Selective Harmonic Elimination via Optimal Control

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Abstract

El problema de *Selective Harmonic Elimination pulse-width modulation*(SHE) es planteado como el problema de control óptimo, con el fin de encontrar soluciones de ondas escalón sin prefijar el número de ángulos de conmutación. De esta manera, la metodología de control óptimo es capaz de encontar la forma de onda óptima y de encontrar la localizaciones de los ángulos de conmutación, incluso sin prefijar el número de conmutaciones. Este es un nuevo enfoque para el problema SHE en concreto y para los sistemas de control con un conjunto finito de controles admisibles en general.

Key words: Selective Harmonic Elimination; Finite Set Control, Piecewise Linear function.

1 Introduction and motivations

Selective Harmonic Elimination (SHE) [**?**] is a well-known methodology in electrical engineering, employed to improve the performances of a converter by controlling the phase and amplitude of the harmonics in its output voltage. As a matter of fact, this technique allows to increase the power of the converter and, at the same time, to reduce its losses.

In broad terms, the process consists in generating a *control signal* with a desired harmonic spectrum, by modulating or eliminating some specific lower order frequencies. This signal is in the shape of a step function and is fully characterized by two features:

1. The *waveform*, i.e. the set of (constant) values the function may assume.
2. The *switching angles*, defining the points in the domain where the function changes from one constant value to another.

Because of the growing complexity of modern electrical networks, consequence for instance of the high penetration of renewable energy sources, the demand in power of electronic converters is day by day increasing. For this and other reasons, SHE has been a preeminent research interest in the electrical engineering community, and a plethora of SHE-based techniques has been developed in recent years. An incomplete bibliography includes [**?**, **?**, **?**].

Nowadays, SHE is mostly based on offline computations to obtain the commutation patterns describing the control signal.

Add references and mention real-time approaches.

The present document is organized as follows. In Section 2, we will present the mathematical formulation of the SHE problem and a classical resolution method based on an optimization process. In Section 3, we will formulate the SHE problem as a controlled dynamical system. In Section 4, we will present the optimal control problem allowing us to obtain the solutions to the SHE problem. In Section 5, we will present our numerical experiments. Finally, in Section 6, we will equation the conclusions of our study.

2 Mathematical formulation of SHE

This section is devoted to the mathematical formulation of the SHE problem. In what follows, with the notation

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$\mathcal{U}$ we will always refer to a finite set of real numbers, contained in the interval $[-1, 1]$

$$\mathcal{U} = \{u_\ell\}_{\ell=1}^{L} \subset [-1, 1], \tag{2.1}$$

with cardinality $|\mathcal{U}| = L$.

In broad terms, our objective is to design a piece-wise constant function $u(\tau) : [0, 2\pi) \to \mathcal{U}$ such that some of its lower-order Fourier coefficients take specific values determined a priori.

Due to the application in power converters, we will focus here on functions with *half-wave symmetry*, i.e.

$$u(\tau + \pi) = -u(\tau) \quad \text{for all } \tau \in [0, \pi).$$

In this way, u is fully determined by its values in the interval $[0, \pi)$ and its Fourier series only involves the odd terms, thus taking the form

$$u(\tau) = \sum_{\substack{i \in \mathbb{N} \\ i \text{ odd}}} a_i \cos(i\tau) + \sum_{\substack{j \in \mathbb{N} \\ j \text{ odd}}} b_j \sin(j\tau), \tag{2.2}$$

where the coefficients a_i and b_j are given by

$$\begin{aligned} a_i &= \frac{2}{\pi} \int_0^\pi u(\tau) \cos(i\tau)\, d\tau, \\ b_j &= \frac{2}{\pi} \int_0^\pi u(\tau) \sin(j\tau)\, d\tau. \end{aligned} \tag{2.3}$$

Because of this half-wave symmetry, in what follows, we will always work with the restriction $u|_{[0,\pi)}$ which, with some abuse of notation, we shall still denote by u. We can then give a general formulation of the SHE problem as follows:

Problem 2.1 (SHE) *Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two sets of odd numbers with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$, respectively, and $\mathcal{U}$ be defined as in (2.1). Given the vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, we look for a piece-wise constant function $u : [0, \pi) \to \mathcal{U}$ such that*

$$\begin{aligned} a_i &= (\boldsymbol{a}_T)_i, \quad \textit{for all } i \in \mathcal{E}_a, \\ b_j &= (\boldsymbol{b}_T)_j, \quad \textit{for all } j \in \mathcal{E}_b, \end{aligned}$$

with $\{a_i\}_{i \in \mathcal{E}_a}$ and $\{b_j\}_{j \in \mathcal{E}_b}$ given by (2.3).

Fig. 1 shows an example of a function u solution of the SHE problem.

As we anticipated in Section 1, the control signal u is fully characterized by its waveform and the switching angles, to which we now give a precise definition.

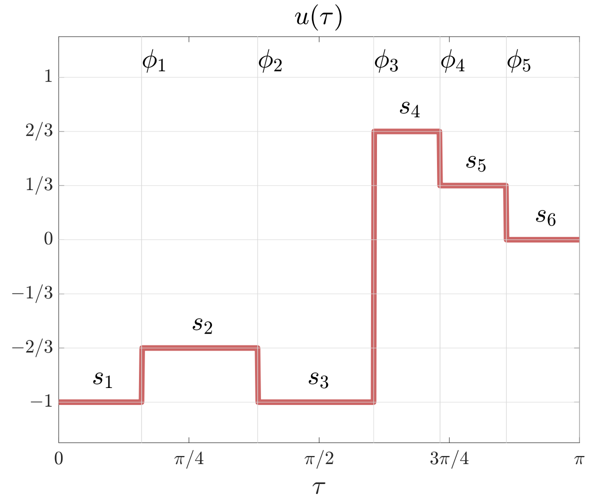

Fig. 1. Possible solution of Problem 2.1, where we considered that $\mathcal{U} = \{-1/3, -2/3, 0, 1/3, 2/3, 1\}$. We show the switching angles $\boldsymbol{\phi}$ and the waveform $\mathcal{S}$ (see Definitions 2.1 and 2.2).

Definition 2.1 (Wave-form) *Given the finite set $\mathcal{U}$ defined in (2.1) and $M \in \mathbb{N}$, we will call a waveform any possible $(M+1)$-tuple $\mathcal{S} = (s_m)_{m=0}^{M+1}$ with $s_m \in \mathcal{U}$ for all $m = 0, \ldots, M+1$.*

Definition 2.2 (Switching angles) *Given the finite set $\mathcal{U}$ defined in (2.1), $M \in \mathbb{N}$ and a piece-wise constant function $u : [0, \pi) \to \mathcal{U}$, we shall refer as switching angles $\boldsymbol{\phi} = \{\phi_m\}_{m=0}^{M+1} \subset [0, \pi]$, with $\phi_0 = 0$ and $\phi_{M+1} = \pi$, to the points in the domain $[0, \pi)$ where u changes its value.*

In view of the above definitions, we can provide the following explicit expression for u:

$$\begin{aligned} u &= \sum_{m=0}^{M+1} s_m \chi_{[\phi_m, \phi_{m+1}]} \\ s_m &\in \mathcal{S}, \ \phi_m \in \boldsymbol{\phi}, \quad \text{for all } m = 0, \ldots, M+1, \end{aligned} \tag{2.4}$$

where we denoted by $\chi_{[\phi_m, \phi_{m+1}]}$ the characteristic function of the interval $[\phi_m, \phi_{m+1}]$.

Besides, taking into account (2.4), a direct computation yields that the Fourier coefficients (2.3) are given by

$$a_i = a_i(\boldsymbol{\phi}) = \frac{2}{i\pi} \sum_{k=1}^{M+1} s_k \Big[\sin(i\phi_k) - \sin(i\phi_{k-1})\Big]$$

$$b_j = b_j(\boldsymbol{\phi}) = \frac{2}{j\pi} \sum_{k=1}^{M+1} s_k \Big[\cos(j\phi_{k-1}) - \cos(j\phi_k)\Big]$$

Given a waveform $\mathcal{S}$, Problem 2.1 then reduces to find the switching locations $\boldsymbol{\phi}$ (see [?, ?, ?]). This can be cast as a minimization problem in the variables $\{\phi_m\}_{m=0}^{M+1}$, where the cost functional is the Euclidean distance between the obtained Fourier coefficients $\{a_i(\boldsymbol{\phi}), b_j(\boldsymbol{\phi})\}$

and the targets $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^{N_a} \times \mathbb{R}^{N_b}$.

Problem 2.2 (Optimization for SHE) *Given a waveform $\mathcal{S}$ and a step function u in the form (2.4), we look for the switching angles $\boldsymbol{\phi}$ by means of the following minimization problem:*

$$\min_{\boldsymbol{\phi} \in [0,\pi]^M} \left(\sum_{i \in \mathcal{E}_a} \|a_T^i - a_i(\boldsymbol{\phi})\|^2 + \sum_{j \in \mathcal{E}_b} \|b_T^j - b_j(\boldsymbol{\phi})\|^2 \right)$$

$$\text{subject to: } 0 = \phi_0 < \phi_1 < \ldots < \phi_M < \phi_{M+1} = \pi \tag{2.5}$$

Since the cardinality of $\mathcal{S}$ is not known a priori, meaning that we do not know how many switches will be necessary to reach the desired values of the Fourier coefficients, a common approach to solve the SHE problem consists in fixing the number of changes and generating all the possible combinations of elements of $\mathcal{S}$, to later solve an optimization problem for each one of them. Nevertheless, taking into account that the number of possible tuples $\mathcal{S}$ is given by $|\mathcal{U}|^{|\mathcal{S}|}$, it is evident that the complexity of the above approach increases rapidly. This problem has been studied in [?] where, through appropriate algebraic transformations, the authors are able to convert the SHE problem into a polynomial system whose solutions' set contains all the possible waveforms for a given set $\mathcal{U}$ and number of elements in the sequence $\mathcal{S}$ which, however, is predetermined.

On the other hand, we shall also mention that the SHE methodology has been developed to provide in real-time different target Fourier coefficients con with a KHz latency. This makes impossible to find real-time solutions by optimization, making then necessary to pre-determine solutions that can later be interpolated. Nevertheless, it is well-known that, fixed a sequence S, the continuity of the switching locations with respect to a continuous variation of the target Fourier coefficients may be quite cumbersome. In the majority of the cases, it is impossible to find a continuous solution in a large interval, an it is necessary to change the waveform S while moving across different solution regions ([?, ?]). This makes difficult the interpolation of solutions and their finding.

In this document, we will present the SHE problem as an optimal control one, where the optimization variable is the signal $u(\tau)$ defined in the entire interval $[0, \pi)$. In particular, we will describe how the Fourier coefficients of the function $u(\tau)$ can be seen as the final state of a system controlled by $u(\tau)$. Hence, the optimization is performed among all the possible functions that satisfy $|u(\tau)| < 1$ and can control the final state at the desired Fourier coefficients. Then we will show how to design a control problem so that the solution is a step function.

In this way, we can reformulate Problem 2.1 as follows. In this formulation, the SHE problem converts in a minimization problem with restrictions which can be solved by well-known techniques. Since the problem has several minimizers, we shall solve it employing global optimizers. Furthermore, since the choice of the waveform is arbitrary, we shall proceed in the same way for each possible waveform.

Tengo que enlazar estas sección con la formulacion de control óptimo.

3 SHE as a dynamical system

As we anticipated in Section 1, the main contribution of the present paper is to provide a novel and alternative approach to the SHE problem, based on optimal control. As we shall see, this methodology will allow us avoiding the choice of the waveform, as the optimization process selects the most convenient one in each case.

To this end, the starting point is to rewrite the expression of the Fourier coefficients (2.3) as the evolution of a dynamical system. This can be easily done by means of the fundamental theorem of differential calculus as follows: for all $i, j \in \mathbb{N}$, let α_i and β_j be the solutions of the following Cauchy problems

$$\begin{cases} \dot{\alpha}_i(\tau) = \dfrac{2}{\pi} u(\tau) \cos(i\tau), & \tau \in [0, \pi) \\ \alpha_i(0) = 0 \end{cases}$$
$$\begin{cases} \dot{\beta}_j(\tau) = \dfrac{2}{\pi} u(\tau) \sin(j\tau), & \tau \in [0, \pi) \\ \beta_j(0) = 0 \end{cases} \tag{3.1}$$

Then

$$\alpha_i(\tau) = \frac{2}{\pi} \int_0^\tau u(\zeta) \cos(i\zeta)\, d\zeta$$
$$\beta_j(\tau) = \frac{2}{\pi} \int_0^\tau u(\zeta) \sin(j\zeta)\, d\zeta$$

and the Fourier coefficients (2.3) are given by $a_i = \alpha_i(\pi)$ and $b_j = \beta_j(\pi)$.

Let now

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \ldots, e_a^{N_a}\}, \quad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \ldots, e_b^{N_b}\}$$

be two sets of odd numbers, and denote

$$\boldsymbol{\alpha} = \{\alpha_i\}_{i \in \mathcal{E}_a}, \quad \boldsymbol{\beta} = \{\beta_j\}_{j \in \mathcal{E}_b}.$$

Then, for any $\tau \in [0, \pi)$, we can define the vectors

$$\boldsymbol{\mathcal{D}}^{\alpha}(\tau) = \begin{bmatrix} \cos(e_a^1 \tau) \\ \cos(e_a^2 \tau) \\ \vdots \\ \cos(e_a^{N_a} \tau) \end{bmatrix}, \quad \boldsymbol{\mathcal{D}}^{\beta}(\tau) = \begin{bmatrix} \sin(e_b^1 \tau) \\ \sin(e_b^2 \tau) \\ \vdots \\ \sin(e_b^{N_b} \tau) \end{bmatrix}$$

with $\boldsymbol{\mathcal{D}}^{\beta}(\tau) \in \mathbb{R}^{N_a}$ and $\boldsymbol{\mathcal{D}}^{\beta}(\tau) \in \mathbb{R}^{N_b}$, and the dynamical systems (3.1) can be rewritten in a vectorial form as:

$$\begin{cases} \dot{\boldsymbol{\alpha}}(\tau) = \dfrac{2}{\pi} \boldsymbol{\mathcal{D}}^{\alpha}(\tau) u(\tau), \quad \tau \in [0, \pi) \\ \boldsymbol{\alpha}(0) = 0 \end{cases}$$

$$\begin{cases} \dot{\boldsymbol{\beta}}(\tau) = \dfrac{2}{\pi} \boldsymbol{\mathcal{D}}^{\beta}(\tau) u(\tau), \quad \tau \in [0, \pi) \\ \boldsymbol{\beta}(0) = 0 \end{cases} \tag{3.2}$$

Finally, compressing the notation even more, we can now denote

$$\boldsymbol{x}(\tau) = \begin{bmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{bmatrix}, \quad \boldsymbol{\mathcal{D}}(\tau) = \begin{bmatrix} \boldsymbol{\mathcal{D}}^{\alpha}(\tau) \\ \boldsymbol{\mathcal{D}}^{\beta}(\tau) \end{bmatrix}$$

so that (3.2) becomes

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = \dfrac{2}{\pi} \boldsymbol{\mathcal{D}}(\tau) u(\tau), \quad \tau \in [0, \pi) \\ \boldsymbol{x}(0) = 0 \end{cases} \tag{3.3}$$

and the target coefficients of the SHE problem will be given by $\boldsymbol{x}_0 := [\boldsymbol{a}_T, \boldsymbol{b}_T]^{\top} = \boldsymbol{x}(\pi)$.

Taking into account this new formulation, as we shall see in more detail in Section 4, Problem 2.1 can now be recast as a control one for the dynamical systems (3.3), in which we look for a control function $u(\tau)$ steering the state $\boldsymbol{x}(\tau)$ from the origin to the target $\boldsymbol{x}_0 := [\boldsymbol{a}_T, \boldsymbol{b}_T]^{\top}$ in time $\tau = \pi$.

Moreover, since most often control problems are designed to drive the state of a given dynamical system to an equilibrium configuration, for instance the zero state, we introduce the change of variables $\boldsymbol{x}(\tau) \mapsto \boldsymbol{x}_T - \boldsymbol{x}(\tau)$ which allows us to reverse the time in (3.3), thus obtaining

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = -\dfrac{2}{\pi} \boldsymbol{\mathcal{D}}(\tau) u(\tau), \quad \tau \in [0, \pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases}, \tag{3.4}$$

In this new configuration, the control function u is now required to steer the solution of (3.4) from the initial datum $\boldsymbol{x}_T$ to zero in time $\tau = \pi$.

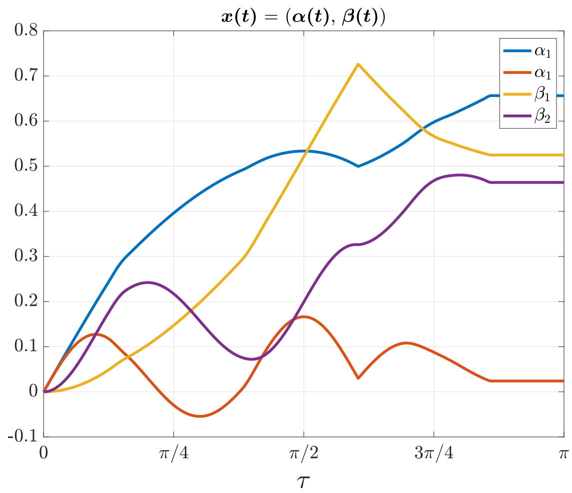

Fig. 2. Evolución del sistema dinámico con los conjuntos $\mathcal{E}_a = \{1, 2\}$ y $\mathcal{E}_b = \{1, 2\}$ considerando el control $u(\tau)$ presentado en la figura 1

We can then formulate the following control problem, which is equivalent to Problem 2.1:

Problem 3.1 *Let $\mathcal{U}$ be defined as in (2.1) and let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two sets of odd numbers with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$, respectively. Given the vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, let us define $\boldsymbol{x}_T = [\boldsymbol{a}_T, \boldsymbol{b}_T]^{\top} \in \mathbb{R}^{N_a \times N_b}$. We look for $u :\in [0, \pi) \to \mathcal{U}$ such that the solution of (3.4) with initial datum $\boldsymbol{x}(0) = \boldsymbol{x}_T$ satisfies $\boldsymbol{x}(\pi) = 0$.*

Remark 3.1 (Quarter-wave symmetry) *We shall mention that, in the SHE literature, it is usual to distinguish among the half-wave symmetry problem (addressed in the present paper) and the quarter-wave symmetry one in which*

$$u\left(\tau + \frac{\pi}{2}\right) = -u(\tau) \quad \textit{for all } \tau \in \left[0, \frac{\pi}{2}\right).$$

In quarter-wave symmetry, the SHE problem simplifies, as the Fourier coefficients $\{a_i\}_{i \in \mathcal{E}_a}$ turn out to be all zero. Hence, only the phases of the converter's signal can be controlled, while the half-wave SHE allows to deal with the amplitudes as well. It is worth to remark nonetheless that our optimal control formulation can be easily adapted to the quarter-wave symmetry problem by simply replacing the Fourier coefficients (2.3) with

$$a_i = 0, \qquad b_j = \frac{4}{\pi} \int_0^{\frac{\pi}{4}} u(\tau) \sin(j\tau) \, d\tau.$$

4 Optimal control for SHE

As we anticipated in Section 3, the SHE problem is equivalent to controlling a dynamical system associated with the Fourier coefficients (2.3). In this section, we present a rigorous formulation of this mentioned control problem and we analyze some relevant properties.

In what follows, for a given vector $\boldsymbol{v} \in \mathbb{R}^d$, we shall always denote by $\|\boldsymbol{v}\|$ the euclidean norm $\|\boldsymbol{v}\|_{\mathbb{R}^d}$.

Problem 4.1 (OCP for SHE) *Let $\mathcal{U}$ be defined as in (2.1). Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinality N_a and N_b, respectively, and given the target $\boldsymbol{x}_T \in \mathbb{R}^{N_a+N_b}$, we look for the function $u(\tau) : [0, \pi) \to \mathcal{U}$ solution of the optimal control problem*

$$\min_{u\in\mathcal{U}} \frac{1}{2}\|\boldsymbol{x}(\pi)\|^2$$

$$\textit{subject to:} \quad \begin{cases} \dot{\boldsymbol{x}}(\tau) = -\dfrac{2}{\pi}\boldsymbol{\mathcal{D}}(\tau)u(\tau), & \tau \in [0,\pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases}$$

The solution of Problem 4.1 may be quite complex to be obtained, due to the restriction on the admissible control values.

Definition 4.1 (Digital control of set $\boldsymbol{\mathcal{U}}$) *A control $u(\tau)$ is called digital if, for each time $\tau \geq 0$, it only takes values in the finite set of real number $\mathcal{U}$.*

¿Igual el nombre de la definicion debe depender de $\mathcal{U}$?, Es decir, ‘Digital control of set $\mathcal{U}$’

Falta caracterizar los instantes de instantes de tiempo donde $u(\tau) \notin \mathcal{U}$. Estos pueden existir pero el conjunto de estos puntos debe ser de medida nula

In order to bypass this difficulty, following a standard optimal control approach, we can formulate an equivalent minimization problem in which, instead of looking for $u \in \mathcal{U}$, we simply require that $|u| < 1$ and we introduce a penalization term to ensure that u is a piece-wise constant function (digital control). This alternative optimal control problem, which can be solved more easily by employing standard tools, reads as follows:

Problem 4.2 (Penalized OCP for SHE) *Fix $\epsilon > 0$. Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ and the target $\boldsymbol{x}_T \in \mathbb{R}^{N_a+N_b}$, we look for the function $u : [0, \pi) \to \mathcal{U}$ as the solution of:*

$$\min_{|u|<1} \left[\Psi(\boldsymbol{x}) + \epsilon \int_0^\pi \mathcal{L}(u(\tau))d\tau \right]$$

$$\Psi(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}(\pi)\|^2,$$

under the dynamics given by (3.4).

In Problem 4.2, the penalization function $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ will be chosen such that the optimal control u^* only takes values in $\mathcal{U}$. Furthermore, the parameter ϵ should be small so that the solution minimizes the distance from the final state and the target.

Next we will study the optimality conditions of the problem, for a general $\mathcal{L}$, and then specify how $\mathcal{L}$ should be so that the optimal control u^* only takes the allowed values in $\mathcal{U}$.

4.1 Optimality conditions

To write the optimality conditions of the problem we will use the Pontryagin minimum principle [**?**, Chapter 2.7]. With this purpose, it is necessary to first introduce the Hamiltonian function

$$H(u, \boldsymbol{p}, \tau) = \epsilon\mathcal{L}(u) - \frac{2}{\pi}\big(\boldsymbol{p}(\tau) \cdot \boldsymbol{\mathcal{D}}(\tau)\big)u(\tau),$$

where $\boldsymbol{p}(\tau)$ is the so-called adjoint state, which is associated with the restriction imposed by the system. This vector has the same dimension of the state $\boldsymbol{x}$, so that

$$\boldsymbol{x}(\tau) = \begin{bmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{bmatrix} \Leftrightarrow \boldsymbol{p}(\tau) = \begin{bmatrix} \boldsymbol{p}^\alpha(\tau) \\ \boldsymbol{p}^\beta(\tau) \end{bmatrix}. \tag{4.1}$$

In what follows, we will enumerate the optimality conditions arising from the Pontryagin principle.

1. **Adjoint equation**: the ODE describing the evolution of the adjoint variable is given by

   $$\dot{\boldsymbol{p}}(\tau) = -\nabla_x H(u(\tau), \boldsymbol{p}(\tau), \tau).$$

   In our case, since the Hamiltonian does not depend on the dynamics, we simply have

   $$\dot{\boldsymbol{p}}(\tau) = 0, \tag{4.2}$$

   that is, the adjoint state is constant in time.
2. **Final condition of the adjoint**: As it is well-known, the adjoint equation is defined backward in time, meaning that its initial condition is actually a final one, posed at $\tau = \pi$. This final condition is given by

   $$\boldsymbol{p}(\pi) = \nabla_{\boldsymbol{x}}\Psi(\boldsymbol{x}) = \boldsymbol{x}(\pi) - \boldsymbol{x}_T.$$

   This, together with (4.2), tells us that

   $$\boldsymbol{p}(\tau) = \boldsymbol{x}(\pi) - \boldsymbol{x}_T, \qquad \text{for all } \tau \in [0, \pi).$$

3. **Optimal Waveform**: We known that

   $$u^* = \arg\min_{|u|<1} H(\tau, \boldsymbol{p}^*, u),$$

so that, in this case, we can write

$$u^*(\tau) = \arg\min_{|u|<1} \left[\epsilon\mathcal{L}(u(\tau)) - \frac{2}{\pi}(\boldsymbol{p}^* \cdot \boldsymbol{\mathcal{D}}(\tau))u(\tau) \right]. \tag{4.3}$$

Therefore, this optimality condition reduces to the optimization of a function in a variable within the interval $[-1, 1]$.

Definition 4.2 *Dado el problema 4.2 definimos una función $\mathcal{H}_m : [-1,1] \to \mathbb{R}$ tal que:*

$$\mathcal{H}_m(u) = \epsilon\mathcal{L}(u) - mu \mid \forall m \in \mathbb{R} \tag{4.4}$$

Es importante notar que la función $\mathcal{H}_m$ es el Hamiltoniano del sistema donde hemos remplazado el valor

$$[\boldsymbol{p}^* \cdot \boldsymbol{\mathcal{D}}(\tau)] = \sum_{i\in\mathcal{E}_a} p^*_\alpha \cos(i\tau) + \sum_{j\in\mathcal{E}_b} p^*_\beta \sin(j\tau) \tag{4.5}$$

por el parámetro m. De manera que el Hamiltoniano evaluado en la trayectoria óptima varía de manera continua en todo el intervalo $\tau \in [0, \pi)]$. Esta es la razón por la que el estudio de la función $\mathcal{H}_m$, una función uni-variable parametrizada por m ,tiene implicaciones en el Problema 4.2.

Definition 4.3 *Dado el Problema 4.2 definimos una función $\mathcal{G} : \mathbb{R} \to [-1, 1]$ tal que:*

$$\mathcal{G}(m) = \arg\min_{u\in[-1,1]} \mathcal{H}_m(u) \tag{4.6}$$

Definition 4.4 *Dado el Problema 4.2 definimos el conjunto $\mathcal{M}$ como:*

$$\mathcal{M} = \{m \in \mathbb{R} \mid \mathcal{G}(m) \notin \mathcal{U}\} \tag{4.7}$$

4.2 Caso Binivel (Control Bang-Bang)

El caso binivel es el que tiene como conjunto admisible de controles $\mathcal{U} = \{-1, 1\}$.

Proposition 4.1 *Dado el Problema 4.2 con el conjunto admisible de control $\mathcal{U} = \{-1, 1\}$. Si la función $\mathcal{L}$ es concava en el intervalo $[-1, 1]$ del Problema 4.2, entonces la solución de problema es un control digital del conjunto $\mathcal{U} = \{-1, 1\}$*
Proof: *Si $\mathcal{L}$ es concava en el intervalo entonces $\mathcal{H}_m$ también lo es. De manera que $G(m)$ solo puede tomar los valores $\{-1, 1\}$.*

4.3 Caso Multi-nivel

Para el caso multi nivel deben existir m para los cuales H_m tenga un mínimo dentro del intervalo $[-1, 1]$. Además este mínimo no puede variar de manera continua con un variación de m.

Proposition 4.2 *Si $\mathcal{L}$ es derivable entonces la solución del Problema 4.2 no es un control digital.*
Proof: *Dado que $\mathcal{L}$ es derivable en todo el intervalo $[-1, 1]$ también lo es $\mathcal{H}_m$. De manera que podemos derivar la función $\mathcal{H}_m$.*

$$\frac{d\mathcal{H}_m}{du} = 0 \to \frac{d\mathcal{L}}{du} = m \tag{4.8}$$

Proposition 4.3 *Si la solución del Problema 4.2 es un control digital, entonces $\mathcal{M}$ es el conjunto vacio o un conjunto finito de elementos.*
Proof: *En el caso que $\mathcal{M} = \{\emptyset\}$ por definición de $\mathcal{M}$ la imagen de $\mathcal{G}(m)$ es el conjunto $\mathcal{U}$ de manera que la solución del Problema 4.2 es un control digital de $\mathcal{U}$.*

Fuera del caso anterior, si suponemos que el conjunto $\mathcal{M}$ contiene algún subintervalo $\mathcal{I}_\mathcal{M} \subset [-1, 1]$ podemos tomar $m_1 \in \mathcal{I}_\mathcal{M}$ y $m_2 = m_1 + \varepsilon$ / $\varepsilon << 1$ de modo que se debe cumplir que $m_2 \in \mathcal{I}_\mathcal{M}$

$$\begin{aligned} \mathcal{G}(m_1) &= \arg\min_{u\in[-1,1]} [\mathcal{H}_{m_1}(u)] && \in \mathcal{U} \\ \mathcal{G}(m_2) &= \arg\min_{u\in[-1,1]} [\mathcal{H}_{m_1}(u) + \varepsilon u] && \in \mathcal{U} \end{aligned} \tag{4.9}$$

¿Eso implica que $\mathcal{U}$ tentría algún subintervalo continuo de $[-1, 1]$? ■.

In what follows, we will show that there are several ways of designing a penalization term $\mathcal{L}$ giving us digital controls.

Theorem 4.1 *Assume that the finite set $\mathcal{U}$ defined in (2.1) is composed by elements in ascending order. Let $\mathcal{Y} = \{y_\ell\}_{\ell=1}^L$ be another finite set, with the same cardinality as $\mathcal{U}$, such that the $L-1$ tuple $d\mathcal{Y} = \{y_\ell - y_{\ell+1}\}_{\ell=1}^{L-1}$ is monotone. Let $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ be a piece-wise continuous function en los intervalos $\{[u_l, u_{l+1})\}_{l=1}^L$ de manera que $\{y_l = \mathcal{L}(u_l)\}_{l=1}^L$. Si la funciones definidas entre cada intervalo $\{[u_l, u_{l+1}]\}_{l=1}^L$ son concavas, entonces la penalización $\mathcal{L}$ en el problema 4.2 da lugar a un control que solo toma valores en $\mathcal{U}$.*

Para que el considerar que el control óptimo del problema solo toma valores digitales los minimos 4.4 para todo valor de m de estar contenido en $\mathcal{U}$. A continuación la prueba

Proof: *Teniendo en cuenta que la función*

4.3.1 Piecewise linear penalization

In this section, we discuss how to design the penalization term $\mathcal{L}(u)$ so that the optimal control is always contained in $\mathcal{U}$.

In more detail, we can choose the affine interpolation of a parabola $\mathcal{L}:[-1,1]\to\mathbb{R}$ as a penalization term. That is

$$\mathcal{L}(u)=\begin{cases}[(u_{k+1}+u_k)(u-u_k)+u_k^2] & \text{if } \ u\in[u_k,u_{k+1}[\\ 1 & \text{if } u=u_{N_u}\end{cases} \tag{4.10}$$

$$\forall k\in\{1,\dots,N_u-1\}$$

Nevertheless, to compute the minimum of $\mathcal{H}_m(u)$, we shall take into account that this function is not differentiable and the optimality condition then requires to work with the subdifferential $\partial\mathcal{L}(u)$, which given by

$$\partial\mathcal{L}(u)=\begin{cases}\{u_1+u_2\} & \text{if } \ u=u_1\\ \{u_k+u_{k+1}\} & \text{if } \ u\in]u_k,u_{k+1}[\quad \dagger\\ [u_k+u_{k-1},u_{k+1}+u_k] & \text{if } u=u_k \quad \ddagger\\ \{u_{N_u}+u_{N_u-1}\} & \text{if } u=u_{N_u}\end{cases} \tag{4.11}$$

$$\dagger\ \forall k\in\{1,\dots,N_u-1\} \quad \ddagger\ \forall k\in\{2,\dots,N_u-1\}$$

Hence, we have $\partial H_m=\epsilon\partial\mathcal{L}-m$. This means that, given $m\in\mathbb{R}$, we look for $u\in[-1,1]$ minimizing $\mathcal{H}_m(u)$. It is then necessary to determine whether zero belongs to $\partial\mathcal{H}_m(u)$.

- **Case 1:** $m\le\epsilon(u_1+u_2)$: since m is less than the minimum of all subdifferentials, then zero does not belong to any of the intervals we defined. Hence, the minimum is in one of the extrema

$$\arg\min_{|u|<1}\mathcal{H}_m(u)=u_1 \tag{4.12}$$

- **Case 2:** $m=\epsilon(u_{k+1}+u_k)$: taking into account that $\forall k\in\{1,\dots,N_u-1\}$,

$$\arg\min_{|u|<1}\mathcal{H}_m(u)=[u_k,u_{k+1}[\tag{4.13}$$

- **Case 3:** $\epsilon(u_k+u_{k-1})<m<\epsilon(u_{k+1}+u_k)$: taking into account that $\forall k\in\{2,\dots,N_u-1\}$,

$$\arg\min_{|u|<1}\mathcal{H}_m(u)=u_k \tag{4.14}$$

- **Case 4:** $m>\epsilon(u_{N_u}+u_{N_u-1})$:

$$\arg\min_{|u|<1}\mathcal{H}_m(u)=u_{N_u} \tag{4.15}$$

In other words, only when $m=\epsilon(u_{k+1}+u_k)$ the minima of the Hamiltonian belong to an interval. For all the other values of $m\in\mathbb{R}$, these minima are contained in $\mathcal{U}$. So that under a continuous variation of m, Case 2 can only occur pointwise. Recalling the optimal control problem $m(\tau)=[\boldsymbol{p}(\tau)\cdot\boldsymbol{\mathcal{D}}(\tau)]$, we can notice that Case 2 corresponds to the instants τ of change of value.

5 Numerical simulations

In this section, we will present several examples in which we solve our optimal control problem through the direct method and the non-linear constrained optimization tool `CasADi` [?].

5.1 Smooth approximation of piece-wise linear penalization

With the final aim of using an optimization software to solve our optimal control problem, we will approximate our piece-wise linear penalization with the help of the Heaviside function $h:\mathbb{R}\to\mathbb{R}$ and its smooth approximation defined as follows:

$$h(x)=\begin{cases}1 & \text{if } x\ge 0\\ 0 & \text{if } x<0\end{cases} \qquad \begin{cases}h^\eta(x)=(1+\tanh(\eta x))/2\\ \eta\to\infty\end{cases} \tag{5.1}$$

Using h, we can define the (smooth) function $\Pi^\eta_{a,b}:\mathbb{R}\to\mathbb{R}$ as:

$$\begin{aligned}\Pi^\eta_{[a,b]}(x)&=-1+h^\eta(x-a)+h^\eta(-x+b)\\ &=\frac{\tanh[\eta(x-a)]+\tanh[\eta(b-x)]}{2}.\end{aligned}$$

In this way, we can define the smooth version of (4.10):

$$\mathcal{L}^\eta(u)=\sum_{k=1}^{N_u-1}\left[(u_{k+1}+u_k)(u-u_k)+u_k^2\right]\Pi^\eta_{[u_k,u_{k+1}]}(u) \tag{5.2}$$

So that, when $\eta\to\infty$, then $\mathcal{L}^\eta\to\mathcal{L}$.

5.2 Direct method for OCP-SHE

To solve the optimal control problem (4.2), we use a direct method. If we consider a partition $\mathcal{P}=\{\tau_0,\tau_1,\dots,\tau_T\}$ of interval $[0,T]$, we can represent a function $\{u(\tau)\mid\tau\in[0,T]\}$ as a vector $\boldsymbol{u}\in\mathbb{R}^T$ where component $u_t=u(\tau_t)$. Then the optimal control problem (4.1) can be written as optimization problem with variable $\boldsymbol{u}\in\mathbb{R}^T$. This problem is a nonlinear programming, for this we use CasADi software to solve. Hence, given a partition of the interval $[0,\pi)$, we can formulate the problem 4.2 as the following one in discrete time

Problem 5.1 (Numerical OCP) *Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively, given the target vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$, so that $\boldsymbol{x}_0 = [\boldsymbol{a}_T, \boldsymbol{b}_T]^T$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$ and a partition $\mathcal{P}_\tau = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of the interval $[0, \pi)$, we search a vector $\boldsymbol{u} \in \mathbb{R}^T$ that minimizes the following function:*

$$\min_{\boldsymbol{u}\in\mathbb{R}^T} \left[||\boldsymbol{x}^T||^2 + \epsilon \sum_{t=0}^{T-1} \mathcal{L}^\eta(u_t)\Delta\tau_t \right] \tag{5.3}$$

suject to:

$$\forall \tau \in \mathcal{P} \begin{cases} \boldsymbol{x}^{t+1} = \boldsymbol{x}^t - (2/\pi)\Delta\tau_t \boldsymbol{\mathcal{D}}(\tau_t)u_t \\ \boldsymbol{x}^0 = \boldsymbol{x}_0 \end{cases} \tag{5.4}$$

5.3 Examples

Todos los ejemplos que presentaremos a continuación tendrán en conmún los siguientes parámetros $\epsilon = 10^{-5}$, $\eta = 10^{-5}$ y una partición $\mathcal{P}_t = \{0.0, 0.1, 0.2, \ldots, \pi\}$.

5.3.1 Bang-Bang

Consideremos el Problema 5.1 con los siguientes parámetros $\mathcal{E}_a = \{1, 5, 7\}$ y $\mathcal{E}_b = \{1, 5, 7\}$ y el control admisible en $\mathcal{U} = \{-1, 1\}$. Además los vectores objetivos son: $\boldsymbol{a}_T = (i_d, 0)^T$ y $\boldsymbol{b}_T = (i_d, 0)^T$ para todo $i_d \in [-1, 1]$.

5.3.2 Bang-off-Bang

Consideremos el Problema 5.1 con los siguientes parámetros $\mathcal{E}_a = \{1, 5\}$ y $\mathcal{E}_b = \{1, 5\}$ y el control admisible en $\mathcal{U} = \{-1, 0, 1\}$. Además los vectores objetivos son: $\boldsymbol{a}_T = (1/2, 0)^T$ y $\boldsymbol{b}_T = (1/2, 0)^T$ para todo $i_d \in [-1, 1]$.

5.3.3 Multi-nivel

Consideremos el Problema 5.1 con los siguientes parámetros $\mathcal{E}_a = \{1, 5\}$ y $\mathcal{E}_b = \{1, 5\}$ y el control admisible en $\mathcal{U} = \{-1, -1/2, 0, 1/2, 1\}$. Además los vectores objetivos son: $\boldsymbol{a}_T = (1/2, 0, 0)^T$ y $\boldsymbol{b}_T = (1/2, 0, 0)^T$ para todo $i_d \in [-1, 1]$.

6 Conclusiones

We presented the SHE problem from the point of view of control theory. This methodology allows obtaining a 10^{-4}–10^{-5} precision in the distance to the target vector. Nevertheless, comparing with methodologies where the commutation number is fixed a priori, our approximation is computationally more expensive. Notwithstanding that, the optimal control provides solutions in the entire range of the modulation index, although the number of solutions or their locations change dramatically.

This methodology for the SHE problem connects control theory with harmonic elimination. In this way, the SHE problem can be solved through classical tools.

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