Multilevel Selective Harmonic Modulation via Optimal Control

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Abstract

We consider the *Selective Harmonic Modulation* (SHM) problem, consisting in the design of a staircase control signal with some prescribed frequency components. In this work, we propose a novel methodology to address SHM as an optimal control problem in which the admissible controls are piecewise constant functions, taking values only in a given finite set. In order to fulfill this constraint, we introduce a cost functional with piecewise affine penalization, which, by means of Pontryagin's maximum principle, makes the optimal control have the desired staircase form. Moreover, the addition of the penalization term for the control provides uniqueness and continuity of the solution with respect to the target frequencies. Another advantage of our approach is that the number of switching angles and the waveform need not be determined a priori. Indeed, the solution to the optimal control problem is the entire control signal, and therefore, it determines the waveform and the location of the switches. We also provide several numerical examples in which the SHM problem is solved by means of our approach.

Key words: Selective Harmonic Modulation; Optimal Control Theory; Finite-Set Control; Pontryagin's maximum principle; Bang-bang control.

1 Introduction and motivations

Selective Harmonic Modulation (SHM) [Sun and Grotstollen, 1992, Sun et al., 1996] is a well-known methodology in power electronics engineering, employed to improve the performances of a converter by controlling the phase and amplitude of the harmonics in its output voltage. As a matter of fact, this technique allows to increase the power of the converter and, at the same time, to reduce its losses.

Because of the growing complexity of modern electrical networks, consequence for instance of the high penetration of renewable energy sources, the demand in power of electronic converters is day by day increasing. For this and other reasons, SHM has been a preeminent research interest in the power electronics community, and a plethora of SHM-based techniques has been developed in recent years.

In broad terms, SHM consists in generating a *control signal* with a desired harmonic spectrum by modulating some specific lower-order Fourier coefficients. In practice, the signal is constructed as a step function with a finite number of switches, taking values only in a given finite set. Such a signal can be fully characterized by two features (see Fig. 1):

1. The *waveform*, i.e. the sequence of values that the function takes in its domain.
2. The *switching angles*, i.e. the sequence of points where the signal switches from one value to following one.

Using this simple characterization of the signal, in many practical situations, the SHM problem is reduced to a finite-dimensional optimization one in which, for a given suitable waveform, the aim is to find the optimal location of the switching angles. However, this approach has the difficulty of choosing a suitable waveform, which may be quite cumbersome in some situations. In fact, even

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determining the number of switching angles is not in general straightforward.

To overcome these difficulties, in this work we propose a new approach to SHM based on control theory: the Fourier coefficients of the signal are identified with the terminal state of a controlled dynamical system, where the control is actually the signal, solution to the SHM problem. We then look for piecewise constant controls, taking values only in a given finite set, and satisfying the prescribed terminal condition (see Section 4 for more details).

One of the main difficulties in our approach is that the constraints on the admissible controls, which must have staircase form, prevent us from applying the classical tools in optimal control theory, such as Pontryagin's maximum principle. para Jesus: decir algo de cmo las restricciones dificultan la utilizacin de mtodos numricos como casadi/ipopt.

In order to bypass this obstruction, we consider a different optimal control problem, without these type of constraints, in which the desired staircase structure is obtained by introducing a suitable penalization term for the control.

The main contributions of the present paper are the following:

1. We reformulate the SHM problem as an optimal control one, with a staircase-form constraint on the control. An advantage of this formulation is that neither the waveform of the solution nor the number of switching angles need to be a priori determined.
2. We introduce a penalization term for the control which implicitly induces the desired staircase property on the solution to the optimal control problem. Different choices of the penalization term can give rise to solutions to the SHM problem with different waveform.
3. For each choice of the penalization term, we prove uniqueness and continuity of the solution with respect to the target frequencies. We point out that the continuity of the solution with respect to the target seems to be a highly desirable property in real applications of SHM, and sometimes, difficult to achieve.
4. We also provide some numerical examples, where we solve the SHM problem by means of our approach. The numerical examples confirm that the solution provided by our methodology is, effectively, continuous with respect to the target frequencies.

This document is structured as follows. In Section 2, we introduce the mathematical formulation of the general SHM problem. In Section 3, we recall the classical methodology casting the SHM problem through finite-dimensional optimization and we show the main criticalities related to this approach. In Section 4, we present the new approach to SHM as an optimal control problem, and state our main results concerning the uniqueness and stability of the solution. Section 5 is devoted to some numerical examples of concrete SHM problems that we have solved by means of our methodology. In Section 6, we give the proofs of the theoretical results presented in Section 4. Finally, in Section 7, we summarize and comment the conclusions of our work.

2 Preliminaries

This section is devoted to the mathematical formulation of the SHM problem and to introduce the notation that will be used throughout the paper. Let

$$\mathcal{U} = \{u_1, \dots, u_L\} \tag{2.1}$$

be a given set of $L \geq 2$ real numbers satisfying

$$u_1 = -1,\ u_L = 1 \ \text{ and } \ u_k < u_{k+1} \quad \forall k \in \{1, \dots, L\}.$$

The goal is to construct a step function

$$u(t) : [0, 2\pi) \to \mathcal{U},$$

with a finite number of switches, such that some of its lower-order Fourier coefficients take specific values prescribed a priori.

Due to applications in power converters, it is typical to only consider functions with *half-wave symmetry*, i.e. satisfying

$$u(t+\pi) = -u(t) \quad \text{for all } t \in [0, \pi). \tag{2.2}$$

In view of (2.2), in what follows, we will only work with the restriction $u|_{[0,\pi)}$, which, with some abuse of notation, we still denote by u.

Moreover, as a consequence of this symmetry, the Fourier series of u only involves the odd terms (as the even terms just vanish), i.e.

$$u(t) = \sum_{\substack{j \in \mathbb{N} \\ j\ odd}} a_j \cos(jt) + \sum_{\substack{j \in \mathbb{N} \\ j\ odd}} b_j \sin(jt),$$

with

$$\begin{aligned} a_j &= \frac{2}{\pi} \int_0^\pi u(\tau) \cos(j\tau)\, d\tau, \\ b_j &= \frac{2}{\pi} \int_0^\pi u(\tau) \sin(j\tau)\, d\tau. \end{aligned} \tag{2.3}$$

As we anticipated, we are only considering piecewise constant functions with a finite number of switches, taking values only in $\mathcal{U}$. In other words, we look for functions $u : [0, \pi) \to \mathcal{U}$ of the form

$$u(t) = \sum_{m=0}^{M} s_m \chi_{[\phi_m, \phi_{m+1})}(t), \quad M \in \mathbb{N} \tag{2.4}$$

for some $\mathcal{S} = \{s_m\}_{m=0}^{M}$ satisfying

$$s_m \in \mathcal{U} \text{ and } s_m \neq s_{m+1} \quad \forall m \in \{0, \dots, M\}$$

and $\Phi = \{\phi_m\}_{m=1}^{M}$ such that

$$0 = \phi_0 < \phi_1 < \dots < \phi_M < \phi_{M+1} = \pi.$$

In (2.4), $\chi_{[\phi_m, \phi_{m+1})}$ denotes the characteristic function of the interval $[\phi_m, \phi_{m+1})$. With these notations, we can define the waveform and the switching angles as follows:

Definition 2.1 *For a function $u : [0, \pi) \to \mathcal{U}$ of the form (2.4), we refer to $\mathcal{S}$ as the* waveform, *and we refer to Φ as the* switching angles.

Observe that any function u of the form (2.4) is fully characterized by its waveform $\mathcal{S}$ and switching angles Φ. An example of such a function is displayed in Fig. 1.

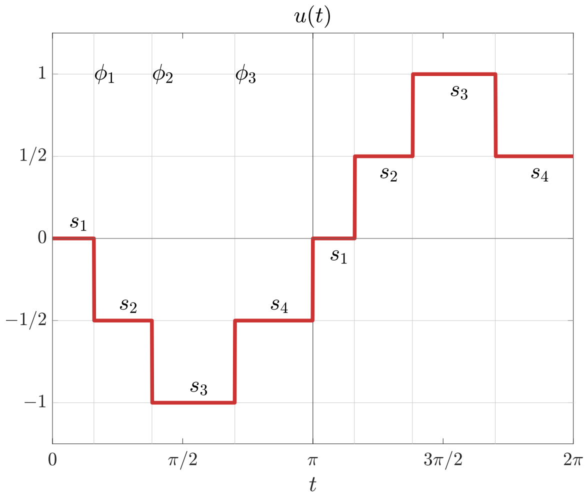

Fig. 1. A possible solution to the SHM Problem, where we considered the control-set as $\mathcal{U} = \{-1, -1/2, 0, 1/2, 1\}$. We show the switching angles Φ and the waveform $\mathcal{S}$ (see Definition 2.1). The function $u(t)$ is displayed on the whole interval $[0, 2\pi)$ to highlight the half-wave symmetry introduced in (2.2).

In the practical engineering applications that motivated our study, due to technical limitations, it is preferable to employ signals taking consecutive values in $\mathcal{U}$. In the sequel, we will refer to this property of the waveform as the *staircase property*. We can rigorously formulate this property as follows:

Definition 2.2 *We say that a signal u of the form (2.4) fulfills the* staircase property *if its waveform $\mathcal{S}$ satisfies*

$$(s_m^{min}, s_m^{max}) \cap \mathcal{U} = \emptyset, \quad \forall m \in \{0, \dots, M-1\}, \tag{2.5}$$

where $s_m^{min} = s_m \wedge s_{m+1}$ and $s_m^{max} = s_m \vee s_{m+1}$.

Note that when $\mathcal{U} = \{-1, 1\}$ (which is known in the SHM literature as the bi-level problem), this property is satisfied for any u of the form (2.4).

We can now formulate the SHM problem as follows:

Problem 2.1 (SHM) *Let $\mathcal{U}$ be given as in (2.1), and let $\mathcal{E}_a$ and $\mathcal{E}_b$ be two finite sets of odd numbers of cardinality $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively. For any two given vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, construct a function $u : [0, \pi) \to \mathcal{U}$ of the form (2.4), satisfying (2.5), such that the vectors $\boldsymbol{a} \in \mathbb{R}^{N_a}$ and $\boldsymbol{b} \in \mathbb{R}^{N_b}$, defined as*

$$\boldsymbol{a} = (a_j)_{j \in \mathcal{E}_a} \quad \text{and} \quad \boldsymbol{b} = (b_j)_{j \in \mathcal{E}_b} \tag{2.6}$$

satisfy

$$\boldsymbol{a} = \boldsymbol{a}_T \quad \text{and} \quad \boldsymbol{b} = \boldsymbol{b}_T,$$

where the coefficients a_j and b_j in (2.6) are given by (2.3).

Remark 2.1 (SHE) *In Problem 2.1, we gave a very general formulation of SHM. This formulation contains also the so-called* Selective Harmonic Elimination (SHE) *problem (see [Sun et al., 1996]), in which the target vectors are such that*

$$\begin{aligned} (a_T)_1 \neq 0 \quad (a_T)_{i \neq 1} = 0 \quad \forall i \in \mathcal{E}_a \\ (b_T)_1 \neq 0 \quad (b_T)_{j \neq 1} = 0 \quad \forall j \in \mathcal{E}_b. \end{aligned}$$

SHE is of great relevance in the electric engineering literature. Its objective is to generate a signal with amplitude $m_1 = \sqrt{a_1^2 + b_1^2}$ and phase $\varphi_1 = \arctan(b_1/a_1)$, removing some specific higher-frequency components. In this way, SHE may be understood as a generator of clean Fourier modes through a staircase signal.

3 SHM as a finite-dimensional optimization problem

A typical approach to address the SHM Problem 2.1 ([Sun et al., 1996, Konstantinou and Agelidis, 2010, Yang et al., 2015]) consists in looking for solutions u having a specific waveform $\mathcal{S}$ a priori determined, optimizing only over the location of the switching angles Φ.

Note that, for a fixed waveform $\mathcal{S}$, the Fourier coefficients of a function u of the form (2.4) can be written in terms of the switching angles Φ in the following way:

$$a_j = a_j(\Phi) = \frac{2}{j\pi}\sum_{m=0}^{M} s_m \Big[\sin(j\phi_{m+1}) - \sin(j\phi_m)\Big]$$

$$b_j = b_j(\Phi) = \frac{2}{j\pi}\sum_{m=0}^{M} s_m \Big[\cos(j\phi_m) - \cos(j\phi_{m+1})\Big]$$

Hence, for two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ as in Problem 2.1, and any fixed waveform $\mathcal{S}$, we can define the functions

$$\begin{aligned} \boldsymbol{a}_{\mathcal{S}}(\Phi) &:= \big(a_j(\Phi)\big)_{j\in\mathcal{E}_a} \in \mathbb{R}^{N_a} \\ \boldsymbol{b}_{\mathcal{S}}(\Phi) &:= \big(b_j(\Phi)\big)_{j\in\mathcal{E}_b} \in \mathbb{R}^{N_b} \end{aligned} \tag{3.1}$$

which associates, to any sequence of switching angles $\{\phi_m\}_{m=1}^{M}$, the corresponding Fourier coefficients. Therefore, SHM can be cast as a finite-dimensional optimization problem in the following way:

Problem 3.1 (Optimization problem for SHM) *Let $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ be given as in Problem 2.1. Let $\mathcal{S} := \{s_m\}_{m=0}^{M}$ be a fixed waveform satisfying (2.5). Find the switching angles $\Phi = \{\phi_m\}_{m=1}^{M}$ solution to the following minimization problem:*

$$\min_{\Phi\in[0,\pi]^M} \left(\|\boldsymbol{a}_{\mathcal{S}}(\Phi) - \boldsymbol{a}_T\|^2 + \|\boldsymbol{b}_{\mathcal{S}}(\Phi) - \boldsymbol{b}_T\|^2\right)$$

$$\textit{subject to: } 0 = \phi_0 < \phi_1 < \ldots < \phi_M < \phi_{M+1} = \pi.$$

where $\boldsymbol{a}_{\mathcal{S}}(\Phi)$ and $\boldsymbol{b}_{\mathcal{S}}(\Phi)$ are defined as in (3.1).

At this regard, it is important to notice that the optimization Problem 3.1 solves the original SHM Problem 2.1 only when the optimal cost is zero. This makes necessary to fully characterize the space of targets $(\boldsymbol{a}_T, \boldsymbol{b}_T)$ for which the solution of Problem 3.1 is a solution of Problem 2.1. With this aim, we will define the the *optimal cost* and the *solvable set* as follows:

Definition 3.1 (optimal cost) *We call* optimal cost *$V_{\mathcal{S}} : \mathbb{R}^{N_a} \times \mathbb{R}^{N_b} \to \mathbb{R}$, the function that takes as input variables the target vectors $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ and returns the optimal cost of the Problem 3.1.*

Definition 3.2 (solvable set) *We define a* solvable set *$\mathcal{R}_{\mathcal{S}}$ as:*

$$\mathcal{R}_{\mathcal{S}} = \left\{(\boldsymbol{a}_T, \boldsymbol{b}_T) \in \mathbb{R}^{N_a+N_b} \mid V_{\mathcal{S}}(\boldsymbol{a}_T, \boldsymbol{b}_T) = 0\right\} \tag{3.2}$$

Furthermore, we define the following *policy* function which maps the solutions of Problem 3.1 into the set $\mathcal{R}_{\mathcal{S}}$.

Definition 3.3 (Policy) *We will call* policy *a function $\Pi_{\mathcal{S}} : \mathcal{R}_{\mathcal{S}} \to [0,\pi]^M$ such that $\Phi^* = \Pi_{\mathcal{S}}(\boldsymbol{a}_T, \boldsymbol{b}_T)$, with Φ^* the optimal switching angles solutions of Problem 2.1 for the targets $(\boldsymbol{a}_T, \boldsymbol{b}_T)$.*

With the aim of reconstructing the policy $\Pi_{\mathcal{S}}$, a typical approach is to solve numerically Problem 3.1 for a limited number of points in $\mathbb{R}^{N_a+N_b}$ and check that the optimal cost is zero. Secondly, one interpolates the function $\Pi_{\mathcal{S}}$ in the convex set generated by the points previously obtained. Nevertheless, this approach has several difficulties and drawbacks.

1. *Combinatory problem*: in practice, one does not dispose of a suitable waveform $\mathcal{S}$ which yields a solution to the Problem 2.1. A common approach to solve the SHM problem consists in fixing the number of switches M, and then solve Problem 3.1 for all the possible combinations of M elements of $\mathcal{U}$. However, taking into account that the number of possible M-tuples in $\mathcal{U}$ is of the order $(L-1)^M$, it is evident that the complexity of the above approach increases rapidly when $L > 1$. This problem has been studied in [Yang et al., 2015] where, through appropriate algebraic transformations, the authors are able to convert the SHM problem into a polynomial system whose solutions' set contains all the possible waveforms $\mathcal{S}$ of M elements in $\mathcal{U}$. As a drawback of this approach, the number of switches M needs to be prefixed. However, in some cases, determining the number of switches which are necessary to reach the desired Fourier coefficients is not a straightforward task.
2. *Solvable set problem*: given a waveform $\mathcal{S}$, the corresponding solvable set $\mathcal{R}_{\mathcal{S}}$ is usually very small, yielding to policies $\Pi_{\mathcal{S}}$ which are not very effective. This issue is typically addressed by solving Problem 3.1 for a set of waveforms $\{\mathcal{S}_l\}_{l=1}^{r}$ and obtaining different policies $\{\Pi_{\mathcal{S}_l}\}_{l=1}^{r}$ and solvable sets $\{\mathcal{R}_{\mathcal{S}_l}\}_{l=1}^{r}$ for each one of them. By gathering them, on then creates a new policy which is applicable in a wider scenario. Nevertheless, due to the fact that in general to different waveforms may correspond disjoint or even overlapping solvable sets, this union of policies is chaotic, providing different solutions for the same target vector $(\boldsymbol{a}_T, \boldsymbol{b}_T)$, or even generating regions with no solution (see Fig. 2).
3. *Policy problem*: due to the complexity of a policy generated by the union of different waveforms, the continuity of the switching angles cannot be guaranteed. This is a well known problem in the SHM community [Agelidis et al., 2008, Dahidah and Agelidis, 2008, Dahidah et al., 2015, Yang et al., 2017] (see Fig. 2).

As we shall see, all these mentioned criticalities will be overcome by our optimal control approach.

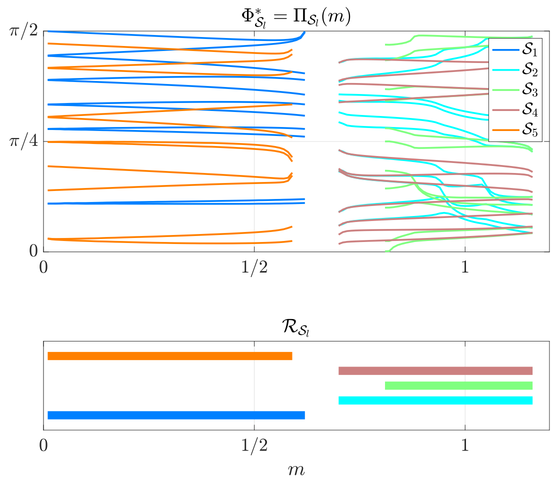

Fig. 2. In the first picture, we display the optimal switching angles $\Phi^*_{\mathcal{S}}$ associated to different waveforms $\{\mathcal{S}_l\}_{l=1}^{7}$ for a SHM problem (see Remark 2.1), considering the sets $\mathcal{E}_a = \{1\}$ and $\mathcal{E}_b = \{1, 5, 7, 11, 13, 17, 19, 23, 25, 29, 31\}$. We chose the target $(a_1, b_1) = (m, 0)$ $\forall m \in [0, 1.2]$. The second figure shows the solvable sets for each waveform we considered.

4 SHM as an optimal control problem

Our main contribution in the present paper consists in formulating the SHM problem as an optimal control problem. In this formulation, the Fourier coefficients of the signal $u(t)$ are identified with the terminal state of a controlled dynamical system of $N_a + N_b$ components defined in the time-interval $[0, \pi]$. The control of the system is precisely the signal $u(t)$, defined as a function $[0, \pi] \to \mathcal{U}$, which has to steer the state from the origin to the desired values of the prescribed Fourier coefficients.

The starting point of this approach is to rewrite the Fourier coefficients of the function $u(t)$ as the final state of a dynamical system controlled by $u(t)$. To this end, let us first note that, in view of (2.3), for all $u \in L^\infty([0, \pi]; \mathbb{R})$ any Fourier coefficient a_j satisfies

$$a_j = y(\pi),$$

with $y \in C([0, \pi]; \mathbb{R})$ defined as

$$y(t) = \frac{2}{\pi} \int_0^t u(\tau) \cos(j\,\tau) d\tau.$$

Besides, as a consequence of the fundamental theorem of calculus, $y(\cdot)$ is the unique solution to the differential equation

$$\begin{cases} \dot{y}(t) = \dfrac{2}{\pi} \cos(j\,t) u(t), & t \in [0, \pi] \\ y(0) = 0. \end{cases} \tag{4.1}$$

Analogously, we can also write the Fourier coefficients b_j, defined in (2.3), as the solution at time $t = \pi$ of a differential equation similar to (4.1).

Hence, for $\mathcal{E}_a$, $\mathcal{E}_b$, $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ given, the SHM Problem 2.1 can be reduced to finding a control function u of the form (2.4), satisfying (2.5), such that the corresponding solution $\boldsymbol{y} \in C([0, \pi]; \mathbb{R}^{N_a + N_b})$ to the dynamical system

$$\begin{cases} \dot{\boldsymbol{y}}(t) = \dfrac{2}{\pi} \boldsymbol{\mathcal{D}}(t) u(t), & t \in [0, \pi] \\ \boldsymbol{y}(0) = 0. \end{cases} \tag{4.2}$$

satisfies

$$\boldsymbol{y}(\pi) = [\boldsymbol{a}_T; \boldsymbol{b}_T]^\top,$$

where

$$\boldsymbol{\mathcal{D}}(t) = \left[\boldsymbol{\mathcal{D}}^a(t); \boldsymbol{\mathcal{D}}^b(t)\right]^\top, \tag{4.3}$$

with $\boldsymbol{\mathcal{D}}^a(t) \in \mathbb{R}^{N_a}$ and $\boldsymbol{\mathcal{D}}^b(t) \in \mathbb{R}^{N_b}$ given by

$$\boldsymbol{\mathcal{D}}^a(t) = \begin{bmatrix} \cos(e_a^1 t) \\ \cos(e_a^2 t) \\ \vdots \\ \cos(e_a^{N_a} t) \end{bmatrix}, \quad \boldsymbol{\mathcal{D}}^b(t) = \begin{bmatrix} \sin(e_b^1 t) \\ \sin(e_b^2 t) \\ \vdots \\ \sin(e_b^{N_b} t) \end{bmatrix} \tag{4.4}$$

Here, e_a^i and e_b^i denote the elements in $\mathcal{E}_a$ and $\mathcal{E}_b$, i.e.

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \dots, e_a^{N_a}\}, \quad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \dots, e_b^{N_b}\}.$$

In the sequel, and in order to simplify the notation, we reverse the time in (4.2) using the transformation $\boldsymbol{x}(t) = \boldsymbol{y}(\pi - t)$. In this way, the SHM problem turns into the following null controllability one, for a dynamical system with initial condition $\boldsymbol{x}(0) = [\boldsymbol{a}_T; \boldsymbol{b}_T]^\top$ (see also Fig. 3).

Problem 4.1 (SHM via null controllability) *Let $\mathcal{U}$ be given as in (2.1). Let $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ be given as in Problem 2.1, we look for a function $u : [0, \pi] \to [-1, 1]$ of the form (2.4), satisfying (2.5), such that the solution to the initial-value problem*

$$\begin{cases} \dot{\boldsymbol{x}}(t) = -\dfrac{2}{\pi} \boldsymbol{\mathcal{D}}(t) u(t), & t \in [0, \pi] \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 := [\boldsymbol{a}_T; \boldsymbol{b}_T] \end{cases} \tag{4.5}$$

satisfies $\boldsymbol{x}(\pi) = 0$, where $\boldsymbol{\mathcal{D}}$ is given by (4.3)–(4.4).

A natural approach for null controllability problems such as Problem 4.1 is to formulate them as an optimal control one, where the cost functional to be minimized is the euclidean distance from the final state $\boldsymbol{x}(\pi)$ to the the origin. In what follows, for a given vector $\boldsymbol{v} \in \mathbb{R}^d$,

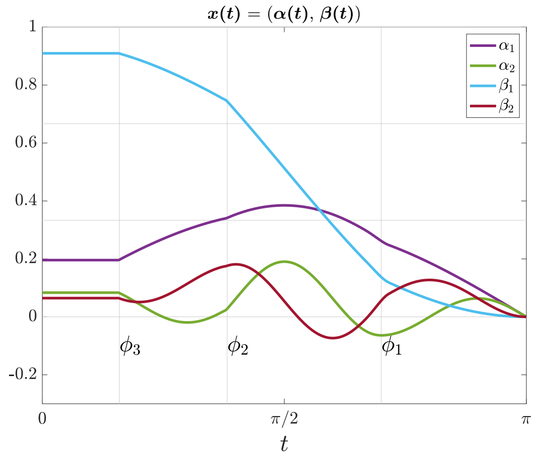

Fig. 3. Evolution of the dynamical system (4.5) with $\mathcal{E}_a = \{1, 2\}$ and $\mathcal{E}_b = \{1, 2\}$ corresponding to the control u in Figure 1. The positions of the switching angles $\boldsymbol{\phi}$ are displayed as well.

we denote by $\|\boldsymbol{v}\|$ the euclidean norm $\|\boldsymbol{v}\|_{\mathbb{R}^d}$. Let us introduce the set of admissible controls.

$$\mathcal{A} := \Big\{ u : [0, \pi) \to [-1, 1] \ \text{ measurable} \Big\}$$
$$\mathcal{A}_{ad} := \Big\{ u \in \mathcal{A} \text{ of the form (2.4) satisfying (2.5)} \Big\}$$

Problem 4.2 (OCP for SHM) *Let $\mathcal{U}$ be a given set as in (2.1). Let $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ be given as in Problem 2.1, we look for an admissible control $u \in \mathcal{A}_{ad}$ solution to the following optimal control problem:*

$$\min_{u \in \mathcal{A}_{ad}} \frac{1}{2}\|\boldsymbol{x}(\pi)\|^2 \quad \text{subject to the dynamics (4.5).}$$

Remark 4.1 *Note that the cost functional in Problem 4.2 is quadratic and, therefore, it always admits at least one minimizer for any target $[\boldsymbol{a}_T; \boldsymbol{b}_T]^\top$. However, this minimizer is a solution to the SHM problem if and only if the minimum is equal to zero. Otherwise, we say that the target $[\boldsymbol{a}_T; \boldsymbol{b}_T]^\top$ is unreachable and the SHM problem 2.1 has no solution. In this work, we will not discuss the reachable set for the control problem (4.2).*

A main feature of the SHM problem is that we are looking for signal functions u of the form (2.4) satisfying (2.5). In principle, this can be guaranteed by by adding directly this constraint on the set of admissible controls $\mathcal{A}_{ad}$, as we did in Problem 4.2. Notwithstanding that, the inclusion of such a constraint makes the optimal control problem extremely difficult, as it prevents us from applying standard arguments such as the Pontryagin's maximum principle, and implementing all the computational techniques developed in the last decades to solve optimal control problems, for instance ???.

In order to bypass this difficulty, we propose a modification of Problem 4.2 by adding to the cost functional, a penalization term for the control.

Problem 4.3 (Penalized OCP for SHM) *Fix $\varepsilon > 0$ and a convex function $\mathcal{L} \in C([-1, 1]; \mathbb{R})$. Let $\mathcal{E}_a$, $\mathcal{E}_b$ and the targets $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ be given as in Problem 2.1. We look for a control $u \in \mathcal{A}$ solution to the following optimal control problem:*

$$\min_{u \in \mathcal{A}} \left(\frac{1}{2}\|\boldsymbol{x}(\pi)\|^2 + \varepsilon \int_0^\pi \mathcal{L}(u(t))dt \right)$$

subject to the dynamics (4.5).

Observe that, in Problem 4.3, we do not impose the constraint that the control has to be of the form (2.4), satisfying the staircase property (2.5). Nevertheless, as we shall see, these features of u will arise naturally in the solution to Problem 4.3, from a suitable choice of the penalization term $\mathcal{L}$. Another important advantage of adding a penalization term for the control is that, as we shall prove in Theorems 4.1 and 4.2, it ensures the uniqueness for the solution, and the continuity of this one with respect to the targets $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$.

On the contrary, one needs to take into account that the penalization term for the control might prevent the optimal trajectory from reaching the target. In other words, even if there exists a control for which the optimal trajectory satisfies $\boldsymbol{x}(\pi) = 0$, the optimal control in Problem 4.3 might not do so, and therefore, the solution to Problem 4.3 would not be a solution to the SHM problem. This issue may be controlled by a proper selection of the weighting parameter ε which allows to tune the precision of the optimal control for the perturbed problem, guaranteeing that the final state of the optimal trajectory is close enough to zero. This is the content of the following proposition.

Proposition 4.1 *Assume that $[\boldsymbol{a}_T, \boldsymbol{b}_T]^\top$ is such that Problem 4.1 admits a solution, and let $u^* \in \mathcal{A}$ be the solution to Problem 4.3. Then the associated trajectory $\boldsymbol{x}^* \in C([0, \pi]; \mathbb{R})$, solution to (4.5), satisfies*

$$\|\boldsymbol{x}^*(\pi)\|^2 \leq 4\varepsilon\pi\|\mathcal{L}\|_\infty,$$

where $\|\cdot\|_\infty$ denotes the max-norm in $C([-1, 1]; \mathbb{R})$.

The proof of Proposition 4.1 is postponed to Section 6. Let us now describe the construction of penalization functions $\mathcal{L}$ which guarantee that any solution to Problem 4.3 has the form (2.4) and satisfies (2.5). To this end, we will distinguish two cases, depending on the cardinality of $\mathcal{U}$.

4.1 Bilevel SHM problem via OCP (Bang-Bang Control)

In this case, the control set $\mathcal{U}$ defined in (2.1) has only two elements, i.e. $\mathcal{U} = \{-1, 1\}$. In the control theory literature, a control taking only two values is known as *bang-bang control*. In the SHM literature, this kind of solution are called *bi-level solutions*. Note that in this case, any u with the form (2.4) trivially satisfies the staircase property (2.5).

Theorem 4.1 *Let $\mathcal{U} = \{-1, 1\}$, and $\boldsymbol{x}_0$ be given. For some $\alpha \in \mathbb{R}$ with $\alpha \neq 0$, consider the Problem 4.3 with $\mathcal{L}(u) = \alpha u$. Then, the optimal control u^*, solution to Problem 4.3 is unique and has a bang-bang structure, i.e. it is of the form (2.4). In addition, the solution u^* to Problem 4.3 is continuous with respect to $\boldsymbol{x_0}$ in the strong topology of $L^1(0, \pi)$.*

The proof of Theorem 4.1 is postponed to Section 6, and follows from the optimality conditions given by the Pontryagin's maximum principle. In particular, the linearity of $\mathcal{L}$ and of the dynamical system (4.5), implies that the associated Hamiltonian is also linear, and then, the minimum is always attained at the limits of the interval $[-1, 1]$.

We point out that, by choosing different penalization functions $\mathcal{L}$, we may obtain solutions to the SHM problem with different waveforms. Typical choices which give rise to bang-bang controls are

$$\mathcal{L}(u) = u, \quad \mathcal{L}(u) = -u.$$

See Fig. 7 for an illustration.

4.2 Multilevel SHM problem via OCP

Inspired by the ideas of the previous subsection, we can address the case when $\mathcal{U}$ contains more than two elements. This is known in the power electronics literature as the *multilevel SHM problem*. Now, the goal is to construct a function $\mathcal{L}$ such that the Hamiltonian associated to Problem 4.3 always attains the minimum at points in $\mathcal{U}$. A way to construct such a function $\mathcal{L}$ is to interpolate a parabola in $[-1, 1]$ by affine functions, considering the elements in $\mathcal{U}$ as the interpolating points. Since, between any two points in $\mathcal{U}$, the function $\mathcal{L}$ is a straight line, the Hamiltonian is a concave function in these intervals, and hence, the minimum is always attained at points in $\mathcal{U}$.

Theorem 4.2 *Let $\boldsymbol{x_0}$ be given, and let $\mathcal{U}$ be a given set as in (2.1). For any $\alpha > 0$ and $\beta \in \mathbb{R}$, set the function*

$$\mathcal{P}(u) = \alpha(u - \beta)^2. \tag{4.6}$$

Consider Problem 4.3 with

$$\mathcal{L}(u) = \begin{cases} \lambda_k(u) & \text{if } u \in [u_k, u_{k+1}) \\ \mathcal{P}(1) & \text{if } u = u_L \end{cases} \tag{4.7}$$

$$\forall k \in \{1, \dots, L-1\},$$

where

$$\lambda_k(u) := \frac{(u - u_k)\mathcal{P}(u_{k+1}) + (u_{k+1} - u)\mathcal{P}(u_k)}{u_{k+1} - u_k}. \tag{4.8}$$

Assume in addition that the function $\mathcal{L}$ has a unique minimum in $[-1, 1]$. Then, the optimal control u^*, solution to Problem 4.3, is unique and has the form (2.4) satisfying (2.5). Moreover, the solution u^* to Problem 4.3 is continuous with respect to $\boldsymbol{x}_0$ in the strong topology of $L^1(0, \pi)$.

The assumption of $\mathcal{L}$ having a unique minimum in $[-1, 1]$ is actually necessary to ensure the staircase form (2.4) for the solution. Not assuming this hypothesis would entail the possibility of having continuous solutions for specific targets. Nevertheless, the assumption of $\mathcal{L}$ having a unique minimizer can be easily ensured by choosing, for instance, $\beta = \pm 1$.

Remark 4.2 *For completeness, we shall mention that in Theorem 4.2 the function $\mathcal{L}$ can actually have a more general form, still yielding to a staircase optimal control u^*. As a matter of fact, as we shall see in Section 6, the proof of Theorem 4.2 does not use the fact that the function $\mathcal{P}$ is a parabola. If we replace this $\mathcal{P}$ with any other strictly convex function, our result remains valid. The choice we made of defining $\mathcal{P}$ as in (4.6) is motivated by the fact that, most often, in optimal control theory the penalization terms are chosen to be quadratic.*

Remark 4.3 (*Bang-off-bang control*) *We note that when $\mathcal{U} = \{-1, 0, 1\}$, we can just use the L^1-norm of the control as penalization, i.e. $\mathcal{L}(u) = |u|$. This yields to the so-called* bang-off-bang *controls, that are widely studied in the literature [Nagahara et al., 2013, Ikeda and Nagahara, 2016]. By taking a different parabola $\mathcal{P}$, one can then obtain different bang-off-bang solutions to the SHM problem.*

We illustrate in Fig. 8 different examples of penalization functions $\mathcal{L}$ giving rise to multilevel solutions to the SHM problem. We point out that, by varying the values of α and β in Theorem 4.2, we can obtain solutions with different waveforms.

5 Numerical simulations

In this section, we present several examples in which we implement the optimal control strategy we proposed to solve the SHM problem. All the simulations we are going to present can be found also in [Oroya, 2021]. Our Experiments were conducted on a personal MacBook Pro laptop (1,4 GHz Quad-Core Intel Core i5, 8GB RAM, Intel Iris Plus Graphics 1536 MB).

To solve our optimal control Problem 4.3, we will employ the direct method [Rao, 2009] which, in broad terms, consists in discretizing the cost functional and the dynamics, and then apply some optimization algorithm.

The dynamics will be approximated with the Euler method, while for solving the discrete minimization problem we will employ the non-linear constrained optimization tool `CasADi` [Andersson et al., 2019]. `CasADi` is an open-source tool for nonlinear optimization and algorithmic differentiation which implements the interior point method via the optimization software `IPOPT` [Wächter and Biegler, 2006]. To be efficiently applied to solve an optimal control problem, we then need the functional we aim to minimize to be smooth. This is not the case of the functional in Problem 4.3 which, due to the piecewise linear penalization, is not differentiable at the points $u_k \in \mathcal{U}$. For this reason, we will first need to build a smooth approximation of the function $\mathcal{L}$ we introduced in (4.7). Once we have this approximation, we will employ the optimal control approach we presented in Section 4 to solve some specific examples of SHM problem.

5.1 Smooth approximation of piecewise linear penalization

As we mentioned, to efficiently employ `CasADi` for solving our optimal control problem, we need to build a smooth approximation of the cost functional. For this reason, we will regularize the piecewise linear penalization defined in (4.7) in the following way.

First of all, for all real $\theta > 0$, we introduce the $C^\infty(\mathbb{R})$ function

$$h^\theta(x) := \frac{1+\tanh(\theta x)}{2}$$

and observe that, as $\theta \to +\infty$, h^θ converges in $L^\infty(\mathbb{R})$ to the Heaviside function

$$h(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases}.$$

Secondly, for all $k \in \{1, \ldots, N_u - 1\}$ we define the (smooth) function $\chi^\theta_{[u_k,u_{k+1})} : \mathbb{R} \to \mathbb{R}$ given by

$$\begin{aligned} \chi^\theta_{[u_k,u_{k+1})}(x) :&= -1 + h^\theta(x-u_k) + h^\theta(-x+u_{k+1}) \\ &= \frac{\tanh[\theta(x-u_k)] + \tanh[\theta(u_{k+1}-x)]}{2} \end{aligned}$$

which, as $\theta \to +\infty$, converges in $L^\infty(\mathbb{R})$ to the characteristic function $\chi_{[u_k,u_{k+1})}$.

Finally, we employ $\chi^\theta_{[u_k,u_{k+1})}$ to define

$$\mathcal{L}^\theta(u) = \sum_{k=1}^{N_u-1} \lambda_k \chi^\theta_{[u_k,u_{k+1})}(u), \tag{5.1}$$

with λ_k given by (4.8), which, as $\theta \to +\infty$, converges in $L^\infty(\mathbb{R})$ to the penalization function $\mathcal{L}$ defined in (4.7).

Finally, notice that this regularization procedure is independent of the function λ_k in (4.7), which is just required to be in the form (4.8). Nevertheless, in our numerical experiments we shall select some specific λ_k. In particular, we will use

$$\lambda_k = (u_{k+1} + u_k)(u - u_k) + u_k^2, \tag{5.2}$$

which corresponds to taking $a = 1$ and $b = 0$ in (4.6).

5.2 Direct method for OCP-SHE

To solve the optimal control Problem 4.3, we use a direct method, whose starting point is to discretize the cost functional and the dynamics.

To this end, let us consider a N_t-points partition of the interval $[0, \pi]$

$$\mathcal{T} = \{t_k\}_{k=1}^{N_t}$$

and denote by $\boldsymbol{u} \in \mathbb{R}^{N_t}$ the vector with components $u_k = u(t_k)$, $k = 1, \ldots, N_t$.

Then the optimal control problem (4.2) can be written as optimization problem with variable $\boldsymbol{u} \in \mathbb{R}^{N_t}$. This problem is a nonlinear programming, for this we use `CasADi` software to solve.

Hence, given a partition of the interval $[0, \pi)$, we can formulate the problem 4.3 as the following one in discrete time.

Problem 5.1 (Numerical OCP) *Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$, respectively, the target vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, and the partition $\mathcal{T}$ of the interval $[0, \pi]$, we look for $\boldsymbol{u} \in \mathbb{R}^{N_t}$ that solves the following minimization problem:*

$$\min_{\boldsymbol{u}\in\mathbb{R}^{N_t}} \left[\|\boldsymbol{x}_{N_t}\|^2 + \varepsilon \sum_{k=1}^{N_t-1} \left[\frac{\mathcal{L}^\theta(u_{t_k}) + \mathcal{L}^\theta(u_{t_{k+1}})}{2} \Delta t_k \right] \right]$$

subject to:

$$\forall \tau \in \mathcal{P} \begin{cases} \boldsymbol{x}_{t_{k+1}} = \boldsymbol{x}_{t_k} - \Delta t_k (2/\pi) \boldsymbol{\mathcal{D}}(t_k) \\ \boldsymbol{x}_{t_1} = \boldsymbol{x}_0 := [\boldsymbol{a}_T, \boldsymbol{b}_T]^\top \end{cases}$$

where

$$\Delta t_k = t_{k+1} - t_k \quad \forall k \in \{1, \dots, N_t - 1\}. \tag{5.3}$$

5.3 Numerical experiments

We now present several numerical experiments to show the effectiveness of our optimal control approach to solve SHM problems. All the examples that we are going to present will share the following common parameter: $\varepsilon = 10^{-5}$, $\theta = 10^5$ and $\mathcal{P}_t = \{0, 0.1, 0.2, \dots, \pi\}$. Moreover, we will consider $\mathcal{E}_a = \mathcal{E}_b = \{1, 5, 7, 11, 13\}$, and the target vectors $\boldsymbol{a}_T = \boldsymbol{b}_T = (m, 0, 0, 0, 0, 0)^\top$ for all values of $m \in [-0.8, 0.8]$.

We will treat all the specific types of controls we mentioned before, that is

1. *bang-bang*, with $\mathcal{U} = \{-1, 1\}$.
2. *bang-off-bang*, with $\mathcal{U} = \{-1, 0, 1\}$.
3. *multilevel*, $\mathcal{U} = \{-1, -1/2, 0, 1/2, 1\}$.

The results of our simulations are displayed in Fig. 4, 5 and 6 where, for all $m \in [-0.8, 0.8]$, we show the *policy* $\Pi(m)$ giving us the optimal control signal u^*. Here the terminology *policy* and the notation $\Pi(m)$ are consistent with the ones of Definition 3.3.

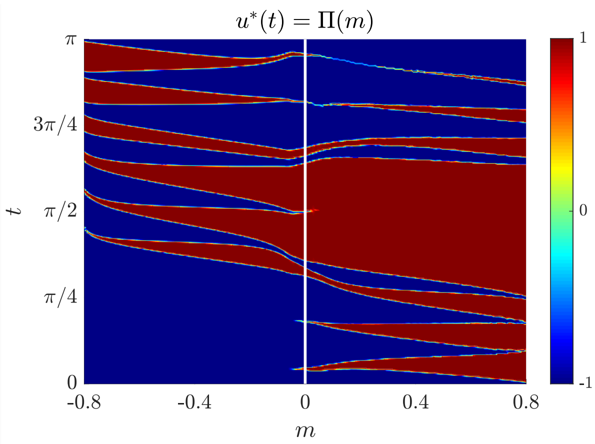

Fig. 4. Bang-bang control for the SHM problem.

In these plots, to each value of the parameter m in the horizontal line, it corresponds an optimal control having a staircase structure and changing value in $\mathcal{U}$ in the correspondence of the change of color. For instance, in Fig. 4, the control will be $u = -1$ in the blue region and $u = 1$ in the red one. This, of course, is in accordance with Theorems 4.1 and 4.2.

In addition to that, if we compare the policies $\Pi(m)$ displayed in Fig. 4, 5 and 6 with the policies $\Pi_{\mathcal{S}}$ of Fig. 2, we can see that the issues we mentioned in Section 3 concerning the solvable set and the continuity of the policy have been overcome by our approach. In particular, formulating the SHM problem via optimal control we are capable of finding solutions in an ample range of the parameter m. This is because, while solving Problem 5.1, we are not restricted to have a specific waveform, as it is Problem 3.1. Instead, for any value of m the best waveform and switching angles to reach the desired targets are automatically obtained through the minimization process to compute the optimal control. In other words, the possibility of changing the waveform during the optimization process can be seen as an extra degree of freedom allowing to have a large solvable set and a continuous policy. Furthermore, also the *combinatory problem* we introduced in Section 3 is solved by our approach, as we do not need anymore to launch the same optimization process for all the possible waveforms of a given set $\mathcal{U}$.

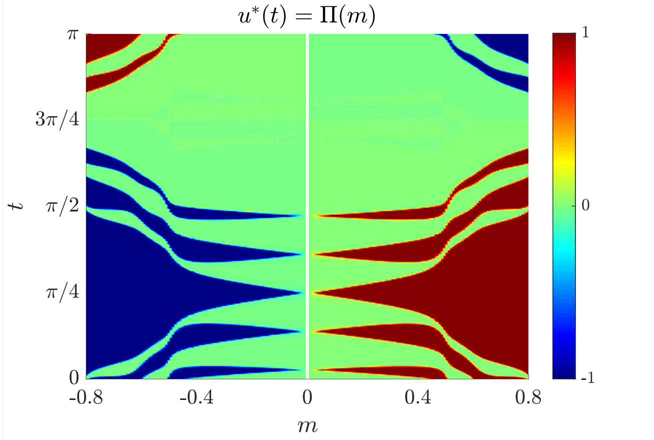

Fig. 5. Bang-off-bang control for the SHM problem.

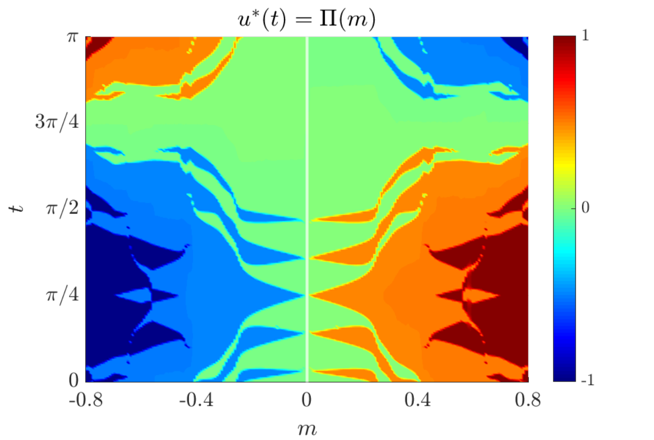

Fig. 6. Multilevel control for the SHM problem.

All these considerations show that the methodology introduced in this paper is theoretically and computationally competitive to solve the SHM problem.

6 Proofs of results in Section 4

This section is devoted to the proofs of the results presented in Section 4, i.e. Proposition 4.1 and Theorems 4.1 and 4.2. For the sake of readability, we organize the proofs as follows: in subsection 6.1 we prove the existence a minimizer for Problem 4.3, and also give the proof of Proposition 4.1; in subsection 6.2, we deduce the necessary optimality conditions from Pontryagin's Maximum Principle; in subsection 6.3 we prove that, when $\mathcal{L}$ is given as in Theorem 4.1, the solutions to Problem 4.3 have a bang-bang form; in subsection 6.4 we prove the analogous result for Theorem 4.2. Finally, in subsection 6.5 we give the proof of uniqueness and continuity of the solution to Problem 4.3 with respect to the initial condition, when the penalization term $\mathcal{L}$ is given as in Theorems 4.1 or 4.2.

6.1 Existence of minimizers

The existence of a minimizer, solution to Problem 4.3, can be easily proved by means of the direct method in calculus of variations. Indeed, observe that the dynamical system (4.5) is linear and the admissible controls in $\mathcal{A}$ are uniformly bounded. In addition, the functional to be minimized is convex with respect to the control, which suffices to ensure the weak lower semicontinuity of the functional, allowing us to pass to the limit in the minimizing sequence.

Let us now give the proof of Proposition 4.1, which provides an upper estimate for the error in the solution to the optimal control problem with the penalization term for the control, in the case when the SHM problem admits a solution.

Proof: **(of Proposition 4.1)** *Since we are supposing that Problem 4.1 has a solution, there exists a control $\tilde{u} \in \mathcal{A}_{ad}$ such that its corresponding trajectory $\tilde{\boldsymbol{x}}$, solution to (4.5), satisfies $\tilde{\boldsymbol{x}}(\pi) = 0$.*

Now, let $u^* \in \mathcal{A}$ be the solution to Problem 4.3, and let $\boldsymbol{x}^*$ be its corresponding trajectory. By the optimality of u^* we have

$$\frac{1}{2}\|\boldsymbol{x}^*(\pi)\|^2 + \varepsilon \int_0^\pi \mathcal{L}(u^*(\tau))d\tau \leq \varepsilon \int_0^\pi \mathcal{L}(\tilde{u}(\tau))d\tau,$$

and hence, we deduce that $\|\boldsymbol{x}^*(\pi)\|^2 \leq 4\varepsilon\pi\|\mathcal{L}\|_\infty$.

6.2 Optimality condditions

The proofs of Theorems 4.1 and 4.2 are based on the optimality conditions for the Optimal Control Problem 4.3, which can be deduced by means of Pontryagin's maximum principle [Bryson, 1975, Chapter 2.7].

To this end, let us first introduce the Hamiltonian associated to the Optimal Control Problem 4.3:

$$\mathcal{H}(t, \boldsymbol{p}, u) = \varepsilon\mathcal{L}(u) - \frac{2}{\pi}\big(\boldsymbol{p} \cdot \boldsymbol{\mathcal{D}}(t)\big)u(t), \tag{6.1}$$

where $\boldsymbol{p} \in \mathbb{R}^{N_a+N_b}$ is the so-called adjoint variable, and arises from the restriction imposed by the dynamical system (4.5). In view of the definition of $\boldsymbol{\mathcal{D}}(t)$ in (4.3)-(4.4), we will sometimes write the state and the adjoint variables using the following notation:

$$\boldsymbol{x}(t) = \begin{bmatrix} \boldsymbol{a}(t) \\ \boldsymbol{b}(t) \end{bmatrix} \quad \text{and} \quad \boldsymbol{p}(t) = \begin{bmatrix} \boldsymbol{p}^a(t) \\ \boldsymbol{p}^b(t) \end{bmatrix}.$$

Now, let us derive the optimality conditions arising from Pontryagin's Maximum Principle.

1. **The adjoint system**: for any $u^* \in \mathcal{A}$, solution to the Problem 4.3, there exists a unique adjoint trajectory $\boldsymbol{p}^* \in C([0,\pi]; \mathbb{R}^{N_a+N_b})$ which satisfies the following terminal-value problem

   $$\begin{cases} \dot{\boldsymbol{p}^*}(t) = -\nabla_x \mathcal{H}(u(t), \boldsymbol{p}^*(t), t), & t \in [0,\pi] \\ \boldsymbol{p}^*(\pi) = \nabla_x \Psi(\boldsymbol{x}^*(\pi)) \end{cases}$$

   where Ψ is the terminal cost of the Optimal Control Problem 4.3. In our case, we have $\Psi(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}\|^2$. Moreover, since the Hamiltonian does not depend on the state variable $\boldsymbol{x}$, we simply have $\dot{\boldsymbol{p}^*}(t) = 0$ for all $t \in [0,\pi]$. We therefore deduce that the adjoint trajectory is constant, and given by

   $$\boldsymbol{p}^*(t) = \boldsymbol{x}^*(\pi), \qquad \forall t \in [0,\pi]. \tag{6.2}$$

2. **The Optimal Control**: now, using the optimal adjoint trajectory, we can deduce the necessary optimality condition for the control, which reads as follows:

   $$u^*(t) \in \underset{|u|\leq 1}{\arg\min}\, \mathcal{H}(t, \boldsymbol{p}^*(t), u), \quad \forall t \in [0,\pi]. \tag{6.3}$$

   As we will see in subsections 6.3 and 6.4, for penalization functions $\mathcal{L}$ as the ones we consider in Theorems 4.1 and 4.2, this argmin is a singleton for almost every $t \in [0,\pi]$. Hence, given the adjoint $\boldsymbol{p}^*$, the condition (6.3) uniquely determines the optimal control almost everywhere in $(0,\pi)$. The only points where the control is not uniquely determined are, precisely, the switching angles, i.e. the points of discontinuity of the solution.

In view of the form of the adjoint trajectory (6.2) associated to the optimal state trajectory $\boldsymbol{x}^*$, let us introduce the function

$$\mu^*(t) := \frac{2}{\pi}\left(\boldsymbol{x}^*(\pi)\cdot \boldsymbol{\mathcal{D}}(t)\right) \tag{6.4}$$
$$= \sum_{j\in\mathcal{E}_a} a_j^*(\pi)\cos(jt) + \sum_{k\in\mathcal{E}_b} b_k^*(\pi)\sin(kt).$$

Then, in view of (6.1) and (6.2), we can write the optimality condition (6.3) as

$$u^*(t) \in \arg\min_{|u|\le 1} \mathcal{J}(u,\mu^*(t)). \tag{6.5}$$

where $\mathcal{J}$ is defined as

$$\mathcal{J}(u,\mu^*(t)) := \varepsilon\mathcal{L}(u) - \mu^*(t)u. \tag{6.6}$$

We are now ready to prove that the solutions to Problem 4.3, when $\mathcal{L}$ is chosen as in Theorems 4.1 and 4.2, have the desired staircase form (2.4)–(2.5).

6.3 *Proof of Theorem 4.1 (bang-bang control)*

We need to prove that, if $\mathcal{L}(u) = \alpha u$ for some $0 \neq \alpha \in \mathbb{R}$, then any optimal control u^* has the form (2.4) with $\mathcal{U} = \{-1, 1\}$. Or in other words, $u^*(t)$ takes values in $\mathcal{U}$ for all $t \in [0,\pi)$, except for a finite number of times.

Let $u^* \in \mathcal{A}$ be a solution to Problem 4.3, and let $\boldsymbol{x}^*$ be its associated optimal trajectory. We just need to notice that, in view of the optimality condition (6.5) and the choice of $\mathcal{L}$, the function u^* satisfies

$$u^*(t) = \begin{cases} -1 & \text{if } \mu^*(t) < \varepsilon\alpha \\ 1 & \text{if } \mu^*(t) > \varepsilon\alpha \end{cases}$$

Observe that, in the case when $\mu^*(t) = 0$, which corresponds only to the cases when $\boldsymbol{x}^*(\pi) = 0$, the optimal control is constant and is just given by $u^*(t) = -\mathrm{sgn}(\alpha)$.

In all the other cases, when $\boldsymbol{x}^*(\pi) \neq 0$, the function $\mu^*(t)$ is a linear combination of sines and cosines, and therefore, the equality $\mu^*(t) = \varepsilon\alpha$ can only hold for a finite number of times $t \in [0,\pi)$, which are the discontinuity points of u^* (the switching angles). Note that the choice of u^* at these points is irrelevant as it represents a set of zero measure.

6.4 *Proof of Theorem 4.2 (multilevel control)*

In this case, we suppose that $\mathcal{U} = \{u_k\}_{k=1}^L$ is a finite set of real numbers in $[-1,1]$ satisfying

$$-1 = u_1 < u_2 < \ldots < u_L = 1, \quad \text{with } L > 2. \tag{6.7}$$

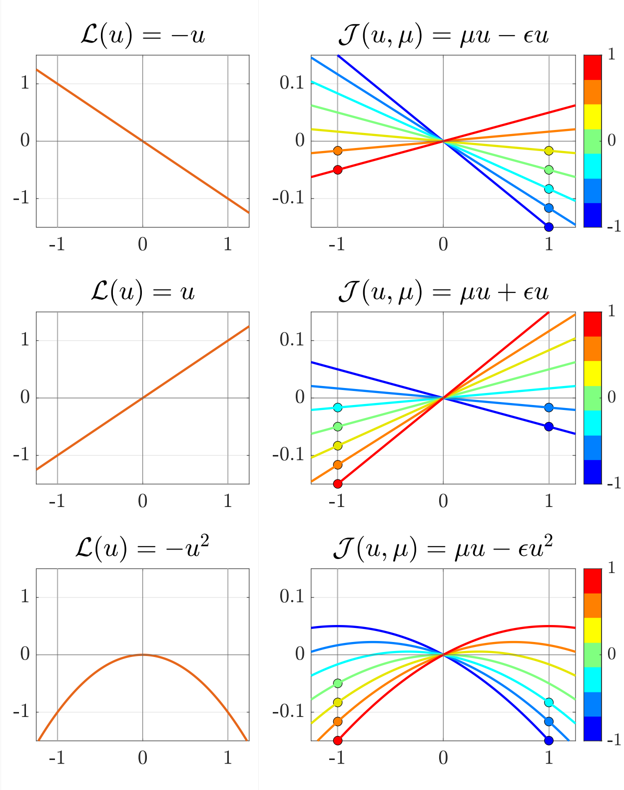

Fig. 7. Bi-level SHE: in the left column we show two examples of functions $\mathcal{L}$ compatible with Theorem 4.1. In the right columns we display the behavior of the corresponding Hamiltonian for different values of μ.

The case $L = 2$ is just the bi-level case. As in the previous proof, our goal is to show that the argmin in (6.5) is a singleton and belongs to $\mathcal{U}$ for every $t \in [0,\pi)$ except for a finite number of points in $[0,\pi)$.

In this case, the study of the minimizers of $\mathcal{J}$ is slightly more involved since the penalization function $\mathcal{L}$ defined in (4.7)-(4.8) is not differentiable at the points $u_k \in \mathcal{U}$.

Since $\mathcal{L}$ is an affine interpolation of a convex function, and therefore, it is Lipschitz and convex, we deduce that also $\mathcal{J}$ is Lipschitz and convex as a function of u. In view of this, we have that u^* minimizes $\mathcal{J}(u,\mu)$ if and only if

$$0 \in \partial_u \mathcal{J}(u^*,\mu), \tag{6.8}$$

where ∂_u denotes the subdifferential with respect to u.

Let us recall the definition of subdifferential from convex analysis:

$$\partial_u\mathcal{J}(u,\mu) = \{c\in\mathbb{R} \quad \text{s.t.}$$
$$\mathcal{J}(v,\mu) - \mathcal{J}(u,\mu) \ge c(v-u)$$
$$\forall v \in [-1,1]\}.$$

In the case of a convex function as $\mathcal{J}(\cdot,\mu)$, one can readily show that the subdifferential at $u \in (-1,1)$ is the nonempty interval $[a,b]$, where a and b are the one-sided derivatives

$$a = \lim_{v\to u^-} \frac{\mathcal{J}(v,\mu)-\mathcal{J}(u,\mu)}{v-u}$$
$$b = \lim_{v\to u^+} \frac{\mathcal{J}(v,\mu)-\mathcal{J}(u,\mu)}{v-u}.$$

Moreover, the subdifferential at $u=-1$ and $u=1$ is given by $(-\infty,b]$ and $[a,+\infty)$ respectively. Notice that, if $\mathcal{J}$ is differentiable at some $u \in (-1,1)$, then the left and the right derivatives coincide, and thus, $\partial_u\mathcal{J}(u,\mu)$ is just the classical derivative.

Using this characterization of the subdifferential, we can compute $\partial_u\mathcal{J}(u,\mu)$ for all $u \in [-1,1]$ in terms of μ. To this end, let us define

$$p_k := \frac{d}{du}\lambda_k(u) = \frac{\mathcal{P}(u_{k+1})-\mathcal{P}(u_k)}{u_{k+1}-u_k}$$

for all $k \in \{1,\ldots,L-1\}$, with $\lambda_k(u)$ given by (4.8). Using the definition of $\mathcal{J}$ in (6.6) and $\mathcal{L}$ in (4.7), we can compute

$$\begin{aligned}
\partial_u\mathcal{J}(-1,\mu) &= (-\infty, \varepsilon p_1 - \mu],\\
\partial_u\mathcal{J}(1,\mu) &= [\varepsilon p_{L-1} - \mu, +\infty),\\
\partial_u\mathcal{J}(u_k,\mu) &= [\varepsilon p_{k-1} - \mu,\, \varepsilon p_k - \mu],
\end{aligned}$$

for all $k \in \{2,\ldots,L-1\}$, and

$$\partial_u\mathcal{J}(u,\mu) = \{\varepsilon p_k - \mu\},$$

for all $u \in (u_k,u_{k+1})$ and all $k \in \{1,\ldots,L-1\}$.

Now, in view of the above computation, we obtain that

$$\begin{aligned}
0 &\in \partial_u\mathcal{J}(-1,\mu) && \text{iff} \quad \mu \le \varepsilon p_1,\\
0 &\in \partial_u\mathcal{J}(1,\mu) && \text{iff} \quad \mu \ge \varepsilon p_{L-1},\\
0 &\in \partial_u\mathcal{J}(u_k,\mu) && \text{iff} \quad \varepsilon p_{k-1} \le \mu \le \varepsilon p_k,
\end{aligned} \tag{6.9}$$

for all $k \in \{2,\ldots,L-1\}$, and

$$0 \in \partial_u\mathcal{J}(u,\mu) \quad \forall u \in [u_k,u_{k+1}] \qquad \text{iff } \mu = \varepsilon p_k \tag{6.10}$$

for all $k \in \{1,\ldots,L-1\}$.

Using (6.9), along with the optimality condition (6.8), we deduce that, for a.e $\mu \in \mathbb{R}$, we have

$$\arg\min_{|u|\le 1} \mathcal{J}(u,\mu) = \{u_k\} \quad \text{for some } u_k \in \mathcal{U}. \tag{6.11}$$

Indeed, (6.11) does not hold if and only if

$$\mu = \varepsilon p_k \qquad \text{for some } k \in \{1,\ldots,L-1\}. \tag{6.12}$$

Observe that, in the case when $\mu^*(t) = 0$, which corresponds only to the cases when $\boldsymbol{x}^*(\pi) = 0$, the optimal control is constant and is just given by $u^*(t) = \arg\min_{|u|\le 1} \mathcal{L}(u)$ which, by hypothesis, is a singleton and belongs to $\mathcal{U}$ (note that between any two consecutive points of $\mathcal{U}$, the function $\mathcal{L}$ is a straight line).

In all the other cases, i.e. when $\boldsymbol{x}^*(\pi) \neq 0$, the function $\mu^*(t)$ is a linear combination of sines and cosines, and therefore, the equality $\mu^*(t) = \varepsilon p_k$ can only hold, for each $k \in \{1,\ldots,L-1\}$, a finite number of times in $[0,\pi)$. These are precisely the discontinuity points of u^* (the switching angles).

We have proved that, for all $t \in [0,\pi)$ except for a finite number of discontinuity points, which are precisely the switching angles $\{\phi_m\}_{m=0}^M$, we have $u^*(t) \equiv u_k$ for some $u_k \in \mathcal{U}$. Observe that, due to the continuity of $\mu^*(t)$, along with (6.9), it is clear that $u^*(t)$ does not change value between two consecutive switching angles. Therefore, u^* is piecewise constant, with a finite number of switches. The choice of u^* at the discontinuity points is irrelevant as it represents a set of zero measure.

Finally, the staircase property of the waveform (2.5) can be deduced from (6.5) and (6.9), along with the continuity of the function $\mu^*(t)$.

6.5 Uniqueness and continuity of solutions

The proofs in this subsection apply to both Theorems 4.1 and 4.2 (the bilevel and the multilevel case).

***Proof:* (uniqueness)** *We first prove that Problem 4.3 admits a unique solution, i.e. for each $\boldsymbol{x}_0 \in \mathbb{R}^N$, there exists a unique $u^* \in \mathcal{A}$ minimizing the functional*

$$F(u,\boldsymbol{x}_0) := \frac{1}{2}\|\boldsymbol{x}(\pi)\|^2 + \varepsilon \int_0^\pi \mathcal{L}(u(t))dt, \tag{6.13}$$

where, for each $u \in \mathcal{A}$, $\boldsymbol{x}(\pi)$ is given by

$$\boldsymbol{x}(\pi) = \boldsymbol{x}_0 - \frac{2}{\pi}\int_0^\pi \boldsymbol{\mathcal{D}}(t)u(t)dt.$$

We argue by contradiction. Suppose that there exist two functions $u_1, u_2 \in \mathcal{A}$ solutions to Problem 4.3, which are different in a set of positive measure. As both of them are optimal, using the arguments in subsections 6.2, 6.3 and 6.4, we deduce that the controls u_1 and u_2 are uniquely determined a. e. in $[0,\pi)$ by the final state of the associated trajectory, i.e. $\boldsymbol{x}_1^*(\pi)$ and $\boldsymbol{x}_2^*(\pi)$respectively. Therefore, if $u_1 \neq u_2$ in a set of positive measure, then we have $\boldsymbol{x}_1^*(\pi) \neq \boldsymbol{x}_2^*(\pi)$.

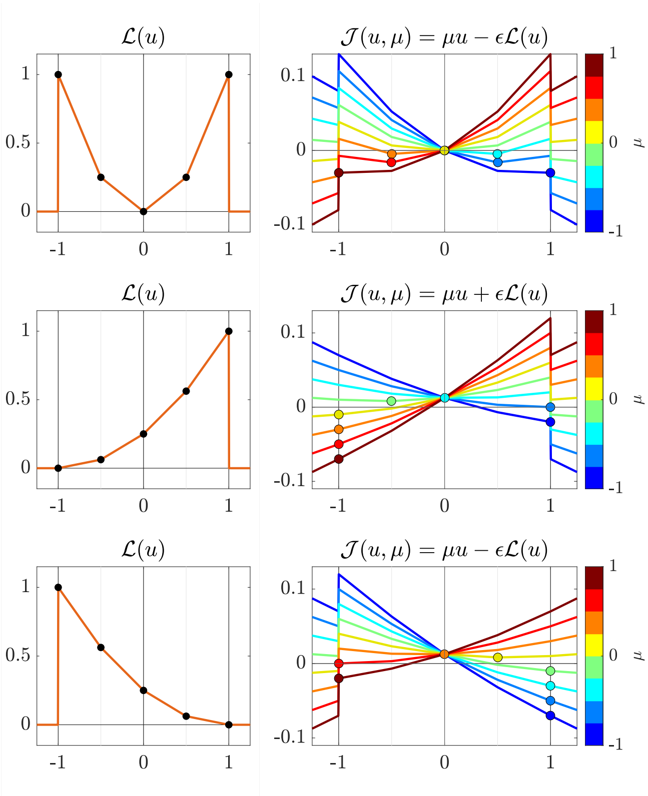

Fig. 8. Multilevel SHM: in the left column we show three types of penalizations which are compatible with our theoretical results. In the right column we show the behavior of the corresponding Hamiltonian for different values of μ.

Let us now consider the control

$$\tilde{u}(t) = \frac{u_1(t) + u_2(t)}{2}.$$

By the linearity of the dynamics (4.5), the convexity of $\mathcal{L}$, and using that $\boldsymbol{x}_1^*(\pi) \neq \boldsymbol{x}_2^*(\pi)$, we obtain

$$\begin{aligned} F(\tilde{u}, \boldsymbol{x}_0) &= \frac{1}{2}\left\| \frac{\boldsymbol{x}_1^*(\pi) + \boldsymbol{x}_2^*(\pi)}{2} \right\|^2 \\ &\quad + \varepsilon \int_0^\pi \mathcal{L}\left(\frac{u_1(t) + u_2(t)}{2}\right) dt \\ &< \frac{F(u_1, \boldsymbol{x}_0) + F(u_2, \boldsymbol{x}_0)}{2} \end{aligned}$$

Hence, using that both u_1 and u_2 minimize the functional $F(\cdot, \boldsymbol{x}_0)$, we obtain

$$F(\tilde{u}, \boldsymbol{x}_0) < F(u_1, \boldsymbol{x}_0),$$

which contradicts the optimality of u_1. We therefore conclude that the $u_1(t) = u_2(t)$ for a.e. $t \in [0, \pi)$.

Proof: **(continuity w.r.t. initial condition)** *Let us now give the proof of the L^1-continuity of the unique solution u^* to Problem 4.3 with respect to the initial condition. Let $\boldsymbol{x}_0$ be fixed. We need to prove that, for all $\gamma > 0$, there exists $\delta > 0$ such that*

$$\|\boldsymbol{x}_1 - \boldsymbol{x}_0\| \leq \delta \quad \textit{implies} \quad \|u_1^* - u_0^*\|_{L^1(0,\pi)} < \gamma,$$

where u_0^* and u_1^* are the optimal controls corresponding to the initial conditions $\boldsymbol{x}_0$ and $\boldsymbol{x}_1$ respectively.

As we have proved in subsections 6.3 and 6.4, for any $\boldsymbol{x}_1$, the optimal control u_1^*, solution to Problem 4.3, is piecewise constant, taking values in $\mathcal{U}$, with a finite number of discontinuity points (switching points).

We claim that the number of switching points is bounded from above by a constant $M^* \in \mathbb{N}$, independent of $\boldsymbol{x}_1 \in \mathbb{R}^N$.

Indeed, as we proved in subsection 6.2, the optimal control u_1^* is determined by the optimality condition (6.5), using the function μ^* defined in (6.4). If $\mu^* \equiv 0$, then u_1^* is constant and there are no switching points. In the other cases, $\mu^*(t)$ is a linear combination of sines and cosines with fixed frequencies. In the bilevel case, in subsection 6.3 we proved that the switching points correspond to the intersection points of $\mu^*(t)$ with $\varepsilon\alpha$. In the multilevel case, we proved in subsection 6.4 that the switching points correspond to the intersections of $\mu^*(t)$ with εp_k, see (6.12). In view of (6.4), as the frequencies are fixed, the number of these intersection points cannot exceed a certain number M^*, independent of the coefficients a_j^* and b_k^* in (6.4). The claim then follows.

Using the fact that, for any $\boldsymbol{x}_1$, the solution u_1^* is piecewise constant, taking values only in $\mathcal{U}$, and with a finite number of switches less than some M^* independent of $\boldsymbol{x}_1$, we deduce that there exists $K > 0$, independent of $\boldsymbol{x}_1$ such that

$$\|u_1^*\|_{BV} \leq K.$$

See (6.19) below for the definition of the BV norm. We then obtain that, for any $\boldsymbol{x_1} \in \mathbb{R}^N$,

$$u_1^* \in \mathcal{A}_K^* := \{u \in \mathcal{A} \,:\, \|u\|_{BV} \leq K\}.$$

Now, for any $\gamma > 0$ fixed, we can apply Lemma 6.1 below, to ensure the existence of $\eta > 0$ such that

$$F(u, \boldsymbol{x}_0) \geq F(u_0^*, \boldsymbol{x_0}) + \eta, \tag{6.14}$$

for all for all $u \in \mathcal{A}_K^*$, with $\|u - u_0^*\|_{L^1(0,\pi)} = \gamma$.

Since the set $\mathcal{A}_K^*$ is convex and u_0^* minimizes $F(\cdot, \boldsymbol{x}_0)$, we can use (6.14) and the convexity of the function $u \mapsto F(u, \boldsymbol{x}_0)$, to deduce that

$$F(u, \boldsymbol{x}_0) \geq F(u_0^*, \boldsymbol{x_0}) + \eta, \tag{6.15}$$

for all $u \in \mathcal{A}_K^*$ such that $\|u - u_0^*\|_{L^1(0,\pi)} \geq \gamma$.

Observe that, for any $u \in \mathcal{A}$, the function $\boldsymbol{x} \mapsto F(u, \boldsymbol{x})$ is locally Lipschitz, and therefore, there exists a constant $C_F > 0$ satisfying

$$|F(u, \boldsymbol{x}_1) - F(u, \boldsymbol{x}_0)| \leq C_F \|\boldsymbol{x}_1 - \boldsymbol{x}_0\| \tag{6.16}$$

for any $\boldsymbol{x}_1$ such that $\|\boldsymbol{x}_1 - \boldsymbol{x}_0\| \leq 1$. Notice that, since $u \in \mathcal{A}$ only takes values in $[-1, 1]$, the constant C_F can be chosen independently of u.

Now, combining (6.15) and (6.16), we obtain

$$\begin{aligned} F(u_0^*, \boldsymbol{x_1}) &\leq F(u_0^*, \boldsymbol{x}_0) + C_F \|\boldsymbol{x}_1 - \boldsymbol{x}_0\| \\ &\leq F(u, \boldsymbol{x}_0) - \eta + C_F \|\boldsymbol{x}_1 - \boldsymbol{x}_0\| \\ &\leq F(u, \boldsymbol{x}_1) - \eta + 2C_F \|\boldsymbol{x}_1 - \boldsymbol{x}_0\| \end{aligned} \tag{6.17}$$

for all $u \in \mathcal{A}_K^*$ such that $\|u - u_0^*\|_{L^1(0,\pi)} \geq \gamma$.

Finally, we can choose $\delta \in (0, 1)$ such that $\delta < \frac{\eta}{4C_F}$, and from (6.17), we deduce that, if $\|\boldsymbol{x}_1 - \boldsymbol{x}_0\| \leq \delta$, then

$$F(u_0^*, \boldsymbol{x_1}) \leq F(u, \boldsymbol{x}_1) - \frac{\eta}{2}$$

for all $u \in \mathcal{A}_K^*$ such that $\|u - u_0^*\|_{L^1(0,\pi)} \geq \gamma$, which then implies that necessarily

$$\|u_1^* - u_0^*\|_{L^1(0,\pi)} \leq \gamma.$$

This concludes the proof of the L^1-continuity of the solution with respect to the initial condition.

Let us conclude the section with the following Lemma, which has been used in the proof of the continuity of the solution with respect to the initial condition.

Lemma 6.1 *Let $\boldsymbol{x}_0 \in \mathbb{R}^N$ be given and let $\mathcal{L}$ be a function as in Theorem 4.1 or 4.2. Let $u_0^* \in \mathcal{A}$ be the unique solution to Problem 4.3. For any $K > 0$, define the set of controls*

$$\mathcal{A}_K^* := \{u \in \mathcal{A} \, : \, \|u\|_{BV} \leq K\}. \tag{6.18}$$

Then, for any $\gamma > 0$, there exists $\eta := \eta(\gamma, K) > 0$ such that

$$F(u_0^*, \boldsymbol{x}_0) \leq F(u, \boldsymbol{x}_0) - \eta,$$

for all $u \in \mathcal{A}_K^*$ such that $\|u - u_0^*\|_{L^1(0,\pi)} = \gamma$.

In the definition of $\mathcal{A}_K^*$, we are considering measurable functions of bounded variation in $(0, \pi)$, i.e. functions whose distributional derivative is a Radon measure in $(0, \pi)$, that we denote by $|Du|$, and such that $|Du|(0, \pi)$ is finite. We recall that the norm $\| \cdot \|_{BV}$ is defined as

$$\|u\|_{BV} := \int_0^\pi |u(t)| dt + |Du|(0, \pi). \tag{6.19}$$

See Chapter 3 in the book [Ambrosio et al., 2000] for further details on the space of functions of bounded variation.

Proof: **(Proof of Lemma 6.1)** *We need to prove that*

$$\mathcal{I}_{\gamma,K} = F(u_0^*, \boldsymbol{x}_0) + \eta,$$

for some $\eta > 0$, where

$$\mathcal{I}_{\gamma,K} := \inf \left\{ F(u, \boldsymbol{x}_0) \, ; \, u \in \mathcal{A}_K^* \text{ s.t. } \|u - u_0^*\|_{L^1(0,\pi)} = \gamma \right\}.$$

The result follows from the fact that the space $BV(0, \pi)$ is compactly embedded in $L^1(0, \pi)$, see Theorem 3.23 in [Ambrosio et al., 2000]. Consider any minimizing sequence $u_n \in \mathcal{A}_K^*$ with $\|u_n - u_0^*\|_{L^1(0,\pi)} = \gamma$, satisfying

$$\lim_{n \to \infty} F(u_n, \boldsymbol{x}_0) = \mathcal{I}_{K,\gamma}.$$

By Theorem 3.23 in [Ambrosio et al., 2000], there exists a subsequence of u_n which converges to some $\tilde{u} \in \mathcal{A}$, strongly in $L^1(0, \pi)$.

From the continuity of the L^1-norm and of the functional $F(\cdot, \boldsymbol{x}_0)$ with respect to the strong L^1-topology, we deduce that the limit $\tilde{u}$ satisfies

$$\|\tilde{u} - u_0^*\|_{L^1(0,\pi)} = \gamma \quad \text{and} \quad F(\tilde{u}, \boldsymbol{x}_0) = \mathcal{I}_{K,\gamma}.$$

Finally, since u_0^* is the unique minimizer of $F(\cdot, \boldsymbol{x}_0)$, we conclude that

$$\mathcal{I}_{\gamma,K} - F(u_0^*, \boldsymbol{x}_0) = F(\tilde{u}, \boldsymbol{x}_0) - F(u_0^*, \boldsymbol{x}_0) = \eta > 0.$$

7 Conclusions

Esta seccin no dice nada. Solo repite lo que ya se dijo en la intro.

In this paper, we have considered the Selective Harmonic Modulation problem, consisting in the construction of a staircase control signal u with a desired harmonic spectrum. We have proposed a novel approach based on an optimal control problem to solve the SHM problem. More precidely, we have shown that the SHM Problem 2.1 can be reformulated in terms of a null-controllability one in which the solution u to SHM plays the role of the control. In this framework, u can be obtained through the minimization of a cost functional over the controls having a particular staircase form.

By adding in the functional a suitable penalization term for the control, the staircase form of the solution can be guaranteed without imposing it directly as a restriction. Moreover, the penalization term for the control provides uniqueness and continuity of the solution with respect to the target frequencies.

Another important advantage of our methodology with respect to the previously existing ones is that, by tackling SHM under the perspective of optimal control, we are able to solve several critical issues (described in detail in Section 3) arising in practical applications.

1. *Combinatory problem*: our optimal control-based approach automatically determines the best waveform and switching angles for the desired control signals. This yields two relevant advances with respect to the existing techniques. On the one hand, the optimal control formulation renders a computationally lighter methodology to solve the SHM problem, as it does not need to repeatedly solve an optimization problem for different waveforms. On the other hand, it bypasses the cumbersome task of determining a priori the number of switches which is necessary to reach the desired Fourier coefficients.
2. *Solvable set problem*: having the liberty of changing the waveform during the optimization process, our approach is capable of covering a more ample solvable set than the existing methodologies.
3. *Policy problem*: the policy obtained through our methodology is not a gathering of several policies for different waveforms to which may correspond disjoint or even overlapping solvable sets. Hence, as we can observe in our simulations in Section 5, the continuity of the switching angles is guaranteed and we do not generate regions with no solution to the SHM problem.

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