_b} \end{aligned} \tag{III.1}$$

which associates, to any sequence of switching angles $\{\phi_m\}_{m=1}^{M}$, the corresponding Fourier coefficients. Therefore, SHM can be cast as a finite-dimensional optimization problem in the following way:

Problem III.1 (Optimization problem for SHM) *Let $\mathcal{E}_a$, $\mathcal{E}_b$, $\boldsymbol{a}_T$, and $\boldsymbol{b}_T$ be given as in Problem II.1. Let $\mathcal{S} := \{s_m\}_{m=0}^{M}$ be a fixed waveform satisfying (II.5). Find the switching angles $\Phi = \{\phi_m\}_{m=1}^{M}$ solution to the following minimization problem:*

$$\min_{\Phi\in[0,\pi]^M} \Big(\|\boldsymbol{a}_{\mathcal{S}}(\Phi) - \boldsymbol{a}_T\|^2 + \|\boldsymbol{b}_{\mathcal{S}}(\Phi) - \boldsymbol{b}_T\|^2\Big)$$

$$\textit{subject to: } 0 = \phi_0 < \phi_1 < \ldots < \phi_M < \phi_{M+1} = \pi.$$

where $\boldsymbol{a}_{\mathcal{S}}(\Phi)$ and $\boldsymbol{b}_{\mathcal{S}}(\Phi)$ are defined as in (III.1).

At this regard, it is important to notice that the optimization Problem III.1 solves the original SHM Problem II.1 only when the optimal cost is zero. This makes necessary to fully characterize the space of targets $(\boldsymbol{a}_T, \boldsymbol{b}_T)$ for which the solution of Problem III.1 is a solution of Problem II.1. With this aim, we will define the the *optimal cost* and the *solvable set* as follows:

Definition III.1 (optimal cost) *We call* optimal cost $V_{\mathcal{S}} : \mathbb{R}^{N_a} \times \mathbb{R}^{N_b} \to \mathbb{R}$, *the function that takes as input variables the target vectors $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ and returns the optimal cost of the Problem III.1.*

Definition III.2 (solvable set) *We define a* solvable set $\mathcal{R}_{\mathcal{S}}$ *as:*

$$\mathcal{R}_{\mathcal{S}} = \Big\{(\boldsymbol{a}_T, \boldsymbol{b}_T) \in \mathbb{R}^{N_a+N_b} \;\big|\; V_{\mathcal{S}}(\boldsymbol{a}_T, \boldsymbol{b}_T) = 0\Big\} \tag{III.2}$$

Furthermore, we define the following *policy* function which maps the solutions of Problem III.1 into the set $\mathcal{R}_{\mathcal{S}}$.

Definition III.3 (Policy) *We will call* policy *a function $\Pi_{\mathcal{S}} : \mathcal{R}_{\mathcal{S}} \to [0,\pi]^M$ such that $\Phi^* = \Pi_{\mathcal{S}}(\boldsymbol{a}_T, \boldsymbol{b}_T)$, with Φ^* the optimal switching angles solutions of Problem II.1 for the targets $(\boldsymbol{a}_T, \boldsymbol{b}_T)$.*

With the aim of reconstructing the policy $\Pi_{\mathcal{S}}$, a typical approach is to solve numerically Problem III.1 for a limited number of points in $\mathbb{R}^{N_a+N_b}$ and check that the optimal cost is zero. Secondly, one interpolates the function $\Pi_{\mathcal{S}}$ in the convex set generated by the points previously obtained. Nevertheless, this approach has several difficulties and drawbacks.

1. *Combinatory problem*: in practice, one does not dispose of a suitable waveform $\mathcal{S}$ which yields a solution to the Problem II.1. A common approach to solve the SHM problem consists in fixing the number of switches M, and then solve Problem III.1 for all the possible combinations of M elements of $\mathcal{U}$. However, taking into account that the number of possible M-tuples in $\mathcal{U}$ is of the order $(L-1)^M$, it is evident that the complexity of the above approach increases rapidly when $L > 1$. This problem has been studied in [4] where, through appropriate algebraic transformations, the authors are able to convert the SHM problem into a polynomial system whose solutions' set contains all the possible waveforms $\mathcal{S}$ of M elements in $\mathcal{U}$. As a drawback of this approach, the number of switches M needs to be prefixed. However, in some cases, determining the number of switches which are necessary to reach the desired Fourier coefficients is not a straightforward task.
2. *Solvable set problem*: given a waveform $\mathcal{S}$, the corresponding solvable set $\mathcal{R}_{\mathcal{S}}$ is usually very small, yielding to policies $\Pi_{\mathcal{S}}$ which are not very effective. This issue is typically addressed by solving Problem III.1 for a set of waveforms $\{\mathcal{S}_l\}_{l=1}^{r}$ and obtaining different policies $\{\Pi_{\mathcal{S}_l}\}_{l=1}^{r}$ and solvable sets $\{\mathcal{R}_{\mathcal{S}_l}\}_{l=1}^{r}$ for each one of them. By gathering them, on then creates a new policy applicable in a wider scenario. However, since to different waveforms may correspond disjoint or even overlapping solvable sets, this union of policies is chaotic, providing different solutions for the same target $(\boldsymbol{a}_T, \boldsymbol{b}_T)$, or even generating regions with no solution (see Fig. 2).

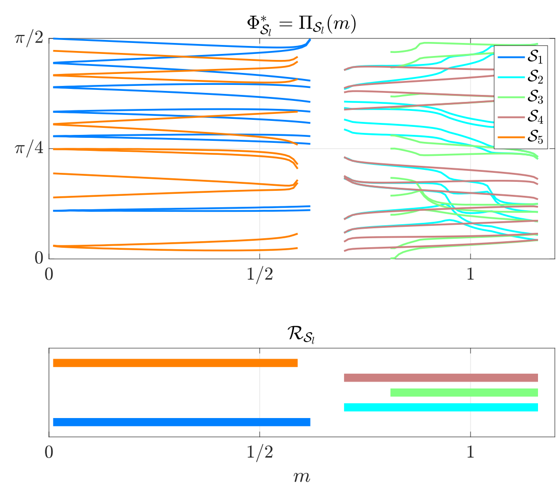

Fig. 2. In the first picture, we display the optimal switching angles $\boldsymbol{\Phi}^*_{\mathcal{S}}$ associated to different waveforms $\{\mathcal{S}_l\}_{l=1}^{7}$ for a SHM problem (see Remark II.1), considering the sets $\mathcal{E}_a = \{1\}$ and $\mathcal{E}_b = \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31\}$. We chose the target $(a_1, b_1) = (m, 0)\ \forall m \in [0, 1.2]$. The second figure shows the solvable sets for each waveform we considered.

3. *Policy problem*: due to the complexity of a policy generated by the union of different waveforms, the continuity of the switching angles cannot be guaranteed. This is a well known problem in the SHM community [5, 6, 7, 8] (see Fig. 2).

As we shall see, all these mentioned criticalities will be overcome by our optimal control approach.

IV. SHM AS AN OPTIMAL CONTROL PROBLEM

Our main contribution in the present paper consists in formulating the SHM problem as an optimal control one. In this formulation, the Fourier coefficients of the signal $u(t)$ are identified with the terminal state of a controlled dynamical system of $N_a + N_b$ components defined in the time-interval $[0, \pi)$. The control of the system is precisely the signal $u(t)$, defined as a function $[0, \pi) \to \mathcal{U}$, which has to steer the state from the origin to the desired values of the prescribed Fourier coefficients.

The starting point of this approach is to rewrite the Fourier coefficients of the function $u(t)$ as the final state of a dynamical system controlled by $u(t)$. To this end, let us first note that, in view of (II.3), for all $u \in L^\infty([0,\pi);\mathbb{R})$ any Fourier coefficient a_j satisfies

$$a_j = y(\pi),$$

with $y \in C([0,\pi);\mathbb{R})$ defined as

$$y(t) = \frac{2}{\pi}\int_0^t u(\tau)\cos(j\,\tau)d\tau.$$

Besides, as a consequence of the fundamental theorem of calculus, $y(\cdot)$ is the unique solution to the differential equation

$$\begin{cases} \dot{y}(t) = \dfrac{2}{\pi}\cos(j\,t)u(t), & t \in [0,\pi) \\ y(0) = 0. \end{cases} \tag{IV.1}$$

Analogously, we can also write the Fourier coefficients b_j, defined in (II.3), as the solution at time $t = \pi$ of a differential equation similar to (IV.1).

Hence, for $\mathcal{E}_a$, $\mathcal{E}_b$, $\boldsymbol{a}_T$, and $\boldsymbol{b}_T$ given, the SHM Problem II.1 can be reduced to finding a control function u of the form (II.4), satisfying (II.5), such that the corresponding solution $\boldsymbol{y} \in C([0,\pi];\mathbb{R}^{N_a+N_b})$ to the dynamical system

$$\begin{cases} \dot{\boldsymbol{y}}(t) = \dfrac{2}{\pi}\boldsymbol{\mathcal{D}}(t)u(t), & t \in [0,\pi) \\ \boldsymbol{y}(0) = 0. \end{cases} \tag{IV.2}$$

satisfies

$$\boldsymbol{y}(\pi) = [\boldsymbol{a}_T; \boldsymbol{b}_T]^\top,$$

where

$$\boldsymbol{\mathcal{D}}(t) = \left[\boldsymbol{\mathcal{D}}^a(t); \boldsymbol{\mathcal{D}}^b(t)\right]^\top, \tag{IV.3}$$

with $\boldsymbol{\mathcal{D}}^a(t) \in \mathbb{R}^{N_a}$ and $\boldsymbol{\mathcal{D}}^b(t) \in \mathbb{R}^{N_b}$ given by

$$\boldsymbol{\mathcal{D}}^a(t) = \begin{bmatrix} \cos(e_a^1 t) \\ \cos(e_a^2 t) \\ \vdots \\ \cos(e_a^{N_a} t) \end{bmatrix}, \quad \boldsymbol{\mathcal{D}}^b(t) = \begin{bmatrix} \sin(e_b^1 t) \\ \sin(e_b^2 t) \\ \vdots \\ \sin(e_b^{N_b} t) \end{bmatrix} \tag{IV.4}$$

Here, e_a^i and e_b^i denote the elements in $\mathcal{E}_a$ and $\mathcal{E}_b$, i.e.

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \dots, e_a^{N_a}\}, \quad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \dots, e_b^{N_b}\}.$$

In the sequel, and in order to simplify the notation, we reverse the time in (IV.2) using the transformation $\boldsymbol{x}(t) = \boldsymbol{y}(\pi - t)$. In this way, the SHM problem turns into the following null controllability one, for a dynamical system with initial condition $\boldsymbol{x}(0) = [\boldsymbol{a}_T; \boldsymbol{b}_T]^\top$ (see also Fig. 3).

Problem IV.1 (SHM via null controllability) *Let $\mathcal{U}$ be given as in (II.1). Let $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ be given as in Problem II.1, we look for a function $u : [0,\pi) \to [-1,1]$ of the form (II.4), satisfying (II.5), such that the solution to the initial-value problem*

$$\begin{cases} \dot{\boldsymbol{x}}(t) = -\dfrac{2}{\pi}\boldsymbol{\mathcal{D}}(t)u(t), & t \in [0,\pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 := [\boldsymbol{a}_T; \boldsymbol{b}_T] \end{cases} \tag{IV.5}$$

satisfies $\boldsymbol{x}(\pi) = 0$, where $\boldsymbol{\mathcal{D}}$ is given by (IV.3)–(IV.4).

A natural approach for null controllability problems such as Problem IV.1 is to formulate them as an optimal control one, where the cost functional to be minimized is the euclidean distance from the final state $\boldsymbol{x}(\pi)$ to the the origin. In what follows, for a given vector $\boldsymbol{v} \in \mathbb{R}^d$, we denote by $\|\boldsymbol{v}\|$ the euclidean norm $\|\boldsymbol{v}\|_{\mathbb{R}^d}$. Let us introduce the set of admissible controls.

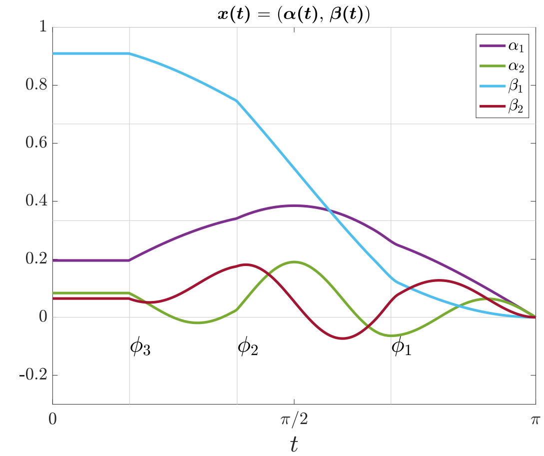

Fig. 3. Evolution of the dynamical system (IV.5) with $\boldsymbol{\mathcal{E}_a} = \{1, 2\}$ and $\boldsymbol{\mathcal{E}_b} = \{1, 2\}$ corresponding to the control $\boldsymbol{u}$ in Figure 1. The positions of the switching angles ϕ are displayed as well.

$$\mathcal{A} := \left\{u : [0,\pi) \to [-1,1] \ \text{ measurable}\right\}$$
$$\mathcal{A}_{ad} := \left\{u \in \mathcal{A} \text{ of the form (II.4) satisfying (II.5)}\right\}$$

Problem IV.2 (OCP for SHM) *Let $\mathcal{U}$ be a given set as in (II.1). Let $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ be given as in Problem II.1. We look for an admissible control $u \in \mathcal{A}_{ad}$ solution to the following optimal control problem:*

$$\min_{u\in\mathcal{A}_{ad}} \frac{1}{2}\|\boldsymbol{x}(\pi)\|^2 \quad \textit{subject to the dynamics (IV.5).}$$

Remark IV.1 *Note that the cost functional in Problem IV.2 is quadratic and, therefore, it always admits at least one minimizer for any target $[\boldsymbol{a}_T; \boldsymbol{b}_T]^\top$. However, this minimizer is a solution to the SHM problem if and only if the minimum is equal to zero. Otherwise, we say that the target $[\boldsymbol{a}_T; \boldsymbol{b}_T]^\top$ is unreachable and the SHM problem II.1 has no solution. In this work, we will not discuss the reachable set for the control problem (IV.2).*

A main feature of the SHM problem is that we are looking for signal functions u of the form (II.4) satisfying (II.5). In principle, this can be guaranteed by by adding directly this constraint on the set of admissible controls $\mathcal{A}_{ad}$, as we did in Problem IV.2. However, considering an optimization problem in a non-convex, null-measure set is strange. It is known that mathematical optimization in general is an NP-hard problem, while for the concrete case of convex optimization, algorithms in polynomial time are known, such as: Interior point method [9], projected gradient descent [10], Penalty method [11], etc.

In order to bypass this difficulty, we propose a modification of Problem IV.2 by adding to the cost functional, a penalization term for the control.

Problem IV.3 (Penalized OCP for SHM) *Fix $\varepsilon > 0$ and a convex function $\mathcal{L} \in C([-1,1];\mathbb{R})$. Let $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ be given as in Problem II.1. We look for a control $u \in \mathcal{A}$ solution to the following optimal control problem:*

$$\min_{u\in\mathcal{A}} \left(\frac{1}{2}\|\boldsymbol{x}(\pi)\|^2 + \varepsilon\int_0^\pi \mathcal{L}(u(t))dt\right)$$

subject to the dynamics (IV.5).

Observe that, in Problem IV.3, we do not impose the constraint that the control has to be of the form (II.4), satisfying the staircase property (II.5). Nevertheless, as we shall see, these features of u will arise naturally in the solution to Problem IV.3, from a suitable choice of the penalization term $\mathcal{L}$. Another important advantage of adding a penalization term for the control is that, as we shall prove in Theorems IV.1 and IV.2, it ensures the uniqueness for the solution, and the continuity of this one with respect to the targets $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$.

On the contrary, one needs to take into account that the penalization term for the control might prevent the optimal trajectory from reaching the target. In other words, even if there exists a control for which the optimal trajectory satisfies $\boldsymbol{x}(\pi) = 0$, the optimal control in Problem IV.3 might not do so, and therefore, the solution to Problem IV.3 would not be a solution to the SHM problem. This issue may be controlled by a proper selection of the weighting parameter ε which allows to tune the precision of the optimal control for the perturbed problem, guaranteeing that the final state of the optimal trajectory is close enough to zero. This is the content of the following proposition.

Proposition IV.1 *Assume that $[\boldsymbol{a}_T, \boldsymbol{b}_T]^\top$ is such that Problem IV.1 admits a solution, and let $u^* \in \mathcal{A}$ be the solution to Problem IV.3. Then the associated trajectory $\boldsymbol{x}^* \in C([0,\pi];\mathbb{R})$, solution to (IV.5), satisfies*

$$\|\boldsymbol{x}^*(\pi)\|^2 \leq 4\varepsilon\pi\|\mathcal{L}\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the max-norm in $C([-1,1];\mathbb{R})$.

The proof of Proposition IV.1 is postponed to Section VI. Let us now describe the construction of penalization functions $\mathcal{L}$ which guarantee that any solution to Problem IV.3 has the form (II.4) and satisfies (II.5). To this end, we will distinguish two cases, depending on the cardinality of $\mathcal{U}$.

A. Bilevel SHM problem via OCP (Bang-Bang Control)

In this case, the control set $\mathcal{U}$ defined in (II.1) has only two elements, i.e. $\mathcal{U} = \{-1, 1\}$. In the control theory literature, a control taking only two values is known as *bang-bang control*. In the SHM literature, this kind of solution are called *bi-level solutions*. Note that in this case, any u with the form (II.4) trivially satisfies the staircase property (II.5).

Theorem IV.1 *Let $\mathcal{U} = \{-1, 1\}$, and $\boldsymbol{x}_0$ be given. For some $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, consider the Problem IV.3 with $\mathcal{L}(u) = \alpha\, u$. Then, the optimal control u^*, solution to Problem IV.3 is unique and has a bang-bang structure, i.e. it is of the form (II.4). In addition to that, the solution u^* to Problem IV.3 is continuous with respect to $\boldsymbol{x}_0$ in the strong topology of $L^1(0,\pi)$.*

The proof of Theorem IV.1 is postponed to Section VI, and follows from the optimality conditions given by the Pontryagin's maximum principle. In particular, the linearity of $\mathcal{L}$ and of the dynamical system (IV.5), implies that the associated Hamiltonian is also linear, and then, the minimum is always attained at the limits of the interval $[-1,1]$.

We point out that, by choosing different penalizations $\mathcal{L}$, we may obtain solutions to the SHM problem with different waveforms due to the change of the Hamiltonian. See for instance Fig. 9, where we have chosen $\mathcal{L}(u) = \pm u$.

B. Multilevel SHM problem via OCP

Inspired by the ideas of the previous subsection, we can address the case when $\mathcal{U}$ contains more than two elements. This is known in the power electronics literature as the *multilevel SHM problem*. Now, the goal is to construct a function $\mathcal{L}$ such that the Hamiltonian associated to Problem IV.3 always attains the minimum at points in $\mathcal{U}$. A way to construct such a function $\mathcal{L}$ is to interpolate a parabola in $[-1,1]$ by affine functions, considering the elements in $\mathcal{U}$ as the interpolating points. Since between any two points in $\mathcal{U}$, the function $\mathcal{L}$ is a straight line, the Hamiltonian is a concave function in these intervals, and hence, the minimum is always attained at points in $\mathcal{U}$.

Theorem IV.2 *Let $\boldsymbol{x}_0$ be given, and let $\mathcal{U}$ be a given set as in (II.1). For any $\alpha > 0$ and $\beta \in \mathbb{R}$, set the function*

$$\mathcal{P}(u) = \alpha(u-\beta)^2. \tag{IV.6}$$

Consider Problem IV.3 with

$$\mathcal{L}(u) = \begin{cases} \lambda_k(u) & \text{if } u \in [u_k, u_{k+1}) \\ \mathcal{P}(1) & \text{if } u = u_L \end{cases} \tag{IV.7}$$

$$\text{for all } k \in \{1, \ldots, L-1\},$$

where

$$\lambda_k(u) := \frac{(u-u_k)\mathcal{P}(u_{k+1}) + (u_{k+1}-u)\mathcal{P}(u_k)}{u_{k+1}-u_k}. \tag{IV.8}$$

Assume in addition that the function $\mathcal{L}$ has a unique minimum in $[-1,1]$. Then, the optimal control u^*, solution to Problem IV.3, is unique and has the form (II.4) satisfying (II.5). Moreover, the solution u^* to Problem IV.3 is continuous with respect to $\boldsymbol{x}_0$ in the strong topology of $L^1(0,\pi)$.

The assumption of $\mathcal{L}$ having a unique minimum in $[-1,1]$ is actually necessary to ensure the staircase form (II.4) for the solution. Not assuming this hypothesis would entail the possibility of having continuous solutions for specific targets. Nevertheless, the assumption of $\mathcal{L}$ having a unique minimizer can be easily ensured by choosing, for instance, $\beta = \pm 1$.

Remark IV.2 *For completeness, we shall mention that in Theorem IV.2 $\mathcal{L}$ can actually have a more general form, still yielding to a staircase optimal control u^*. Indeed, as we shall see in Section VI, the proof of Theorem IV.2 does not use the fact that the function $\mathcal{P}$ is a parabola. If we replace this $\mathcal{P}$ with any other strictly convex function, our result remains valid. The choice we made of defining $\mathcal{P}$ as in (IV.6) is motivated by the fact that, most often, in optimal control theory the penalization terms are chosen to be quadratic.*

Remark IV.3 (*Bang-off-bang control*) *We note that when $\mathcal{U} = \{-1, 0, 1\}$, we can just use the L^1-norm of the control as penalization, i.e. $\mathcal{L}(u) = |u|$. This yields to the so-called* bang-off-bang *controls, that are widely studied in the literature [12, 13]. By taking a different parabola $\mathcal{P}$, one can then obtain different bang-off-bang solutions to the SHM problem.*

We illustrate in Fig. 4 different examples of penalization functions $\mathcal{L}$ giving rise to multilevel solutions to the SHM problem. We point out that, by varying the values of α and β in Theorem IV.2, we can obtain solutions with different waveforms.

V. Numerical simulations

In this section, we present several examples in which we implement the optimal control strategy we proposed to solve the SHM problem. All the simulations we are going to present can be found also in [14]. Our Experiments were conducted on a personal MacBook Pro laptop (1,4 GHz Quad-Core Intel Core i5, 8GB RAM, Intel Iris Plus Graphics 1536 MB).

To solve our optimal control Problem IV.3, we will employ the direct method [15] which, in broad terms, consists in discretizing the cost functional and the dynamics, and then apply some optimization algorithm.

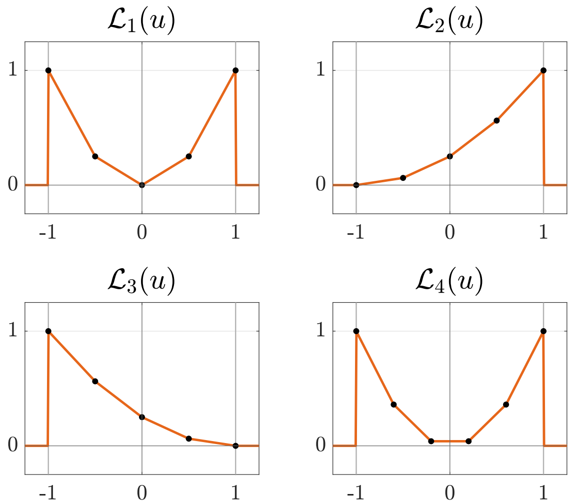

Fig. 4. The some penalization functions. The $\mathcal{L}_1$, $\mathcal{L}_2$ and $\mathcal{L}_3$ functions produce a control on $\mathcal{A}_{ad}$. As we will see in the ref sec: proof: PLP section, $\mathcal{L}_4$, as it does not have a unique minimum, is not capable of generating an admissible control.

The dynamics will be approximated with the Euler method, while for solving the discrete minimization problem we will employ the non-linear constrained optimization tool `CasADi` [16]. `CasADi` is an open-source tool for nonlinear optimization and algorithmic differentiation which implements the interior point method via the optimization software `IPOPT` [17]. To be efficiently applied to solve an optimal control problem, we then need the functional we aim to minimize to be smooth. While this is clearly true in the bi-level case of Problem IV.2, the functional in Problem IV.3, due to the piece-wise linear penalization, is not differentiable at the points $u_k \in \mathcal{U}$. For this reason, when treating the multilevel case, we will first need to build a smooth approximation of the function $\mathcal{L}$ we introduced in (IV.7). Once we have this approximation, we will employ the optimal control approach we presented in Section IV to solve some specific examples of SHM problem.

A. Smooth approximation of $\mathcal{L}$ for multilevel control

As we mentioned, to efficiently employ `CasADi` for solving our optimal control problem in the multilevel case, we need to build a smooth approximation of the cost functional. For this reason, we will regularize the piece-wise linear penalization defined in (IV.7) in the following way.

First of all, for all real parameter $\theta > 0$, we introduce the $C^\infty(\mathbb{R})$ function

$$h^\theta(x) := \frac{1 + \tanh(\theta x)}{2}$$

and observe that, as $\theta \to +\infty$, h^θ converges in $L^\infty(\mathbb{R})$ to the Heaviside function

$$h(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}.$$

Secondly, for all $k \in \{1, \ldots, N_u - 1\}$ we define the (smooth) function $\chi^\theta_{[u_k, u_{k+1})} : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} \chi^\theta_{[u_k, u_{k+1})}(x) :&= -1 + h^\theta(x - u_k) + h^\theta(-x + u_{k+1}) \\ &= \frac{\tanh[\theta(x - u_k)] + \tanh[\theta(u_{k+1} - x)]}{2} \end{aligned}$$

which, as $\theta \to +\infty$, converges in $L^\infty(\mathbb{R})$ to the characteristic function $\chi_{[u_k, u_{k+1})}$. Finally, we employ $\chi^\theta_{[u_k, u_{k+1})}$ to define

$$\mathcal{L}^\theta(u) = \sum_{k=1}^{N_u - 1} \lambda_k \chi^\theta_{[u_k, u_{k+1})}(u), \tag{V.1}$$

with λ_k given by (IV.8), which, as $\theta \to +\infty$, converges in $L^\infty(\mathbb{R})$ to the penalization function $\mathcal{L}$ defined in (IV.7).

Notice that this regularization procedure is independent of the function λ_k in (IV.7), which is just required to be in the form (IV.8). Nevertheless, in our numerical experiments we shall select some specific λ_k. In particular, we will use

$$\lambda_k = (u_{k+1} + u_k)(u - u_k) + u_k^2, \tag{V.2}$$

which corresponds to taking $\alpha = 1$ and $\beta = 0$ in (IV.6).

B. Direct method for OCP-SHE

To solve Problem IV.3, we use a direct method, whose starting point is to discretize the cost functional and the dynamics. To this end, let us consider a N_t-points partition of the interval $[0, \pi]$

$$\mathcal{T} = \{t_k\}_{k=1}^{N_t}$$

and denote by $\boldsymbol{u} \in \mathbb{R}^{N_t}$ the vector with components $u_k = u(t_k)$, $k = 1, \ldots, N_t$.

Then the optimal control problem (IV.2) can be written as optimization one with variable $\boldsymbol{u} \in \mathbb{R}^{N_t}$. In more detail, we can formulate the problem IV.3 as the following one in discrete time.

Problem V.1 (Numerical OCP) *Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$, respectively, the target vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, and the partition $\mathcal{T}$ of the interval $[0, \pi]$, we look for $\boldsymbol{u} \in \mathbb{R}^{N_t}$ that solves the following minimization problem:*

$$\min_{\boldsymbol{u} \in \mathbb{R}^{N_t}} \left[\|\boldsymbol{x}_{N_t}\|^2 + \varepsilon \sum_{k=1}^{N_t - 1} \left[\frac{\mathcal{L}^\theta(u_{t_k}) + \mathcal{L}^\theta(u_{t_{k+1}})}{2} \Delta t_k \right] \right]$$

$$\textit{subject to: } \begin{cases} \boldsymbol{x}_{t_{k+1}} = \boldsymbol{x}_{t_k} - \Delta t_k (2/\pi) \boldsymbol{\mathcal{D}}(t_k) \\ \boldsymbol{x}_{t_1} = \boldsymbol{x}_0 := [\boldsymbol{a}_T, \boldsymbol{b}_T]^\top \end{cases}$$

where

$$\Delta t_k = t_{k+1} - t_k \quad \forall k \in \{1, \ldots, N_t - 1\}. \tag{V.3}$$

C. Numerical experiments

We now present several numerical experiments to show the effectiveness of our optimal control approach to solve SHM problems. All the examples will share the common parameters $\varepsilon = 10^{-5}$, $\theta = 10^5$, and $\mathcal{P}_t = \{0, 0.1, 0.2, \ldots, \pi\}$. We will consider $\mathcal{E}_a = \mathcal{E}_b = \{1, 5, 7, 11, 13\}$, and the target vectors $\boldsymbol{a}_T = \boldsymbol{b}_T = (m, 0, 0, 0, 0, 0)^\top$ for all values of $m \in [-0.8, 0.8]$. Moreover, we will treat all the specific types of controls we mentioned before, that is

1. *bang-bang*, with $\mathcal{U} = \{-1, 1\}$.
2. *bang-off-bang*, with $\mathcal{U} = \{-1, 0, 1\}$.
3. *odd multilevel*, $\mathcal{U} = \{-1, -1/2, 0, 1/2, 1\}$.
4. *even multilevel*, $\mathcal{U} = \{-1, -3/5, -1/5, 1/5, 3/5, 1\}$.

The results of our simulations are displayed in Fig. 5, 6 and 7 where, for all $m \in [-0.8, 0.8]$, we show the *policy* $\Pi(m)$ giving us the optimal control signal u^*. Here the terminology *policy* refers to the function $\Pi : \mathbb{R}^{N_a + N_b} \to \mathbb{R}^{N_t}$, which takes as the input value the initial condition $\boldsymbol{x}_0$ and returns the solution $\boldsymbol{u}$ of the Problem V.1. This function is analogous to the function defined in III.3 for the finite dimensional Problem III.1, however this function does not depend on any specific waveform.

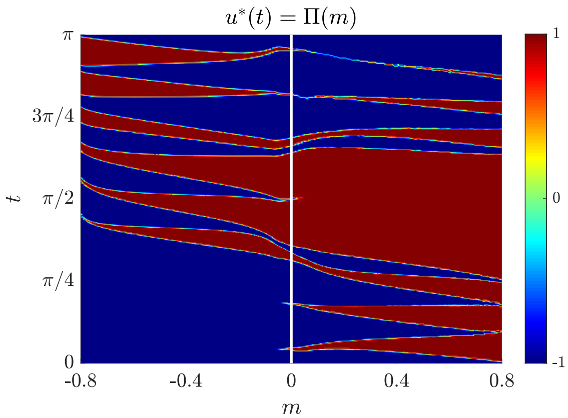

Fig. 5. Bang-bang control for the SHM problem.

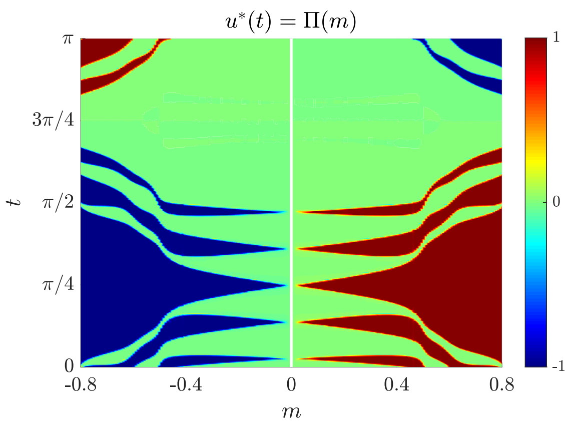

Fig. 6. Bang-off-bang control for the SHM problem.

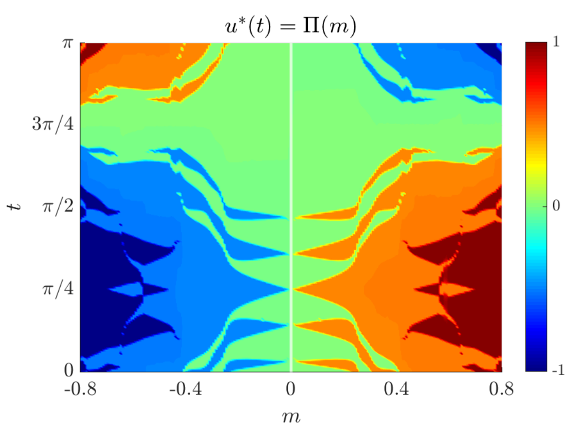

Fig. 7. Odd multilevel control for the SHM problem.

In these plots, to each value of the parameter m in the horizontal line, it corresponds an optimal control having a staircase structure and changing value in $\mathcal{U}$ in the correspondence of the change of color. For instance, in Fig. 5, the control will be $u = -1$ in the blue region and $u = 1$ in the red one. This, of course, is in accordance with Theorems IV.1 and IV.2.

In addition to that, if we compare the policies $\Pi(m)$ displayed in Fig. 5, 6 and 7 with the policies $\Pi_{\mathcal{S}}$ of Fig. 2, we can see that the issues we mentioned in Section III concerning the solvable set and the continuity of the policy have been overcome by our approach. In particular, formulating the SHM problem via optimal control we are capable of finding solutions in an ample range of the parameter m. This is because, while solving Problem V.1, we are not restricted to have a specific waveform, as it is Problem III.1. Instead, for any value of m the best waveform and switching angles to reach the desired targets are automatically obtained through the minimization process to compute the optimal control. In other words, the possibility of changing the waveform during the optimization process can be seen as an extra degree of freedom allowing to have a large solvable set and a continuous policy. Furthermore, also the *combinatory problem* we introduced in Section III is solved by our approach, as we do not need anymore to launch the same optimization process for all the possible waveforms of a given set $\mathcal{U}$.

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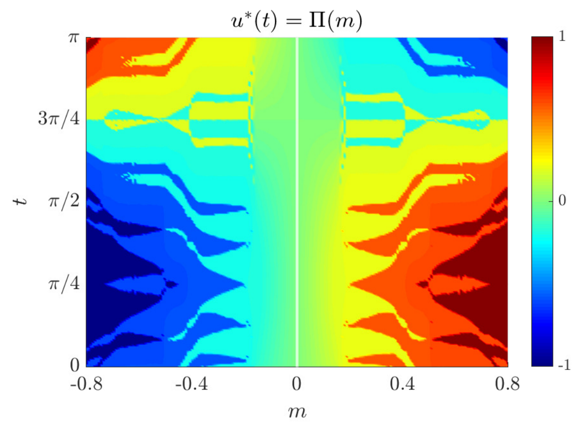

Fig. 8. Even multilevel control for the SHM problem.

VI. Proofs of results in Section IV

This section is devoted to the proofs of the results presented in Section IV, i.e. Proposition IV.1 and Theorems IV.1 and IV.2. For the sake of readability, we organize the proofs as follows: in subsection VI-A we prove the existence a minimizer for Problem IV.3, and also give the proof of Proposition IV.1; in subsection VI-B, we deduce the necessary optimality conditions from Pontryagin's Maximum Principle; in subsection VI-C, we prove that, when $\mathcal{L}$ is given as in Theorem IV.1, the solutions to Problem IV.3 are bang-bang; in subsection VI-D we prove the analogous result for Theorem IV.2. Finally, in subsection VI-E we give the proof of uniqueness and continuity of the solution to Problem IV.3 with respect to the initial condition, when the penalization term $\mathcal{L}$ is given as in Theorems IV.1 or IV.2.

A. Existence of minimizers

The existence of a minimizer, solution to Problem IV.3, can be easily proved by means of the direct method in calculus of variations. Indeed, observe that the dynamical system (IV.5) is linear and the admissible controls in $\mathcal{A}$ are uniformly bounded. In addition to that, the functional to be minimized is convex with respect to the control, which suffices to ensure the weak lower semicontinuity of the functional, allowing us to pass to the limit in the minimizing sequence.

Let us now give the proof of Proposition IV.1, which provides an upper estimate for the error in the solution to the optimal control problem with the penalization term for the control, in the case when the SHM problem admits a solution.

Proof: [of Proposition IV.1] Since we are supposing that Problem IV.1 has a solution, there exists a control $\tilde{u} \in \mathcal{A}_{ad}$ such that its corresponding trajectory $\tilde{\boldsymbol{x}}$, solution to (IV.5), satisfies $\tilde{\boldsymbol{x}}(\pi) = 0$.

Now, let $u^* \in \mathcal{A}$ be the solution to Problem IV.3, and let $\boldsymbol{x}^*$ be its corresponding trajectory. By the optimality of u^* we have

$$\frac{1}{2}\|\boldsymbol{x}^*(\pi)\|^2 + \varepsilon \int_0^\pi \mathcal{L}(u^*(\tau))d\tau \leq \varepsilon \int_0^\pi \mathcal{L}(\tilde{u}(\tau))d\tau,$$

and hence, we deduce that $\|\boldsymbol{x}^*(\pi)\|^2 \leq 4\varepsilon\pi\|\mathcal{L}\|_\infty$. ■

B. Optimality conditions

The proofs of Theorems IV.1 and IV.2 are based on the optimality conditions for Problem IV.3, which can be deduced by means of Pontryagin's maximum principle [18, Chapter 2.7].

To this end, let us first introduce the Hamiltonian associated to the Optimal Control Problem IV.3:

$$\mathcal{H}(t, \boldsymbol{p}, u) = \varepsilon\mathcal{L}(u) - \frac{2}{\pi}\big(\boldsymbol{p} \cdot \boldsymbol{\mathcal{D}}(t)\big)u(t), \tag{VI.1}$$

where $\boldsymbol{p} \in \mathbb{R}^{N_a+N_b}$ is the so-called adjoint variable, and arises from the restriction imposed by the dynamical system (IV.5). In view of the definition of $\boldsymbol{\mathcal{D}}(t)$ in (IV.3)-(IV.4), we will sometimes write the state and the adjoint variables using the following notation:

$$\boldsymbol{x}(t) = \big[\boldsymbol{a}(t), \boldsymbol{b}(t)\big]^\top \quad \text{and} \quad \boldsymbol{p}(t) = \Big[\boldsymbol{p}^a(t), \boldsymbol{p}^b(t)\Big]^\top.$$

Now, let us derive the optimality conditions arising from Pontryagin's Maximum Principle.

1. **The adjoint system**: for any $u^* \in \mathcal{A}$ solution to Problem IV.3, there exists a unique adjoint trajectory $\boldsymbol{p}^* \in C([0, \pi]; \mathbb{R}^{N_a+N_b})$ which satisfies the following terminal-value problem
$$\begin{cases} \dot{\boldsymbol{p}}^*(t) = -\nabla_x\mathcal{H}(u(t), \boldsymbol{p}^*(t), t), & t \in [0, \pi] \\ \boldsymbol{p}^*(\pi) = \nabla_x\Psi(\boldsymbol{x}^*(\pi)) \end{cases}$$
where $\Psi(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}\|^2$ is the terminal cost. Moreover, since the Hamiltonian does not depend on the state variable $\boldsymbol{x}$, we simply have $\dot{\boldsymbol{p}}^*(t) = 0$ for all $t \in [0, \pi)$. We therefore deduce that the adjoint trajectory is constant, and given by
$$\boldsymbol{p}^*(t) = \boldsymbol{x}^*(\pi), \qquad \text{for all } t \in [0, \pi). \tag{VI.2}$$
2. **The Optimal Control**: now, using the optimal adjoint trajectory, we can deduce the necessary optimality condition for the control, which reads as follows:
$$u^*(t) \in \operatorname*{arg\,min}_{|u|\leq 1} \mathcal{H}(t, \boldsymbol{p}^*(t), u), \qquad \text{for all } t \in [0, \pi). \tag{VI.3}$$
As we will see in subsections VI-C and VI-D, for penalization functions $\mathcal{L}$ as the ones we consider in Theorems IV.1 and IV.2, this argmin is a singleton for almost every $t \in [0, \pi)$. Hence, given the adjoint $\boldsymbol{p}^*$, the condition (VI.3) uniquely determines the optimal control almost everywhere in $[0, \pi)$. The only points where the control is not uniquely determined are, precisely, the switching angles, i.e. the points of discontinuity of the solution.

In view of the form of the adjoint trajectory (VI.2) associated to the optimal state trajectory $\boldsymbol{x}^*$, let us introduce the function

$$\begin{aligned} \mu^*(t) :=& \frac{2}{\pi}\big(\boldsymbol{x}^*(\pi) \cdot \boldsymbol{\mathcal{D}}(t)\big) \\ =& \sum_{j\in\mathcal{E}_a} a_j^*(\pi)\cos(jt) + \sum_{k\in\mathcal{E}_b} b_k^*(\pi)\sin(kt). \end{aligned} \tag{VI.4}$$

Then, in view of (VI.1) and (VI.2), we can write the optimality condition (VI.3) as

$$u^*(t) \in \operatorname*{arg\,min}_{|u|\leq 1} \mathcal{J}(u, \mu^*(t)). \tag{VI.5}$$

where $\mathcal{J}$ is defined as

$$\mathcal{J}(u, \mu^*(t)) := \varepsilon\mathcal{L}(u) - \mu^*(t)u. \tag{VI.6}$$

We are now ready to prove that the solutions to Problem IV.3, when $\mathcal{L}$ is chosen as in Theorems IV.1 and IV.2, have the desired staircase form (II.4)–(II.5).

C. Proof of Theorem IV.1 (bang-bang control)

We need to prove that, if $\mathcal{L}(u) = \alpha u$ for some $0 \neq \alpha \in \mathbb{R}$, then any optimal control u^* has the form (II.4) with $\mathcal{U} = \{-1, 1\}$. Or in other words, $u^*(t)$ takes values in $\mathcal{U}$ for all $t \in [0, \pi)$, except for a finite number of times.

Let $u^* \in \mathcal{A}$ be a solution to Problem IV.3, and let $\boldsymbol{x}^*$ be its associated optimal trajectory. We just need to notice that, due to (VI.5) and the choice of $\mathcal{L}$, the function u^* satisfies

$$u^*(t) = \begin{cases} -1 & \text{if } \mu^*(t) < \varepsilon\alpha \\ 1 & \text{if } \mu^*(t) > \varepsilon\alpha \end{cases}.$$

Observe that, in the case when $\mu^*(t) = 0$, which corresponds only to the cases when $\boldsymbol{x}^*(\pi) = 0$, the optimal control is constant and is just given by $u^*(t) = -\text{sgn}(\alpha)$.

In all the other cases, when $\boldsymbol{x}^*(\pi) \neq 0$, the function $\mu^*(t)$ is a linear combination of sines and cosines, and therefore, the equality $\mu^*(t) = \varepsilon\alpha$ can only hold for a finite number of times $t \in [0, \pi)$, which are the discontinuity points of u^* (the switching angles). Note that the choice of u^* at these points is irrelevant as it represents a set of zero measure. We can see a graphical interpretation of this result in Figure 9.

D. Proof of Theorem IV.2 (multilevel control)

In this case, we suppose that $\mathcal{U} = \{u_k\}_{k=1}^L$ is a finite set of real numbers in $[-1, 1]$ satisfying

$$-1 = u_1 < u_2 < \ldots < u_L = 1, \quad \text{with } L > 2. \tag{VI.7}$$

The case $L = 2$ is just the bi-level case. As in the previous proof, our goal is to show that the argmin in (VI.5) is a singleton and belongs to $\mathcal{U}$ for every $t \in [0, \pi)$ except for a finite number of points in $[0, \pi)$.

In this case, the study of the minimizers of $\mathcal{J}$ is slightly more involved since the penalization function $\mathcal{L}$ defined in (IV.7)-(IV.8) is not differentiable at the points $u_k \in \mathcal{U}$.

Since $\mathcal{L}$ is an affine interpolation of a convex function and, therefore, it is Lipschitz and convex, we deduce that also $\mathcal{J}$ is Lipschitz and convex as a function of u. In view of this, we have that u^* minimizes $\mathcal{J}(u, \mu)$ if and only if

$$0 \in \partial_u\mathcal{J}(u^*, \mu), \tag{VI.8}$$

where ∂_u denotes the subdifferential with respect to u.

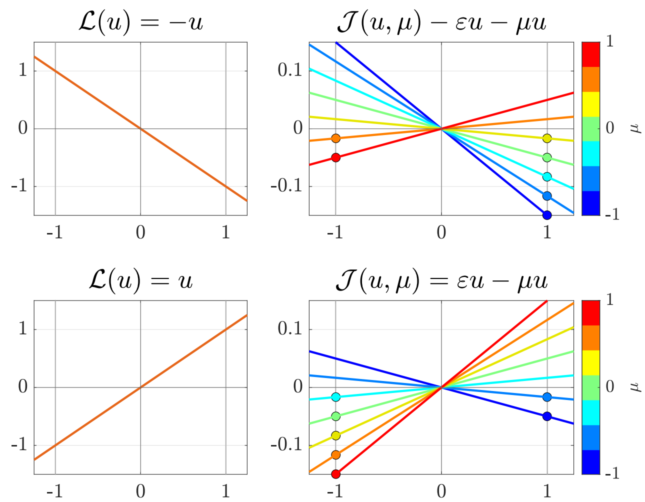

Fig. 9. Bi-level SHE: in the left column we show two examples of functions $\mathcal{L}$ compatible with Theorem IV.1. In the right columns we display the behavior of the corresponding Hamiltonian for different values of $\boldsymbol{\mu}$.We draw the minumun as a point for every $\boldsymbol{\mu}$, and we can see how only take the value in $\boldsymbol{\mathcal{U}}$.

Let us recall below the definition of subdifferential from convex analysis:

$$\partial_u \mathcal{J}(u,\mu) = \{c \in \mathbb{R} \quad \text{s.t.} \quad \mathcal{J}(v,\mu) - \mathcal{J}(u,\mu) \geq c(v-u), \quad \forall v \in [-1,1]\}.$$

In the case of a convex function as $\mathcal{J}(\cdot,\mu)$, one can readily show that the subdifferential at $u \in (-1,1)$ is the nonempty interval $[a,b]$, where a and b are the one-sided derivatives

$$a = \lim_{v\to u^-} \frac{\mathcal{J}(v,\mu)-\mathcal{J}(u,\mu)}{v-u}, \quad b = \lim_{v\to u^+} \frac{\mathcal{J}(v,\mu)-\mathcal{J}(u,\mu)}{v-u}.$$

Moreover, the subdifferential at $u=-1$ and $u=1$ is given by $(-\infty,b]$ and $[a,+\infty)$ respectively. Notice that, if $\mathcal{J}$ is differentiable at some $u \in (-1,1)$, then the left and the right derivatives coincide, and thus, $\partial_u\mathcal{J}(u,\mu)$ is just the classical derivative.

Using this characterization of the subdifferential, we can compute $\partial_u\mathcal{J}(u,\mu)$ for all $u\in[-1,1]$ in terms of μ. To this end, let us define

$$p_k := \frac{d}{du}\lambda_k(u) = \frac{\mathcal{P}(u_{k+1})-\mathcal{P}(u_k)}{u_{k+1}-u_k}$$

for all $k \in \{1,\dots,L-1\}$, with $\lambda_k(u)$ given by (IV.8). Using the definition of $\mathcal{J}$ in (VI.6) and $\mathcal{L}$ in (IV.7), we can compute

$$\partial_u\mathcal{J}(-1,\mu) = (-\infty, \varepsilon p_1 - \mu],$$

$$\partial_u\mathcal{J}(1,\mu) = [\varepsilon p_{L-1} - \mu, +\infty),$$

$$\partial_u\mathcal{J}(u_k,\mu) = [\varepsilon p_{k-1} - \mu,\, \varepsilon p_k - \mu],$$

for all $k \in \{2,\dots,L-1\}$, and

$$\partial_u\mathcal{J}(u,\mu) = \{\varepsilon p_k - \mu\},$$

for all $u \in (u_k,u_{k+1})$ and all $k\in\{1,\dots,L-1\}$. In view of the above computation, we obtain that

$$\begin{aligned} 0 &\in \partial_u\mathcal{J}(-1,\mu) && \text{iff} \quad \mu \leq \varepsilon p_1,\\ 0 &\in \partial_u\mathcal{J}(1,\mu) && \text{iff} \quad \mu \geq \varepsilon p_{L-1},\\ 0 &\in \partial_u\mathcal{J}(u_k,\mu) && \text{iff} \quad \varepsilon p_{k-1} \leq \mu \leq \varepsilon p_k, \end{aligned} \tag{VI.9}$$

for all $k\in\{2,\dots,L-1\}$, and

$$0 \in \partial_u\mathcal{J}(u,\mu) \quad \forall u \in [u_k,u_{k+1}] \qquad \text{iff } \mu = \varepsilon p_k \tag{VI.10}$$

for all $k\in\{1,\dots,L-1\}$.

Using (VI.9), along with the optimality condition (VI.8), we deduce that, for a.e $\mu\in\mathbb{R}$, we have

$$\operatorname*{arg\,min}_{|u|\leq 1} \mathcal{J}(u,\mu) = \{u_k\} \quad \text{for some } u_k \in \mathcal{U}. \tag{VI.11}$$

Indeed, (VI.11) does not hold if and only if

$$\mu = \varepsilon p_k \qquad \text{for some } k \in \{1,\dots,L-1\}. \tag{VI.12}$$

Observe that, in the case when $\mu^*(t)=0$, which corresponds only to $\boldsymbol{x}^*(\pi)=0$, the optimal control is constant and is just given by $u^*(t) = \operatorname{arg\,min}_{|u|\leq 1}\mathcal{L}(u)$ which, by hypothesis, is a singleton and belongs to $\mathcal{U}$ (note that between any two consecutive points of $\mathcal{U}$, the function $\mathcal{L}$ is a straight line).

In all the other cases, i.e. when $\boldsymbol{x}^*(\pi)\neq 0$, the function $\mu^*(t)$ is a linear combination of sines and cosines, and therefore, the equality $\mu^*(t) = \varepsilon p_k$ can only hold, for each $k\in\{1,\dots,L-1\}$, a finite number of times in $[0,\pi)$. These are precisely the discontinuity points of u^* (the switching angles).

We have proved that, for all $t\in[0,\pi)$ except for a finite number of discontinuity points, which are precisely the switching angles $\{\phi_m\}_{m=0}^M$, we have $u^*(t)\equiv u_k$ for some $u_k\in\mathcal{U}$. Observe that, due to the continuity of $\mu^*(t)$, along with (VI.9), it is clear that $u^*(t)$ does not change value between two consecutive switching angles. Therefore, u^* is piecewise constant, with a finite number of switches. The choice of u^* at the discontinuity points is irrelevant as it represents a set of zero measure.

Finally, the staircase property of the waveform (II.5) can be deduced from (VI.5) and (VI.9), along with the continuity of the function $\mu^*(t)$. Also, we the same philosophy of Figure 9 in Bang-Bang case, we can see a graphical interpretation of this results in Figure 10.

E. Uniqueness and continuity of solutions

The proofs in this subsection apply to both Theorems IV.1 and IV.2 (the bilevel and the multilevel case).

Proof: [uniqueness] We first prove that Problem IV.3 admits a unique solution, i.e. for each $\boldsymbol{x}_0\in\mathbb{R}^N$, there exists a unique $u^*\in\mathcal{A}$ minimizing the functional

$$F(u,\boldsymbol{x}_0) := \frac{1}{2}\|\boldsymbol{x}(\pi)\|^2 + \varepsilon\int_0^\pi \mathcal{L}(u(t))dt, \tag{VI.13}$$

where, for each $u\in\mathcal{A}$, $\boldsymbol{x}(\pi)$ is given by

$$\boldsymbol{x}(\pi) = \boldsymbol{x}_0 - \frac{2}{\pi}\int_0^\pi \boldsymbol{\mathcal{D}}(t)u(t)dt.$$

We argue by contradiction. Suppose that there exist two functions $u_1,u_2\in\mathcal{A}$ solutions to Problem IV.3, which are different in a set of positive measure. As both of them are optimal, using the arguments in subsections VI-B, VI-C and VI-D, we deduce that the controls u_1 and u_2 are uniquely determined a. e. in $[0,\pi)$ by the final state of the associated trajectory, i.e. $\boldsymbol{x}_1^*(\pi)$ and $\boldsymbol{x}_2^*(\pi)$, respectively. Therefore, if $u_1\neq u_2$ in a set of positive measure, then we have $\boldsymbol{x}_1^*(\pi)\neq\boldsymbol{x}_2^*(\pi)$.

Let us now consider the control

$$\tilde{u}(t) = \frac{u_1(t)+u_2(t)}{2}.$$

By the linearity of the dynamics (IV.5), the convexity of $\mathcal{L}$, and using that $\boldsymbol{x}_1^*(\pi)\neq\boldsymbol{x}_2^*(\pi)$, we obtain

$$F(\tilde{u},\boldsymbol{x}_0) = \frac{1}{2}\left\|\frac{\boldsymbol{x}_1^*(\pi)+\boldsymbol{x}_2^*(\pi)}{2}\right\|^2 + \varepsilon\int_0^\pi \mathcal{L}\left(\frac{u_1(t)+u_2(t)}{2}\right)dt$$
$$< \frac{F(u_1,\boldsymbol{x}_0)+F(u_2,\boldsymbol{x}_0)}{2}.$$

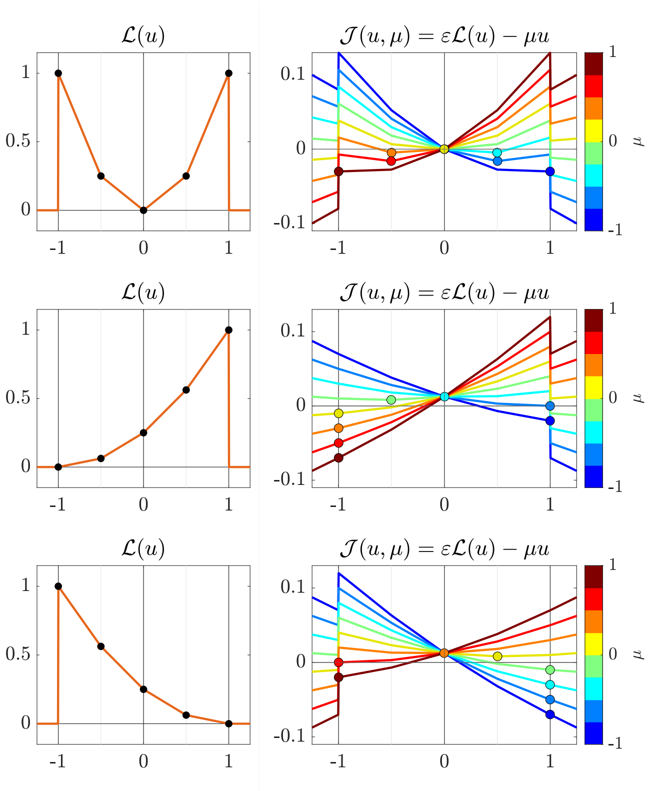

Fig. 10. Multilevel SHM: in the left column we show three types of penalizations which are compatible with our theoretical results. In the right column we show the behavior of the corresponding Hamiltonian for different values of $\boldsymbol{\mu}$. We draw the minumun as a point for every $\boldsymbol{\mu}$, and we can see how only take the value in $\boldsymbol{\mathcal{U}}$.

Hence, using that both u_1 and u_2 minimize the functional $F(\cdot,\boldsymbol{x}_0)$, we obtain

$$F(\tilde{u},\boldsymbol{x}_0)<F(u_1,\boldsymbol{x}_0),$$

which contradicts the optimality of u_1. We therefore conclude that the $u_1(t)=u_2(t)$ for a.e. $t\in[0,\pi)$. ∎

Proof: [continuity w.r.t. initial condition] Let us now give the proof of the L^1-continuity of the unique solution u^* to Problem IV.3 with respect to the initial condition.

Let $\boldsymbol{x}_0$ be fixed. We need to prove that, for all $\gamma>0$, there exists $\delta>0$ such that

$$\|\boldsymbol{x}_1-\boldsymbol{x}_0\|\leq\delta \quad \text{implies} \quad \|u_1^*-u_0^*\|_{L^1(0,\pi)}<\gamma,$$

where u_0^* and u_1^* are the optimal controls corresponding to the initial conditions $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ respectively.

As we have proved in subsections VI-C and VI-D, for any $\boldsymbol{x}_1$, the optimal control u_1^*, solution to Problem IV.3, is piece-wise constant, taking values in $\mathcal{U}$, with a finite number of discontinuity points (switching points).

We claim that the number of switching points is bounded from above by a constant $M^*\in\mathbb{N}$, independent of $\boldsymbol{x}_1\in\mathbb{R}^N$.

Indeed, as we proved in subsection VI-B, the optimal control u_1^* is determined by the optimality condition (VI.5), using the function μ^* defined in (VI.4). If $\mu^*\equiv 0$, then u_1^* is constant and there are no switching points. In the other cases, $\mu^*(t)$ is a linear combination of sines and cosines with fixed frequencies. In the bilevel case, in subsection VI-C we proved that the switching points correspond to the intersection points of $\mu^*(t)$ with $\varepsilon\alpha$. In the multilevel case, we proved in subsection VI-D that the switching points correspond to the intersections of $\mu^*(t)$ with εp_k, see (VI.12). In view of (VI.4), as the frequencies are fixed, the number of these intersection points in the interval $[0,\pi)$ cannot exceed a certain number M^*, independent of the coefficients a_j^* and b_k^* in (VI.4). Actually, M^* only depends on $\max\{\mathcal{E}_a,\mathcal{E}_b\}$ and the cardinality of $\mathcal{U}$. The claim then follows.

Using the fact that, for any $\boldsymbol{x}_1$, the solution u_1^* is piece-wise constant taking values only in $\mathcal{U}$, and with a finite number of switches less than some M^* independent of $\boldsymbol{x}_1$, we deduce that there exists $K>0$, independent of $\boldsymbol{x}_1$ such that

$$\|u_1^*\|_{BV}\leq K.$$

See (VI.19) below for the definition of the BV norm. We then obtain that, for any $\boldsymbol{x_1}\in\mathbb{R}^N$,

$$u_1^*\in\mathcal{A}_K^*:=\{u\in\mathcal{A}\ :\ \|u\|_{BV}\leq K\}.$$

Now, for any $\gamma>0$ fixed, we can apply Lemma VI.1 below, to ensure the existence of $\eta>0$ such that

$$F(u,\boldsymbol{x}_0)\geq F(u_0^*,\boldsymbol{x_0})+\eta, \tag{VI.14}$$

for all for all $u\in\mathcal{A}_K^*$, with $\|u-u_0^*\|_{L^1(0,\pi)}=\gamma$.

Since the set $\mathcal{A}_K^*$ is convex and u_0^* minimizes $F(\cdot,\boldsymbol{x}_0)$, we can use (VI.14) and the convexity of the function $u\mapsto F(u,\boldsymbol{x}_0)$, to deduce that

$$F(u,\boldsymbol{x}_0)\geq F(u_0^*,\boldsymbol{x_0})+\eta, \tag{VI.15}$$

for all $u\in\mathcal{A}_K^*$ such that $\|u-u_0^*\|_{L^1(0,\pi)}\geq\gamma$.

Observe that, for any $u\in\mathcal{A}$, the function $\boldsymbol{x}\mapsto F(u,\boldsymbol{x})$ is locally Lipschitz, and therefore, there exists a constant $C_F>0$ satisfying

$$|F(u,\boldsymbol{x}_1)-F(u,\boldsymbol{x}_0)|\leq C_F\|\boldsymbol{x}_1-\boldsymbol{x}_0\| \tag{VI.16}$$

for any $\boldsymbol{x}_1$ such that $\|\boldsymbol{x}_1-\boldsymbol{x}_0\|\leq 1$. Notice that, since $u\in\mathcal{A}$ only takes values in $[-1,1]$, C_F can be chosen independently of u.

Now, combining (VI.15) and (VI.16), for all $u\in\mathcal{A}_K^*$ such that $\|u-u_0^*\|_{L^1(0,\pi)}\geq\gamma$, we obtain

$$\begin{aligned}F(u_0^*,\boldsymbol{x_1})&\leq F(u_0^*,\boldsymbol{x}_0)+C_F\|\boldsymbol{x}_1-\boldsymbol{x}_0\|\\&\leq F(u,\boldsymbol{x}_0)-\eta+C_F\|\boldsymbol{x}_1-\boldsymbol{x}_0\|\\&\leq F(u,\boldsymbol{x}_1)-\eta+2C_F\|\boldsymbol{x}_1-\boldsymbol{x}_0\|\end{aligned} \tag{VI.17}$$

Finally, we can choose $\delta\in(0,1)$ such that $\delta<\frac{\eta}{4C_F}$, and from (VI.17), we deduce that, if $\|\boldsymbol{x}_1-\boldsymbol{x}_0\|\leq\delta$, then

$$F(u_0^*,\boldsymbol{x_1})\leq F(u,\boldsymbol{x}_1)-\frac{\eta}{2}$$

for all $u\in\mathcal{A}_K^*$ such that $\|u-u_0^*\|_{L^1(0,\pi)}\geq\gamma$, which then implies that necessarily

$$\|u_1^*-u_0^*\|_{L^1(0,\pi)}\leq\gamma.$$

This concludes the proof of the L^1-continuity of the solution with respect to the initial condition. ∎

Let us conclude the section with the following Lemma, which has been used in the previous proof.

Lemma VI.1 *Let $\boldsymbol{x}_0\in\mathbb{R}^N$ be given and let $\mathcal{L}$ be a function as in Theorem IV.1 or IV.2. Let $u_0^*\in\mathcal{A}$ be the unique solution to Problem IV.3. For any $K>0$, define the set of controls*

$$\mathcal{A}_K^*:=\{u\in\mathcal{A}\ :\ \|u\|_{BV}\leq K\}. \tag{VI.18}$$

Then, for any $\gamma>0$, there exists $\eta:=\eta(\gamma,K)>0$ such that

$$F(u_0^*,\boldsymbol{x}_0)\leq F(u,\boldsymbol{x}_0)-\eta,$$

for all $u\in\mathcal{A}_K^*$ such that $\|u-u_0^*\|_{L^1(0,\pi)}=\gamma$.

In the definition of $\mathcal{A}_K^*$, we are considering measurable functions of bounded variation in $(0,\pi)$, i.e. functions whose distributional derivative is a Radon measure in $(0, \pi)$, that we denote by $|Du|$, and such that $|Du|(0, \pi)$ is finite. We recall that the norm $\|\cdot\|_{BV}$ is defined as

$$\|u\|_{BV} := \int_0^{\pi} |u(t)|dt + |Du|(0, \pi). \tag{VI.19}$$

See [19, Chapter 3] for further details on the space of functions of bounded variation.

Proof: [Proof of Lemma VI.1] We need to prove that

$$\mathcal{I}_{\gamma,K} = F(u_0^*, \boldsymbol{x}_0) + \eta,$$

for some $\eta > 0$, where

$$\mathcal{I}_{\gamma,K} := \inf \left\{ F(u, \boldsymbol{x}_0) \,:\, u \in \mathcal{A}_K^* \text{ with } \|u - u_0^*\|_{L^1(0,\pi)} = \gamma \right\}.$$

The result follows from the fact that the space $BV(0, \pi)$ is compactly embedded in $L^1(0, \pi)$, see [19, Theorem 3.23]. Consider any minimizing sequence $u_n \in \mathcal{A}_K^*$ with $\|u_n - u_0^*\|_{L^1(0,\pi)} = \gamma$, satisfying

$$\lim_{n\to\infty} F(u_n, \boldsymbol{x}_0) = \mathcal{I}_{K,\gamma}.$$

By [19, Theorem 3.23], there exists a subsequence of u_n which converges to some $\tilde{u} \in \mathcal{A}$, strongly in $L^1(0, \pi)$.

From the continuity of the L^1-norm and of the functional $F(\cdot, \boldsymbol{x}_0)$ with respect to the strong L^1-topology, we deduce that the limit $\tilde{u}$ satisfies

$$\|\tilde{u} - u_0^*\|_{L^1(0,\pi)} = \gamma \quad \text{and} \quad F(\tilde{u}, \boldsymbol{x}_0) = \mathcal{I}_{K,\gamma}.$$

Finally, since u_0^* is the unique minimizer of $F(\cdot, \boldsymbol{x}_0)$, we conclude that

$$\mathcal{I}_{\gamma,K} - F(u_0^*, \boldsymbol{x}_0) = F(\tilde{u}, \boldsymbol{x}_0) - F(u_0^*, \boldsymbol{x}_0) = \eta > 0.$$

■

VII. Conclusions

In this paper, we have proposed a novel optimal control based approach to the Selective Harmonic Modulation problem. More precisely, we have described how the SHM Problem II.1 can be reformulated in terms of a null-controllability one for which the solution u plays the role of the control and can be obtained through the minimization of a suitable cost functional. Besides, we have shown both theoretically and through numerical simulations that our methodology we are able to solve several critical issues (described in detail in Section III) arising in practical power electronic engineering applications.

1. *Combinatory problem*: our approach automatically determines the best waveform and switching angles for the desired control signals. This yields two relevant advances with respect to the existing techniques. On the one hand, this renders a computationally lighter methodology to solve the SHM problem, as it does not need to repeatedly solve an optimization problem for different waveforms. On the other hand, it bypasses the cumbersome task of estimating a priori the number of switches which is necessary to reach the desired Fourier coefficients.
2. *Solvable set problem*: having the liberty of changing the waveform during the optimization process, our approach is capable of covering an ample solvable set.
3. *Policy problem*: the policy obtained through our methodology is not a gathering of several policies to which may correspond disjoint or even overlapping solvable sets. Hence, the continuity of the switching angles is guaranteed and we do not generate regions with no solution to the SHM problem.

In conclusion, this work provides an exhaustive analysis of the SHM problem under the perspective of optimal control. However, some relevant issues have not been completely covered by our study, and will be considered in future works:

1. **Minimal number of switching angles.** In practical applications, to optimize the converters' performances, it is required to maintain the number of switches in the SHM signal the lowest possible. It then become very relevant to determine which is the minimum number of switches allowing to reach the desired target Fourier coefficients.
2. **Continuity in the $L^\infty(0, \pi)$ topology.** Related to the previous point, we observe in our numerical simulations in Section V that, although the optimal control u^* is continuous with respect to the initial datum in the strong topology of $L^1(0, \pi)$, when varying the parameter m, the waveform may change with the appearance and disappearance of switching angles. It is then desirable a finer analysis of the continuity properties of u^*, to understand why and when the aforementioned phenomenon occurs.
3. **Characterization of the solvable set.** It would be very interesting to fully characterize the solvable set for the SHM problem, thus determining the entire range of Fourier coefficients which can be reached through our computed optimal control.
4. **Analysis of th computational cost.** In this paper, for the sake of brevity, we have not fully discussed the computational efficiency of our proposed methodology. This, however, is a very relevant issue when facing practical applications. It would then be interesting to analyze which could be the best performing algorithm to numerically solve our optimal control problem, and to compare our strategy with other existing ones in the SHM literature.

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