Selective Harmonic Elimination: a control theory approach

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Abstract

El problema de *Selective Harmonic Elimination pulse-width modulation*(SHE) es planteado como el problema de control óptimo, con el fin de encontrar soluciones de ondas escalón sin prefijar el número de ángulos de conmutación. De esta manera, la metodología de control óptimo es capaz de encontar la forma de onda óptima y de encontrar la localizaciones de los ángulos de conmutación, incluso sin prefijar el número de conmutaciones. Este es un nuevo enfoque para el problema SHE en concreto y para los sistemas de control con un conjunto finito de controles admisibles en general.

Contents

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1 Introducción

Selective Harmonic Elimination (SHE) [1] is a well-known methodology allowing to generate signals in the form of a step function with a desired harmonic spectrum.

In broad terms, this means to obtain a function $u(\tau)$ defined on $[0, 2\pi)$ whose values in the whole interval can only belong to a finite set of real numbers $\mathcal{U}$ and which, in addition to that, has certain predetermined Fourier coefficients.

Due to the application in power converters, we will focus on functions with half-wave symmetry, i. e. $u(\tau + \pi) = -u(\tau)$ for all $\tau \in [0, \pi)$. In this way, the function $u(\tau)$ is determined by its values in the interval $[0, \pi)$ and its Fourier series expansion takes the form

$$u(\tau) = \sum_{\substack{i \in \mathbb{N} \\ i\ odd}} a_i \cos(i\tau) + \sum_{\substack{j \in \mathbb{N} \\ j\ odd}} b_j \sin(j\tau), \tag{1.1}$$ {?}

where the coefficients a_i and b_j are given by

$$a_i = \frac{2}{\pi} \int_0^{\pi} u(\tau) \cos(i\tau)\, d\tau, \tag{1.2}$$?an?

$$b_j = \frac{2}{\pi} \int_0^{\pi} u(\tau) \sin(j\tau)\, d\tau. \tag{1.3}$$?bn?

In view of this, the SHE problem can be formulated as:

Problem 1.1 (SHE) Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ with cardinalities $|\mathcal{E}_a| = N_a$ and $|\mathcal{E}_b| = N_b$ respectively, and given the target vectors $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$ and $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$, we look for a function $u(\tau)$, $\tau \in [0, \pi)$, such that $u(\tau)$ can only take values within $\mathcal{U}$ a finite subset of the interval $[-1, 1]$ and whose Fourier coefficients satisfy:

$$\begin{aligned} a_i &= (\boldsymbol{a}_T)_i, \qquad \text{for all } i \in \mathcal{E}_a, \\ b_j &= (\boldsymbol{b}_T)_j, \qquad \text{for all } j \in \mathcal{E}_b. \end{aligned}$$

Roughly speaking, this problem can be understood as finding a sequence $S \subset \mathcal{U}$ defining the values that the function $u(\tau)$ will assume and in which order they will appear (see [2]).

In this way, given the sequence S we can focus in finding the exact locations where the function $u(\tau)$ changes its values.

Following the terminology introduced in the SHE literature ([3, 2, 4]), we will refer to these locations of the changes in the values of the waveform $\mathcal{S}$ as *switching angles*.

To finding of the switching angles given the sequence S can be addressed as a minimization problem where the variables are the angles while the cost functional is the Euclidean distance between the obtained Fourier coefficients and the desired ones.

Since the cardinality of S is not known a priori, meaning that we do not know how many switches will be necessary, it appears that the only solution to the SHE problem consists in fix the number of changes and counting all the possible combinations, to later solve an optimization problem for each one of them.

Taking into account that the number of possible sequences S is given by $|\mathcal{U}|^{|S|}$, it is evident that the complexity of the above approach increases rapidly with the cardinality of S.

This problem has been studied in [3] where, through appropriate algebraic transformations, the authors are able to convert the SHE problem into a polynomial system whose solutions' set contains all the possible waveforms for a given set $\mathcal{U}$ and number of elements in the sequence S which, however, is predetermined.

On the other hand, we shall also mention that the SHE methodology has been developed to provide in real-time different target Fourier coefficients con with a KHz latency.

This makes impossible to find real-time solutions by optimization, making then necessary to pre-determine solutions that can later be interpolated.

Nevertheless, it is well-known that, fixed a sequence S, the continuity of the switching locations with respect to a continuous variation of the target Fourier coefficients may be quite cumbersome.

In the majority of the cases, it is impossible to find a continuous solution in a large interval, an it is necessary to change the waveform S while moving across different solution regions ([3, 5]). This makes difficult the interpolation of solutions and their finding.

In this document, we will present the SHE problem as an optimal control one, where the optimization variable is the signal $u(\tau)$ defined in the entire interval $[0, \pi)$.

In particular, we will describe how the Fourier coefficients of the function $u(\tau)$ can be seen as the final state of a system controlled by $u(\tau)$. Hence, the optimization is performed among all the possible functions that satisfy $|u(\tau)| < 1$ and can control the final state at the desired Fourier coefficients. Then we will show how to design a control problem so that the solution is a step function.

The present document is organized as follows. In Section 2, we will present the classical formulations of the SHE problem os an optimization one, introducing some seminal concept and discussing its advantages and disadvantages. In Section 3, we will formulate the SHE problem as a controlled dynamical system. In Section 4, we will present the optimal control problem allowing us to obtain the solutions to the SHE problem. In Section 5, we will present our numerical experiments. Finally, in Section 6, we will gather the conclusions of our study.

2 Classical approach

The classical approximation uses the piece-wise definition of the function $u(\tau)$ and tries to exploit this information to simplify the problem. In this definition, $u(\tau)$ can be fully characterized by the switching angles and the constant values it may assume. We define these two concepts as follows.

Definition 2.1 (Switching angles) Given a function $u : [0, \pi] \to \mathcal{U}$, the switching locations are the values of $\tau \in [0, \pi]$ where $u(\tau)$ changes its value discontinuously. We will denote the commutation angles as $\boldsymbol{\phi} = \{\phi_0, \phi_1, \ldots, \phi_M, \phi_{M+1}\}$, where we have taken $\phi_0 = 0$ and $\phi_{M+1} = \pi$.

Definition 2.2 (Waveform) Given $\mathcal{U}$ a finite subset of $[-1, 1]$, we will call a waveform a finite set $S = \{s_1, s_2, s_3, \ldots, s_{M+1}\}$ of elements of $\mathcal{U}$ with repetition.

Then a waveform S indicates the values that the function will take and in which order they will appear within the interval $[0, \pi)$, while $\boldsymbol{\phi}$ indicates the switching locations. Considering these two elements, we can rewrite the Fourier coefficients as

$$a_i = \frac{2}{\pi} \int_0^\pi u(\tau) \cos(i\tau) d\tau = \frac{2}{\pi} \sum_{k=1}^{M+1} \int_{\phi_{k-1}}^{\phi_k} s_k \cos(i\tau) d\tau = \frac{2}{i\pi} \sum_{k=1}^{M+1} s_k \sin(i\tau) \Big|_{\phi_{k-1}}^{\phi_k} \tag{2.1} \{?\}$$

$$b_j = \frac{2}{\pi}\int_0^\pi u(\tau)\sin(j\tau)d\tau = \frac{2}{\pi}\sum_{k=1}^{M+1}\int_{\phi_{k-1}}^{\phi_k} s_k \sin(j\tau)d\tau = -\frac{2}{j\pi}\sum_{k=1}^{M+1} s_k \cos(j\tau)\Big|_{\phi_{k-1}}^{\phi_k} \tag{2.2} \{?\}$$

Hence

$$a_i(\boldsymbol{\phi}) = \frac{2}{i\pi}\sum_{k=1}^{M+1} s_k\Big[\sin(i\phi_k) - \sin(i\phi_{k-1})\Big], \tag{2.3} \{?\}$$

$$b_j(\boldsymbol{\phi}) = -\frac{2}{j\pi}\sum_{k=1}^{M+1} s_k\Big[\cos(j\phi_k) - \cos(j\phi_{k-1})\Big]. \tag{2.4} \{?\}$$

In this way, we can reformulate Problem 1.1 as follows.

Problem 2.1 (optimization for SHE) Given a waveform S, we look for the switching angles $\boldsymbol{\phi}$ by means of the following minimization problem:

$$\min_{\boldsymbol{\phi}\in[0,\pi]^M}\left[\sum_{i\in\mathcal{E}_a}\|a_T^i - a_i(\boldsymbol{\phi})\|^2 + \sum_{j\in\mathcal{E}_b}\|b_T^j - b_j(\boldsymbol{\phi})\|^2\right] \tag{2.5} \{?\}$$

subject to:

$$0 < \phi_1 < \phi_2 < \cdots < \phi_{M-1} < \phi_M < \pi \tag{2.6} \{?\}$$

In this formulation, the SHE problem converts in a minimization problem with restrictions which can be solved by well-known techniques. Since the problem has several minimizers, we shall solve it employing global optimizers. Furthermore, since the choice of the waveform is arbitrary, we shall proceed in the same way for each possible waveform.

3 Selective Harmonic Elimination as dynamical system

Inspired by the continuous nature of the optimization variable $u(\tau)$, we propose in this document the formulation from the optimal control. In this way we avoid the choice of the waveform, so that the optimization problem chooses the most convenient one in each case. So we look for $u(\tau) \in \mathcal{U}$, $\tau \in [0,\pi)]$, that has the desired Fourier coefficients.

We will use the fundamental theorem of differential calculus to rewrite the expression of the Fourier coefficients (1.2) and (1.3) as the evolution of a dynamical system. That is to say

$$\alpha_i(\tau) = \frac{2}{\pi}\int_0^\tau u(\tau)\cos(i\tau)d\tau \Rightarrow \begin{cases}\dot{\alpha}_i(\tau) & = \frac{2}{\pi}u(\tau)\cos(i\tau)\\ \alpha_i(0) & = 0\end{cases} \quad \text{for all } i \text{ odd} \tag{3.1} ?\underline{\texttt{ode}}?$$

$$\beta_j(\tau) = \frac{2}{\pi}\int_0^\tau u(\tau)\sin(j\tau)d\tau \Rightarrow \begin{cases}\dot{\beta}_j(\tau) & = \frac{2}{\pi}u(\tau)\sin(j\tau)\\ \beta_j(0) & = 0\end{cases} \quad \text{for all } j \text{ odd} \tag{3.2} ?\underline{\texttt{ode}}?$$

The evolution of the dynamical systems $\alpha_i(\tau)$ and $\beta_j(\tau)$ from the time $\tau = 0$ to $\tau = \pi$ gives us the coefficients a_i and b_j.

We introduce notation to refer to vectors $\boldsymbol{\alpha} = \{\alpha_i\}_{i\in\mathcal{E}_a}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j\in\mathcal{E}_b}$.

In this way, the general SHE problem (1.1) can be formulated as a control problem for a dynamical system where $\boldsymbol{\alpha}(\tau)$ and $\boldsymbol{\beta}(\tau)$ are the states and $u(\tau)$ is the control variable, and whose objective will be to bring the states from the origin to the objective vectors $\boldsymbol{a}_T$ and $\boldsymbol{b}_T$ in time $\tau = \pi$.

In order to obtain a compact expression of the problem that simplifies our understanding of it, we will introduce notation. So if we consider a problem with sets of odd numbers:

$$\mathcal{E}_a = \{e_a^1, e_a^2, e_a^3, \dots, e_a^{N_a}\} \qquad \mathcal{E}_b = \{e_b^1, e_b^2, e_b^3, \dots, e_b^{N_b}\} \tag{3.3 {?}}$$

then we can define the vectors $\boldsymbol{\mathcal{D}}^\beta(\tau) \in \mathbb{R}^{N_a}$ y $\boldsymbol{\mathcal{D}}^\beta(\tau) \in \mathbb{R}^{N_b} \mid \forall \tau \in (0, \pi]$ such that:

$$\boldsymbol{\mathcal{D}}^\alpha(\tau) = \frac{2}{\pi}\begin{bmatrix} \cos(e_a^1\tau) \\ \cos(e_a^2\tau) \\ \dots \\ \cos(e_a^{N_a}\tau) \end{bmatrix} \qquad \boldsymbol{\mathcal{D}}^\beta(\tau) = \frac{2}{\pi}\begin{bmatrix} \sin(e_b^1\tau) \\ \sin(e_b^2\tau) \\ \dots \\ \sin(e_b^{N_b}\tau) \end{bmatrix} \tag{3.4 {?}}$$

So the dynamical system can be written as:

$$\begin{cases} \dot{\boldsymbol{\alpha}}(\tau) = \boldsymbol{\mathcal{D}}^\alpha(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{\alpha}(0) = 0 \end{cases} \qquad \begin{cases} \dot{\boldsymbol{\beta}}(\tau) = \boldsymbol{\mathcal{D}}^\beta(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{\beta}(0) = 0 \end{cases} \tag{3.5 {?}}$$

Compressing the notation even more we can call the total state of the system $\boldsymbol{x}(\tau)$ to the concatenation of the states $\boldsymbol{\alpha}(\tau)$ and $\boldsymbol{\beta}(\tau)$ so that:

$$\boldsymbol{x}(\tau) = \begin{bmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{bmatrix} \qquad \boldsymbol{x}_0 = \begin{bmatrix} \boldsymbol{a}_T \\ \boldsymbol{b}_T \end{bmatrix} \qquad \boldsymbol{\mathcal{D}}(\tau) = \begin{bmatrix} \boldsymbol{\mathcal{D}}^\alpha(\tau) \\ \boldsymbol{\mathcal{D}}^\beta(\tau) \end{bmatrix} \tag{3.6 {?}}$$

So for a pair of sets $\mathcal{E}_a$ and $\mathcal{E}_b$ we have the following associated dynamical system:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = \boldsymbol{\mathcal{D}}(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = 0 \end{cases} \tag{3.7 {?}}$$

Then we look for a function $u(\tau)$ such that it leads the dynamical system to the point $\boldsymbol{x}_T$, that is, the final state $\boldsymbol{x}(\pi)$ is $\boldsymbol{x}_0$. Debido a que en la teoría de control es más típico llevar el sistema dinámico al origen de coordenadas desde una condición de initical no nula, realizaremos el siguiente cambio de variables: $\boldsymbol{x}'(\tau) = \boldsymbol{x}(\tau) - \boldsymbol{x}_0$. Haciendo un abuso de notación $\boldsymbol{x}'(\tau) \to \boldsymbol{x}(\tau)$ nuestro sistema dinámico se puede escribir como:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = -\boldsymbol{\mathcal{D}}(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \tag{3.8 {?}}$$

De manera que si el sistema dinámico en tiempo $\tau = \pi$ se encuentra en el origen de coordenadas, entonces los coeficientes de Fourier del control que $u(\tau)$ que le ha llevado allí son los los asociados a la condición inicial $\boldsymbol{x}_T$.

4 Optimal Control for SHE

Since the SHE problem is equivalent to controlling a dynamic system from the origin of coordinates to a specific point, we must formulate a control problem that solves this task but also complies with the restrictions on the values that the control can take. It is necessary to set a finite subset $\mathcal{U}$ of the interval $[-1, 1]$, the optimal control is such that it can only take the values allowed in the discretization. In other words, the control problem is:

Problem 4.1 (OCP for SHE) Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ and given the target vector $\boldsymbol{x}_T \in \mathbb{R}^{N_a+N_b}$, also given a set $\mathcal{U}$ of admissible controls, we look for the function $u(\tau) \mid \tau \in [0, \pi)$ such that:

$$\min_{u(\tau)\in\mathcal{U}} \frac{1}{2}||\boldsymbol{x}(\pi)||^2 \tag{4.1} \{?\}$$

suject to:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = -\boldsymbol{\mathcal{D}}(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \tag{4.2} \{?\}$$

The solution of this control problem is complex due to the restriction on the admissible control values. In order to obtain a problem that can be solved by classical control theory we can formulate the following control problem:

Problem 4.2 (Regularized OCP for SHE) Given two sets of odd numbers $\mathcal{E}_a$ and $\mathcal{E}_b$ and given the target vector $\boldsymbol{x}_T \in \mathbb{R}^{N_a+N_b}$, we look for the function $|u(\tau)| < 1 \mid \tau \in [0, \pi)$ such that:

$$\min_{|u(\tau)|<1} \left[\frac{1}{2}||\boldsymbol{x}(\pi)||^2 + \epsilon \int_0^\pi \mathcal{L}(u(\tau))d\tau\right] \tag{4.3} \{?\}$$

suject to:

$$\begin{cases} \dot{\boldsymbol{x}}(\tau) = -\boldsymbol{\mathcal{D}}(\tau)u(\tau) & \tau \in [0, \pi) \\ \boldsymbol{x}(0) = \boldsymbol{x}_0 \end{cases} \tag{4.4} \{?\}$$

Where we will choose $\mathcal{L} : \mathbb{R} \to \mathbb{R}$ such that the optimal control u^* only takes values in the discretization $\mathcal{U}$ of the interval $[-1.1]$. Furthermore, the parameter ϵ should be small so that the solution minimizes the distance from the final state and the target. Next we will study the optimality conditions of the problem, for a general $\mathcal{L}$ function, and then specify how $\mathcal{L}$ should be so that the optimal control u^* only takes the allowed values in $\mathcal{U}$.

4.1 Optimality conditions

To write the optimility conditions of the problem we will use the principle of the Pontryagin minimum [6, Chapter 2.7]. For them it is necessary to define the Hamiltonian of the system, which in this case is:

$$H(u, \boldsymbol{p}, \tau) = \epsilon\mathcal{L}(u) - [\boldsymbol{p}^T(\tau) \cdot \boldsymbol{\mathcal{D}}(\tau)]u(\tau) \tag{4.5} ?\underline{\texttt{hamil}}?$$

Where the variable $\boldsymbol{p}(\tau)$ called adjoint state is introduced, which is associated with the restriction imposed by the system. Este tiene la misma dimensión del estado, de manera que

$$\boldsymbol{x}(\tau) = \begin{bmatrix} \boldsymbol{\alpha}(\tau) \\ \boldsymbol{\beta}(\tau) \end{bmatrix} \Leftrightarrow \boldsymbol{p}(\tau) = \begin{bmatrix} \boldsymbol{p}^{\alpha}(\tau) \\ \boldsymbol{p}^{\beta}(\tau) \end{bmatrix} \tag{4.6} \{?\}$$

A continuación enumeraremos las condiciones de optimalidad provenientes del prinicio de mínimo de Pontryagin.

1. **Final condition of the adjoint**: This optimiality condition is obtained from the cost in the final time $\tau = \pi$ of the control problem in this case $\Psi(\boldsymbol{x}) = \frac{1}{2}\|\boldsymbol{x}(\pi) - \boldsymbol{x}_T\|^2$.

$$\boldsymbol{p}(\pi) = \nabla_{\boldsymbol{x}} \Psi(\boldsymbol{x}) = (\boldsymbol{x}(\pi) - \boldsymbol{x}_T) \tag{4.7} \{?\}$$

2. **Adjoint evolution equation**:

$$\dot{\boldsymbol{p}}(\tau) = -\nabla_x H(u(\tau), \boldsymbol{p}(\tau), \tau) = 0 \tag{4.8} \{?\}$$

   From where it can be deduced that $\boldsymbol{p}(\tau)$ is a constant so that $\boldsymbol{p}(\tau) = \boldsymbol{x}(\pi) - \boldsymbol{x}_T \mid \forall \tau \in [0, \pi)$ so from now on we will refer to it simply as $\boldsymbol{p}$, noting that it is invariant in time.

3. **Optimal control shape**: We known that $u^* = \arg\min_{|u|<1} H(\tau, \boldsymbol{p}^*, u)$, so in this case we can write:

$$u^*(\tau) = \underset{|u|<1}{\arg\min} \left[\epsilon \mathcal{L}(u(\tau)) - [\boldsymbol{p}^{*T} \cdot \boldsymbol{\mathcal{D}}(\tau)] u(\tau) \right] \tag{4.9} \{?\}$$

   Therefore, this optimality condition reduces to the optimization of a function in a variable within the interval $[-1, 1]$.

Es importante recordar que

$$[\boldsymbol{p}^{*T} \cdot \boldsymbol{\mathcal{D}}(\tau)] = \sum_{i \in \mathcal{E}_a} p_\alpha^* \cos(i\tau) + \sum_{j \in \mathcal{E}_b} p_\beta^* \sin(j\tau) \mid \forall \tau \in [0, \pi) \tag{4.10} \{?\}$$

we build a function $\mathcal{H}_m : [-1, 1] \to \mathbb{R}$ such that:

$$\mathcal{H}_m(u) = \epsilon \mathcal{L}(u) - mu \mid \forall m \in \mathbb{R} \tag{4.11} \{?\}$$

Donde hemos remplazado el término $[\boldsymbol{p}^{*T} \cdot \boldsymbol{\mathcal{D}}(\tau)]$ por un parámetro m que puede variar en todo la recta real. Es decir, el problema para diseñar un problema de control óptimo que solo pueda tomar valores en $\mathcal{U}$ se reduce a diseñar una función unidimensional com parámertro m cuyos mínimos sean los elementos de $\mathcal{U}$ para cualquier valor de m.

4.2 Piecewise linear penalization

En esta subsección presentaremos como diseñar el término de penalización $\mathcal{L}(u)$ para que el control óptimo en cualquier caso este contenida en $\mathcal{U}$. De manera más concreta podemos elegir la interpolación afín de una parábola $\mathcal{L} : [-1, 1] \to \mathbb{R}$ como término de penalización. Es decir:

$$\mathcal{L}(u) = \begin{cases} \left[(u_{k+1} + u_k)(u - u_k) + u_k^2\right] & \text{if } u \in [u_k, u_{k+1}[\\ 1 & \text{if } u = u_{N_u} \end{cases} \tag{4.12} ?\underline{\text{PLP}}?$$
$$\forall k \in \{1, \ldots, N_u - 1\}$$

Entonces el Hamiltoniano queda como:

$$\mathcal{H}_m(u) = \begin{cases} \epsilon\big[(u_{k+1}+u_k)(u-u_k)+u_k^2\big] - mu & \text{if } u \in [u_k, u_{k+1}[\\ \epsilon - mu & \text{if } u = u_{N_u} \end{cases} \qquad (4.13)\ \{?\}$$
$$\forall k \in \{1, \ldots, N_u - 1\}$$

De manera que para calcular el mínimo de $\mathcal{H}_m(u)$ deberemos considerar que esta función no es diferenciable por lo que la condicion de optimizalidad no puede realizarse con la difereciación típica sino con la subdiferencial.

Entonces calulremos la subdiferecial $\partial\mathcal{H}_m$ todo

$$\partial\mathcal{H}_m(u) = \begin{cases} \{\epsilon(u_1+u_2)-m\} & \text{if } u = u_1 \\ \{\epsilon(u_k+u_{k+1})-m\} & \text{if } u \in]u_k, u_{k+1}[\quad \dagger \\ [\epsilon(u_k+u_{k-1})-m\ ,\ \epsilon(u_{k+1}+u_k)-m] & \text{if } u = u_k \qquad \dagger\,\dagger \\ \{\epsilon(u_{N_u}+u_{N_u-1})-m\} & \text{if } u = u_{N_u} \end{cases} \qquad (4.14)\ \{?\}$$
$$\dagger\ \forall k \in \{1, \ldots, N_u - 1\} \qquad \dagger\,\dagger\ \forall k \in \{2, \ldots, N_u - 1\}$$

Entonces dado $m \in \mathbb{R}$ busamos $u \in [-1, 1]$ que minimiza $\mathcal{H}_m(u)$. Es necesario comprobar que cero perteneca al subdiferecial $\partial\mathcal{H}_m(u)$

- **Case 1:** $m \leq \epsilon(u_1 + u_2)$: Dado que m es menor que el menor de todos los subdiferenciales entonces el cero no se encuentra dentro de ninguna de los intervalos definidos para la subdiferencial. De manera que el mínimo esta en un extremo

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_1 \qquad (4.15)\ \{?\}$$

- **Case 2:** $m = \epsilon(u_{k+1} + u_k)$: Teniendo en cuenta que $\forall k \in \{1, \ldots, N_u - 1\}$.

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = [u_k, u_{k+1}[\qquad (4.16)\ \{?\}$$

- **Case 3:** $\epsilon(u_k + u_{k-1}) < m < \epsilon(u_{k+1} + u_k)$: Teniendo en cuenta que $\forall k \in \{2, \ldots, N_u - 1\}$.

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_k \qquad (4.17)\ \{?\}$$

- **Case 4:** $m > \epsilon(u_{N_u} + u_{N_u-1})$:

$$\arg\min_{|u|<1} \mathcal{H}_m(u) = u_{N_u} \qquad (4.18)\ \{?\}$$

Es decir solo cuando $m = \epsilon(u_{k+1}+u_k)$ los mínimos del Hamiltoniano esta dentro de un intervalo, para otros valores de m los mínimos del Hamiltoniano están contenidos en $\mathcal{U}$. Así que ante una variación continua de m el caso 2 solo ocurrirá de manera puntal. Recordando en el problema de control óptimo $m(\tau) = [\boldsymbol{p}^T(\tau) \cdot \boldsymbol{\mathcal{D}}(\tau)]$, podemos notar que el caso 2 corresponde a los instantes τ de cambio de valor.

5 Numerical simulations

En esta sección mostraremos algunos ejemplo resolviendo el problema de control óptimo mediante el método directo y la herramienta de optimización no lineal bajo constrain: CasADi [7].

5.1 Smooth approximation of piecewise linear penalization

Con el fin de utilizar softwre de optimización para resolver el problema de control óptimo planteado aproximaremos penalización lineal a trozos mediante con ayuda de función escalón de Heaviside $h : \mathbb{R} \to \mathbb{R}$ y su aproximación suave $h^\eta : \mathbb{R} \to \mathbb{R}$ definida de la siguiente manera:

$$h(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases} \qquad \begin{cases} h^\eta(x) = (1 + \tanh(\eta x))/2 \\ \eta \to \infty \end{cases} \tag{5.1 {?}}$$

Con ayuda de la función h definimos la función ventana suave $\Pi^\eta_{a,b} : \mathbb{R} \to \mathbb{R}$ como:

$$\Pi^\eta_{[a,b]}(x) = -1 + h^\eta(x-a) + h^\eta(-x+b) \tag{5.2 {?}}$$

De manera simplificada:

$$\Pi^\eta_{[a,b]}(x) = \frac{\tanh[\eta(x-a)] + \tanh[\eta(b-x)]}{2} \tag{5.3 {?}}$$

De esta menera podemos escribir la versión suave de (4.12):

$$\mathcal{L}^\eta(u) = \sum_{k=1}^{N_u - 1} \left[(u_{k+1} + u_k)(u - u_k) + u_k^2\right] \Pi^\eta_{[u_k, u_{k+1}]}(u) \tag{5.4 {?}}$$

De manera que cuando $\eta \to \infty$ entonces $\mathcal{L}^\eta \to \mathcal{L}$

5.2 Direct method for OCP-SHE

To solve the optimal control problem (4.2), we use a direct method. If we consider a partition $\mathcal{P} = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of interval $[0, T]$, we can represent a function $\{u(\tau) \mid \tau \in [0, T]\}$ as a vector $\boldsymbol{f} \in \mathbb{R}^T$ where component $f_t = u(\tau_t)$. Then the optimal control problem (4.1) can be written as optimization problem with variable $\boldsymbol{f} \in \mathbb{R}^T$. This problem is a nonlinear programming, for this we use CasADi software to solve. Entonces, dado una partición del intervalo $[0, \pi)$ podemos reformular el problema (4.1) como el siguiente problema en tiempo discreto:

Problem 5.1 (OCP numérico) Dados dos conjuntos de números impares $\mathcal{E}_a$ and $\mathcal{E}_b$ con cardinalidades $|\mathcal{E}_a| = N_a$ y $|\mathcal{E}_b| = N_b$ respectivamente, dados los vectores objetivos $\boldsymbol{a}_T \in \mathbb{R}^{N_a}$, de manera que $\boldsymbol{x}_0 = [\boldsymbol{a}_T, \boldsymbol{b}_T]^T$; y $\boldsymbol{b}_T \in \mathbb{R}^{N_b}$ y una partition $\mathcal{P} = \{\tau_0, \tau_1, \ldots, \tau_T\}$ of interval $[0, \pi)$. We search a vector $\boldsymbol{u} \in \mathbb{R}^T$ that minimize the following function:

$$\min_{\boldsymbol{u} \in \mathbb{R}^T} \left[||\boldsymbol{x}^T||^2 + \epsilon \sum_{t=0}^{T-1} \mathcal{L}^\eta(u_t) \Delta\tau_t \right] \tag{5.5 {?}}$$

suject to:

$$\forall \tau \in \mathcal{P} \begin{cases} \boldsymbol{x}^{t+1} = \boldsymbol{x}^t - \Delta\tau_t \boldsymbol{\mathcal{D}}(\tau_t) u_t \\ \boldsymbol{x}^0 = \boldsymbol{x}_0 \end{cases} \tag{5.6 {?}}$$

5.3 Resultados

Cabe mencionar que todas las simulaciones se han realizado con un ordenador de mesa con $8Gb$ de ram, y el tiempo de ejecución para la búsqueda de soluciones dado un vector objetivo es del orden del segundo. A continuación listaremos cada uno de los resultados numéricos obtenidos:

1. **OCP con simetría de cuarto de onda**: Consideraremos el problema con un conjunto de números impares $\mathcal{E}_b = \{1, 5\}$ con una discretización del intervalo $[0, \pi/2]$ de $T = 200$. Mostramos las soluciones para los vectores objetivo $b_T^1 = \{(-0.4, -0.3, \dots, 0.3, 0.4)\}$ manteniendo $b_T^5 = 0$ para todos los casos. Mostramos las trayectorias óptimas obtenidas en la figura (**??**), donde se puede ver una continuidad en las soluciones con respecto a vector objetivo.

2. **OCP con simetría de cuarto onda para un intervalo del** b_1: Para este ejemplo consideramos el siguiente conjunto de números impares: $\mathcal{E}_b = \{1, 5, 7, 11, 13\}$. Ademas consideramos el vector objetivo $\boldsymbol{b}_T = [m_a, 0, 0, 0, 0]$, donde $m_a \in [-1, 1]$ es un parámetro. with three penalization terms: $\mathcal{L}(u) = -f$, $\mathcal{L}(u) = +f$ and $\mathcal{L}(u) = -f^2$ obtained by direct method with uniform partition of interval $[0, \pi/2]$ with $T = 400$ and penalization parameter $\epsilon = 10^{-5}$. Para cada uno de los términos de penalización utilizados la distancia entre los coeficientes de Fourier se encuentra en el orden de 10^{-4}. Sin embargo, cuando el término de penalización es $\mathcal{L}(u) = -f^2$ la solución no presenta continudad con respecto al vector objetivo. Por otra parte, es importante mencionar que las soluciones para los términos de penalización $\mathcal{L}(u) = -f$ y $\mathcal{L}(u) = f$ cumplen una simetría por lo que invirtiendo las soluciones con respecto al origen y invirtiendo el signo de las soluciones se puede ver que ambas soluciones son la misma.

3. **SHE para tres niveles**: Podemos ver que en el caso en el que el control $u(\tau)$ solo pueda tomar valores entre $[0, 1]$ obtenemos señales que pueden tomar tres niveles en el intervalo $[0, 2\pi]$ gracias a la simetría de cuarto de onda. Si resolvemos el problema de control óptimo pero esta vez cambiando las restricciones $|u(\tau)| < 1$ por $\{0 < u(\tau) < 1\}$. Se ha realizado el mismo procedimiento que en el caso anterior, obteniendo soluciones para los mismo términos de penalización obteniendo la figura (**??**). Allí se muestra la continudad de las soluciones y que estas se encuentran en el orden de 10^{-4}.

4. **Cambio en el número de conmunationes**: Gracias a la formulación de control óptimo para el problema SHE podemos variar el número de ángulos de conmuntación. Este es el cado del siguientes ejemplo, donde hemos tomado como conjunto de números pares $\mathcal{E}_b = \{1, 3, 9, 13, 17\}$, además consideramos el vector objetivo $\boldsymbol{b}_T = [m_a, 0, 0, 0, 0]$, donde $m_a \in [0, 1]$ es un parámetro. En este problema hemos utilizado una penalización tipo $\mathcal{L} = f$ con un parémetro de penalización $\epsilon = 10^{-4}$. Podemos ver en la figura (**??**) como el problema de control óptimo es versátil y es capaz de mover entre varios conjuntos de soluciones.

5. **OCP para SHE con simetría de media onda**: Se ha relizado el caso de control óptimo de media onda con con $\mathcal{E}_a = \{1, 3, 5\}$ y $\mathcal{E}_b = \{1, 3, 5, 9\}$, donde $\boldsymbol{a}_T = [m_a, 0, 0]$, $\boldsymbol{b}_T = [m_a, m_a, 0, 0]$ y $m_a \in [-0.6, 0.6]$. Se ha elegido la penalización $L(u) = +f$

6 Conclusiones

Se ha presentado el problema SHE desde un punto de vista de la teoría de control. Esta metodología es efectiva para llega a presiciones $10^{-4} - 10^{-5}$ en la distancia al vector objetivo. Sin embargo en comparación con metodologías donde el número de conmutaciones es prefijado, nuestra aproximación es más costosa. Sin enbargo, el control óptimo asegura soluciones en todo el rango de indice de modulación, aunque el número de soluciones o la localización de estas cambien abruptamente.

Este plantamiento del problema SHE enlaza la teoría de control con la eliminación de harmónicos. De esta manera el problema SHE se puede resolver mediante herramientras clásicas.

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