Selective Harmonic Elimination via Optimal Control Theory

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Abstract

En este documento formularemos el problema de *Selective Harmonic Elimination pulse-width modulation*(SHE-PWM) como el problema de control óptimo, con el fin de encontrar soluciones de ondas cuadradas sin prefijar el número de ángulos de conmunatación. Esta nueva perspectiva nos permite realizar un análisis sobre la continuidad de soluciones.

Contents

1 Introducción

The *Selective Harmonic Elimination* (SHE) problem is a modulation method that allows you to generate step signals[1]. with a desired harmonic spectrum. That is, given some Fourier coefficients, the SHE problem looks for the step waveform $\{u(\tau)|\tau \in (0, 2\pi]\}$ whose Fourier coefficients are as required. In general, half-wave symmetry, that is $u(\tau + \pi) = -u(\tau)$, is required so in this work we will focus on this case. In this context, if the waveform and number of switching are fixed, the

[1]It should be noted that the step signals we are talking about are discontinuous functions defined in the interval $[0, 2\pi]$ and that they can only take values in a small and finite set of values.

problem SHE can be formulated as an optimization problem where the variables to be optimized are the switching locations. Then we look for the switching locations that minimize the Euclidean distance between coefficients of the sought function and the given coefficients.

Although the problem SHE given a preset waveform is easily solvable, there are several difficulties in its application. It is not possible to calculate the switching locations by means of a real-time optimization, which is why the solutions for different values of objective Fourier coefficients are pre-calculated. When a non-precalculated solution is required, interpolations are carried out with the help of the other solutions to obtain it. However, it is known that the space of solutions at the commutation angles is discontinuous with respect to a continuous variation of objective Fourier coefficients. This is the reason why the interpolation of solutions is complex and sometimes impossible.

The nature of the appearance and fading of solutions through a continuous variation of the Fourier coefficients is unknown. By calculating of the solutions it is known that the more switching locations are considered, the larger the continuous region of solutions. This tells us that the number of switching locations required along a region of the solution space could change so that this formulation is not very flexible for a continuous description of the solutions.

In this document we will present the SHE problem as an optimal control problem, where the optimization variable is the signal $u(\tau)$ defined in the entire interval $[0, 2\pi)$. Thus we will describe how in the problem of the Fourier coefficients of the function $u(\tau)$ they can be seen as the final state of a system controlled by $u(\tau)$. So the optimization is performed among all the possible functions that satisfy $|u(\tau)| < 1$ that can control the final state at the desired Fourier coefficients. Then we will show how to design a control problem so that the solution is a step function. Finally, we will show solutions to the SHE problem by formulating the optimal control, seeing how this methodology is versatile in the face of the variation in the number of commutations.

2 Classical approach

Next, we will mention the classic formulation of the SHE problem, mentioning its strengths and limitations in order to emphasize the advantages of considering it as a control problem. It is known that the description of the function $u(\tau)$ is determined with its values in the interval $\tau \in [0, \pi)$. In this way, we will refer to a function $\{u(\tau) | \tau \in [0, \pi)\}$ whose Fourier series expansion can be written as:

$$u(\tau) = \sum_{i \in odd} a_i \cos(i\tau) + \sum_{j \in odd} b_j \sin(j\tau) \tag{2.1}$$

Wher a_i and b_j are:

$$a_i = \frac{2}{\pi} \int_0^\pi u(\tau) \cos(i\tau) d\tau \mid \forall i \ odd \tag{2.2}$$

$$b_j = \frac{2}{\pi} \int_0^\pi u(\tau) \sin(j\tau) d\tau \mid \forall j \ odd \tag{2.3}$$

So the SHE problem can be formulated more precisely as:

Problem 2.1 (SHE) Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinalities $|\mathcal{E}_a| = N_a$ y $|\mathcal{E}_b| = N_b$ respectively, and given the target vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, we look for a function $\{u(\tau) \mid \tau \in [0,\pi)\}$ such that $u(\tau)$ can only take the values within $\mathcal{U}$ a finite subset of the interval $[-1,1]$. Also whose Fourier coefficients satisfy: $a_i = (\boldsymbol{a}_T)_i \mid \forall i \in \mathcal{E}_a$ y $b_j = (\boldsymbol{b}_T)_j \mid \forall j \in \mathcal{E}_b$.

So in order to solve this problem through the classical formulation we need in addition to the elements of this statement a waveform that they can optimize.

Definition 2.1 (Switching locations) Given a function $u : [0,\pi] \to \mathcal{U}$ the switching locations are the values of τ where the function $u(\tau)$ changes its value discontinuously. We will denote the commutation angles as $\boldsymbol{\phi} = \{\phi_0, \phi_1, \dots, \phi_M, \phi_{M+1}\}$ where we have taken $\phi_0 = 0$ and $\phi_{M+1} = \pi$,

Definition 2.2 (Waveform) Given $\mathcal{U}$ a finite subset of $[-1,1]$ we will call a waveform a finite set $S = \{s_1, s_2, s_3, \dots, s_{M+1}\}$ elements of $\mathcal{U}$ with repetition.

Then a waveform S indicates the values that the function will take and in what order within the interval $[0,\pi)$, while the switching locations $\boldsymbol{\phi}$ indicates the switching locations. Considering these two elements we can rewrite the Fourier coefficients as:

$$a_i = \frac{2}{\pi}\int_0^{\pi} u(\tau)\cos(i\tau)d\tau = \frac{2}{\pi}\sum_{k=1}^{M+1}\int_{\phi_{k-1}}^{\phi_k} s_k\cos(i\tau)d\tau = \frac{2}{i\pi}\sum_{k=1}^{M+1} s_k \sin(i\tau)\Big|_{\phi_{k-1}}^{\phi_k} \tag{2.4}$$

$$b_j = \frac{2}{\pi}\int_0^{\pi} u(\tau)\sin(j\tau)d\tau = \frac{2}{\pi}\sum_{k=1}^{M+1}\int_{\phi_{k-1}}^{\phi_k} s_k\sin(j\tau)d\tau = \frac{2}{j\pi}\sum_{k=1}^{M+1} s_k \cos(j\tau)\Big|_{\phi_{k-1}}^{\phi_k} \tag{2.5}$$

So that:

$$a_i(\boldsymbol{\phi}) = +\frac{2}{i\pi}\sum_{k=1}^{M+1} s_k\Big[\sin(i\phi_k) - \sin(i\phi_{k-1})\Big] \tag{2.6}$$

$$b_j(\boldsymbol{\phi}) = -\frac{2}{j\pi}\sum_{k=1}^{M+1} s_k\Big[\cos(j\phi_k) - \cos(j\phi_{k-1})\Big] \tag{2.7}$$

Problem 2.2 (optimization for SHE) Then given a waveform S we look for the locations of change by means of the following minimization problem:

$$\min_{\boldsymbol{\phi}\in[0,\pi]^M}\left[\sum_{i\in\mathcal{E}_a}\|a_T^i - a_i(\boldsymbol{\phi})\|^2 + \sum_{j\in\mathcal{E}_b}\|b_T^j - b_j(\boldsymbol{\phi})\|^2\right] \tag{2.8}$$

$$\text{suject to:}$$

$$0 < \phi_1 < \phi_2 < \dots < \phi_{M-1} < \phi_M < \pi \tag{2.9}$$

There are well differentiated special cases in the SHE literature. These are:

- Forma de onda binivel. Si consideramos una forma de onda tal que $S = \{+1, -1, +1, \dots\}$, entonces podemos llamar a problema SHE asociado como SHE de dos niveles.
- Forma de onda trinivel. Si consideramos una forma de onda tal que $S = \{+1, 0, +1, 0, \dots\}$, entonces podemos llamar a problema SHE asociado como SHE de dos niveles.

3 Selective Harmonic Elimination as dynamical system

Inspired by the continuous nature of the optimization variable $u(\tau)$ we propose in this document the formulation from the optimal control. In this way we avoid the choice of the waveform, so that the optimization problem chooses the most convenient in each case. So we look for $\{u(\tau) \in \mathcal{U} \mid \forall \tau \in [0, \pi)]\}$ that has the desired Fourier coefficients. We will use the fundamental theorem of differential calculus to rewrite the expression of the Fourier coefficients (2.2) and (2.3) as the evolution of a dynamical system. That is to say:

$$\alpha_i(\tau) = \frac{2}{\pi}\int_0^\tau u(\tau)\cos(i\tau)d\tau \Rightarrow \begin{cases} \dot{\alpha}_i(\tau) & = \frac{2}{\pi}u(\tau)\cos(i\tau) \\ \alpha_i(0) & = 0 \end{cases} \tag{3.1}$$

$$\beta_j(\tau) = \frac{2}{\pi}\int_0^\tau u(\tau)\sin(j\tau)d\tau \Rightarrow \begin{cases} \dot{\beta}_j(\tau) & = \frac{2}{\pi}u(\tau)\sin(j\tau) \\ \beta_j(0) & = 0 \end{cases} \tag{3.2}$$

The evolution of the dynamical systems $\alpha_i(\tau)$ and $\beta_j(\tau)$ from the time $\tau = 0$ to $\tau = \pi$ gives us the coefficients a_i y b_j. We introduce notation to refer to vectors $\boldsymbol{\alpha} = \{\alpha_i\}_{i\in\mathcal{E}_a}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathcal{E}_b}$. In this way, the general SHE problem (2.1) can be formulated as a control problem of a dynamic system where $\boldsymbol{\alpha}(\tau)$ and $\boldsymbol{\beta}(\tau)$ are the states of the system and where $u(\tau)$ is the control variable, and whose objective will be to bring the states from the origin of coordinates to the objective vectors $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ in time $\tau = \pi$.

In order to obtain a compact expression of the problem that simplifies our understanding of it, we will introduce notation. So if we consider a problem with sets of odd numbers:

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \ldots, e_a^{N_a}\} \qquad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \ldots, e_b^{N_b}\} \tag{3.3}$$

then we can define the vectors $\boldsymbol{\mathcal{D}}^\beta(\tau) \in \mathbb{R}^{N_a}$ y $\boldsymbol{\mathcal{D}}^\beta(\tau) \in \mathbb{R}^{N_b} \mid \forall \tau \in (0, \pi]$ such that:

$$\boldsymbol{\mathcal{D}}^\alpha(\tau) = \frac{2}{\pi}\begin{bmatrix} \cos(e_a^1\tau) \\ \cos(e_a^2\tau) \\ \cdots \\ \cos(e_a^{N_a}\tau) \end{bmatrix} \qquad \boldsymbol{\mathcal{D}}^\beta(\tau) = \frac{2}{\pi}\begin{bmatrix} \sin(e_b^1\tau) \\ \sin(e_b^2\tau) \\ \cdots \\ \sin(e_b^{N_b}\tau) \end{bmatrix} \tag{3.4}$$

So the dynamical system can be written as:

$$\begin{cases} \dot{\boldsymbol{\alpha}}(\tau) = \boldsymbol{\mathcal{D}}^\alpha(\tau)u(\tau) \quad \tau \in [0, \pi) \\ \boldsymbol{\alpha}(0) = 0 \end{cases} \qquad \begin{cases} \dot{\boldsymbol{\beta}}(\tau) = \boldsymbol{\mathcal{D}}^\beta(\tau)u(\tau) \quad \tau \in [0, \pi) \\ \boldsymbol{\beta}(0) = 0 \end{cases} \tag{3.5}$$

Compressing the notation even more we can call the total state of the system $\boldsymbol{x}(\tau)$ to the concatenation of the states $\boldsymbol{\alpha}(\tau)$ and $\boldsymbol{\beta}(\tau)$ so that:

$$\boldsymbol{x}(\tau) = \begin{bmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{bmatrix} \qquad \boldsymbol{x}_T = \begin{bmatrix} \boldsymbol{a}_T \\ \boldsymbol{b}_T \end{bmatrix} \qquad \boldsymbol{\mathcal{D}}(\tau) = \begin{bmatrix} \boldsymbol{\mathcal{D}}^\alpha(\tau) \\ \boldsymbol{\mathcal{D}}^\beta(\tau) \end{bmatrix} \tag{3.6}$$

So for a pair of sets $\mathcal{E}_a$ and $\mathcal{E}_b$ we have the following associated dynamical system:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = \boldsymbol{\mathcal{D}}(\tau)u(\tau) & \tau \in [0,\pi) \\ \boldsymbol{x}(0) = 0 \end{cases} \tag{3.7}$$

Then we look for a function $u(\tau)$ such that it leads the dynamical system to the point $\boldsymbol{x}_T$, that is, the final state $\boldsymbol{x}(\pi)$ is $\boldsymbol{x}_T$.

4 Optimal Control for SHE

Since the SHE problem is equivalent to controlling a dynamic system from the origin of coordinates to a specific point, we must formulate a control problem that solves this task but also complies with the restrictions on the values that the control can take. It is necessary to set a finite subset $\mathcal{U}$ of the interval $[-1, 1]$, the optimal control is such that it can only take the values allowed in the discretization. In other words, the control problem is:

Problem 4.1 (OCP for SHE) Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ and given the target vector $\boldsymbol{x}_T \in \mathbb{R}^{N_a+N_b}$, also given a set $\mathcal{U}$ of admissible controls, we look for the function $u(\tau) \mid \tau \in [0, \pi)$ such that:

$$\min_{u(\tau)\in\mathcal{U}} \left[\frac{1}{2} ||\boldsymbol{x}(\pi) - \boldsymbol{x}_T||^2 \right] \tag{4.1}$$

suject to:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = \boldsymbol{\mathcal{D}}(\tau)u(\tau) & \tau \in [0,\pi) \\ \boldsymbol{x}(0) = 0 \end{cases} \tag{4.2}$$

The solution of this control problem is complex due to the restriction on the admissible control values. In order to obtain a problem that can be solved by classical control theory we can formulate the following control problem:

Problem 4.2 (Regularized OCP for SHE) Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ and given the target vector $\boldsymbol{x}_T \in \mathbb{R}^{N_a+N_b}$, we look for the function $|u(\tau)| < 1 \mid \tau \in [0, \pi)$ such that:

$$\min_{|u(\tau)|<1} \left[\frac{1}{2} ||\boldsymbol{x}(\pi) - \boldsymbol{x}_T||^2 + \epsilon \int_0^\pi \mathcal{L}(u(\tau))d\tau \right] \tag{4.3}$$

suject to:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = \boldsymbol{\mathcal{D}}(\tau)u(\tau) & \tau \in [0,\pi) \\ \boldsymbol{x}(0) = 0 \end{cases} \tag{4.4}$$

Where we will choose $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ such that the optimal control u^* only takes values in the discretization $\mathcal{U}$ of the interval $[-1.1]$. Furthermore, the parameter ϵ should be small so that the solution minimizes the distance from the final state and the target. Next we will study the optimality conditions of the problem, for a general $\mathcal{L}$ function, and then specify how $\mathcal{L}$ should be so that the optimal control u^* only takes the allowed values in $\mathcal{U}$.

4.1 Optimality conditions

To write the optimility conditions of the problem we will use the principle of the Pontryagin minimum. For them it is necessary to define the Hamiltonian of the system, which in this case is:

$$H(u, \boldsymbol{p}, \tau) = \epsilon \mathcal{L}(u) + [\boldsymbol{p}^T(\tau) \cdot \boldsymbol{\mathcal{D}}(\tau)]u(\tau) \tag{4.5}$$

Where the variable $\boldsymbol{p}(\tau)$ called attached state is introduced, which is associated with the restriction imposed by the system. From the Hamiltonian, the following optimality conditions can be written:

1. **Final condition of the adjoint**: This optimiality condition is obtained from the final cost of the control problem in this case $\Psi(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}(\pi) - \boldsymbol{x}_T\|^2$.

$$\boldsymbol{p}(\pi) = \nabla_{\boldsymbol{x}} \Psi(\boldsymbol{x}) = (\boldsymbol{x}(\pi) - \boldsymbol{x}_T) \tag{4.6}$$

2. **Adjoint evolution equation**:

$$\dot{\boldsymbol{p}}(\tau) = -\nabla_x H(u(\tau), \boldsymbol{p}(\tau), \tau) = 0 \tag{4.7}$$

From where it can be deduced that $\boldsymbol{p}(\tau) = \text{cte}$ so that $\boldsymbol{p}(\tau) = \boldsymbol{x}(\pi) - \boldsymbol{x}_T \mid \forall \tau \in [0, \pi)$ so from now on we will refer to it simply as $\boldsymbol{p}$, noting that it is invariant in time.

3. **Optimal control shape**: We known that $u^* = \arg\min_{|u|<1} H(\tau, \boldsymbol{p}^*, u)$, so in this case we can write:

$$u^*(\tau) = \underset{|u|<1}{\arg\min} \left[\epsilon \mathcal{L}(u(\tau)) + [\boldsymbol{p}^{*T} \cdot \boldsymbol{\mathcal{D}}(\tau)]u(\tau) \right] \tag{4.8}$$

Therefore, this optimality condition reduces to the optimization of a function in a variable within the interval $[-1, 1]$. We need note that the value of $[\boldsymbol{p}^{*T} \cdot \boldsymbol{\mathcal{D}}(\tau)]$ is unknow but we know that this value can change the sign. So we need design the $\mathcal{L}$ to the optimal control u^* must be a some element of $\mathcal{U}$ for every value of $[\boldsymbol{p}^{*T} \cdot \boldsymbol{\mathcal{D}}(\tau)]$. For this study, we build a function $H^* : [-1, 1] \to \mathbb{R}$ such that:

$$H^*(u) = \epsilon \mathcal{L}(u) + mu \mid \forall m \in \mathbb{R} \tag{4.9}$$

If we found a function $\mathcal{L}$ such that its minimums in interval $[-1, 1]$ are in $\mathcal{U}$ the solutions of regularized optimal control are in $\mathcal{U}$.

4.2 Conditions for bi-level and tri-level controls

Since we need the optimal control u^* to be *bang-bang*, we must design $H^*(u)$ so that its minimum is at the extremes of the interval $[-1, 1]$. In the case of a variable (this case) we can only achieve this behavior if there is no minimum within the interval. One way to ensure this behavior is to take the function $H^*(u)$ concave, by choosing the term $\mathcal{L}(u)$. This is because the concavity of the term $\mathcal{L}(u)$ determines the concavity of the Hamiltonian $H^*(u)$ thanks to the fact that the second derivative of $H^*(u)$ is proportional to the second derivative of the penalty term, that is:

$$\frac{d^2 H^*}{du^2} = \epsilon \frac{d^2 \mathcal{L}}{du^2} \tag{4.10}$$

In this way whenever we choose a penalty term such that:

$$\frac{d^2\mathcal{L}}{du^2} \leq 0 \tag{4.11}$$

we will obtain a concave $H^*(u)$ function within a convex interval giving rise to an optimal control u^* emph bang-bang. We can see an illustration of this behavior in the figure (1).

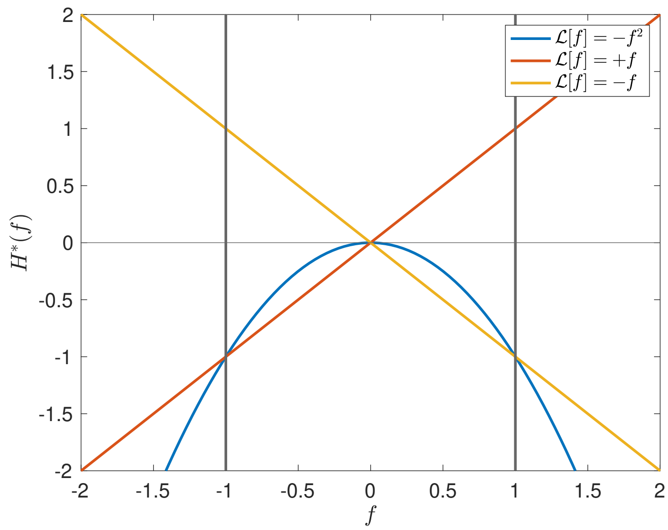


Figure 1: Ilustración sobre el comportamiento de la función $H^*(u)$ para distintos términos de penalización $\mathcal{L}(u)$. El problema de minimización de $H^*(u)$ con respecto a f para las distintas curvas presentadas en la figura siempre tiene minimos en los extremos del intervalo $[-1, 1]$ de manera que el control óptimo en los tres casos, será *bang-bang*.

4.3 Condiciones para la obtención de controles multi-nivel

Dado que necestiamos que existan mínimos de la función $H^*(u)$ en el interior del intervalo $[-1, 1]$ la función $\mathcal{L}$ no podrá ser convexa. Además si queremos que los minimimos para cualquier valor de $m \in \mathbb{R}$ solo tome valores finitos esta función de ser no diferenciable. De manera que un variación infinitesimal de m mantenga constante el mínimo de $\mathcal{L}(u)$. Proponemos una función $\mathcal{L}$ como una aproximación afin tomando como vértices los puntos pertenecientes al conjunto $\mathcal{U}$.

5 Experimentos numéricos

En esta sección mostraremos soluciones obtenidas numéricamente del problema de control presentado. Para ello primero explicaremos como la metodología que utilizamos para resolver los problemas de control (4.2) y (**??**), y luego mostraremos casos concretos mostrando la precisión obenida y algunos ejemplos donde el número de ángulos varía con respecto al valor de los coeficientes de Fourier considerados.

5.1 Numerical solution of OCP-SHE

To solve the optimal control problem (4.2), we use a direct method. If we consider a partition $\mathcal{P} = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of interval $[0, T]$, we can represent a function $\{u(\tau) \mid \tau \in [0, T]\}$ as a vector $\boldsymbol{f} \in \mathbb{R}^T$ where component $f_t = u(\tau_t)$. Then the optimal control problem (4.2) can be written as optimization problem with variable $\boldsymbol{f} \in \mathbb{R}^T$. This problem is a nonlinear programming, for this we use CasADi software to solve. Entonces, dado una partición del intervalo $[0, \pi)$ podemos reformular el problema (4.2) como el siguiente problema en tiempo discreto:

Problem 5.1 (OCP numérico con simetría de media onda) Dados dos conjuntos de números impares $\mathcal{E}_a$ and $\mathcal{E}_b$ con cardinalidades $|\mathcal{E}_a| = N_a$ y $|\mathcal{E}_b| = N_b$ respectivamente , dados los vectores objetivos $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ y $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$ y una partition $\mathcal{P} = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of interval $[0, \pi)$. We search a vector $\boldsymbol{f} \in \mathbb{R}^T$ that minimize the following function:

$$\min_{\boldsymbol{f} \in \mathbb{R}^T} \left[||\boldsymbol{a}_T - \boldsymbol{\alpha}^T||^2 + ||\boldsymbol{b}_T - \boldsymbol{\beta}^T||^2 + \epsilon \sum_{t=0}^{T-1} \mathcal{L}(u_t) \Delta\tau_t \right] \tag{5.1}$$

$$\text{suject to:}$$

$$i \in \mathcal{E}_a \quad \begin{cases} \alpha_i^{t+1} = \alpha_i^t + \Delta\tau_t (2/\pi) \cos(i\tau_t) f_t \\ \alpha_i^0 = 0 \end{cases}$$

$$j \in \mathcal{E}_b \quad \begin{cases} \beta_j^{t+1} = \beta_j^t + \Delta\tau_t (2/\pi) \sin(j\tau_t) f_t \\ \beta_j^0 = 0 \end{cases}$$

$$|f_t| < 1 \mid \Delta\tau_t = \tau_{t+1} - \tau_t \mid \forall t \in \{1, \ldots, T-1\}$$

En el caso del problema con simetría de cuarto de onda siguiendo la misma linea argumental anterior podemos formular el siguiente problema de optimización:

Problem 5.2 (OCP numérico con simetría de cuarto de onda) Given a set of odd numbers $\mathcal{E}_b$ with carinality $|\mathcal{E}_b| = N_b$, given the target vector $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$ and partition $\mathcal{P} = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of interval $[0, \pi/2]$. We search a vector $\boldsymbol{f} \in \mathbb{R}^T$ that minimize the following function:

$$\min_{\boldsymbol{f} \in \mathbb{R}^T} \left[||\boldsymbol{b}_T - \boldsymbol{\beta}^T||^2 + \epsilon \sum_{t=0}^{T-1} \mathcal{L}(u_t) \Delta\tau_t \right] \tag{5.2}$$

$$\text{suject to:}$$

$$j \in \mathcal{E}_b \quad \begin{cases} \beta_j^{t+1} = \beta_j^t + \Delta\tau_t (4/\pi) \sin(j\tau_t) f_t \\ \beta_j^0 = 0 \end{cases}$$

$$|f_t| < 1 \mid \Delta\tau_t = \tau_{t+1} - \tau_t \mid \forall t \in \{1, \ldots, T-1\}$$

5.2 Resultados

Cabe mencionar que todas las simulaciones se han realizado con un ordenador de mesa con $8Gb$ de ram, y el tiempo de ejecución para la búsqueda de soluciones dado un vector objetivo es del orden del segundo. A continuación listaremos cada uno de los resultados numéricos obtenidos:

1. **OCP con simetría de cuarto de onda**: Consideraremos el problema con un conjunto de números impares $\mathcal{E}_b = \{1, 5\}$ con una discretización del intervalo $[0, \pi/2]$ de $T = 200$. Mostramos las soluciones para los vectores objetivo $b_T^1 = \{(-0.4, -0.3, \dots, 0.3, 0.4)\}$ manteniendo $b_T^5 = 0$ para todos los casos. Mostramos las trayectorias óptimas obtenidas en la figura (3), donde se puede ver una continuidad en las soluciones con respecto a vector objetivo.

2. **OCP con simetría de cuarto onda para un intervalo del** b_1: Para este ejemplo consideramos el siguiente conjunto de números impares: $\mathcal{E}_b = \{1, 5, 7, 11, 13\}$. Ademas consideramos el vector objetivo $\boldsymbol{b}_T = [m_a, 0, 0, 0, 0]$, donde $m_a \in [-1, 1]$ es un parámetro. with three penalization terms: $\mathcal{L}(u) = -f$, $\mathcal{L}(u) = +f$ and $\mathcal{L}(u) = -f^2$ obtained by direct method with uniform partition of interval $[0, \pi/2]$ with $T = 400$ and penalization parameter $\epsilon = 10^{-5}$. Para cada uno de los términos de penalización utilizados la distancia entre los coeficientes de Fourier se encuentra en el orden de 10^{-4}. Sin embargo, cuando el término de penalización es $\mathcal{L}(u) = -f^2$ la solución no presenta continudad con respecto al vector objetivo. Por otra parte, es importante mencionar que las soluciones para los términos de penalización $\mathcal{L}(u) = -f$ y $\mathcal{L}(u) = f$ cumplen una simetría por lo que invirtiendo las soluciones con respecto al origen y invirtiendo el signo de las soluciones se puede ver que ambas soluciones son la misma.

3. **SHE para tres niveles**: Podemos ver que en el caso en el que el control $u(\tau)$ solo pueda tomar valores entre $[0, 1]$ obtenemos señales que pueden tomar tres niveles en el intervalo $[0, 2\pi]$ gracias a la simetría de cuarto de onda. Si resolvemos el problema de control óptimo pero esta vez cambiando las restricciones $|u(\tau)| < 1$ por $\{0 < u(\tau) < 1\}$. Se ha realizado el mismo procedimiento que en el caso anterior, obteniendo soluciones para los mismo términos de penalización obteniendo la figura (7). Allí se muestra la continudad de las soluciones y que estas se encuentran en el orden de 10^{-4}.

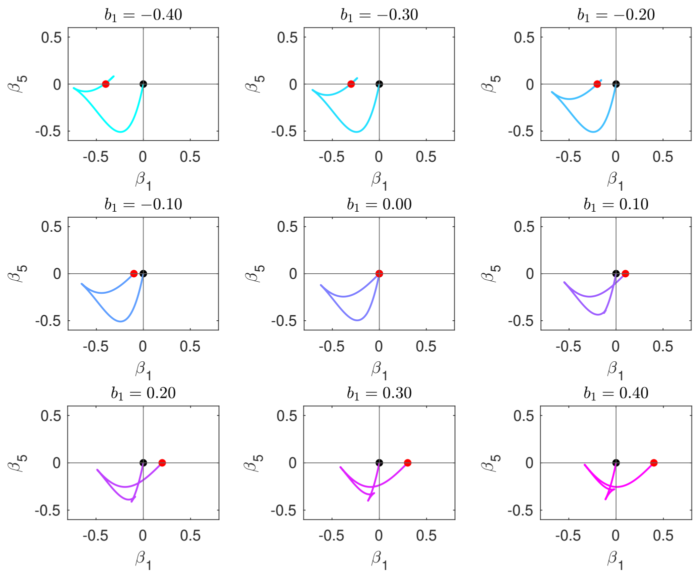
4. **Cambio en el número de conmunationes**: Gracias a la formulación de control óptimo para el problema SHE podemos variar el número de ángulos de conmuntación. Este es el cado del siguientes ejemplo, donde hemos tomado como conjunto de números pares $\mathcal{E}_b = \{1, 3, 9, 13, 17\}$, además consideramos el vector objetivo $\boldsymbol{b}_T = [m_a, 0, 0, 0, 0]$, donde $m_a \in [0, 1]$ es un parámetro. En este problema hemos utilizado una penalización tipo $\mathcal{L} = f$ con un parémetro de penalización $\epsilon = 10^{-4}$. Podemos ver en la figura (5) como el problema de control óptimo es versátil y es capaz de mover entre varios conjuntos de soluciones.

5. **OCP para SHE con simetría de media onda**: Se ha relizado el caso de control óptimo de media onda con con $\mathcal{E}_a = \{1, 3, 5\}$ y $\mathcal{E}_b = \{1, 3, 5, 9\}$, donde $\boldsymbol{a}_T = [m_a, 0, 0]$, $\boldsymbol{b}_T = [m_a, m_a, 0, 0]$ y $m_a \in [-0.6, 0.6]$. Se ha elegido la penalización $L(u) = +f$

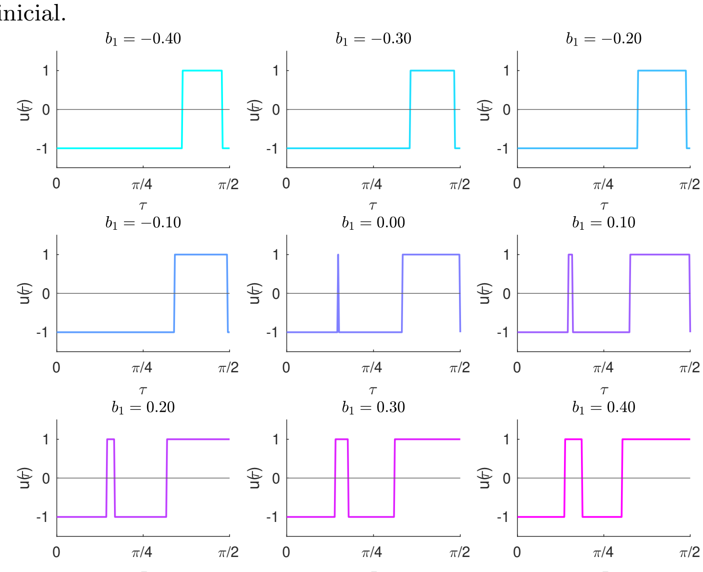
6 Conclusiones

Se ha presentado el problema SHE desde un punto de vista de la teoría de control. Esta metodología es efectiva para llega a presiciones $10^{-4} - 10^{-5}$ en la distancia al vector objetivo. Sin embargo en comparación con metodologías donde el número de conmutaciones es prefijado, nuestra aproximación es más costosa. Sin enbargo, el control óptimo asegura soluciones en todo el rango de indice de modulación, aunque el número de soluciones o la localización de estas cambien abruptamente.

Este plantamiento del problema SHE enlaza la teoría de control con la eliminación de harmónicos. De esta manera el problema SHE se puede resolver mediante herramientras clásicas.



(a) Dynamical System: el punto rojo hace referencia al punto final mientras que el punto negro hace referencia al punto inicial.

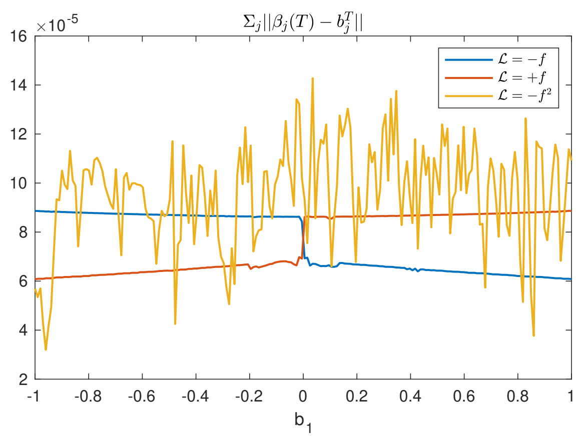


(b) Control

Figure 2: Mostramos las trayectorias óptimas y controles óptimos para distintos vectores objetivo.

$\mathcal{L} = -f$ $\mathcal{L} = +f$ $\mathcal{L} = -f^2$

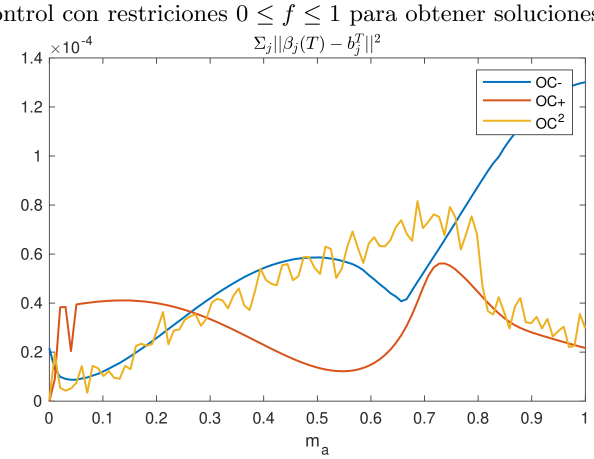
(a) Comparison of solutions for different values of m_a. El problema con penalización $L(u) = -f$ y $L(u) = +f$ son equivalentes bajo una tranformación de inversión conrespecto a origen de coordendas y un cambio de signo.



(b) Error

Figure 3: The order of magnitud of the distance of target

(a) Soluciones para un control con restriciones $0 \leq f \leq 1$ para obtener soluciones de tres niveles.



(b) error

Figure 4: Errors of solution in three level solutions

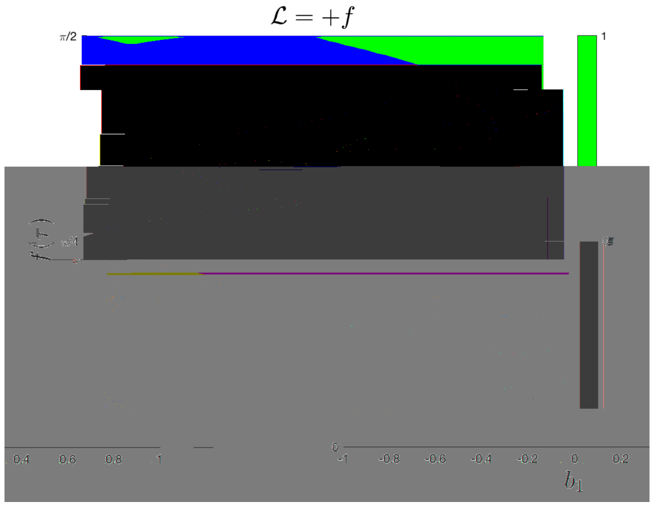


Figure 5: Discontinuidad en las soluciones obtenidas. Soluciones para SHE con simetría cuarto de onda con $\mathcal{E}_b = \{1, 3, 9, 13, 17\}$, donde $\boldsymbol{b}_T = [m_a, 0, 0, 0, 0]$ y $m_a \in [0, 1]$

References

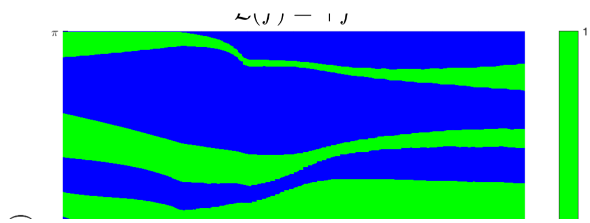


Figure 6: Discontinuidad en las soluciones obtenidas. Soluciones para SHE con simetría media de onda con $\mathcal{E}_a = \{1,3,5\}$ y $\mathcal{E}_b = \{1,3,5,9\}$, donde $\boldsymbol{a}_T = [m_a, 0, 0]$, $\boldsymbol{b}_T = [m_a, m_a, 0, 0]$ y $m_a \in [-0.6, 0.6]$

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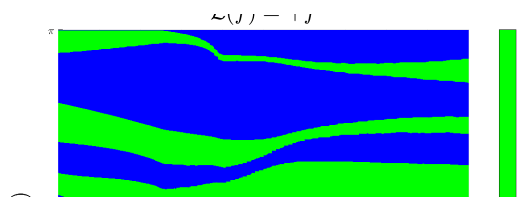
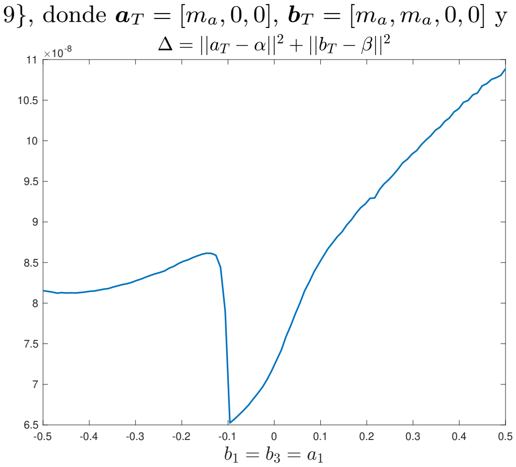


(a) Discontinuidad en las soluciones obtenidas. Soluciones para SHE con simetría media de onda con $\mathcal{E}_a = \{1,3,5\}$ y $\mathcal{E}_b = \{1,3,5,9\}$, donde $\boldsymbol{a}_T = [m_a, 0, 0]$, $\boldsymbol{b}_T = [m_a, m_a, 0, 0]$ y $m_a \in [0, 1]$



(b) error

Figure 7: Errors of solution in three level solutions