Selective Harmonic Elimination via Optimal Control

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Abstract

El problema de *Selective Harmonic Elimination pulse-width modulation*(SHE) es planteado como el problema de control óptimo, con el fin de encontrar soluciones de ondas escalón sin prefijar el número de ángulos de conmutación. De esta manera, la metodología de control óptimo es capaz de encontar la forma de onda óptima y de encontrar la localizaciones de los ángulos de conmutación, incluso sin prefijar el número de conmutaciones. Este es un nuevo enfoque para el problema SHE en concreto y para los sistemas de control con un conjunto finito de controles admisibles en general.

Key words: Selective Harmonic Elimination; Finite Set Control, Piecewise Linear function.

1 Introduction and motivations

Selective Harmonic Elimination (SHE) [Rodríguez et al., 2002] is a well-known methodology in electrical engineering, employed to improve the performances of a converter by controlling the phase and amplitude of the harmonics in its output voltage. As a matter of fact, this technique allows to increase the power of the converter and, at the same time, to reduce its losses.

In broad terms, the process consists in generating a *control signal* with a desired harmonic spectrum, by modulating or eliminating some specific lower order frequencies. This signal is in the shape of a step function and is fully characterized by two features (see Figure 1):

1. The *waveform*, i.e. the set of (constant) values the function may assume.
2. The *switching angles*, defining the points in the domain where the function changes from one constant value to another.

Because of the growing complexity of modern electrical networks, consequence for instance of the high penetration of renewable energy sources, the demand in power of electronic converters is day by day increasing. For this and other reasons, SHE has been a preeminent research interest in the electrical engineering community, and a plethora of SHE-based techniques has been developed in recent years. An incomplete bibliography includes [Duranay and Guldemir, 2017, Janabi et al., 2020, Yang et al., 2017].

Nowadays, SHE is mostly based on offline computations to obtain the commutation patterns describing the control signal.

Add references and mention real-time approaches.

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2 Mathematical formulation of SHE

This section is devoted to the mathematical formulation of the SHE problem. In what follows, with the notation $\mathcal{U}$ we will always refer to a finite set of real numbers, contained in the interval $[-1, 1]$

$$\mathcal{U} = \{u_\ell\}_{\ell=1}^{L} \subset [-1, 1], \tag{2.1}$$

with cardinality $|\mathcal{U}| = L$.

In broad terms, our objective is to design a piece-wise constant function $u(\tau) : [0, 2\pi) \to \mathcal{U}$ such that some of

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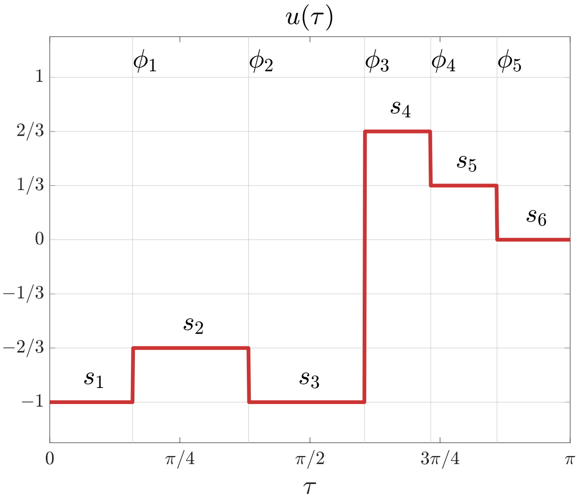

Fig. 1. A Step function and possible solution of Problem 2.1, where we considered a possible finite set of control $\mathcal{U} = \{-1/3, -2/3, 0, 1/3, 2/3, 1\}$. We show the switching angles $\boldsymbol{\phi}$ and the waveform $\mathcal{S}$ (see Definitions 3.1 and 3.2). tengo que darle la vuelta en el tiempo

its lower-order Fourier coefficients take specific values determined a priori.

Due to the application in power converters, we will focus here on functions with *half-wave symmetry*, i.e.

$$u(\tau + \pi) = -u(\tau) \quad \text{for all } \tau \in [0, \pi).$$

In this way, u is fully determined by its values in the interval $[0, \pi)$ and its Fourier series only involves the odd terms, thus taking the form

$$u(\tau) = \sum_{\substack{i \in \mathbb{N} \\ i\ odd}} a_i \cos(i\tau) + \sum_{\substack{j \in \mathbb{N} \\ j\ odd}} b_j \sin(j\tau), \tag{2.2}$$

where the coefficients a_i and b_j are given by

$$\begin{aligned} a_i &= \frac{2}{\pi} \int_0^\pi u(\tau) \cos(i\tau)\, d\tau, \\ b_j &= \frac{2}{\pi} \int_0^\pi u(\tau) \sin(j\tau)\, d\tau. \end{aligned} \tag{2.3}$$

Because of this half-wave symmetry, in what follows, we will always work with the restriction $u|_{[0,\pi)}$ which, with some abuse of notation, we shall still denote by u. We can then give a general formulation of the SHE problem as follows:

Problem 2.1 (SHE) *Let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two sets of odd numbers with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$, respectively, and $\mathcal{U}$ be defined as in (2.1). Given the vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, we look for a piece-wise constant function $u : [0, \pi) \to \mathcal{U}$ such that*

$$\begin{aligned} a_i &= (\boldsymbol{a}_T)_i, \quad \textit{for all } i \in \mathcal{E}_a, \\ b_j &= (\boldsymbol{b}_T)_j, \quad \textit{for all } j \in \mathcal{E}_b, \end{aligned}$$

with $\{a_i\}_{i \in \mathcal{E}_a}$ and $\{b_j\}_{j \in \mathcal{E}_b}$ given by (2.3).

Fig. 1 shows an example of a function u solution of the SHE problem.

3 Classical approach to the SHE problem

En esta sección presentaremos la aproximación actual que se utiliza para abordar el problema, y mencionaremos sus problemas asociado. As we anticipated in Section 1, the control signal u is fully characterized by its waveform and the switching angles, to which we now give a precise definition.

Definition 3.1 (Wave-form) *Given the finite set $\mathcal{U}$ defined in (2.1) and $M \in \mathbb{N}$, we will call a waveform any possible $(M+1)$-tuple $\mathcal{S} = (s_m)_{m=1}^{M+1}$ with $s_m \in \mathcal{U}$ for all $m = 1, \ldots, M+1$.*

Definition 3.2 (Switching angles) *Given the finite set $\mathcal{U}$ defined in (2.1), $M \in \mathbb{N}$ and a piece-wise constant function $u : [0, \pi) \to \mathcal{U}$, we shall refer as switching angles $\boldsymbol{\phi} = \{\phi_m\}_{m=0}^{M+1} \subset [0, \pi]$, with $\phi_0 = 0$ and $\phi_{M+1} = \pi$, to the points in the domain $[0, \pi)$ where u changes its value.*

In view of the above definitions, we can provide the following explicit expression for u:

$$\begin{aligned} u &= \sum_{m=1}^{M+1} s_m \chi_{[\phi_m, \phi_{m+1}]} \\ s_m &\in \mathcal{S},\ \phi_m \in \boldsymbol{\phi}, \quad \text{for all } m = 1, \ldots, M+1, \end{aligned} \tag{3.1}$$

where we denoted by $\chi_{[\phi_m, \phi_{m+1}]}$ the characteristic function of the interval $[\phi_m, \phi_{m+1}]$.

Besides, taking into account (3.1), a direct computation yields that the Fourier coefficients (2.3) are given by

$$a_i = a_i(\boldsymbol{\phi}) = \frac{2}{i\pi} \sum_{k=1}^{M+1} s_k \Big[\sin(i\phi_k) - \sin(i\phi_{k-1}) \Big]$$

$$b_j = b_j(\boldsymbol{\phi}) = \frac{2}{j\pi} \sum_{k=1}^{M+1} s_k \Big[\cos(j\phi_{k-1}) - \cos(j\phi_k) \Big]$$

Given a waveform $\mathcal{S}$, Problem 2.1 then reduces to find the switching locations $\boldsymbol{\phi}$ (see [Yang et al., 2015, Konstantinou and Agelidis, 2010, Sun et al., 1996]). This can be cast as a minimization problem in the variables $\{\phi_m\}_{m=0}^{M+1}$, where the cost functional is the Euclidean distance between the obtained Fourier coefficients

$\{a_i(\boldsymbol{\phi}), b_j(\boldsymbol{\phi})\}$ and the targets $(\boldsymbol{a}, \boldsymbol{b}) \in \mathbb{R}^{N_a} \times \mathbb{R}^{N_b}$.

Problem 3.1 (Optimization for SHE) *Given a waveform $\mathcal{S}$ and a step function u in the form (3.1), we look for the switching angles locations $\boldsymbol{\phi}$ by means of the following minimization problem:*

$$\min_{\boldsymbol{\phi}\in[0,\pi]^M} \left(\sum_{i\in\mathcal{E}_a} \|a_T^i - a_i(\boldsymbol{\phi})\|^2 + \sum_{j\in\mathcal{E}_b} \|b_T^j - b_j(\boldsymbol{\phi})\|^2 \right)$$

subject to: $0 = \phi_0 < \phi_1 < \ldots < \phi_M < \phi_{M+1} = \pi$

A continuación mecionaremos algunos problemas de esta formulación:

(1) **Combinatory problem**: Since the cardinality of $\mathcal{S}$ is not known a priori, meaning that we do not know how many switches will be necessary to reach the desired values of the Fourier coefficients, a common approach to solve the SHE problem consists in fixing the number of changes and generating all the possible combinations of elements of $\mathcal{S}$, to later solve an optimization problem for each one of them. Nevertheless, taking into account that the number of possible tuples $\mathcal{S}$ is given by $|\mathcal{U}|^{|\mathcal{S}|}$, it is evident that the complexity of the above approach increases rapidly. This problem has been studied in [Yang et al., 2015] where, through appropriate algebraic transformations, the authors are able to convert the SHE problem into a polynomial system whose solutions' set contains all the possible waveforms for a given set $\mathcal{U}$ and number of elements in the sequence $\mathcal{S}$. Sin embargo, en este caso el número de cambios es prefijado.

(2) **Continuity problem**: It is well-known that, fixed a sequence S, the continuity of the switching locations with respect to a continuous variation of the target Fourier coefficients may be quite cumbersome. De manera que, encontrar solución donde el número de ángulos pueda cambiar con el vector objetivo, nos puede dar más libertad para obtener soluciones más predecibles.

4 Our Contributions

We will present the SHE problem as an optimal control one, where the optimization variable is the signal $u(\tau)$ defined in the entire interval $[0, \pi)$. In particular, we will describe how the Fourier coefficients of the function $u(\tau)$ can be seen as the final state of a system controlled by $u(\tau)$. Además este control deberá ser una función definida a trozos tal que $u(\tau) : [0, \pi) \to \mathcal{U}$.

Con el fin de aligerar la notación defeniremos control digital como:

Definition 4.1 (Digital control of $\mathcal{U}$) *A control $u(\tau)$ is called digital if, for each time $\tau \geq 0$, it only takes values in the finite set of real number $\mathcal{U}$ except a finite set of values (switching angles).*

Hence, the optimization is performed among all the possible functions that satisfy $|u(\tau)| < 1$ and can control the final state at the desired Fourier coefficients. Then we will show how to design a control problem so that the solution is a digital control of $\mathcal{U}$.

In this formulation, the SHE problem converts in a minimization problem with restrictions which can be solved by well-known techniques. Since the problem has several minimizers, we shall solve it employing global optimizers. Furthermore, since the choice of the waveform is arbitrary, we shall proceed in the same way for each possible waveform.

A continuación enumeraremos nuestras aportaciones, y luego explicaremos cada una de ella en detalle en las siguientes subsecciones:

(1) Reformulación del Problema 2.1 como un problema de control óptimo.
(2) Una función penalización en el problema de control óptimo de manera que podemos conseguir controles considerando el conjunto admisible $\mathcal{U} = \{-1, 1\}$(*bang-bang controls*).
(3) Una función de penalización en funcional de coste del problema de control óptimo de manera que podemos conseguir conjuntos $\mathcal{U}$ más generales.
(4) Condiciones suficientes de las funciones de penalización para obtener controles.

4.1 Reformulation of SHE problem as optimal control problem

Teniendo en cuenta tenemos todos los elementos considerados en la definición del Problema 2.1, introducimos el siguiente sistema dinámico:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = -\dfrac{2}{\pi}\boldsymbol{\mathcal{D}}(\tau)u(\tau), & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases}, \tag{4.1}$$

Where $\boldsymbol{x}(t) \in \mathbb{R}^{N_a+N_b}$ and $\boldsymbol{\mathcal{D}}(\tau) \in \mathbb{R}^{N_a+N_b}$ are:

$$\boldsymbol{x}(\tau) = \begin{bmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{bmatrix}, \quad \boldsymbol{\mathcal{D}}(\tau) = \begin{bmatrix} \boldsymbol{\mathcal{D}}^\alpha(\tau) \\ \boldsymbol{\mathcal{D}}^\beta(\tau) \end{bmatrix}$$

y donde:

$$\mathcal{D}^{\alpha}(\tau) = \begin{bmatrix} \cos(e_a^1\tau) \\ \cos(e_a^2\tau) \\ \vdots \\ \cos(e_a^{N_a}\tau) \end{bmatrix}, \quad \mathcal{D}^{\beta}(\tau) = \begin{bmatrix} \sin(e_b^1\tau) \\ \sin(e_b^2\tau) \\ \vdots \\ \sin(e_b^{N_b}\tau) \end{bmatrix}$$

with $\mathcal{D}^{\beta}(\tau) \in \mathbb{R}^{N_a}$ and $\mathcal{D}^{\beta}(\tau) \in \mathbb{R}^{N_b}$, and where the set $\mathcal{E}_a$ and $\mathcal{E}_b$ are:

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \dots, e_a^{N_a}\}, \quad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \dots, e_b^{N_b}\}$$

Dado este sistema podemos considerar el siguiente problema de control

Problem 4.1 *Let $\mathcal{U}$ be defined as in (2.1) and let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two sets of odd numbers with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$, respectively. Given the vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, let us define $\boldsymbol{x}_0 = [\boldsymbol{a}_T, \boldsymbol{b}_T]^\top \in \mathbb{R}^{N_a} \times \mathbb{R}^{N_b}$. We look for $u :\in [0, \pi) \to \mathcal{U}$ such that the solution of (6.4) with initial datum $\boldsymbol{x}(0) = \boldsymbol{x}_0$ satisfies $\boldsymbol{x}(\pi) = 0$.*

Theorem 4.1 *El control óptimo $u(\tau)$, solución del Problema 4.1, es el función constante a trozos que resuleve el Problema 2.1.* ■

Esta proposición es una consecuencia directa del teorema funamental del cálculo. En donde se ha considerado la formula integral de los coeficientes de Fourier (2.3) en su forma diferencial, de manera que la integración desde $\tau = 0$ hasta $\tau = \pi/2$ es equivalente a la evolución del sistema (6.4) en el mismo intervalo temporal.

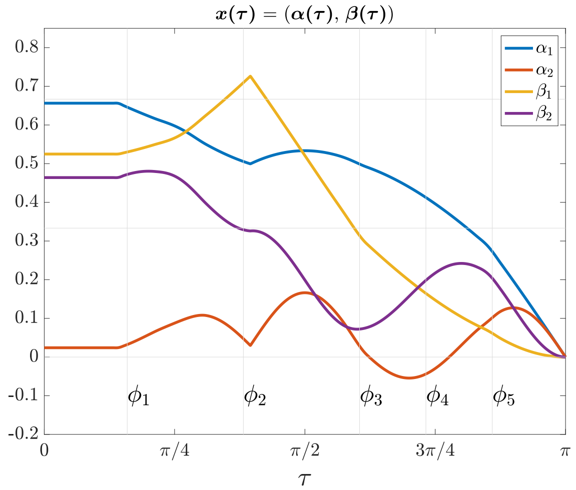

Fig. 2. Evolución del sistema dinámico con los conjuntos $\mathcal{E}_a = \{1, 2\}$ y $\mathcal{E}_b = \{1, 2\}$ considerando el control $u(\tau)$ presentado en la Figura 1. Además mostramos las posiciones de los ángulos de conmutación $\boldsymbol{\phi}$.

Gracias a este resultado podemos plantearnos el problema de control óptimo asociado al Problema 4.1.

In what follows, for a given vector $\boldsymbol{v} \in \mathbb{R}^d$, we shall always denote by $\|\boldsymbol{v}\|$ the euclidean norm $\|\boldsymbol{v}\|_{\mathbb{R}^d}$.

Problem 4.2 (OCP for SHE) *Let $\mathcal{U}$ be defined as in (2.1). Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinality N_a and N_b, respectively, and given the target $\boldsymbol{x}_T \in \mathbb{R}^{N_a+N_b}$, we look for the function $u(\tau) : [0, \pi) \to \mathcal{U}$ solution of the optimal control problem*

$$\min_{u \in \mathcal{U}} \frac{1}{2}\|\boldsymbol{x}(\pi)\|^2$$

$$\textit{subject to:} \quad \begin{cases} \dot{\boldsymbol{x}}(\tau) = -\frac{2}{\pi}\mathcal{D}(\tau)u(\tau), & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases}$$

The solution of Problem 4.2 may be quite complex to be obtained, due to the restriction on the admissible control values.

In order to bypass this difficulty, following a standard optimal control approach, we can formulate an equivalent minimization problem in which, instead of looking for $u \in \mathcal{U}$, we simply require that $|u| < 1$ and we introduce a penalization term to ensure that u is digital control of $\mathcal{U}$.

This alternative optimal control problem, which can be solved more easily by employing standard tools, reads as follows:

Problem 4.3 (Penalized OCP for SHE) *Fix $\epsilon > 0$. Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ and the target $\boldsymbol{x}_T \in \mathbb{R}^{N_a+N_b}$, we look a digital control of $\mathcal{U}$ as the solution of:*

$$\min_{|u|<1} \left[\frac{1}{2}\|\boldsymbol{x}(\pi)\|^2 + \epsilon \int_0^\pi \mathcal{L}(u(\tau))d\tau \right]$$

under the dynamics given by (6.4).

In Problem 4.3, the penalization function $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ will be chosen such that the optimal control u^* only takes values in $\mathcal{U}$. Furthermore, the parameter ϵ should be small so that the solution minimizes the distance from the final state and the target.

En general, un función de penalización que nos de un control digital de $\mathcal{U}$ no es evidente de conseguir. Si consideramos las condiciones de optimilidad del sistema (Principio del mínimo de Pontryagin) podemos obtener una función $\mathcal{H}_m(u)$ (la función Hamiltoniana) que caracteriza el comportamiento de la solución del Problema 5.1.

Definition 4.2 (Hamiltoniano) *Dado el problema 4.3 definimos una función $\mathcal{H}_m : [-1, 1] \to \mathbb{R}$ tal que:*

$$\mathcal{H}_m(u) = \epsilon\mathcal{L}(u) - mu \mid \forall m \in \mathbb{R} \tag{4.2}$$

De esta manera podemos decir que:

Proposition 4.1 *Considerando el Problema 4.3 y su función Hamiltoniana $\mathcal{H}_m$. Si para todo $m \in \mathbb{R} - \mathcal{M}$ todos los mínimos de la función Hamiltoniana están en $\mathcal{U}$, entonces la solución del Problema 4.3 es un control digital de $\mathcal{U}$. Donde $\mathcal{M}$ pueder ser el conjunto vacio o un conjunto de elementos finitos.* ■

Gracias a la proposición 4.1. Podemos diseñar las funiones de penalización $\mathcal{L}$ de manera que la función Hamiltonina $\mathcal{H}_m$ solo tenga mínimos en $\mathcal{U}$.

4.2 SHE bi-nivel via OCP (Bang-Bang Control)

En esta Subsección consideraremos el caso donde el conjunto de controles admisibles es $\mathcal{U} = \{-1, 1\}$. En la literatura de SHE, este tipo de soluciones son llamados como formas de onda bi-nivel, mientras que el la literatura de la teoría de control este es conocido como controles *bang-bang*.

Theorem 4.2 *Dado el Problema 4.3 con el conjunto admisible de control $\mathcal{U} = \{-1, 1\}$. Si la función $\mathcal{L}$ es concava en el intervalo $[-1, 1]$ del Problema 4.3, entonces la solución de problema es un control digital del conjunto $\mathcal{U} = \{-1, 1\}$.* ■

Si aceptamos la Proposición 4.2 es evidente que para obtener un control digital de $\mathcal{U} = \{-1, 1\}$ los mímimos Hamiltoniano $\mathcal{H}_m$ para todo $m \in \mathbb{R} - \mathcal{M}$ deben encontrarse en los bordes del intervalo $[-1, 1]$ de manera que es suficiente considerar una función concava en el interior de este intervalo.

Ilustramos esta idea en la Figura 3, de manera que en la primera columna presentamos tres funciones de penalización compatible con la Proposición 4.2 ($\mathcal{L}(u) = u$, $\mathcal{L}(u) = -u$, $\mathcal{L}(u) = -u^2$). Además mostramos a la derecha de cada una de ellas mostramos el comportamiento del Hamiltoniano para distintos valores de m en diferentes colores y para cada una de ellas marcamos el mímimo del mismo color. Podemos ver que los mínimos en cada caso siempre siempre esta en $\mathcal{U} = \{-1, 1\}$.

4.3 SHE multi-nivel via Piecewise linear penalization

Gracias al caso bi-nivel podemos notar que si definimos una función de penalización $\mathcal{L}$ que entre dos valores u_1 y u_2 no tenga mímimos, es decir sea concava entre esos

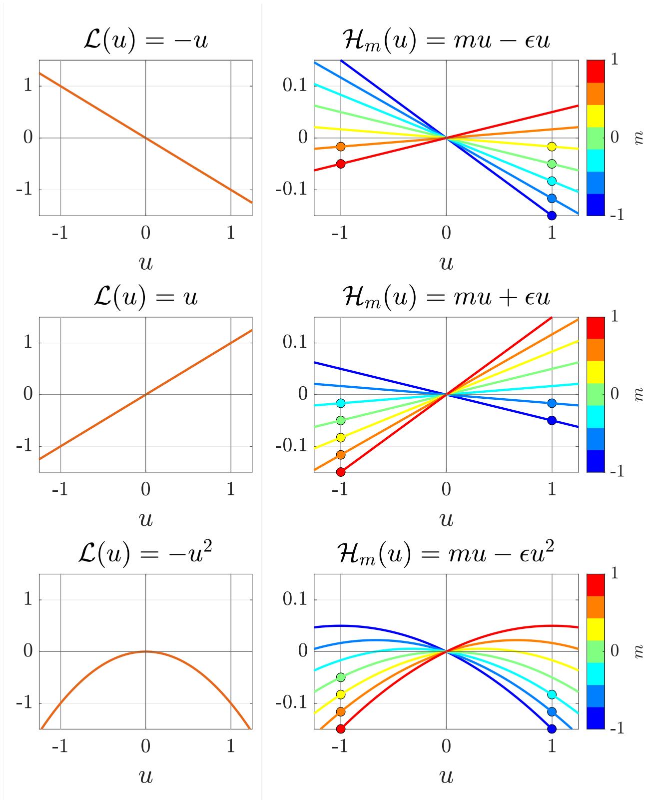

Fig. 3. SHE bi-nivel.
En la primera columna mostramos tres tipos de penalización concavas compatibles con la Proposición 4.2. A la derecha de cada una de ellas mostramos el comportamiento del Hamiltoniano para cada distintos valores de m (colores).

dos puntos entonces la función Hamiltoniana no tendrá el mímimos en $u \in (u_1, u_2)$ más que para un valor de concreto de m. Gracias a esta observación podemos afirmar que:

Theorem 4.3 *Given a set $\mathcal{U}$, we can choose the affine interpolation of a parabola $\mathcal{L}^{p_!} : [-1, 1] \to \mathbb{R}$ as a penalization term. That is*

$$\mathcal{L}^{p_1}(u) = \begin{cases} \lambda_k^{p_1}(u) & \text{if } u \in [u_k, u_{k+1}[\\ 1 & \text{if } u = u_{N_u} \end{cases} \tag{4.3}$$
$$\forall k \in \{1, \ldots, N_u - 1\}$$

Donde

$$\lambda_k^{p_1}(u) = (u_{k+1} + u_k)(u - u_k) + u_k^2 \tag{4.4}$$

De modo que el problema de 4.3 con la penalización $\mathcal{L}(u)$ presentada antes tiene como solución un control digital del conjunto $\mathcal{U}$ ■

Remark 4.1 (*Bang-off-bang*) *En este caso, podemos notar que cuando consideramos un conjunto de controles admisibles como $\mathcal{U} = \{-1, 0, 1\}$ recuperamos la penalización de norma L^1. Este tipo de penalización añadido* *a la restricción $|u| < 1$ nos da controles* bang-off-bang, *bien conocidos en la literatura de control añadir referencias.*

De la misma forma que en caso anterior en la Figura 4 ilustramos como la penalización (4.3) logra cumplir las condiciones de la proposición 4.1. Y por tanto esta penalización da lugar a una control óptimo digital de $\mathcal{U}$.

4.4 General conditions for SHE multi-nivel

Como podemos observar el caso anterior no es más que una caso concrerto de condiciones más generales. De manera que podemos afirmar que:

Theorem 4.4 *Assume that the finite set $\mathcal{U}$ defined in (2.1) is composed by elements in ascending order. Let $\mathcal{Y} = \{y_\ell\}_{\ell=1}^{L}$ be another finite set, with the same cardinality as $\mathcal{U}$, such that the $L-1$ tuple*

$$\frac{\Delta\mathcal{Y}}{\Delta\mathcal{U}} = \left(\frac{y_\ell - y_{\ell+1}}{u_\ell - u_{\ell+1}}\right)_{\ell=1}^{L-1} \tag{4.5}$$

is monotone. Let $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ be a piece-wise continuous function:

$$\mathcal{L}(u) = \begin{cases} \lambda_k(u) & \text{if } u \in [u_k, u_{k+1}[\\ 1 & \text{if } u = u_{N_u} \end{cases} \tag{4.6}$$
$$\forall k \in \{1, \dots, N_u - 1\}$$

tal que $\{y_l = \mathcal{L}(u_l)\}_{l=1}^{L}$.

Entonces, si las funciones λ_k para todo $k \in \{1, \dots, N_u - 1\}$ son funciones concavas, entonces la penalización $\mathcal{L}$ en el problema 4.3 da lugar a un control digital de $\mathcal{U}$. ■

En la Figura 4 se presenta otras dos funciones compatibles con el Teorema 4.4. Estos son:

(1) Aproximación de una parabola desplazada:

$$\lambda_k^{p_2}(u) = -4u^2 + 2(u_k + u_{k+1}) - 2u_k \tag{4.7}$$

(2) Unión de funciones concavas:

$$\lambda_k^{p_3}(u) = \frac{1}{4}[(u_{k+1} + u_k)(u - u_k - 1) + u_k^2] \tag{4.8}$$

Al igual que en la Figura 3, en la Figura 4 se muestra en la primera columna las posibles funciones de penalización. A la derecha de cada una de ellas se muestra el comportamiento del Hamiltoniano asociada a cada una de ellas. Se puede ver como los mimimos para todos los casos, solo toma los valores $\mathcal{U} = \{-1, -1/2, 0, 1/2, 1\}$.

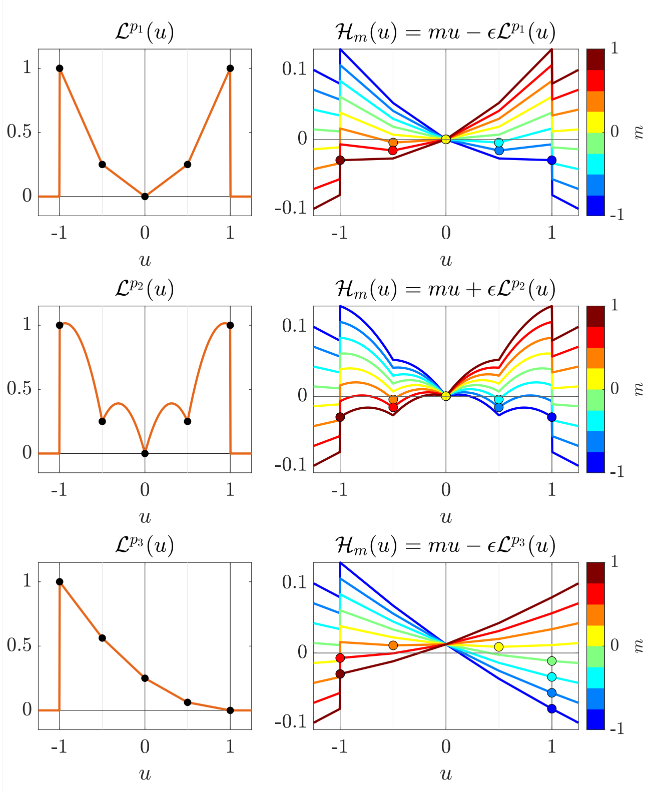

Fig. 4. SHE Multi-nivel.
En la primera columna mostramos tres tipos de penalización concavas compatibles con la Proposición 4.2. A la derecha de cada una de ellas mostramos el comportamiento del Hamiltoniano para cada distintos valores de m (colores).

5 Numerical simulations

In this section, we will present several examples in which we solve our optimal control problem through the direct method and the non-linear constrained optimization tool `CasADi` [Andersson et al., 2019].

5.1 Smooth approximation of piece-wise linear penalization

With the final aim of using an optimization software to solve our optimal control problem, we will approximate our piece-wise linear penalization with the help of the Heaviside function $h : \mathbb{R} \to \mathbb{R}$ and its smooth approximation defined as follows:

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \begin{cases} h^\eta(x) = (1 + \tanh(\eta x))/2 \\ \eta \to \infty \end{cases} \tag{5.1}$$

Using h, we can define the (smooth) function $\Pi^{\eta}_{a,b} : \mathbb{R} \to \mathbb{R}$ as:

$$\begin{aligned}\Pi^{\eta}_{[a,b]}(x) &= -1 + h^{\eta}(x-a) + h^{\eta}(-x+b) \\ &= \frac{\tanh[\eta(x-a)] + \tanh[\eta(b-x)]}{2}.\end{aligned}$$

In this way, we can define the smooth version of (6.12):

$$\mathcal{L}^{\eta}(u) = \sum_{k=1}^{N_u-1} \left[(u_{k+1}+u_k)(u-u_k) + u_k^2\right] \Pi^{\eta}_{[u_k,u_{k+1}]}(u) \tag{5.2}$$

So that, when $\eta \to \infty$, then $\mathcal{L}^{\eta} \to \mathcal{L}$.

5.2 *Direct method for OCP-SHE*

To solve the optimal control problem (4.3), we use a direct method. If we consider a partition $\mathcal{P} = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of interval $[0, T]$, we can represent a function $\{u(\tau) \mid \tau \in [0, T]\}$ as a vector $\boldsymbol{u} \in \mathbb{R}^T$ where component $u_t = u(\tau_t)$. Then the optimal control problem (4.2) can be written as optimization problem with variable $\boldsymbol{u} \in \mathbb{R}^T$. This problem is a nonlinear programming, for this we use CasADi software to solve. Hence, given a partition of the interval $[0, \pi)$, we can formulate the problem 4.3 as the following one in discrete time

Problem 5.1 (Numerical OCP) *Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively, given the target vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$, so that $\boldsymbol{x}_0 = [\boldsymbol{a}_T, \boldsymbol{b}_T]^T$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$ and a partition $\mathcal{P}_\tau = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of the interval $[0, \pi)$, we search a vector $\boldsymbol{u} \in \mathbb{R}^T$ that minimizes the following function:*

$$\min_{\boldsymbol{u} \in \mathbb{R}^T} \left[||\boldsymbol{x}^T||^2 + \epsilon \sum_{t=0}^{T-1} \mathcal{L}^{\eta}(u_t)\Delta\tau_t \right] \tag{5.3}$$

suject to:

$$\forall \tau \in \mathcal{P} \begin{cases} \boldsymbol{x}^{t+1} = \boldsymbol{x}^t - (2/\pi)\Delta\tau_t \boldsymbol{\mathcal{D}}(\tau_t) u_t \\ \boldsymbol{x}^0 = \boldsymbol{x}_0 \end{cases} \tag{5.4}$$

5.3 *Examples*

Todos los ejemplos que presentaremos a continuación tendrán en conmún los siguientes parámetros $\epsilon = 10^{-5}$, $\eta = 10^{-5}$ y una partición $\mathcal{P}_t = \{0.0, 0.1, 0.2, \ldots, \pi\}$.

5.3.1 *Bang-Bang*

Consideremos el Problema 5.1 con los siguientes parámetros $\mathcal{E}_a = \{1, 5, 7\}$ y $\mathcal{E}_b = \{1, 5, 7\}$ y el control admisible en $\mathcal{U} = \{-1, 1\}$. Además los vectores objetivos son: $\boldsymbol{a}_T = (i_d, 0)^T$ y $\boldsymbol{b}_T = (i_d, 0)^T$ para todo $i_d \in [-1, 1]$.

5.3.2 *Bang-off-Bang*

Consideremos el Problema 5.1 con los siguientes parámetros $\mathcal{E}_a = \{1, 5\}$ y $\mathcal{E}_b = \{1, 5\}$ y el control admisible en $\mathcal{U} = \{-1, 0, 1\}$. Además los vectores objetivos son: $\boldsymbol{a}_T = (1/2, 0)^T$ y $\boldsymbol{b}_T = (1/2, 0)^T$ para todo $i_d \in [-1, 1]$.

5.3.3 *Multi-nivel*

Consideremos el Problema 5.1 con los siguientes parámetros $\mathcal{E}_a = \{1, 5\}$ y $\mathcal{E}_b = \{1, 5\}$ y el control admisible en $\mathcal{U} = \{-1, -1/2, 0, 1/2, 1\}$. Además los vectores objetivos son: $\boldsymbol{a}_T = (1/2, 0, 0)^T$ y $\boldsymbol{b}_T = (1/2, 0, 0)^T$ para todo $i_d \in [-1, 1]$.

6 Proofs

6.1 *Proof of Theorem 4.1 (SHE as dinamical system)*

To this end, the starting point is to rewrite the expression of the Fourier coefficients (2.3) as the evolution of a dynamical system. This can be easily done by means of the fundamental theorem of differential calculus as follows: for all $i, j \in \mathbb{N}$, let α_i and β_j be the solutions of the following Cauchy problems

$$\begin{cases} \dot{\alpha}_i(\tau) = \dfrac{2}{\pi} u(\tau)\cos(i\tau), & \tau \in [0, \pi) \\ \alpha_i(0) = 0 \end{cases}$$
$$\begin{cases} \dot{\beta}_j(\tau) = \dfrac{2}{\pi} u(\tau)\sin(j\tau), & \tau \in [0, \pi) \\ \beta_j(0) = 0 \end{cases} \tag{6.1}$$

Then

$$\alpha_i(\tau) = \frac{2}{\pi} \int_0^{\tau} u(\zeta)\cos(i\zeta)\, d\zeta$$
$$\beta_j(\tau) = \frac{2}{\pi} \int_0^{\tau} u(\zeta)\sin(j\zeta)\, d\zeta$$

and the Fourier coefficients (2.3) are given by $a_i = \alpha_i(\pi)$ and $b_j = \beta_j(\pi)$.

Let now

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \ldots, e_a^{N_a}\}, \quad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \ldots, e_b^{N_b}\}$$

be two sets of odd numbers, and denote

$$\boldsymbol{\alpha} = \{\alpha_i\}_{i \in \mathcal{E}_a}, \quad \boldsymbol{\beta} = \{\beta_j\}_{j \in \mathcal{E}_b}.$$

Then, for any $\tau \in [0, \pi)$, we can define the vectors

$$\boldsymbol{\mathcal{D}}^{\alpha}(\tau) = \begin{bmatrix} \cos(e_a^1 \tau) \\ \cos(e_a^2 \tau) \\ \vdots \\ \cos(e_a^{N_a} \tau) \end{bmatrix}, \quad \boldsymbol{\mathcal{D}}^{\beta}(\tau) = \begin{bmatrix} \sin(e_b^1 \tau) \\ \sin(e_b^2 \tau) \\ \vdots \\ \sin(e_b^{N_b} \tau) \end{bmatrix}$$

with $\boldsymbol{\mathcal{D}}^{\beta}(\tau) \in \mathbb{R}^{N_a}$ and $\boldsymbol{\mathcal{D}}^{\beta}(\tau) \in \mathbb{R}^{N_b}$, and the dynamical systems (6.1) can be rewritten in a vectorial form as:

$$\begin{cases} \dot{\boldsymbol{\alpha}}(\tau) = \dfrac{2}{\pi} \boldsymbol{\mathcal{D}}^{\alpha}(\tau) u(\tau), & \tau \in [0, \pi) \\ \boldsymbol{\alpha}(0) = 0 \end{cases}$$
$$\begin{cases} \dot{\boldsymbol{\beta}}(\tau) = \dfrac{2}{\pi} \boldsymbol{\mathcal{D}}^{\beta}(\tau) u(\tau), & \tau \in [0, \pi) \\ \boldsymbol{\beta}(0) = 0 \end{cases} \tag{6.2}$$

Finally, compressing the notation even more, we can now denote

$$\boldsymbol{x}(\tau) = \begin{bmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{bmatrix}, \quad \boldsymbol{\mathcal{D}}(\tau) = \begin{bmatrix} \boldsymbol{\mathcal{D}}^{\alpha}(\tau) \\ \boldsymbol{\mathcal{D}}^{\beta}(\tau) \end{bmatrix}$$

so that (6.2) becomes

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = \dfrac{2}{\pi} \boldsymbol{\mathcal{D}}(\tau) u(\tau), & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = 0 \end{cases} \tag{6.3}$$

and the target coefficients of the SHE problem will be given by $\boldsymbol{x}_0 := [\boldsymbol{a}_T, \boldsymbol{b}_T]^{\top} = \boldsymbol{x}(\pi)$.

Moreover, since most often control problems are designed to drive the state of a given dynamical system to an equilibrium configuration, for instance the zero state, we introduce the change of variables $\boldsymbol{x}(\tau) \mapsto \boldsymbol{x}_T - \boldsymbol{x}(\tau)$ which allows us to reverse the time in (6.3), thus obtaining

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = -\dfrac{2}{\pi} \boldsymbol{\mathcal{D}}(\tau) u(\tau), & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases}, \tag{6.4}$$

In this new configuration, the control function u is now required to steer the solution of (6.4) from the initial datum $\boldsymbol{x}_T$ to zero in time $\tau = \pi$.

6.2 *Proof of Propsition 4.1 (Hamiltonian Function)*

In Problem 4.3, the penalization function $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ will be chosen such that the optimal control u^* only takes values in $\mathcal{U}$. Furthermore, the parameter ϵ should be small so that the solution minimizes the distance from the final state and the target.

Next we will study the optimality conditions of the problem, for a general $\mathcal{L}$, and then specify how $\mathcal{L}$ should be so that the optimal control u^* only takes the allowed values in $\mathcal{U}$.

6.2.1 *Optimality conditions*

To write the optimality conditions of the problem we will use the Pontryagin minimum principle [Bryson, 1975, Chapter 2.7]. With this purpose, it is necessary to first introduce the Hamiltonian function

$$H(u, \boldsymbol{p}, \tau) = \epsilon \mathcal{L}(u) - \frac{2}{\pi} \big(\boldsymbol{p}(\tau) \cdot \boldsymbol{\mathcal{D}}(\tau) \big) u(\tau),$$

where $\boldsymbol{p}(\tau)$ is the so-called adjoint state, which is associated with the restriction imposed by the system. This vector has the same dimension of the state $\boldsymbol{x}$, so that

$$\boldsymbol{x}(\tau) = \begin{bmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{bmatrix} \Leftrightarrow \boldsymbol{p}(\tau) = \begin{bmatrix} \boldsymbol{p}^{\alpha}(\tau) \\ \boldsymbol{p}^{\beta}(\tau) \end{bmatrix}. \tag{6.5}$$

In what follows, we will enumerate the optimality conditions arising from the Pontryagin principle.

1. **Adjoint equation**: the ODE describing the evolution of the adjoint variable is given by

   $$\dot{\boldsymbol{p}}(\tau) = -\nabla_x H(u(\tau), \boldsymbol{p}(\tau), \tau).$$

   In our case, since the Hamiltonian does not depend on the dynamics, we simply have

   $$\dot{\boldsymbol{p}}(\tau) = 0, \tag{6.6}$$

   that is, the adjoint state is constant in time.
2. **Final condition of the adjoint**: As it is well-known, the adjoint equation is defined backward in time, meaning that its initial condition is actually a final one, posed at $\tau = \pi$. This final condition is given by

   $$\boldsymbol{p}(\pi) = \nabla_{\boldsymbol{x}} \Psi(\boldsymbol{x}) = \boldsymbol{x}(\pi) - \boldsymbol{x}_T.$$

   This, together with (6.6), tells us that

   $$\boldsymbol{p}(\tau) = \boldsymbol{x}(\pi) - \boldsymbol{x}_T, \quad \text{for all } \tau \in [0, \pi).$$

3. **Optimal Waveform**: We known that

   $$u^* = \arg\min_{|u|<1} H(\tau, \boldsymbol{p}^*, u),$$

so that, in this case, we can write

$$u^*(\tau) = \arg\min_{|u|<1} \left[\epsilon\mathcal{L}(u(\tau)) - \frac{2}{\pi}\big(\boldsymbol{p}^* \cdot \boldsymbol{\mathcal{D}}(\tau)\big)u(\tau) \right]. \tag{6.7}$$

Therefore, this optimality condition reduces to the optimization of a function in a variable within the interval $[-1,1]$.

Definition 6.1 *Dado el problema 4.3 definimos una función $\mathcal{H}_m : [-1,1] \to \mathbb{R}$ tal que:*

$$\mathcal{H}_m(u) = \epsilon\mathcal{L}(u) - mu \mid \forall m \in \mathbb{R} \tag{6.8}$$

Es importante notar que la función $\mathcal{H}_m$ es el Hamiltoniano del sistema donde hemos remplazado el valor

$$[\boldsymbol{p}^* \cdot \boldsymbol{\mathcal{D}}(\tau)] = \sum_{i\in\mathcal{E}_a} p^*_\alpha \cos(i\tau) + \sum_{j\in\mathcal{E}_b} p^*_\beta \sin(j\tau) \tag{6.9}$$

por el parámetro m. De manera que el Hamiltoniano evaluado en la trayectoria óptima varía de manera continua en todo el intervalo $\tau \in [0,\pi)]$. Esta es la razón por la que el estudio de la función $\mathcal{H}_m$, una función uni-variable parametrizada por m ,tiene implicaciones en el Problema 4.3.

Definition 6.2 *Dado el Problema 4.3 definimos una función $\mathcal{G} : \mathbb{R} \to [-1,1]$ tal que:*

$$\mathcal{G}(m) = \arg\min_{u\in[-1,1]} \mathcal{H}_m(u) \tag{6.10}$$

Definition 6.3 *Dado el Problema 4.3 definimos el conjunto $\mathcal{M}$ como:*

$$\mathcal{M} = \{m \in \mathbb{R} \mid \mathcal{G}(m) \notin \mathcal{U}\} \tag{6.11}$$

6.3 Proof of Theorem 4.2

El caso binivel es el que tiene como conjunto admisible de controles $\mathcal{U} = \{-1,1\}$. Si $\mathcal{L}$ es concava en el intervalo

entonces $\mathcal{H}_m$ también lo es. De manera que $G(m)$ solo puede tomar los valores $\{-1,1\}$.

6.4 Proof of Theorem 4.3

In this section, we discuss how to design the penalization term $\mathcal{L}(u)$ so that the optimal control is always contained in $\mathcal{U}$.

In more detail, we can choose the affine interpolation of a parabola $\mathcal{L} : [-1,1] \to \mathbb{R}$ as a penalization term. That is

$$\mathcal{L}(u) = \begin{cases} [(u_{k+1}+u_k)(u-u_k)+u_k^2] & \text{if } \ u \in [u_k, u_{k+1}[\\ 1 & \text{if } u = u_{N_u} \end{cases} \tag{6.12}$$

$$\forall k \in \{1,\ldots,N_u-1\}$$

Nevertheless, to compute the minimum of $\mathcal{H}_m(u)$, we shall take into account that this function is not differentiable and the optimality condition then requires to work with the subdifferential $\partial\mathcal{L}(u)$, which given by

$$\partial\mathcal{L}(u) = \begin{cases} \{u_1+u_2\} & \text{if } \ u = u_1 \\ \{u_k+u_{k+1}\} & \text{if } \ u \in]u_k, u_{k+1}[\quad \dagger \\ [u_k+u_{k-1}, u_{k+1}+u_k] & \text{if } u = u_k \quad \ddagger \\ \{u_{N_u}+u_{N_u-1}\} & \text{if } u = u_{N_u} \end{cases} \tag{6.13}$$

$$\dagger\ \forall k \in \{1,\ldots,N_u-1\} \qquad \ddagger\ \forall k \in \{2,\ldots,N_u-1\}$$

Hence, we have $\partial H_m = \epsilon\partial\mathcal{L} - m$. This means that, given $m \in \mathbb{R}$, we look for $u \in [-1,1]$ minimizing $\mathcal{H}_m(u)$. It is then necessary to determine whether zero belongs to $\partial\mathcal{H}_m(u)$.

- **Case 1:** $m \leq \epsilon(u_1+u_2)$: since m is less than the minimum of all subdifferentials, then zero does not belong to any of the intervals we defined. Hence, the minimum is in one of the extrema

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_1 \tag{6.14}$$

- **Case 2:** $m = \epsilon(u_{k+1}+u_k)$: taking into account that $\forall k \in \{1,\ldots,N_u-1\}$,

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = [u_k, u_{k+1}[\tag{6.15}$$

- **Case 3:** $\epsilon(u_k+u_{k-1}) < m < \epsilon(u_{k+1}+u_k)$: taking into account that $\forall k \in \{2,\ldots,N_u-1\}$,

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_k \tag{6.16}$$

- **Case 4:** $m > \epsilon(u_{N_u}+u_{N_u-1})$:

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_{N_u} \tag{6.17}$$

In other words, only when $m = \epsilon(u_{k+1}+u_k)$ the minima of the Hamiltonian belong to an interval. For all the other values of $m \in \mathbb{R}$, these minima are contained in $\mathcal{U}$. So that under a continuous variation of m, Case 2 can only occur pointwise. Recalling the optimal control problem $m(\tau) = [\boldsymbol{p}(\tau) \cdot \boldsymbol{\mathcal{D}}(\tau)]$, we can notice that Case 2 corresponds to the instants τ of change of value.

6.5 *Proof of Theorem 4.4*

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7 Conclusiones

We presented the SHE problem from the point of view of control theory. This methodology allows obtaining a $10^{-4}-10^{-5}$ precision in the distance to the target vector. Nevertheless, comparing with methodologies where the commutation number is fixed a priori, our approximation is computationally more expensive. Notwithstanding that, the optimal control provides solutions in the entire range of the modulation index, although the number of solutions or their locations change dramatically.

This methodology for the SHE problem connects control theory with harmonic elimination. In this way, the SHE problem can be solved through classical tools.

Remark 7.1 (Quarter-wave symmetry) *We shall mention that, in the SHE literature, it is usual to distinguish among the half-wave symmetry problem (addressed in the present paper) and the quarter-wave symmetry one in which*

$$u\left(\tau+\frac{\pi}{2}\right)=-u(\tau)\quad \textit{for all } \tau\in\left[0,\frac{\pi}{2}\right).$$

In quarter-wave symmetry, the SHE problem simplifies, as the Fourier coefficients $\{a_i\}_{i\in\mathcal{E}_a}$ turn out to be all zero. Hence, only the phases of the converter's signal can be controlled, while the half-wave SHE allows to deal with the amplitudes as well. It is worth to remark nonetheless that our optimal control formulation can be easily adapted to the quarter-wave symmetry problem by simply replacing the Fourier coefficients (2.3) with

$$a_i=0,\qquad b_j=\frac{4}{\pi}\int_0^{\frac{\pi}{4}}u(\tau)\sin(j\tau)\,d\tau.$$

Entonces podemos introducir el siguiente sistema dinámico:

$$\begin{cases}\dot{\boldsymbol{\beta}}(\tau)=\dfrac{2}{\pi}\boldsymbol{\mathcal{D}}^{\beta}(\tau)u(\tau), & \tau\in[0,\pi/2)\\ \boldsymbol{\beta}(0)=\boldsymbol{b}_T\end{cases}\tag{7.1}$$

Además del siguiente problema de control:

Problem 7.1 *Let $\mathcal{U}$ be defined as in (2.1) and let $\mathcal{E}_b$ be a set of odd numbers with cardinality $|\mathcal{E}_b|=N_b$. Given the vector $\boldsymbol{b}_T\in\mathbb{R}^{N_b}$. We look for $u:\in[0,\pi/2)\to\mathcal{U}$ such that the solution of (7.1) with initial datum $\boldsymbol{x}(0)=\boldsymbol{x}_0$ satisfies $\boldsymbol{x}(\pi)=0$.*

Donde de la misma forma que en el problema con simetría de media onda la solución de este problema es también solución del problema SHE con simetría de cuarto de onda.

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