Selective Harmonic Elimination via Optimal Control

Umberto Biccari [a,b], Carlos Esteve [c,a], Jesús Oroya [b]

[a] *Chair of Computational Mathematics, Fundación Deusto, Avenida de las Universidades 24, 48007 Bilbao, Basque Country, Spain.*

[b] *Universidad de Deusto, Avenida de las Universidades 24, 48007 Bilbao, Basque Country, Spain.*

[c] *Departamento de Matemáticas, Universidad Autónoma de Madrid, 28049 Madrid, Spain.*

Abstract

El problema de *Selective Harmonic Elimination pulse-width modulation* (SHE) es planteado como el problema de control ptimo, con el fin de encontrar soluciones de ondas escaln sin prefijar el nmero de ngulos de conmutacin. De esta manera, la metodologa de control ptimo es capaz de encontar la forma de onda ptima y de encontrar la localizaciones de los ngulos de conmutacin, incluso sin prefijar el nmero de conmutaciones. Este es un nuevo enfoque para el problema SHE en concreto y para los sistemas de control con un conjunto finito de controles admisibles en general.

Key words: Selective Harmonic Elimination; Finite Set Control, Piecewise Linear function.

1 Introduction and motivations

Selective Harmonic Elimination (SHE) [Rodríguez et al., 2002] is a well-known methodology in electrical engineering, employed to improve the performances of a converter by controlling the phase and amplitude of the harmonics in its output voltage. As a matter of fact, this technique allows to increase the power of the converter and, at the same time, to reduce its losses.

In broad terms, the process consists in generating a *control signal* with a desired harmonic spectrum, by modulating or eliminating some specific lower order frequencies. This signal is in the shape of a step function and is fully characterized by two features:

1. the *waveform*, i.e. the set of (constant) values the function may assume.
2. the *switching angles*, defining the points in the domain where the function changes from one constant value to another.

Because of the growing complexity of modern electrical networks, consequence for instance of the high penetration of renewable energy sources, the demand in power of electronic converters is day by day increasing. For this and other reasons, SHE has been a preeminent research interest in the electrical engineering community, and a plethora of SHE-based techniques has been developed in recent years. An incomplete bibliography includes [Duranay and Guldemir, 2017, Janabi et al., 2020, Yang et al., 2017].

Nowadays, SHE is mostly based on offline computations to obtain the commutation patterns describing the control signal.

Add references and mention real-time approaches.

The present document is organized as follows. In Section 2, we will present the mathematical formulation of the SHE problem and a classical resolution method based on an optimization process. In Section 3, we will formulate the SHE problem as a controlled dynamical system. In Section 4, we will present the optimal control problem allowing us to obtain the solutions to the SHE problem. In Section 5, we will present our numerical experiments. Finally, in Section 6, we will equation the conclusions of our study.

2 Mathematical formulation of SHE

This section is devoted to the mathematical formulation of the SHE problem. In what follows, we with the nota-

⋆ This paper was not presented at any IFAC meeting.

Email addresses: umberto.biccari@deusto.es (Umberto Biccari), carlos.esteve@uam.es (Carlos Esteve), djoroya@deusto.es (Jesús Oroya).

tion $\mathcal{U}$ we will always refer to a finite set of real numbers contained in the interval $[-1, 1]$:

$$\mathcal{U} = \{u_\ell\}_{\ell=1}^{L} \subset [-1, 1] \tag{2.1}$$

Our objective is to design a piece-wise constant function $u(\tau) : [0, 2\pi) \to \mathcal{U}$ such that some of its lower-order Fourier coefficients take specific values determined a priori.

Due to the application in power converters, we will focus here on functions with *half-wave symmetry*, i. e.

$$u(\tau + \pi) = -u(\tau) \quad \text{for all } \tau \in [0, \pi).$$

In this way, u is fully determined by its values in the interval $[0, \pi)$, and its Fourier series expansion only involves the odd terms taking the form

$$u(\tau) = \sum_{\substack{i \in \mathbb{N} \\ i \text{ odd}}} a_i \cos(i\tau) + \sum_{\substack{j \in \mathbb{N} \\ j \text{ odd}}} b_j \sin(j\tau), \tag{2.2}$$

where the coefficients a_i and b_j are given by

$$\begin{aligned} a_i &= \frac{2}{\pi} \int_0^\pi u(\tau) \cos(i\tau)\, d\tau, \\ b_j &= \frac{2}{\pi} \int_0^\pi u(\tau) \sin(j\tau)\, d\tau. \end{aligned} \tag{2.3}$$

We can then give a general formulation of the SHE problem as follows:

Problem 2.1 (SHE) *Let $\mathcal{U}$ be defined as in (2.1) and let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two sets of odd numbers with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$, respectively. Given the vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, we look for $u :\in [0, \pi) \to \mathcal{U}$ such that*

$$\begin{aligned} a_i &= (\boldsymbol{a}_T)_i, \quad &\text{for all } i \in \mathcal{E}_a, \\ b_j &= (\boldsymbol{b}_T)_j, \quad &\text{for all } j \in \mathcal{E}_b, \end{aligned}$$

with $\{a_i\}_{i \in \mathcal{E}_a}$ and $\{b_j\}_{j \in \mathcal{E}_b}$ given by (2.3).

Figure **??** shows an example of a function u solution of the SHE problem.

Include here a picture of a possible solution u.

As we anticipated in Section 1, the control signal u is fully characterized by its waveform and the switching angles, to which we give a precise definition as follows:

Definition 2.1 (Wave-form) *Given a finite set of real numbers $\mathcal{U}$ defined as in (2.1), we will call a waveform any possible tuple $\mathcal{S} = (s_m)_{m=0}^{M+1}$ with $s_m \in \mathcal{U}$ for all $m = 0, \ldots, M+1$.*

Definition 2.2 (Switching angles) *Given a finite set of real numbers $\mathcal{U}$ defined as in (2.1) and a piece-wise constant function $u : [0, \pi) \to \mathcal{U}$, we shall refer as switching angles $\boldsymbol{\phi} = \{\phi_m\}_{m=0}^{M+1} \subset [0, \pi]$, with $\phi_0 = 0$ and $\phi_{M+1} = \pi$, to the points in the domain $[0, \pi)$ where u changes its value.*

In view of the above definitions, we can provide the following explicit expression for the function u:

$$\begin{aligned} u &= \sum_{m=0}^{M+1} s_m \chi_{[\phi_m, \phi_{m+1}]} \\ s_m &\in \mathcal{S},\ \phi_m \in \boldsymbol{\phi}, \quad \text{for all } m = 0, \ldots, M+1, \end{aligned} \tag{2.4}$$

where we denoted by $\chi_{[\phi_m, \phi_{m+1}]}$ the characteristic function of the interval $[\phi_m, \phi_{m+1}]$.

Besides, taking into account (2.4), we can rewrite the Fourier coefficients (2.3) as

$$\begin{aligned} a_i = a_i(\boldsymbol{\phi}) &= \frac{2}{i\pi} \sum_{k=1}^{M+1} s_k \Big[\sin(i\phi_k) - \sin(i\phi_{k-1}) \Big] \\ b_j = b_j(\boldsymbol{\phi}) &= \frac{2}{j\pi} \sum_{k=1}^{M+1} s_k \Big[\cos(j\phi_{k-1}) - \cos(j\phi_k) \Big] \end{aligned}$$

Given a waveform $\mathcal{S}$, Problem 2.1 then reduces to find the switching locations $\boldsymbol{\phi}$ (see [Yang et al., 2015, Konstantinou and Agelidis, 2010, Sun et al., 1996]). This can be cast as a minimization problem in the variables $\{\phi_m\}_{m=0}^{M+1}$, where the cost functional is the Euclidean distance between the obtained Fourier coefficients $\{a_i(\boldsymbol{\phi}), b_j(\boldsymbol{\phi})\}$ and the targets $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^{N_a} \times \mathbb{R}^{N_b}$.

Problem 2.2 (Optimization for SHE) *Given a waveform $\mathcal{S}$ and a step function u in the form (2.4), we look for the switching angles $\boldsymbol{\phi}$ by means of the following minimization problem:*

$$\begin{aligned} &\min_{\boldsymbol{\phi} \in [0,\pi]^M} \left(\sum_{i \in \mathcal{E}_a} \|a_T^i - a_i(\boldsymbol{\phi})\|^2 + \sum_{j \in \mathcal{E}_b} \|b_T^j - b_j(\boldsymbol{\phi})\|^2 \right) \\ &\textit{subject to: } 0 = \phi_0 < \phi_1 < \ldots < \phi_M < \phi_{M+1} = \pi \end{aligned} \tag{2.5}$$

Since the cardinality of $\mathcal{S}$ is not known a priori, meaning that we do not know how many switches will be necessary, it appears that the only solution to the SHE problem consists in fix the number of changes and counting all the possible combinations, to later solve an optimization problem for each one of them. Taking into account that the number of possible multi-set S is given by $|\mathcal{U}|^{|\mathcal{S}|}$, it is evident that the complexity of the above approach increases rapidly. This problem has been studied in [Yang et al., 2015] where, through appropriate algebraic transformations, the authors are able to convert the SHE problem into a polynomial system whose solutions' set contains all the possible waveforms for a given set $\mathcal{U}$ and number of elements in the sequence $\mathcal{S}$ which, however, is predetermined.

On the other hand, we shall also mention that the SHE methodology has been developed to provide in real-time different target Fourier coefficients con with a KHz latency. This makes impossible to find real-time solutions by optimization, making then necessary to pre-determine solutions that can later be interpolated. Nevertheless, it is well-known that, fixed a sequence S, the continuity of the switching locations with respect to a continuous variation of the target Fourier coefficients may be quite cumbersome. In the majority of the cases, it is impossible to find a continuous solution in a large interval, an it is necessary to change the waveform S while moving across different solution regions ([Yang et al., 2015, Yang et al., 2017]). This makes difficult the interpolation of solutions and their finding.

In this document, we will present the SHE problem as an optimal control one, where the optimization variable is the signal $u(\tau)$ defined in the entire interval $[0, \pi)$. In particular, we will describe how the Fourier coefficients of the function $u(\tau)$ can be seen as the final state of a system controlled by $u(\tau)$. Hence, the optimization is performed among all the possible functions that satisfy $|u(\tau)| < 1$ and can control the final state at the desired Fourier coefficients. Then we will show how to design a control problem so that the solution is a step function.

In this way, we can reformulate Problem 2.1 as follows. In this formulation, the SHE problem converts in a minimization problem with restrictions which can be solved by well-known techniques. Since the problem has several minimizers, we shall solve it employing global optimizers. Furthermore, since the choice of the waveform is arbitrary, we shall proceed in the same way for each possible waveform.

Tengo que enlazar estas seccin con la formulacion de control ptimo.

3 SHE as a dynamical system

As we anticipated in Section 1, the main contribution of the present paper is to provide a novel and alternative approach to the SHE problem, based on the optimal control. As we shall see, this methodology will allow us avoiding the choice of the waveform, as the optimization process chooses the most convenient one in each case.

To this end, the starting point is to rewrite the expression of the Fourier coefficients (2.3) as the evolution of a dynamical system. This can be easily done by means of the fundamental theorem of differential calculus as follows: for all $i, j \in \mathbb{N}$, let α_i and β_j be the solutions of the following Cauchy problems

$$\begin{cases} \dot{\alpha}_i(\tau) = \dfrac{2}{\pi} u(\tau)\cos(i\tau), & \tau \in [0, \pi) \\ \alpha_i(0) = 0 \end{cases}$$
$$\begin{cases} \dot{\beta}_j(\tau) = \dfrac{2}{\pi} u(\tau)\sin(j\tau), & \tau \in [0, \pi) \\ \beta_j(0) = 0 \end{cases} \tag{3.1}$$

Then

$$\alpha_i(\tau) := \frac{2}{\pi}\int_0^\tau u(\zeta)\cos(i\sigma)\, d\zeta$$
$$\beta_j(\tau) = \frac{2}{\pi}\int_0^\tau u(\zeta)\sin(j\zeta)\, d\zeta$$

and the Fourier coefficients (2.3) are given by $a_i = \alpha_i(\pi)$ and $b_j = \beta_j(\pi)$.

Let now

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \dots, e_a^{N_a}\}, \quad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \dots, e_b^{N_b}\}$$

be two sets of odd numbers, and denote

$$\boldsymbol{\alpha} = \{\alpha_i\}_{i\in\mathcal{E}_a}, \quad \boldsymbol{\beta} = \{\beta_j\}_{j\in\mathcal{E}_b}.$$

Then, for any $\tau \in [0, \pi)$, we can define the vectors $\boldsymbol{\mathcal{D}}^\beta(\tau) \in \mathbb{R}^{N_a}$ and $\boldsymbol{\mathcal{D}}^\beta(\tau) \in \mathbb{R}^{N_b}$ as:

$$\boldsymbol{\mathcal{D}}^\alpha(\tau) = \begin{bmatrix} \cos(e_a^1\tau) \\ \cos(e_a^2\tau) \\ \vdots \\ \cos(e_a^{N_a}\tau) \end{bmatrix}, \quad \boldsymbol{\mathcal{D}}^\beta(\tau) = \begin{bmatrix} \sin(e_b^1\tau) \\ \sin(e_b^2\tau) \\ \vdots \\ \sin(e_b^{N_b}\tau) \end{bmatrix}$$

and the dynamical systems (3.1) can be rewritten in a vectorial form as:

$$\begin{cases} \dot{\boldsymbol{\alpha}}(\tau) = \dfrac{2}{\pi} \boldsymbol{\mathcal{D}}^\alpha(\tau) u(\tau), & \tau \in [0, \pi) \\ \boldsymbol{\alpha}(0) = 0 \end{cases}$$
$$\begin{cases} \dot{\boldsymbol{\beta}}(\tau) = \dfrac{2}{\pi} \boldsymbol{\mathcal{D}}^\beta(\tau) u(\tau), & \tau \in [0, \pi) \\ \boldsymbol{\beta}(0) = 0 \end{cases} \tag{3.2}$$

Compressing the notation even more, we can now denote

$$\boldsymbol{x}(\tau)=\begin{bmatrix}\boldsymbol{\alpha}(\tau)\\ \boldsymbol{\beta}(\tau)\end{bmatrix},\quad \boldsymbol{\mathcal{D}}(\tau)=\begin{bmatrix}\boldsymbol{\mathcal{D}}^{\alpha}(\tau)\\ \boldsymbol{\mathcal{D}}^{\beta}(\tau)\end{bmatrix}$$

so that (3.2) becomes

$$\begin{cases}\dot{\boldsymbol{x}}(\tau)=\frac{2}{\pi}\boldsymbol{\mathcal{D}}(\tau)u(\tau), & \tau\in[0,\pi)\\ \boldsymbol{x}(0)=0\end{cases}\tag{3.3}$$

and the target coefficients of the SHE problem will be given by $\boldsymbol{x}_T:=[\boldsymbol{a}_T,\boldsymbol{b}_T]^\top=\boldsymbol{x}(\pi)$.

Taking into account this new formulation, as we shall see in more detail in Section 4, Problem 2.1 can now be recast as a control one for the dynamical systems (3.3), in which we look for a control function $u(\tau)$ steering the state $\boldsymbol{x}(\tau)$ from the origin to the target $\boldsymbol{x}_T:=[\boldsymbol{a}_T,\boldsymbol{b}_T]^\top$ in time $\tau=\pi$.

Moreover, since most often control problems are designed to drive the state of a given dynamical system to an equilibrium configuration, for instance the zero state, we introduce the change of variables $\boldsymbol{x}(\tau)\mapsto\boldsymbol{x}_T-\boldsymbol{x}(\tau)$ which allows us to reverse the time in (3.3), thus obtaining

$$\begin{cases}\dot{\boldsymbol{x}}(\tau)=-\frac{2}{\pi}\boldsymbol{\mathcal{D}}(\tau)u(\tau), & \tau\in[0,\pi)\\ \boldsymbol{x}(0)=\boldsymbol{x}_T\end{cases},\tag{3.4}$$

In this new configuration, the control function u is now required to steer the solution of (3.4) from the initial datum $\boldsymbol{x}_T$ to zero in time $\tau=\pi$.

We can then formulate the following control problem, which is equivalent to Problem 2.1:

Problem 3.1 *Let $\mathcal{U}$ be defined as in (2.1) and let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two sets of odd numbers with cardinalities $|\mathcal{E}_a|=N_a$ and $|\mathcal{E}_b|=N_b$, respectively. Given the vectors $\boldsymbol{a}_T\in\mathbb{R}^{N_a}$ and $\boldsymbol{b}_T\in\mathbb{R}^{N_b}$, let us define $\boldsymbol{x}_T=[\boldsymbol{a}_T,\boldsymbol{b}_T]^\top\in\mathbb{R}^{N_a\times N_b}$. We look for $u:\in[0,\pi)\to\mathcal{U}$ such that the solution of (3.4) with initial datum $\boldsymbol{x}(0)=\boldsymbol{x}_T$ satisfies $\boldsymbol{x}(\pi)=0$.*

Remark 3.1 (Quarter-wave symmetry) *We shall mention that, in the SHE literature, it is usual to distinguish among the half-wave symmetry problem (addressed in the present paper) and the quarter-wave symmetry one in which*

$$u\left(\tau+\frac{\pi}{2}\right)=-u(\tau)\quad \textit{for all } \tau\in\left[0,\frac{\pi}{2}\right).$$

In quarter-wave symmetry, the SHE problem simplifies, as the Fourier coefficients $\{a_i\}_{i\in\mathcal{E}_a}$ turn out to be all zero. Hence, only the phases of the converter's signal can be controlled, while the half-wave SHE allows to deal with the amplitudes as well. It is worth to remark nonetheless that our optimal control formulation can be easily adapted to the quarter-wave symmetry problem by simply replacing the Fourier coefficients (2.3) with

$$a_i=0,\qquad b_j=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}u(\tau)\sin(j\tau)\,d\tau.$$

4 Optimal control for SHE

As we anticipated in Section 3, the SHE problem is equivalent to controlling a dynamical system associated with the Fourier coefficients (2.3). In this section, we present a rigorous formulation of this mentioned control problem and we analyze some relevant properties.

Problem 4.1 (OCP for SHE) *Let $\mathcal{U}$ be defined as in (2.1). Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinality N_a and N_b, respectively, and given the target vector $\boldsymbol{x}_T\in\mathbb{R}^{N_a+N_b}$, we look for the function $u(\tau):[0,\pi)\to\mathcal{U}$ solution of the optimal control problem*

$$\min_{u\in\mathcal{U}}\frac{1}{2}\|\boldsymbol{x}(\pi)\|^2_{\mathbb{R}^{N_a+N_b}}$$
$$\textit{subject to:}\quad\begin{cases}\dot{\boldsymbol{x}}(\tau)=-\frac{2}{\pi}\boldsymbol{\mathcal{D}}(\tau)u(\tau), & \tau\in[0,\pi)\\ \boldsymbol{x}(0)=\boldsymbol{x}_0\end{cases}$$

The solution of Problem 4.1 may be quite complex to be obtained, due to the restriction on the admissible control values. In order to bypass this difficulty, following a standard optimal control approach, we can formulate an equivalent minimization problem in which, instead of looking for $u\in\mathcal{U}$, we simply require that $|u|<1$ and we introduce a penalization term to ensure that u is a piecewise constant function. This alternative optimal control problem, which can be solved more easily by employing standard tools, reads as follows:

Problem 4.2 (Penalized OCP for SHE) *Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ and the target vector $\boldsymbol{x}_T\in\mathbb{R}^{N_a+N_b}$, we look for the function u as the solution of:*

$$\min_{|u|<1}\left[\frac{1}{2}\|\boldsymbol{x}(\pi)\|^2_{\mathbb{R}^{N_a+N_b}}+\epsilon\int_0^{\pi}\mathcal{L}(u(\tau))d\tau\right]$$

under the dynamics given by (3.4).

Where we will choose $\mathcal{L}:\mathbb{R}\to\mathbb{R}$ such that the optimal control u^* only takes values in the discretization $\mathcal{U}$ of the interval $[-1.1]$. Furthermore, the parameter ϵ should be small so that the solution minimizes the distance from the final state and the target. Next we will study the optimality conditions of the problem, for a general $\mathcal{L}$ function, and then specify how $\mathcal{L}$ should be so that the optimal control u^* only takes the allowed values in $\mathcal{U}$.

4.1 Optimality conditions

To write the optimality conditions of the problem we will use the Pontryagin minimum principle [Bryson, 1975, Chapter 2.7]. With this purpose, it is necessary to first introduce define the Hamiltonian function

$$H(u, \boldsymbol{p}, \tau) = \epsilon\mathcal{L}(u) - \frac{2}{\pi}\left[\boldsymbol{p}^\top(\tau)\cdot\boldsymbol{\mathcal{D}}(\tau)\right]u(\tau),$$

where $\boldsymbol{p}(\tau)$ is the so-called adjoint state, which is associated with the restriction imposed by the system. This vector has the same dimension of the state $\boldsymbol{x}$, so that

$$\boldsymbol{x}(\tau) = \begin{bmatrix}\boldsymbol{\alpha}(\tau)\\ \boldsymbol{\beta}(\tau)\end{bmatrix} \Leftrightarrow \boldsymbol{p}(\tau) = \begin{bmatrix}\boldsymbol{p}^\alpha(\tau)\\ \boldsymbol{p}^\beta(\tau)\end{bmatrix}. \tag{4.1}$$

In what follows, we will enumerate the optimality conditions arising from the Pontryagin principle.

1. **Adjoint equation**: the ODE describing the evolution of the adjoint variable is given by

$$\dot{\boldsymbol{p}}(\tau) = -\nabla_x H(u(\tau), \boldsymbol{p}(\tau), \tau) = 0. \tag{4.2}$$

2. **Final condition of the adjoint**: This optimiality condition is obtained from the cost in the final time $\tau = \pi$ of the control problem in this case $\Psi(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}(\pi) - \boldsymbol{x}_T\|^2$.

$$\boldsymbol{p}(\pi) = \nabla_{\boldsymbol{x}}\Psi(\boldsymbol{x}) = (\boldsymbol{x}(\pi) - \boldsymbol{x}_T) \tag{4.3}$$

3. **Optimal Waveform**: We known that

$$u^* = \operatorname*{arg\,min}_{|u|<1} H(\tau, \boldsymbol{p}^*, u),$$

so that, in this case, we can write

$$u^*(\tau) = \operatorname*{arg\,min}_{|u|<1}\left[\epsilon\mathcal{L}(u(\tau)) - \frac{2}{\pi}\left(\boldsymbol{p}^{*\top}\cdot\boldsymbol{\mathcal{D}}(\tau)\right)u(\tau)\right]. \tag{4.4}$$

Therefore, this optimality condition reduces to the optimization of a function in a variable within the interval $[-1, 1]$.

From where it can be deduced that $\boldsymbol{p}(\tau)$ is a constant so that $\boldsymbol{p}(\tau) = \boldsymbol{x}(\pi) - \boldsymbol{x}_T \mid \forall\tau \in [0, \pi)$ so from now on we will refer to it simply as $\boldsymbol{p}$, noting that it is invariant in time.

It is important to recall that

$$[\boldsymbol{p}^{*T}\cdot\boldsymbol{\mathcal{D}}(\tau)] = \sum_{i\in\mathcal{E}_a} p^*_\alpha\cos(i\tau) + \sum_{j\in\mathcal{E}_b} p^*_\beta\sin(j\tau) \tag{4.5}$$

we build a function $\mathcal{H}_m : [-1, 1] \to \mathbb{R}$ such that:

$$\mathcal{H}_m(u) = \epsilon\mathcal{L}(u) - mu|\forall m \in \mathbb{R} \tag{4.6}$$

where we replaced the term $(2/\pi)[\boldsymbol{p}^{*T}\cdot\boldsymbol{\mathcal{D}}(\tau)]$ with a parameter m which may assume all real values. In other words, the problem of designing an optimal control which may only take real values in $\mathcal{U}$ is reduced to construct a one-dimensional function with parameter m whose minima are the elements in $\mathcal{U}$ for all values of m.

4.2 Piecewise linear penalization

In this section, we discuss how to design the penalization term $\mathcal{L}(u)$ so that the optimal control is always contained in $\mathcal{U}$.

In more detail, we can choose the affine interpolation of a parabola $\mathcal{L} : [-1, 1] \to \mathbb{R}$ as a penalization term. That is

$$\mathcal{L}(u) = \begin{cases}\left[(u_{k+1} + u_k)(u - u_k) + u_k^2\right] & \text{if} u \in [u_k, u_{k+1}[\\ 1 & \text{if } u = u_{N_u}\end{cases} \tag{4.7}$$
$$\forall k \in \{1, \ldots, N_u - 1\}$$

Nevertheless, to compute the minimum of $\mathcal{H}_m(u)$, we shall take into account that this function is not differentiable and the optimality condition then requires to work with the subdifferential $\partial\mathcal{L}(u)$, which given by

$$\partial\mathcal{L}(u) = \begin{cases}\{u_1 + u_2\} & \text{if } u = u_1\\ \{u_k + u_{k+1}\} & \text{if } u \in]u_k, u_{k+1}[\quad \dagger\\ [u_k + u_{k-1}, u_{k+1} + u_k] & \text{if } u = u_k \quad \ddagger\\ \{u_{N_u} + u_{N_u-1}\} & \text{if } u = u_{N_u}\end{cases} \tag{4.8}$$
$$\dagger\ \forall k \in \{1, \ldots, N_u - 1\} \qquad \ddagger\ \forall k \in \{2, \ldots, N_u - 1\}$$

Hence, we have $\partial H_m = \epsilon\partial\mathcal{L} - m$. This means that, given $m \in \mathbb{R}$, we look for $u \in [-1, 1]$ minimizing $\mathcal{H}_m(u)$. It is then necessary to determine whether zero belongs to $\partial\mathcal{H}_m(u)$.

- **Case 1:** $m \leq \epsilon(u_1 + u_2)$: since m is less than the minimum of all subdifferentials, then zero does not belong to any of the intervals we defined. Hence, the minimum is in one of the extrema

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_1 \tag{4.9}$$

- **Case 2:** $m = \epsilon(u_{k+1} + u_k)$: taking into account that $\forall k \in \{1, \dots, N_u - 1\}$,

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = [u_k, u_{k+1}[\tag{4.10}$$

- **Case 3:** $\epsilon(u_k + u_{k-1}) < m < \epsilon(u_{k+1} + u_k)$: taking into account that $\forall k \in \{2, \dots, N_u - 1\}$,

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_k \tag{4.11}$$

- **Case 4:** $m > \epsilon(u_{N_u} + u_{N_u - 1})$:

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_{N_u} \tag{4.12}$$

In other words, only when $m = \epsilon(u_{k+1}+u_k)$ the minima of the Hamiltonian belong to an interval. For all the other values of $m \in \mathbb{R}$, these minima are contained in $\mathcal{U}$. So that under a continuous variation of m, Case 2 can only occur pointwise. Recalling the optimal control problem $m(\tau) = [\boldsymbol{p}^T(\tau) \cdot \boldsymbol{\mathcal{D}}(\tau)]$, we can notice that Case 2 corresponds to the instants τ of change of value.

5 Numerical simulations

In this section, we will present several examples in which we solve our optimal control problem through the direct method and the non-linear constrained optimization tool `CasADi` [Andersson et al., 2019].

5.1 *Smooth approximation of piece-wise linear penalization*

With the final aim of using an optimization software to solve our optimal control problem, we will approximate our piece-wise linear penalization with the help of the Heaviside function $h : \mathbb{R} \to \mathbb{R}$ and its smooth approximation defined as follows:

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \begin{cases} h^\eta(x) = (1 + \tanh(\eta x))/2 \\ \eta \to \infty \end{cases} \tag{5.1}$$

Using h, we can define the (smooth) function $\Pi^\eta_{a,b} : \mathbb{R} \to \mathbb{R}$ as:

$$\Pi^\eta_{[a,b]}(x) = -1 + h^\eta(x-a) + h^\eta(-x+b) = \frac{\tanh[\eta(x-a)] + \tanh[\eta(b-x)]}{2}.$$

In this way, we can define the smooth version of (4.7):

$$\mathcal{L}^\eta(u) = \sum_{k=1}^{N_u - 1} \left[(u_{k+1} + u_k)(u - u_k) + u_k^2\right] \Pi^\eta_{[u_k, u_{k+1}]}(u) \tag{5.2}$$

So that, when $\eta \to \infty$, then $\mathcal{L}^\eta \to \mathcal{L}$.

5.2 *Direct method for OCP-SHE*

To solve the optimal control problem (**??**), we use a direct method. If we consider a partition $\mathcal{P} = \{\tau_0, \tau_1, \dots, \tau_T\}$ of interval $[0, T]$, we can represent a function $\{u(\tau) \mid \tau \in [0, T]\}$ as a vector $\boldsymbol{u} \in \mathbb{R}^T$ where component $u_t = u(\tau_t)$. Then the optimal control problem (**??**) can be written as optimization problem with variable $\boldsymbol{u} \in \mathbb{R}^T$. This problem is a nonlinear programming, for this we use CasADi software to solve. Hence, given a partition of the interval $[0, \pi)$, we can formulate the problem (**??**) as the following one in discrete time

Problem 5.1 (Numerical OCP) *Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively, given the target vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$, so that $\boldsymbol{x}_0 = [\boldsymbol{a}_T, \boldsymbol{b}_T]^T$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$ and a partition $\mathcal{P}_\tau = \{\tau_0, \tau_1, \dots, \tau_T\}$ of the interval $[0, \pi)$, we search a vector $\boldsymbol{u} \in \mathbb{R}^T$ that minimizes the following function:*

$$\min_{\boldsymbol{u} \in \mathbb{R}^T} \left[||\boldsymbol{x}^T||^2 + \epsilon \sum_{t=0}^{T-1} \mathcal{L}^\eta(u_t) \Delta\tau_t \right] \tag{5.3}$$

suject to:

$$\forall \tau \in \mathcal{P} \begin{cases} \boldsymbol{x}^{t+1} = \boldsymbol{x}^t - (2/\pi)\Delta\tau_t \boldsymbol{\mathcal{D}}(\tau_t) u_t \\ \boldsymbol{x}^0 = \boldsymbol{x}_0 \end{cases} \tag{5.4}$$

5.3 *Resultados*

All our simulations have been performed with a laptop with $8Gb$ of RAM memory and the execution time for finding the solution given a target vector is os the order of $1s$. IN what follows, we will discuss all the numerical results we have obtained

(1) **OCP with quarter-wave symmetry**: we will consider the problem with a set of odd numbers $\mathcal{E}_b = \{1, 5\}$ and a discretization of $[0, \pi/2]$ with $T = 200$. We show the solutions for the target vectors $b_T^1 = \{(-0.4, -0.3, \dots, 0.3, 0.4)\}$ keeping $b_T^5 = 0$ for each case. We display the optimal trajectories we obtained in Figure **??**, where we can see a continuity of the solutions with respect to the target vector.

(2) **OCP with quarter-wave symmetry for an interval of** b_1: for this example we consider the following set of odd numbers: $\mathcal{E}_b = \{1, 5, 7, 11, 13\}$. Moreover, we consider the target vector $\boldsymbol{b}_T =$

$[m_a, 0, 0, 0, 0]$, where $m_a \in [-1, 1]$ is a parameter, with three penalization terms: $\mathcal{L}(u) = -f$, $\mathcal{L}(u) = +f$ and $\mathcal{L}(u) = -f^2$ obtained by direct method with uniform partition of interval $[0, \pi/2]$ with $T = 400$ and penalization parameter $\epsilon = 10^{-5}$. For each one of the penalization terms we will employ, the distance between the Fourier coefficients is of the order 10^{-4}. Nevertheless, when the penalization term is $\mathcal{L}(u) = -f^2$, the solution does not present continuity with respect to the target vector. On the other hand, it is important to mention that the solutions for the penalization terms $\mathcal{L}(u) = -f$ y $\mathcal{L}(u) = f$ are symmetric, so that inverting those solutions with respect to the origin and inverting their sign, it can be observed that they are the same.

(3) **SHE with three levels**: we can see that in the case in which the control $u(\tau)$ can only take values in $[0, 1]$, we obtain signals which can take three levels in the interval $[0, 2\pi]$ due to the quarter-wave symmetry. If we solve the optimal control problem this time with restrictions $\{0 < u(\tau) < 1\}$. We repeat the same procedure as before, thus obtaining solutions for the same penalization terms and obtaining Figure **??**. There we show the continuity of the solutions and that they are of the order 10^{-4}.

(4) **Changes in the commutations number**: thanks to the optimal control formulation of SHE, we can vary the number of commutation angles. This is illustrated in the following example, where we considered the set of odd numbers $\mathcal{E}_b = \{1, 3, 9, 13, 17\}$. Moreover, we consider the target vector $\boldsymbol{b}_T = [m_a, 0, 0, 0, 0]$, where $m_a \in [0, 1]$ is a parameter. In this problem, we used the penalization $\mathcal{L} = f$ with a penalization parameter $\epsilon = 10^{-4}$. We can see in Figure **??** that the optimal control problem is capable of moving among different solution sets.

(5) **OCP for SHE with half-wave symmetry**: we considered the optimal control problem in half-wave with $\mathcal{E}_a = \{1, 3, 5\}$ and $\mathcal{E}_b = \{1, 3, 5, 9\}$, where $\boldsymbol{a}_T = [m_a, 0, 0]$, $\boldsymbol{b}_T = [m_a, m_a, 0, 0]$ and $m_a \in [-0.6, 0.6]$. We chose the penalization $L(u) = +f$

6 Conclusiones

We presented the SHE problem from the point of view of control theory. This methodology allows obtaining a $10^{-4} - 10^{-5}$ precision in the distance to the target vector. Nevertheless, comparing with methodologies where the commutation number is fixed a priori, our approximation is computationally more expensive. Notwithstanding that, the optimal control provides solutions in the entire range of the modulation index, although the number of solutions or their locations change dramatically.

This methodology for the SHE problem connects control theory with harmonic elimination. In this way, the SHE problem can be solved through classical tools.

Acknowledgements

Partially supported by the Roman Senate.

References

[Andersson et al., 2019] Andersson, J. A. E., Gillis, J., Horn, G., Rawlings, J. B., and Diehl, M. (2019). CasADi – A software framework for nonlinear optimization and optimal control. *Math. Program. Comput.*, 11(1):1–36.

[Bryson, 1975] Bryson, A. E. (1975). *Applied optimal control: optimization, estimation and control.* CRC Press.

[Duranay and Guldemir, 2017] Duranay, Z. B. and Guldemir, H. (2017). Selective harmonic eliminated v/f speed control of single-phase induction motor. *IET Power Electron.*, 11(3):477–483.

[Janabi et al., 2020] Janabi, A., Wang, B., and Czarkowski, D. (2020). Generalized chudnovsky algorithm for real-time pwm selective harmonic elimination/modulation: two-level vsi example. *IEEE Trans. Power Electron.*, 35(5):5437–5446.

[Konstantinou and Agelidis, 2010] Konstantinou, G. S. and Agelidis, V. G. (2010). Bipolar switching waveform: ovel solution sets to the selective harmonic elimination problem. *Proc. IEEE International Conference on Industrial Technology*, pages 696–701.

[Rodríguez et al., 2002] Rodríguez, J., Lai, J. S., and Peng, F. Z. (2002). Multilevel inverters: a survey of topologies, controls, and applications. *IEEE Trans. Ind. Electron.*, 49(4):724–738.

[Sun et al., 1996] Sun, J., Beineke, S., and Grotstollen, H. (1996). Optimal pwm based on real-time solution of harmonic elimination equations. *IEEE Trans. Power Electron.*, 11(4):612–621.

[Yang et al., 2015] Yang, K., Yuan, Z., Yuan, R., Yu, W., Yuan, J., and Wang, J. (2015). A groebner bases theory-based method for selective harmonic elimination. *IEEE Trans. Power Electron.*, 30(12):6581–6592.

[Yang et al., 2017] Yang, K., Zhang, Q., Zhang, J., Yuan, R., Guan, Q., Yu, W., and Wang, J. (2017). Unified selective harmonic elimination for multilevel converters. *IEEE Trans. Power Electron.*, 32(2):1579–1590.