Piece-wise penalization in Optimal Control to Selective Harmonic Elimination

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Abstract

El problema de *Selective Harmonic Elimination pulse-width modulation*(SHE) es planteado como el problema de control óptimo, con el fin de encontrar soluciones de ondas escalón sin prefijar el número de ángulos de conmutación. De esta manera, la metodología de control óptimo es capaz de encontar la forma de onda óptima y de encontrar la localizaciones de los ángulos de conmutación, incluso sin prefijar el número de conmutaciones. Este es un nuevo enfoque para el problema SHE en concreto y para los sistemas de control con un conjunto finito de controles admisibles en general.

Key words: Selective Harmonic Elimination; Finite Set Control, Piecewise Linear function.

1 Introduction and motivations

Selective Harmonic Elimination (SHE) [Rodríguez et al., 2002] is a well-known methodology in electrical engineering, employed to improve the performances of a converter by controlling the phase and amplitude of the harmonics in its output voltage. As a matter of fact, this technique allows to increase the power of the converter and, at the same time, to reduce its losses.

Because of the growing complexity of modern electrical networks, consequence for instance of the high penetration of renewable energy sources, the demand in power of electronic converters is day by day increasing. For this and other reasons, SHE has been a preeminent research interest in the electrical engineering community, and a plethora of SHE-based techniques has been developed in recent years. An incomplete bibliography includes [Duranay and Guldemir, 2017, Janabi et al., 2020, Yang et al., 2017].

In broad terms, the process consists in generating a *control signal* with a desired harmonic spectrum, by modulating or eliminating some specific lower order frequencies. This signal is piece-wise constant function and in this way is fully characterized by two features (see Figure 1):

1. The *waveform*, i.e. A secuence of values that the signal will take.
2. The *switching angles*, defining the points in the domain where the function changes from one constant value to another.

These two main features can be exploited to formulate a minimization problem. Let a waveform, we can use this information to reduce the formula of Fourier coefficients and then define a cost function that depends on a locations of switching angles. However, this formulation have a some problems, asociated to assume a concrete waveform (Section 3).

In this document is show a new formulation of SHE problem, in terms of control theory. In this way, we search a signal without assume a concrete waveform and expect that the own control problem be able to find a optimal waveform and the optimal locations of switching angles. In the following sections we will explain how the SHE problem can be see as control problem, however that optimal control have to be a piece-wise constant function is problematic and atypically in control problem. The clas-

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sical control theory give a strategies to design problem whose only can take a two posible values. This type of optimal control are named *bang-bang* controls and they are well known in community of control thoery. Nevertheless, if we want obtain a optimal control as piece-wise constant function. we don't have a clear strategies to this objective. In this paper, we tackle this problem with goal solve the SHE problem, however this idea can be generalizated for other systems.

In this document, we follow the next structure. In the Section 2, we introduce a mathematical formulation of a general SHE problem. In the Section 3, we show the clasical formulation in SHE literature, and we show two main problems because of this aproach. In the Section 4, we show our contributions in this problem, where expalin the SHE problem as control problem and the strategies to obtain a piece-wise function as optimal control. In the Section 5 we show a concrete examples solve by control theory. In the Section 6, we summaries the proof associated to our contribution. By last, we finished with some conclusion and open problems.

2 Mathematical formulation of SHE

This section is devoted to the mathematical formulation of the SHE problem. In what follows, with the notation $\mathcal{U}$ we will always refer to a finite set of real numbers, contained in the interval $[-1, 1]$

$$\mathcal{U} = \{u_\ell\}_{\ell=1}^{L} \subset [-1, 1], \tag{2.1}$$

with cardinality $|\mathcal{U}| = L$.

In broad terms, our objective is to design a piece-wise constant function $u(\tau) : [0, 2\pi) \to \mathcal{U}$ such that some of its lower-order Fourier coefficients take specific values determined a priori. Due to the application in power converters, we will focus here on functions with *half-wave symmetry*, i.e.

$$u(\tau + \pi) = -u(\tau) \quad \text{for all } \tau \in [0, \pi).$$

Because of this half-wave symmetry, in what follows, we will always work with the restriction $u|_{[0,\pi)}$ which, with some abuse of notation, we shall still denote by u.

In this way, u is fully determined by its values in the interval $[0, \pi)$ and its Fourier series only involves the odd terms, thus taking the form

$$u(\tau) = \sum_{\substack{i \in \mathbb{N} \\ i \text{ odd}}} a_i \cos(i\tau) + \sum_{\substack{j \in \mathbb{N} \\ j \text{ odd}}} b_j \sin(j\tau), \tag{2.2}$$

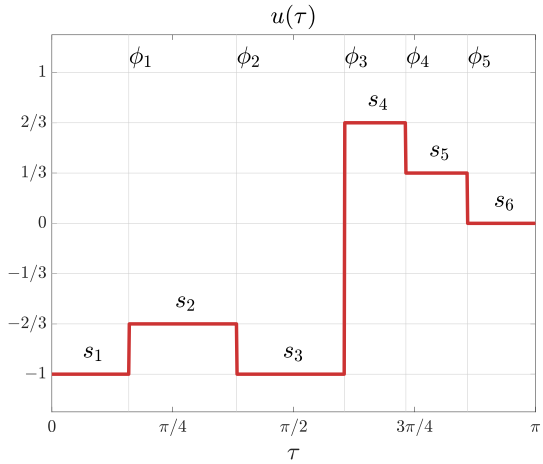

Fig. 1. A Step function and possible solution of Problem 2.1, where we considered a possible finite set of control $\mathcal{U} = \{-1/3, -2/3, 0, 1/3, 2/3, 1\}$. We show the switching angles $\boldsymbol{\phi}$ and the waveform $\mathcal{S}$ (see Definitions 3.1 and 3.2).

where the coefficients a_i and b_j are given by

$$\begin{aligned} a_i &= \frac{2}{\pi} \int_0^\pi u(\tau) \cos(i\tau)\, d\tau, \\ b_j &= \frac{2}{\pi} \int_0^\pi u(\tau) \sin(j\tau)\, d\tau. \end{aligned} \tag{2.3}$$

So considering this preliminaries, we can then give a general formulation of the SHE problem as follows:

Problem 2.1 (SHE) *Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two sets of odd numbers with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$, respectively, and $\mathcal{U}$ be defined as in (2.1). Given the vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, we look for a piece-wise constant function $u : [0, \pi) \to \mathcal{U}$ such that*

$$\begin{aligned} a_i &= (\boldsymbol{a}_T)_i, \quad && \text{for all } i \in \mathcal{E}_a, \\ b_j &= (\boldsymbol{b}_T)_j, \quad && \text{for all } j \in \mathcal{E}_b, \end{aligned}$$

with $\{a_i\}_{i \in \mathcal{E}_a}$ and $\{b_j\}_{j \in \mathcal{E}_b}$ given by (2.3).

Fig. 1 shows an example of a function u solution of the SHE problem.

3 Classical approach to the SHE problem

As we anticipated in Section 1, the control signal u is fully characterized by its waveform and the switching angles, to which we now give a precise definition.

Definition 3.1 (Wave-form) *Given the finite set $\mathcal{U}$ defined in (2.1) and $M \in \mathbb{N}$, we will call a waveform any possible $(M+1)$-tuple $\mathcal{S} = (s_m)_{m=1}^{M+1}$ with $s_m \in \mathcal{U}$ for all $m = 1, \ldots, M+1$.*

Definition 3.2 (Switching angles) *Given the finite set $\mathcal{U}$ defined in (2.1), $M \in \mathbb{N}$ and a piece-wise constant function $u : [0, \pi) \to \mathcal{U}$, we shall refer as switching angles $\boldsymbol{\phi} = \{\phi_m\}_{m=0}^{M+1} \subset [0, \pi]$, with $\phi_0 = 0$ and $\phi_{M+1} = \pi$, to the points in the domain $[0, \pi)$ where u changes its value.*

The idea of classical aproach of SHE problem if exploit these feature to reduce the complexity of the problem. In view of the above definitions, we can provide the following explicit expression for u:

$$u = \sum_{m=1}^{M+1} s_m \chi_{[\phi_m, \phi_{m+1}]} \tag{3.1}$$
$$s_m \in \mathcal{S},\ \phi_m \in \boldsymbol{\phi}, \quad \text{for all } m = 1, \ldots, M+1,$$

where we denoted by $\chi_{[\phi_m, \phi_{m+1}]}$ the characteristic function of the interval $[\phi_m, \phi_{m+1}]$.

Besides, taking into account (3.1), a direct computation yields that the Fourier coefficients (2.3) are given by

$$a_i = a_i(\boldsymbol{\phi}) = \frac{2}{i\pi} \sum_{k=1}^{M+1} s_k \Big[\sin(i\phi_k) - \sin(i\phi_{k-1}) \Big]$$
$$b_j = b_j(\boldsymbol{\phi}) = \frac{2}{j\pi} \sum_{k=1}^{M+1} s_k \Big[\cos(j\phi_{k-1}) - \cos(j\phi_k) \Big]$$

Given a waveform $\mathcal{S}$, Problem 2.1 then reduces to find the switching locations $\boldsymbol{\phi}$ (see [Yang et al., 2015, Konstantinou and Agelidis, 2010, Sun et al., 1996]). This can be cast as a minimization problem in the variables $\{\phi_m\}_{m=0}^{M+1}$, where the cost functional is the Euclidean distance between the obtained Fourier coefficients $\{a_i(\boldsymbol{\phi}), b_j(\boldsymbol{\phi})\}$ and the targets $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^{N_a} \times \mathbb{R}^{N_b}$.

Problem 3.1 (Optimization for SHE) *Given a waveform $\mathcal{S}$ and a step function u in the form (3.1), we look for the switching angles locations $\boldsymbol{\phi}$ by means of the following minimization problem:*

$$\min_{\boldsymbol{\phi} \in [0,\pi]^{M+1}} \left(\sum_{i \in \mathcal{E}_a} \|a_T^i - a_i(\boldsymbol{\phi})\|^2 + \sum_{j \in \mathcal{E}_b} \|b_T^j - b_j(\boldsymbol{\phi})\|^2 \right)$$

subject to: $0 = \phi_0 < \phi_1 < \ldots < \phi_M < \phi_{M+1} = \pi$

Next we will mention some problems of this formulation:

(1) **Combinatory problem**: Since the cardinality of $\mathcal{S}$ is not known a priori, meaning that we do not know how many switches will be necessary to reach the desired values of the Fourier coefficients, a common approach to solve the SHE problem consists in fixing the number of changes and generating all the possible combinations of elements of $\mathcal{S}$, to later solve an optimization problem for each one of them. Nevertheless, taking into account that the number of possible tuples $\mathcal{S}$ is given by $|\mathcal{U}|^{|\mathcal{S}|}$, it is evident that the complexity of the above approach increases rapidly. This problem has been studied in [Yang et al., 2015] where, through appropriate algebraic transformations, the authors are able to convert the SHE problem into a polynomial system whose solutions' set contains all the possible waveforms for a given set $\mathcal{U}$ and number of elements in the sequence $\mathcal{S}$. However, the number of posibles switches is prefixed. In our case, we extend this idea and signal is totally free.

(2) **Continuity problem**: It is well-known that, fixed a waveform S, the continuity of the switching locations with respect to a continuous variation of the target Fourier coefficients may be quite cumbersome [Yang et al., 2017]. This feature makes the search very dificult, because if you take a some inverval where you can solve the SHE problem, you need to assure you that the solution exist in this interval. In this case, the SHE as control problem can change the waveform under a small variation of the target Fourier coefficients.

4 Our Contributions

We will present the SHE problem as an optimal control one, where the optimization variable is the signal $u(\tau)$ defined in the entire interval $[0, \pi)$. In particular, we will describe how the Fourier coefficients of the function $u(\tau)$ can be seen as the final state of a system controlled by $u(\tau)$, and where the control must be a piece-wise constant function. In order to simplify the reading, we will define this type of controls as:

Definition 4.1 (Digital control of $\mathcal{U}$) *A control $u(\tau)$ is called digital if, for each time $\tau \geq 0$, it only takes values in the finite set of real number $\mathcal{U}$ except a finite set of values (switching angles).*

We briefly mention our contributions below, ad then explain each of them in detail:

(1) Re-formulation of Problem 2.1 as a control problem.
(2) Formulate a optimal control problem with penalization term order to obtain as solution a piece-wise function which must be take only two posible values ($\mathcal{U} \in \{-1, 1\}$)(*bang-bang controls*).
(3) Formulate a optimal control problem with penalization term order to obtain as solution a piece-wise function.
(4) Describe a sufficients conditions of some penalization in optimal control problem to obtain a constant piece-wise function as solution

4.1 Reformulation of SHE problem as optimal control problem

Taking into account that we have all the elements considered in the definition of the Problem 2.1, we introduce the following dynamic system:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = -\dfrac{2}{\pi}\boldsymbol{\mathcal{D}}(\tau)u(\tau), & \tau \in [0,\pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases}, \tag{4.1}$$

Where $\boldsymbol{x}(t) \in \mathbb{R}^{N_a+N_b}$ and $\boldsymbol{\mathcal{D}}(\tau) \in \mathbb{R}^{N_a+N_b}$ is:

$$\boldsymbol{\mathcal{D}}(\tau) = \begin{bmatrix} \boldsymbol{\mathcal{D}}^{\alpha}(\tau) \\ \boldsymbol{\mathcal{D}}^{\beta}(\tau) \end{bmatrix}$$

and where $\boldsymbol{\mathcal{D}}^{\beta}(\tau) \in \mathbb{R}^{N_a}$ and $\boldsymbol{\mathcal{D}}^{\beta}(\tau) \in \mathbb{R}^{N_b}$ are,

$$\boldsymbol{\mathcal{D}}^{\alpha}(\tau) = \begin{bmatrix} \cos(e_a^1\tau) \\ \cos(e_a^2\tau) \\ \vdots \\ \cos(e_a^{N_a}\tau) \end{bmatrix}, \quad \boldsymbol{\mathcal{D}}^{\beta}(\tau) = \begin{bmatrix} \sin(e_b^1\tau) \\ \sin(e_b^2\tau) \\ \vdots \\ \sin(e_b^{N_b}\tau) \end{bmatrix} \tag{4.2}$$

with set $\mathcal{E}_a$ and $\mathcal{E}_b$ are:

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \dots, e_a^{N_a}\}, \quad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \dots, e_b^{N_b}\}$$

Let this system, we can consider the following control problem:

Problem 4.1 *Let $\mathcal{U}$ be defined as in (2.1) and let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two sets of odd numbers with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$, respectively. Given the vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, let us define $\boldsymbol{x}_0 = [\boldsymbol{a}_T, \boldsymbol{b}_T]^\top \in \mathbb{R}^{N_a} \times \mathbb{R}^{N_b}$. We look for $u :\in [0,\pi) \to \mathcal{U}$ such that the solution of (4.1) with initial datum $\boldsymbol{x}(0) = \boldsymbol{x}_0$ satisfies $\boldsymbol{x}(\pi) = 0$.*

And given this control problem, we can formalate the following theorem:

Theorem 4.1 *The optimal control $u(\tau)$, the solution of Problem 4.1, is the piece-wise constante function that solve the Problem 2.1.*

The Theorem 4.1 is a directly consecuence of fundamental theorem of calculus. We have consider the formula integral of Fourier coefficients in its diferential form. In this way, the intragration from $\tau = 0$ to $\tau = \pi$ can be seen as a evolution of a dinamical system (4.1) in the time interval $[0,\pi]$.

Thank to this result, we can formulate a optimal control problem associated to the problem 4.1.

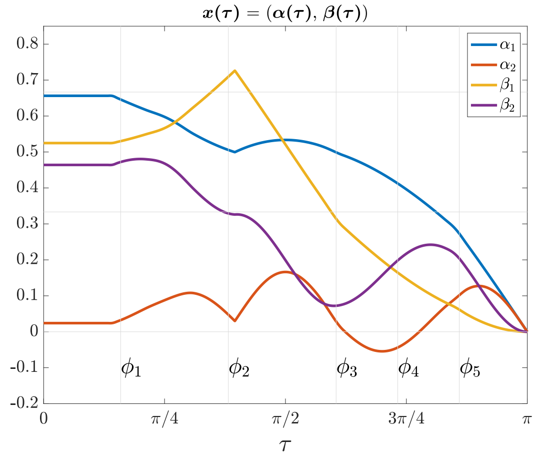

Fig. 2. Evolution of ddinamical system (4.1)$\mathcal{E}_a = \{1,2\}$ y $\mathcal{E}_b = \{1,2\}$ considerando el control $u(\tau)$ presentado en la Figura 1. Además mostramos las posiciones de los ángulos de conmutación $\boldsymbol{\phi}$.

In what follows, for a given vector $\boldsymbol{v} \in \mathbb{R}^d$, we shall always denote by $\|\boldsymbol{v}\|$ the euclidean norm $\|\boldsymbol{v}\|_{\mathbb{R}^d}$.

Problem 4.2 (OCP for SHE) *Let $\mathcal{U}$ be defined as in (2.1). Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinality N_a and N_b, respectively, and given initial condition $\boldsymbol{x}_0 \in \mathbb{R}^{N_a+N_b}$, we look for the function $u(\tau) : [0,\pi) \to \mathcal{U}$ solution of the optimal control problem*

$$\min_{u\in\mathcal{U}} \frac{1}{2}\|\boldsymbol{x}(\pi)\|^2$$

$$\textit{subject to:} \quad \begin{cases} \dot{\boldsymbol{x}}(\tau) = -\dfrac{2}{\pi}\boldsymbol{\mathcal{D}}(\tau)u(\tau), & \tau \in [0,\pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases}$$

The solution of Problem 4.2 may be quite complex to be obtained, due to the restriction on the admissible control values.

In order to bypass this difficulty, following a standard optimal control approach, we can formulate an equivalent minimization problem in which, instead of looking for $u \in \mathcal{U}$, we simply require that $|u| < 1$ and we introduce a penalization term to ensure that u is digital control of $\mathcal{U}$.

This alternative optimal control problem, which can be solved more easily by employing standard tools, reads as follows:

Problem 4.3 (Penalized OCP for SHE) *Fix $\epsilon > 0$. Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ and the target $\boldsymbol{x}_T \in \mathbb{R}^{N_a+N_b}$, we look a digital control of $\mathcal{U}$ as the* *solution of:*

$$\min_{|u|<1} \left[\frac{1}{2}\|\boldsymbol{x}(\pi)\|^2 + \epsilon \int_0^\pi \mathcal{L}(u(\tau))d\tau \right]$$

under the dynamics given by (4.1)*.*

In Problem 4.3, the penalization function $\mathcal{L}:\mathbb{R}\to\mathbb{R}$ will be chosen such that the optimal control u^* is a digital control of $\mathcal{U}$. Furthermore, the parameter ϵ should be small so that the solution minimizes the distance from the final state and the target.

In general, a penalization function that gives us a digital control of $\mathcal{U}$ is not obvious to achieve. If we consider the optimality conditions of the system (4.1) (Pontryagin's minimum principle) we can obtain a function $\mathcal{H}_m(u)$ (the Hamiltonian function) that characterizes the behavior of the solution of the Problem 5.1.

Definition 4.2 (Hamiltonian) *Let a Problem 4.3, we define a function $\mathcal{H}_m : [-1,1] \to \mathbb{R}$ such that:*

$$\mathcal{H}_m(u) = \epsilon\mathcal{L}(u) - mu \mid \forall m \in \mathbb{R} \tag{4.3}$$

Already defined the Hamiltonian function we can say that:

Proposition 4.1 *Let the Problem 4.3 and his Hamiltonian function $\mathcal{H}_m$. If for all $m \in \mathbb{R} - \mathcal{M}$ all minimuns are in $\mathcal{U}$, so the solution of the Problem 4.3 is a digital control of $\mathcal{U}$. Where $\mathcal{M}$ can be the empty set or a finite set of real values.*

Thanks to the Proposition 4.1, we can design the penalization functions such that the Hamiltonian $\mathcal{H}_m$ only have minimuns in $\mathcal{U}$.

4.2 SHE bi-nivel via OCP (Bang-Bang Control)

Now, we consider the case which the set of admisible controls is $\mathcal{U} = \{-1,1\}$. The reader can note that this type of solutions in control theory literature is named *bang-bang control*, while in Selective Harmonic Elimination literature is named bi-level solutions.

Theorem 4.2 *Let the Problem 4.3 with the admisible control set $\mathcal{U} = \{-1,1\}$. If the function $\mathcal{L}$ is concave in the interval $[-1,1]$, so the optimal control is a digital control of $\mathcal{U} = \{-1,1\}$.*

Si aceptamos la Proposición 4.2 es evidente que para obtener un control digital de $\mathcal{U} = \{-1,1\}$ los mímimos Hamiltoniano $\mathcal{H}_m$ para todo $m \in \mathbb{R} - \mathcal{M}$ deben encontrarse en los bordes del intervalo $[-1,1]$ de manera que es suficiente considerar una función concava en el interior de este intervalo.

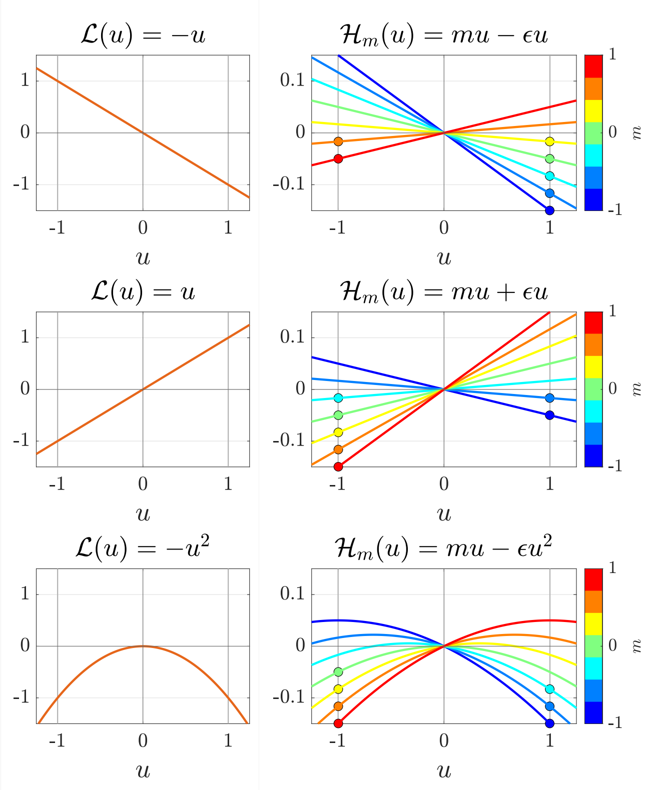

Fig. 3. SHE bi-level.
En la primera columna mostramos tres tipos de penalización concavas compatibles con la Proposición 4.2. A la derecha de cada una de ellas mostramos el comportamiento del Hamiltoniano para cada distintos valores de m (colores).

Ilustramos esta idea en la Figura 3, de manera que en la primera columna presentamos tres funciones de penalización compatible con la Proposición 4.2 ($\mathcal{L}(u) = u$, $\mathcal{L}(u) = -u$, $\mathcal{L}(u) = -u^2$). Además mostramos a la derecha de cada una de ellas mostramos el comportamiento del Hamiltoniano para distintos valores de m en diferentes colores y para cada una de ellas marcamos el mímimo del mismo color. Podemos ver que los mínimos en cada caso siempre siempre esta en $\mathcal{U} = \{-1,1\}$.

4.3 SHE multi-nivel via Piecewise linear penalization

Gracias al caso bi-nivel podemos notar que si definimos una función de penalización $\mathcal{L}$ que entre dos valores u_1 y u_2 no tenga mímimos, es decir sea concava entre esos dos puntos entonces la función Hamiltoniana no tendrá el mímimos en $u \in (u_1,u_2)$ más que para un valor de concreto de m. Gracias a esta observación podemos afirmar que:

Theorem 4.3 *Given a set $\mathcal{U}$, we can choose the affine interpolation of a parabola $\mathcal{L}^{pl} : [-1,1] \to \mathbb{R}$ as a penal-*

ization term. That is

$$\mathcal{L}^{p_1}(u)=\begin{cases}\lambda_k^{p_1}(u) & \text{if } u\in[u_k,u_{k+1}[\\ 1 & \text{if } u=u_{N_u}\end{cases} \quad \forall k\in\{1,\dots,N_u-1\} \tag{4.4}$$

Donde

$$\lambda_k^{p_1}(u)=(u_{k+1}+u_k)(u-u_k)+u_k^2 \tag{4.5}$$

De modo que el problema de 4.3 con la penalización $\mathcal{L}(u)$ presentada antes tiene como solución un control digital del conjunto $\mathcal{U}$.

Remark 4.1 (*Bang-off-bang*) *En este caso, podemos notar que cuando consideramos un conjunto de controles admisibles como $\mathcal{U}=\{-1,0,1\}$ recuperamos la penalización de norma L^1. Este tipo de penalización añadido a la restricción $|u|<1$ nos da controles* bang-off-bang, *bien conocidos en la literatura de control [Wang and Topputo, 2020]*

De la misma forma que en caso anterior en la Figura 4 ilustramos como la penalización (4.4) logra cumplir las condiciones de la proposición 4.1. Y por tanto esta penalización da lugar a una control óptimo digital de $\mathcal{U}$.

4.4 General conditions for SHE multi-nivel

Como podemos observar el caso anterior no es más que una caso concrerto de condiciones más generales. De manera que podemos afirmar que:

Theorem 4.4 *Assume that the finite set $\mathcal{U}$ defined in (2.1) is composed by elements in ascending order. Let $\mathcal{Y}=\{y_\ell\}_{\ell=1}^L$ be another finite set, with the same cardinality as $\mathcal{U}$, such that the $L-1$ tuple*

$$\frac{\Delta\mathcal{Y}}{\Delta\mathcal{U}}=\left(\frac{y_\ell-y_{\ell+1}}{u_\ell-u_{\ell+1}}\right)_{\ell=1}^{L-1} \tag{4.6}$$

is monotone. Let $\mathcal{L}:\mathbb{R}\to\mathbb{R}$ be a piece-wise continuous function:

$$\mathcal{L}(u)=\begin{cases}\lambda_k(u) & \text{if } u\in[u_k,u_{k+1}[\\ 1 & \text{if } u=u_{N_u}\end{cases} \quad \forall k\in\{1,\dots,N_u-1\} \tag{4.7}$$

tal que $\{y_l=\mathcal{L}(u_l)\}_{l=1}^L$.

Entonces, si las funciones λ_k para todo $k\in\{1,\dots,N_u-1\}$ son funciones concavas, entonces la penalización $\mathcal{L}$ en el problema 4.3 da lugar a un control digital de $\mathcal{U}$.

En la Figura 4 se presenta otras dos funciones compatibles con el Teorema 4.4. Estos son:

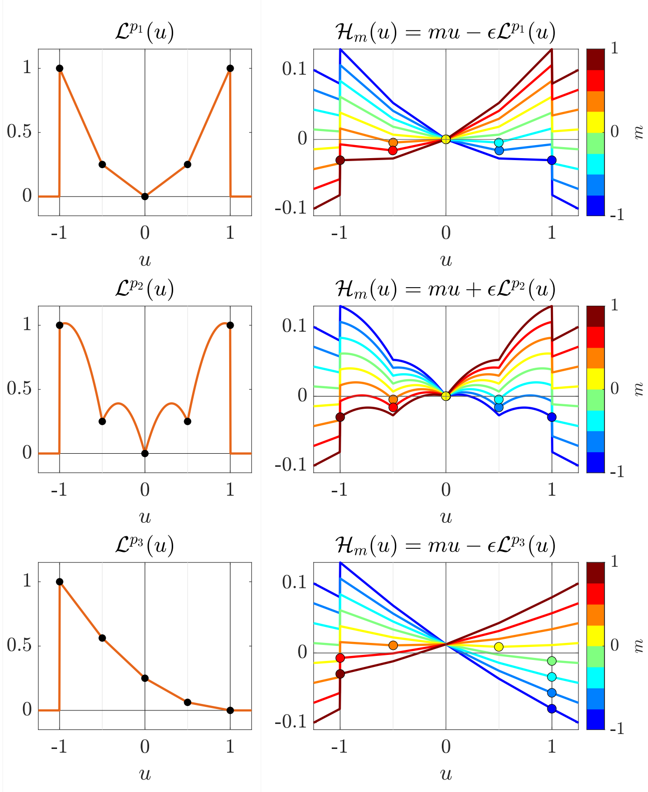

Fig. 4. SHE Multi-nivel.
En la primera columna mostramos tres tipos de penalización concavas compatibles con la Proposición 4.2. A la derecha de cada una de ellas mostramos el comportamiento del Hamiltoniano para cada distintos valores de m (colores).

(1) A union of concave functions:

$$\lambda_k^{p_2}(u)=-4u^2+2(u_k+u_{k+1})-2u_k \tag{4.8}$$

(2) Linear aproximation of shifted parabola

$$\lambda_k^{p_3}(u)=\frac{1}{4}[(u_{k+1}+u_k)(u-u_k-1)+u_k^2] \tag{4.9}$$

Al igual que en la Figura 3, en la Figura 4 se muestra en la primera columna las posibles funciones de penalización. A la derecha de cada una de ellas se muestra el comportamiento del Hamiltoniano asociada a cada una de ellas. Se puede ver como los mimimos para todos los casos, solo toma los valores $\mathcal{U}=\{-1,-1/2,0,1/2,1\}$.

5 Numerical simulations

In this section, we will present several examples in which we solve our optimal control problem through the direct method and the non-linear constrained optimization tool `CasADi` [Andersson et al., 2019]. Además cada una de las simulaciones presentadas se pueden ver en [1]

5.1 Smooth approximation of piece-wise linear penalization

With the final aim of using an optimization software to solve our optimal control problem, we will approximate our piece-wise linear penalization with the help of the Heaviside function $h : \mathbb{R} \to \mathbb{R}$ and its smooth approximation defined as follows:

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \begin{cases} h^\eta(x) = (1 + \tanh(\eta x))/2 \\ \eta \to \infty \end{cases} \tag{5.1}$$

Using h, we can define the (smooth) function $\Pi^\eta_{a,b} : \mathbb{R} \to \mathbb{R}$ as:

$$\begin{aligned}\Pi^\eta_{[a,b]}(x) &= -1 + h^\eta(x-a) + h^\eta(-x+b) \\ &= \frac{\tanh[\eta(x-a)] + \tanh[\eta(b-x)]}{2}.\end{aligned}$$

In this way, we can define the smooth version of (4.4):

$$\mathcal{L}^\eta(u) = \sum_{k=1}^{N_u - 1} \left[(u_{k+1} + u_k)(u - u_k) + u_k^2\right] \Pi^\eta_{[u_k, u_{k+1}]}(u) \tag{5.2}$$

So that, when $\eta \to \infty$, then $\mathcal{L}^\eta \to \mathcal{L}$.

5.2 Direct method for OCP-SHE

To solve the optimal control problem (4.3), we use a direct method. If we consider a partition $\mathcal{P} = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of interval $[0, T]$, we can represent a function $\{u(\tau) \mid \tau \in [0, T]\}$ as a vector $\boldsymbol{u} \in \mathbb{R}^T$ where component $u_t = u(\tau_t)$. Then the optimal control problem (4.2) can be written as optimization problem with variable $\boldsymbol{u} \in \mathbb{R}^T$. This problem is a nonlinear programming, for this we use CasADi software to solve. Hence, given a partition of the interval $[0, \pi)$, we can formulate the problem 4.3 as the following one in discrete time

Problem 5.1 (Numerical OCP) *Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively, given the target vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$, so that $\boldsymbol{x}_0 = [\boldsymbol{a}_T, \boldsymbol{b}_T]^T$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$ and a partition $\mathcal{P}_\tau = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of the interval $[0, \pi)$, we search a vector $\boldsymbol{u} \in \mathbb{R}^T$ that minimizes the following function:*

$$\min_{\boldsymbol{u} \in \mathbb{R}^T} \left[||\boldsymbol{x}^T||^2 + \epsilon \sum_{t=0}^{T-1} \left[\frac{\mathcal{L}^\eta(u_t) + \mathcal{L}^\eta(u_{t+1})}{2} \Delta\tau_t \right] \right] \tag{5.3}$$

suject to:

$$\forall \tau \in \mathcal{P} \begin{cases} \boldsymbol{x}^{t+1} = \boldsymbol{x}^t - (2/\pi)\Delta\tau_t \boldsymbol{\mathcal{D}}(\tau_t) u_t \\ \boldsymbol{x}^0 = \boldsymbol{x}_0 \end{cases} \tag{5.4}$$

5.3 Examples

Todos los ejemplos que presentaremos a continuación tendrán en conmún los siguientes parámetros $\epsilon = 10^{-5}$, $\eta = 10^{-5}$ y una partición $\mathcal{P}_t = \{0.0, 0.1, 0.2, \ldots, \pi\}$. Además consideraremos $\mathcal{E}_a = \{1, 5, 7\}$ y $\mathcal{E}_b = \{1, 5, 7\}$ y los vectores objetivos: $\boldsymbol{a}_T = (i_d, 0, 0)^T$ y $\boldsymbol{b}_T = (i_d, 0, 0)^T$ para todo $i_d \in [-0.8, 0.8]$. Consideraremos a continuación el control *bang-bang*, el control *bang-off-bang* y el control para SHE multi-nivel.

Simulation 5.1 (Bang-Bang) *We consider the Problem 5.1 with a set of admisible controls:*

$$\mathcal{U} = \{-1, 1\} \tag{5.5}$$

Gracias al Teorema 4.2, podemos ver en la Figura 5 como para todos los valores de $i_a \in [-0.8, 0.8]$ el control óptimo solo toma los valores $\{-1, 1\}$ representado en la figura como los colores azul y rojo respectivamente.

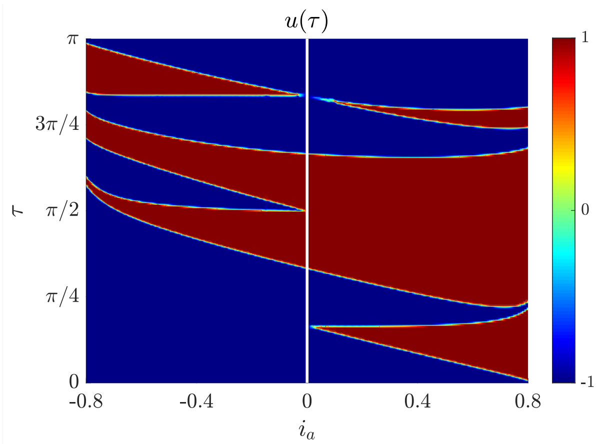

Fig. 5. Results of simulation of Simulation 5.1.

Simulation 5.2 (Bang-off-Bang) *We consider the Problem 5.1 with a set of admisible controls:*

$$\mathcal{U} = \{-1, 0, 1\} \tag{5.6}$$

De la misma forma, cuando consideramos tres valores posibles $\{-1, 0, 1\}$como el conjunto de controles admisibles, recuperamos los controles *bang-off-bang* (Figura 6).

[1] https://github.com/djoroya/SHE-Optimal-Control-paper

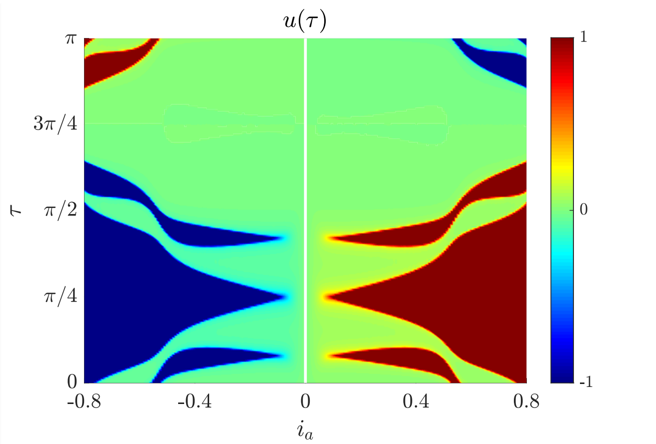

Fig. 6. Results of simulation of Simulation 5.2.

Simulation 5.3 (Multi-level) *We consider the Problem 5.1 with a set of admisible controls:*

$$\mathcal{U} = \{-1, -1/2, 0, 1/2, 1\} \tag{5.7}$$

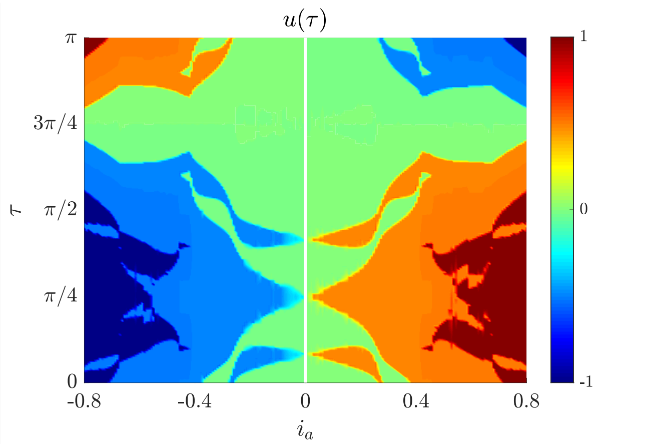

Fig. 7. Results of simulation of Simulation 5.3.

Por último, podemos ver en la Figura 7 el mismo comportamiento cuando consideramos de el conjunto de controles admisibles $\mathcal{U} = \{-1, 1\}$.

This methodology allows obtaining a $10^{-4} - 10^{-5}$ precision in the distance to the target vector

6 Proofs

6.1 Proof of Theorem 4.1 (SHE as dinamical system)

To this end, the starting point is to rewrite the expression of the Fourier coefficients (2.3) as the evolution of a dynamical system. This can be easily done by means of the fundamental theorem of differential calculus as follows: for all $i, j \in \mathbb{N}$, let α_i and β_j be the solutions of the following Cauchy problems

$$\begin{cases} \dot{\alpha}_i(\tau) = \dfrac{2}{\pi} u(\tau)\cos(i\tau), & \tau \in [0, \pi) \\ \alpha_i(0) = 0 \end{cases}$$
$$\begin{cases} \dot{\beta}_j(\tau) = \dfrac{2}{\pi} u(\tau)\sin(j\tau), & \tau \in [0, \pi) \\ \beta_j(0) = 0 \end{cases} \tag{6.1}$$

Then

$$\alpha_i(\tau) = \frac{2}{\pi}\int_0^\tau u(\zeta)\cos(i\zeta)\, d\zeta$$
$$\beta_j(\tau) = \frac{2}{\pi}\int_0^\tau u(\zeta)\sin(j\zeta)\, d\zeta$$

and the Fourier coefficients (2.3) are given by $a_i = \alpha_i(\pi)$ and $b_j = \beta_j(\pi)$.

Let now $\mathcal{E}_a$ and $\mathcal{E}_b$ be two sets of odd numbers, and denote

$$\boldsymbol{\alpha} = \{\alpha_i\}_{i\in\mathcal{E}_a}, \quad \boldsymbol{\beta} = \{\beta_j\}_{j\in\mathcal{E}_b}.$$

The dynamical systems (6.1) can be rewritten in a vectorial form as:

$$\begin{cases} \dot{\boldsymbol{\alpha}}(\tau) = \dfrac{2}{\pi}\boldsymbol{\mathcal{D}}^\alpha(\tau) u(\tau), & \tau \in [0, \pi) \\ \boldsymbol{\alpha}(0) = 0 \end{cases}$$
$$\begin{cases} \dot{\boldsymbol{\beta}}(\tau) = \dfrac{2}{\pi}\boldsymbol{\mathcal{D}}^\beta(\tau) u(\tau), & \tau \in [0, \pi) \\ \boldsymbol{\beta}(0) = 0 \end{cases} \tag{6.2}$$

where we use with $\boldsymbol{\mathcal{D}}^\beta(\tau) \in \mathbb{R}^{N_a}$ and $\boldsymbol{\mathcal{D}}^\beta(\tau) \in \mathbb{R}^{N_b}$ define in (4.2). Finally, compressing the notation even more, we can now denote

$$\boldsymbol{x}(\tau) = \begin{bmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{bmatrix}, \quad \boldsymbol{\mathcal{D}}(\tau) = \begin{bmatrix} \boldsymbol{\mathcal{D}}^\alpha(\tau) \\ \boldsymbol{\mathcal{D}}^\beta(\tau) \end{bmatrix}$$

so that (6.2) becomes

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = \dfrac{2}{\pi}\boldsymbol{\mathcal{D}}(\tau) u(\tau), & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = 0 \end{cases} \tag{6.3}$$

and the target coefficients of the SHE problem will be given by $\boldsymbol{x}_0 := [\boldsymbol{a}_T, \boldsymbol{b}_T]^\top = \boldsymbol{x}(\pi)$.

Moreover, since most often control problems are designed to drive the state of a given dynamical system to an equilibrium configuration, for instance the zero state, we introduce the change of variables $\boldsymbol{x}(\tau) \mapsto \boldsymbol{x}_0 - \boldsymbol{x}(\tau)$ which allows us to reverse the time in (6.3), thus obtaining the system show in (4.1). In this new configuration, the control function u is now required to steer the solution of (4.1) from the initial datum $\boldsymbol{x}_0$ to zero in time $\tau = \pi$.

6.2 Proof of Propsition 4.1 (Hamiltonian Function)

Para demostrar porque la función Hamiltoniano caracteriza el problema de control, es necesario analizar las condiciones de optimialidad. Utilizaremos el principio del mímimo de Pontryagin [Bryson, 1975, Chapter 2.7].

6.2.1 Optimality conditions

First, we introduce the Hamiltonian function

$$\mathcal{H}(u, \boldsymbol{p}, \tau) = \epsilon \mathcal{L}(u) - \frac{2}{\pi}\big(\boldsymbol{p}(\tau) \cdot \boldsymbol{\mathcal{D}}(\tau)\big)u(\tau),$$

where $\boldsymbol{p}(\tau)$ is the so-called adjoint state, which is associated with the restriction imposed by the system (4.1). This vector has the same dimension of the state $\boldsymbol{x}$, so that

$$\boldsymbol{x}(\tau) = \begin{bmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{bmatrix} \Leftrightarrow \boldsymbol{p}(\tau) = \begin{bmatrix} \boldsymbol{p}^{\alpha}(\tau) \\ \boldsymbol{p}^{\beta}(\tau) \end{bmatrix}. \tag{6.4}$$

In what follows, we will enumerate the optimality conditions arising from the Pontryagin principle.

1. **Adjoint system**: the ODE describing the evolution of the adjoint variable is given by

   $$\dot{\boldsymbol{p}}(\tau) = -\nabla_x \mathcal{H}(u(\tau), \boldsymbol{p}(\tau), \tau).$$

   In our case, since the Hamiltonian does not depend on the dynamics, we simply have

   $$\dot{\boldsymbol{p}}(\tau) = 0, \tag{6.5}$$

   that is, the adjoint state is constant in time.
2. **Final condition of the adjoint system**: As it is well-known, the adjoint equation is defined backward in time, meaning that its initial condition is actually a final one, posed at $\tau = \pi$. This final condition is given by

   $$\boldsymbol{p}(\pi) = |\nabla_{\boldsymbol{x}} \Psi(\boldsymbol{x})|_{\boldsymbol{x}=\pi}$$

   Where final cost $\Psi(\boldsymbol{x})$ in our case is $\Psi(\boldsymbol{x}) = \frac{1}{2}||\boldsymbol{x}||^2$, so we can obtain:

   $$\boldsymbol{p}(\pi) = \boldsymbol{x}(\pi) \tag{6.6}$$

   This, together with (6.5), tells us that

   $$\boldsymbol{p}(\tau) = \boldsymbol{x}(\pi), \qquad \text{for all } \tau \in [0, \pi).$$
3. **Optimal Control**: We known that

   $$u^* = \arg\min_{|u|<1} H(\tau, \boldsymbol{p}^*, u),$$

   so that, in this case, we can write

   $$u^*(\tau) = \arg\min_{|u|<1} \left[\epsilon \mathcal{L}(u(\tau)) - \frac{2}{\pi}\big(\boldsymbol{p}^* \cdot \boldsymbol{\mathcal{D}}(\tau)\big)u(\tau)\right]. \tag{6.7}$$

Therefore, this optimality condition reduces to the optimization of a function in a variable within the interval $[-1, 1]$. It important note that the function $\mathcal{H}_m$ (Definition 4.2) is the Hamiltonian of system where we have replaced:

$$[\boldsymbol{p}^* \cdot \boldsymbol{\mathcal{D}}(\tau)] = \sum_{i \in \mathcal{E}_a} p_\alpha^* \cos(i\tau) + \sum_{j \in \mathcal{E}_b} p_\beta^* \sin(j\tau) \tag{6.8}$$

for parameter m. De manera que el Hamiltoniano evaluado en la trayectoria óptima varía de manera continua en todo el intervalo $\tau \in [0, \pi)]$. Es decir, si casi todo $m = [\boldsymbol{p}^* \cdot \boldsymbol{\mathcal{D}}(\tau)]$ los mímimos del Hamiltoniano solo estan en $\mathcal{U}$ entonces el control óptimo solo podrá tomar los valores contenidos en $\mathcal{U}$. El conjunto de valores de $m = [\boldsymbol{p}^* \cdot \boldsymbol{\mathcal{D}}(\tau)]$ tal que u^* no esta en $\mathcal{U}$ deberán ser puntos en $\mathcal{R}$ de manera que eso solo pueda suceder en instantes de tiempo concreto. Estos instantes de tiempos son los tiempos de cambio de valor.

6.3 Proof of Theorem 4.2

El caso binivel es el que tiene como conjunto admisible de controles $\mathcal{U} = \{-1, 1\}$. Si $\mathcal{L}$ es concava en el intervalo entonces $\mathcal{H}_m$ también lo es. De manera que $G(m)$ solo puede tomar los valores $\{-1, 1\}$.

6.4 Proof of Theorem 4.3

To compute the minimum of $\mathcal{H}_m(u)$, we shall take into account that this function is not differentiable and the optimality condition then requires to work with the subdifferential $\partial \mathcal{L}(u)$, which given by

$$\partial \mathcal{L}(u) = \begin{cases} \{u_1 + u_2\} & \text{if } \ u = u_1 \\ \{u_k + u_{k+1}\} & \text{if } \ u \in]u_k, u_{k+1}[\quad \dagger \\ [u_k + u_{k-1}, u_{k+1} + u_k] & \text{if } u = u_k \quad \ddagger \\ \{u_{N_u} + u_{N_u - 1}\} & \text{if } u = u_{N_u} \end{cases} \tag{6.9}$$

$\dagger \ \forall k \in \{1, \ldots, N_u - 1\} \quad \ddagger \ \forall k \in \{2, \ldots, N_u - 1\}$

Hence, we have $\partial H_m = \epsilon\partial\mathcal{L} - m$. This means that, given $m \in \mathbb{R}$, we look for $u \in [-1, 1]$ minimizing $\mathcal{H}_m(u)$. It is then necessary to determine whether zero belongs to $\partial\mathcal{H}_m(u)$.

- **Case 1:** $m \leq \epsilon(u_1 + u_2)$: since m is less than the minimum of all subdifferentials, then zero does not belong to any of the intervals we defined. Hence, the minimum is in one of the extrema

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_1 \tag{6.10}$$

- **Case 2:** $m = \epsilon(u_{k+1} + u_k)$: taking into account that $\forall k \in \{1, \ldots, N_u - 1\}$,

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = [u_k, u_{k+1}[\tag{6.11}$$

- **Case 3:** $\epsilon(u_k + u_{k-1}) < m < \epsilon(u_{k+1} + u_k)$: taking into account that $\forall k \in \{2, \ldots, N_u - 1\}$,

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_k \tag{6.12}$$

- **Case 4:** $m > \epsilon(u_{N_u} + u_{N_u - 1})$:

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_{N_u} \tag{6.13}$$

In other words, only when $m = \epsilon(u_{k+1} + u_k)$ the minima of the Hamiltonian belong to an interval. For all the other values of $m \in \mathbb{R}$, these minima are contained in $\mathcal{U}$. So that under a continuous variation of m, Case 2 can only occur pointwise. Recalling the optimal control problem $m(\tau) = [\boldsymbol{p}(\tau) \cdot \boldsymbol{\mathcal{D}}(\tau)]$, we can notice that Case 2 corresponds to the instants τ of change of value.

6.5 Proof of Theorem 4.4

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7 Conclusiones

We presented the SHE problem from the point of view of control theory. Nevertheless, comparing with methodologies where the commutation number is fixed a priori, our approximation is computationally more expensive. Notwithstanding that, the optimal control provides solutions in the entire range of the modulation index, although the number of solutions or their locations change dramatically.

This methodology for the SHE problem connects control theory with harmonic elimination. In this way, the SHE problem can be solved through classical tools.

7.1 Quarter wave symetry

We shall mention that, in the SHE literature [Wu, 2009], it is usual to distinguish among the half-wave symmetry problem (addressed in the present paper) and the quarter-wave symmetry one in which

$$u\left(\tau + \frac{\pi}{2}\right) = -u(\tau) \quad \text{for all } \tau \in \left[0, \frac{\pi}{2}\right).$$

In quarter-wave symmetry, the SHE problem simplifies, as the Fourier coefficients $\{a_i\}_{i \in \mathcal{E}_a}$ turn out to be all zero. Hence, only the phases of the converter's signal can be controlled, while the half-wave SHE allows to deal with the amplitudes as well. It is worth to remark nonetheless that our optimal control formulation can be easily adapted to the quarter-wave symmetry problem by simply replacing the Fourier coefficients (2.3) with

$$a_i = 0, \qquad b_j = \frac{4}{\pi}\int_0^{\frac{\pi}{4}} u(\tau)\sin(j\tau)\,d\tau.$$

Entonces podemos introducir el siguiente sistema dinámico:

$$\begin{cases} \dot{\boldsymbol{\beta}}(\tau) = \dfrac{2}{\pi}\boldsymbol{\mathcal{D}}^{\beta}(\tau)u(\tau), & \tau \in [0, \pi/2) \\ \boldsymbol{\beta}(0) = \boldsymbol{b}_T \end{cases} \tag{7.1}$$

Además del siguiente problema de control:

Problem 7.1 *Let $\mathcal{U}$ be defined as in (2.1) and let $\mathcal{E}_b$ be a set of odd numbers with cardinality $|\mathcal{E}_b| = N_b$. Given the vector $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$. We look for $u :\in [0, \pi/2) \to \mathcal{U}$ such that the solution of (7.1) with initial datum $\boldsymbol{x}(0) = \boldsymbol{x}_0$ satisfies $\boldsymbol{x}(\pi) = 0$.*

Donde de la misma forma que en el problema con simetría de media onda la solución de este problema es también solución del problema SHE con simetría de cuarto de onda.

7.2 Generalizations

Consideramos que el tipo de penalizaciones utilizados en este problema puedes ser extendido a sistemas LTI:

$$\min_{u \in \mathcal{U}} \frac{1}{2}\|\boldsymbol{x}(T)\|^2 + \int_0^T \mathcal{L}(u(\tau))d\tau \tag{7.2}$$

$$\text{subject to:} \quad \begin{cases} \dot{\boldsymbol{x}}(\tau) = A(\tau)x(\tau) + B(\tau)u(\tau), & \tau \in [0, T) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases}$$

De manera que dado un término de penalización compatible con el Teorema 4.4, sea capaz de condicionar el control óptimo como un control digital.

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