

# Feedback with Carry Shift Registers and Bent Sequences

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## Abstract

A stream cipher uses pseudorandom sequences to mimic the security of a one-time pad. This paper will investigate how Bent functions can be used to generate Bent sequences with large 2-adic span for use in feedback with carry shift registers (FCSR). The linear complexity of an FCSR, a commonly used device in stream ciphers, directly relates to the magnitude of the 2-adic span of the sequence generated. This affects the resistance of the stream cipher against register synthesis attacks. In this paper, it is shown that it is possible to compute the 2-adic value of a Bent sequence.

## 1 Introduction

Bytes of data, or short sequences of 1s and 0s, are exchanged between computer systems each day on public channels. Because gentlemen do read each other's mail, it is necessary to secure the communication of private data sent over public channels. The solution to this problem is solved by *cryptography*, the designing of systems to secure data exchanges over public channels.

The following example is the classical communication scenario presented in many books on cryptography. Let there be two parties, Alice and Bob, who wish to communicate with one another. A third party, Eve, is a potential eavesdropper. Alice wants to send a message, known as the *plaintext*, to Bob. To accomplish this without Eve knowing what the message is before it is received by Bob, Alice must *encrypt* her message by some prearranged method, usually involving an *encryption key*, to generate a related message called *ciphertext*. The idea is that the sent ciphertext, even if it is intercepted by Eve, will be too difficult to interpret and will conceal the plaintext message. Upon receipt of the ciphertext, Bob will *decrypt* the message, usually involving a *decryption key*, similar to the encryption of the message, and obtain the plaintext message. A visual of this scenario is presented in 1.

This scenario is the standard example found in many different introductory cryptography references. Cryptographers have created numerous encryption and decryption schemes, or *cryptosystems*, to secure the messages sent between Alice and Bob. Many of these systems have been broken because of the amount

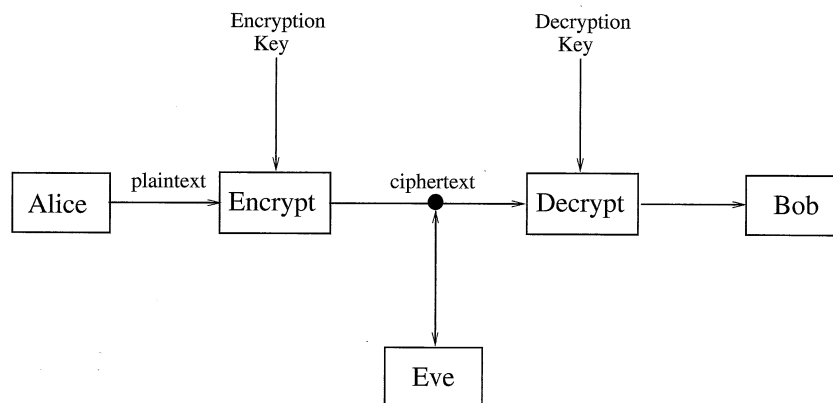


Figure 1: The Basic Communication Scenario for Cryptography [19]

of work that goes into the study of breaking cryptosystems, called *cryptanalysis*. A constant battle exists between the designers and breakers of cryptosystems, strengthening designs and attacks every day. In fact, designing a strong cryptosystem typically requires knowledge of cryptography and cryptanalysis.

Before diving into the topic of this paper, there is an important assumption that must be presented. When designing a cryptosystem, every cryptographer assumes *Kerckhoffs' principle* [19]: “In assessing the security of a cryptosystem, one should always assume the enemy knows the method being used.” The security of a cryptosystem cannot be based on the concealment of the encryption and decryption algorithms. In practice, the enemy can obtain the algorithms in many ways, including the defection or capture of people. The security must be based solely on the key.

Imagine that Alice’s message to Bob is a sequence of 1s and 0s, as it would be in the real world. Consider a cryptosystem which encrypts each bit in the sequence separately. This would be done by what is called a *stream cipher*, which, according to Rueppel, divides bit sequences into individual bits and enciphers each bit with a time-varying function whose time-dependency is governed by the internal state of the stream cipher [16]. The stream cipher can also be thought of in terms of a *keystream* which is a sequence of 1s and 0s the same length of the message that is added to the message using addition in  $\mathbb{F}_2$  (also known as XOR). If the keystream was perfectly random, then the cryptosystem would be unbreakable, or *perfectly secret*, as discovered by Claude Shannon in his famous paper “Communication Theory of Secrecy Systems,” written in 1945 and published in 1949. This cryptosystem is known as the *one-time pad*. Though it is perfectly secret, it can be difficult to implement because of the inability to produce perfectly random keystreams. Constructing a method to produce perfectly random sequences is a contradiction in itself.

Though it is impossible to create perfectly random sequences, it is possible

to get close. This type of sequence is called a *pseudorandom sequence*. These sequences have extremely long periods but are statistically indistinguishable from random sequences. This paper will discuss the construction of pseudorandom sequences using a particular finite state machine called the *feedback with carry shift register*. It will also investigate the possibility of incorporating *Bent functions*, or perfectly non-linear Boolean functions, into the construction of pseudorandom sequences in an attempt to resist synthesizing algorithms.

## 2 Boolean Functions

This section will establish the definition of a *Boolean function* and how to write these functions as polynomials. First, the finite field with two elements, denoted  $\mathbb{F}_2$ , is defined. The definitions and notations will follow those found in [3].

### 2.1 Review of Boolean Functions

The two element field  $(\mathbb{F}_2, \oplus, \cdot)$  is the set  $\{0, 1\}$  with defined binary operations  $\oplus$  and  $\cdot$ , also commonly referred to as the *XOR* and *AND* operators, respectively.

XOR	AND
$0 \oplus 0 := 0$	$0 \cdot 0 := 0$
$0 \oplus 1 := 1$	$0 \cdot 1 := 0$
$1 \oplus 0 := 1$	$1 \cdot 0 := 0$
$1 \oplus 1 := 0$	$1 \cdot 1 := 1$

Table 1: Binary Operations for  $\mathbb{F}_2$

It should be clear that  $(\mathbb{F}_2, \oplus, \cdot)$  is a commutative ring with an identity. Additionally, the only non-zero element 1 is its own inverse. Therefore  $(\mathbb{F}_2, \oplus, \cdot)$  is a finite field, which will now be denoted by  $\mathbb{F}_2$ . The  $n$ -dimensional vector space over  $\mathbb{F}_2$  will be denoted by  $\mathbb{F}_2^n$ , with the usual inner product. Components of these vectors, the individual 1s and 0s, will be known as *bits*. When a bit is *flipped* this means it changed from a 1 to a 0, or vice versa. For two vectors  $x, y \in \mathbb{F}_2^n$  where  $x = (x_0, \dots, x_{n-1})$  and  $y = (y_0, \dots, y_{n-1})$ , the inner product in  $\mathbb{F}_2^n$  will be defined as  $x \cdot y := x_0 \cdot y_0 \oplus \dots \oplus x_{n-1} \cdot y_{n-1}$ .

**Example 1.** Let  $a, b \in \mathbb{F}_2^3$  such that  $a = (1, 0, 1)$  and  $b = (0, 1, 1)$  then

$$\begin{aligned} a + b &= (1 \oplus 0, 0 \oplus 1, 1 \oplus 1) = (1, 1, 0) \\ a \cdot b &= 1 \cdot 0 \oplus 0 \cdot 1 \oplus 1 \cdot 1 = 1 \end{aligned}$$

Each vector in  $\mathbb{F}_2^n$  can be uniquely represented by an integer between 0 and  $2^n - 1$ . To do this, the components of each vector in  $\mathbb{F}_2^n$  are trivially mapped to the integers 0 and 1, and then used in the one-to-one binary representation

function  $B$ :

$$B : \mathbb{F}_2^n \rightarrow \mathbb{N} \cup \{0\} \text{ such that } B(u) := \sum_{i=0}^{n-1} u_i \cdot 2^i. \quad (1)$$

This definition of  $B$  has created the convention where the *least significant bit* appears on the left and the *most significant bit* appears on the right. There are two important functions which are used extensively when comparing vectors in  $\mathbb{F}_2^n$ . These functions count the number of 1s in a vector, locate where the 1s occur in a vector, and count the number of differences between two vectors.

**Definition 2.1.** Let  $x, y \in \mathbb{F}_2^n$ . Then  $wt : \mathbb{F}_2^n \rightarrow \mathbb{N} \cup \{0\}$  is defined by

$$wt(x) := \sum_{i=0}^{n-1} x_i$$

and  $d : \mathbb{F}_2^n \times \mathbb{F}_2^n \rightarrow \mathbb{N} \cup \{0\}$  is defined by

$$d(x, y) := w(x + y).$$

Then  $wt(x)$  is the *Hamming weight* of  $x$  and  $d(x, y)$  is the *Hamming distance* between  $x$  and  $y$ .

*Remark 2.1.* The Hamming weight of a vector  $x \in \mathbb{F}_2^n$  is the number of 1s in the vector, and the Hamming distance between two vectors  $x, y \in \mathbb{F}_2^n$  is the number of bit differences between the two vectors.

**Definition 2.2.** Let  $x \in \mathbb{F}_2^n$ . Then  $supp : \mathbb{F}_2^n \rightarrow \mathcal{P}(\mathbb{N} \cup \{0\})$  is defined by

$$supp(x) := \{i \in \mathbb{N} \cup \{0\} : x_i = 1\}$$

**Example 2.** Let  $a, b, c \in \mathbb{F}_2^5$  such that

$$a = (0, 1, 1, 0, 1), \quad b = (1, 1, 1, 0, 0), \quad \text{and} \quad c = (0, 0, 1, 1, 0).$$

Then,

$$\begin{array}{lll} wt(a) = 3 & supp(a) = \{1, 2, 4\} & d(a, b) = 2 \\ wt(b) = 3 & supp(b) = \{0, 1, 2\} & d(a, c) = 3 \\ wt(c) = 2 & supp(c) = \{2, 3\} & d(b, c) = 3. \end{array}$$

If we define the vectors  $v_i \in \mathbb{F}_2^n$  such that  $v_i = B^{-1}(i)$  for  $0 \leq i \leq 2^n - 1$ , then the sequence  $(v_0, v_1, \dots, v_{2^n-1})$  is said to be in *lexicographical order*. Using the convention of lexicographical ordering and the inner product in  $\mathbb{F}_2^n$ , a  $2^n \times 2^n$  matrix can be written which captures all of the possible inner products of two vectors in  $\mathbb{F}_2^n$ . This is a matrix where  $x_i \cdot x_j$  appears in the  $i$ th row and  $j$ th column. This particular matrix turns out to be a *Hadamard matrix* which will be studied later.

There is an interesting orthogonality property in the vector space  $\mathbb{F}_2^n$  known as the *orthogonality principle* that every non-zero vector in  $\mathbb{F}_2^n$  is orthogonal to exactly half of the vectors in the vectorspace.

**Theorem 2.1.** *Let  $x \in \mathbb{F}_2^n$ . Then*

$$\begin{aligned} \sum_{y \in \mathbb{F}_2^n} (-1)^{x \cdot y} &= 2^n \text{ for } x = 0 \\ &= 0 \text{ otherwise.} \end{aligned}$$

*Proof.* Let  $x = 0 \in \mathbb{F}_2^n$ . Then  $\forall y \in \mathbb{F}_2^n$ ,  $x \cdot y = 0$ , so  $(-1)^{x \cdot y} = 1$ . Therefore,  $\sum_{y \in \mathbb{F}_2^n} (-1)^{x \cdot y} = |\mathbb{F}_2^n| = 2^n$ .

Let  $x \in \mathbb{F}_2^n$  where  $x \neq 0$ . Assume the  $i$ th bit of  $x$  is non-zero and define  $e_i \in \mathbb{F}_2^n$  as a vector with all zero bits except for the  $i$ th bit which is 1. Then

$$\begin{aligned} \sum_{y \in \mathbb{F}_2^n} (-1)^{x \cdot y} &= \sum_{y \in \mathbb{F}_2^n} (-1)^{x \cdot (y + e_i)} \\ &= \sum_{y \in \mathbb{F}_2^n} (-1)^{x \cdot y} (-1)^{x \cdot e_i} \\ &= - \sum_{y \in \mathbb{F}_2^n} (-1)^{x \cdot y}. \end{aligned}$$

Therefore,  $\sum_{y \in \mathbb{F}_2^n} (-1)^{x \cdot y} = - \sum_{y \in \mathbb{F}_2^n} (-1)^{x \cdot y}$ , which implies  $\sum_{y \in \mathbb{F}_2^n} (-1)^{x \cdot y} = 0$ .  $\square$

When introduced to a new vector space, it is natural to begin looking at functions in that field. The particular function of interest here will be what is known as a *Boolean function*.

**Definition 2.3.** Any function  $f$  defined such that

$$f : \mathbb{F}_2^n \rightarrow \mathbb{F}_2$$

is a *Boolean function*. The set of all Boolean functions on  $n$  variables will be denoted by  $\mathcal{BF}_n$ .

The number of Boolean functions increases extremely rapidly as the number of variables increases.<sup>1</sup>

$$|\mathcal{BF}_n| = 2^{2^n} \tag{2}$$

The Boolean function  $f$  is presented in a *truth table* in Table 2.1. The Hamming weight of  $f$  is the number of 1s that  $f$  has when evaluated at every point in the  $\mathbb{F}_2^n$ :

$$wt(f) = |\{u \in \mathbb{F}_2^n : f(u) = 1\}|.$$

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<sup>1</sup>With today's fastest supercomputer operating at 10.51 petaflops (the K computer in Japan), if one floating point operation was expended visiting every Boolean function of 7 variables, it would take over a thousand trillion years to complete the process. This length of time is roughly 70,000 times the age of the universe. Though every symmetric cryptosystem in use can be broken down into several Boolean functions of several variables, it would be infeasible to brute force search through all of the possibilities of Boolean functions which reconstruct the cryptosystem.

$x_0$	$x_1$	$x_2$	$x_3$	$f(x_0, x_1, x_2, x_3)$
0	0	0	0	0
1	0	0	0	1
0	1	0	0	1
1	1	0	0	0
0	0	1	0	1
1	0	1	0	0
0	1	1	0	1
1	1	1	0	0
0	0	0	1	0
1	0	0	1	0
0	1	0	1	1
1	1	0	1	0
0	0	1	1	0
1	0	1	1	0
0	1	1	1	1
1	1	1	1	1

Table 2: Truth Table of  $f$

## 2.2 Boolean Polynomials

A truth table is not a very compact method to define a Boolean function. It is much more efficient and easier to implement a Boolean function when written as a formula. This can be done by writing a Boolean function algebraically. As a first step toward defining  $f$  algebraically a one-to-one and onto function will be defined which maps every  $f$  in  $\mathcal{BF}_n$  to a vector in  $\mathbb{F}_2^{2^n}$ . This will be the function  $V : \mathcal{BF}_n \rightarrow \mathbb{F}_2^{2^n}$  such that

$$V(f)(v_0, \dots, v_{2^n-1}) := (f(v_0), \dots, f(v_{2^n-1})) \text{ where } v_i = B^{-1}(i). \quad (3)$$

It is trivial to show that addition is homomorphic under  $V$ ,

$$V(f_1 \oplus f_2) = V(f_1) \oplus V(f_2).$$

Then the standard basis of  $\mathbb{F}_2^{2^n}$  can be used to pull back an equivalent basis of  $\mathcal{BF}_n$ . Let  $e_i \in \mathbb{F}_2^{2^n}$  be defined so that

$$\begin{aligned} e_0 &= (1, 0, \dots) \\ e_1 &= (0, 1, \dots) \\ &\vdots \\ e_{2^n-1} &= (0, \dots, 0, 1). \end{aligned}$$

The *atomic Boolean functions* will be defined as the  $f_i \in \mathcal{BF}_n$  where there exists an  $e_i \in \mathbb{F}_2^{2^n}$  such that  $V(f_i) = e_i$ . Every vector of  $\mathbb{F}_2^{2^n}$  can be written as a linear combination of the standard basis vectors, and equivalently, every

Boolean function is a linear combination of atomic Boolean functions. This means for every  $u \in \mathbb{F}_2^{2^n}$  there exists a set of  $c_i \in \mathbb{F}_2$  such that

$$\begin{aligned} u &= c_0 e_0 \oplus \cdots \oplus c_{2^n-1} e_{2^n-1} \\ \Leftrightarrow f &= c_0 f_0 \oplus \cdots \oplus c_{2^n-1} f_{2^n-1}. \end{aligned}$$

where  $V(f) = u$ .

The function  $f$  defined in Table 2.1 can be written as a linear combination of the atomic Boolean functions in  $\mathcal{BF}_4$ . Since the coefficients in the linear combinations are either 0 or 1, it is true that every Boolean function can be written as the sum of  $wt(f)$  atomic Boolean functions.

$x_0$	$x_1$	$x_2$	$x_3$	$f$	$f_1$	$f_2$	$f_4$	$f_6$	$f_{10}$	$f_{14}$	$f_{15}$
0	0	0	0	0	0	0	0	0	0	0	0
1	0	0	0	1	1	0	0	0	0	0	0
0	1	0	0	1	0	1	0	0	0	0	0
1	1	0	0	0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	1	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0
0	1	1	0	1	0	0	0	1	0	0	0
1	1	1	0	0	0	0	0	0	0	0	0
0	0	0	1	0	0	0	0	0	0	0	0
1	0	0	1	0	0	0	0	0	0	0	0
0	1	0	1	1	0	0	0	0	1	0	0
1	1	0	1	0	0	0	0	0	0	0	0
0	0	1	1	0	0	0	0	0	0	0	0
1	0	1	1	0	0	0	0	0	0	0	0
0	1	1	1	1	0	0	0	0	0	1	0
1	1	1	1	0	0	0	0	0	0	0	1

Table 3:  $f$  broken into atomic Boolean function in  $\mathcal{BF}_4$

The equation  $f = f_1 + f_2 + f_4 + f_6 + f_{10} + f_{14} + f_{15}$  should be clear from Table 2.2. Knowing how to write the atomic Boolean functions as polynomials would lead to knowing how to write any Boolean function as a polynomial. The polynomials representing the Boolean functions will belong to the polynomial ring  $\mathbb{F}_2[x_0, \dots, x_{n-1}]/(x_0^2 \oplus x_0, \dots, x_{n-1}^2 \oplus x_{n-1})$ . To properly represent an atomic Boolean function, a polynomial must equal 1 at only one vector  $(x_0, \dots, x_{n-1})$ . Recall the support function from Definition 2.2. Then the polynomial representing each Boolean function is as follows:

$$f_i = \left( \prod_{j \in \text{supp}(B^{-1}(i))} x_j \right) \left( \prod_{j \notin \text{supp}(B^{-1}(i))} (1 \oplus x_j) \right). \quad (4)$$

*Proof.* Let  $x = B^{-1}(i)$  where  $x = (x_0, \dots, x_{n-1})$ . Then  $\{x_i : x_i = 1\} = \{x_i :$

$i \in \text{supp}(x)\}$ . Therefore,

$$f_i(x) = \left( \prod_{j \in \text{supp}(B^{-1}(i))} x_j \right) \left( \prod_{j \notin \text{supp}(B^{-1}(i))} (1 \oplus x_j) \right) = 1.$$

Let  $x \neq B^{-1}(i)$ . Then,

$$\begin{aligned} \left( \prod_{j \in \text{supp}(B^{-1}(i))} x_j \right) &= 0 \\ \therefore f_i(x) &= 0. \end{aligned}$$

□

Now the function  $f$  from Table 2.1 can be written as the sum of the following atomic polynomials:

$$\begin{aligned} f_1 &= (1 \oplus x_3)(1 \oplus x_2)(1 \oplus x_1)x_0 \\ &= x_0 \oplus x_1x_0 \oplus x_2x_0 \oplus x_2x_1x_0 \oplus x_3x_0 \oplus x_3x_1x_0 \oplus x_3x_2x_0 \oplus x_3x_2x_1x_0 \\ f_2 &= (1 \oplus x_3)(1 \oplus x_2)x_1(1 \oplus x_0) \\ f_4 &= (1 \oplus x_3)x_2(1 \oplus x_1)(1 \oplus x_0) \\ f_6 &= (1 \oplus x_3)x_2x_1(1 \oplus x_0) \\ f_{10} &= x_3(1 \oplus x_2)x_1(1 \oplus x_0) \\ f_{14} &= x_3x_2x_1(1 \oplus x_0) \\ f_{15} &= x_3x_2x_1x_0 \end{aligned}$$

After summing the atomic polynomials upon multiplying them out,

$$f = x_0x_1x_2x_3 \oplus x_0x_1x_3 \oplus x_0x_3 \oplus x_0 \oplus x_1x_2x_3 \oplus x_1x_2 \oplus x_1 \oplus x_2x_3 \oplus x_2.$$

Now the uniqueness of the polynomial representation for each Boolean function is considered. This is easily seen by considering the uniqueness of each Boolean function and the size of the polynomial ring.

**Theorem 2.2.** *Each  $n$ -variable Boolean function is uniquely represented as a polynomial in the polynomial ring  $\mathbb{F}_2[x_0, \dots, x_{n-1}]/(x_0^2 \oplus x_0, \dots, x_{n-1}^2 \oplus x_{n-1})$ .*

$$f(x) = \sum_{I \in \mathcal{P}(\mathbb{N} \cup \{0\})} a_I \left( \prod_{i \in I} x_i \right) \quad (5)$$

where  $a_I \in \mathbb{F}_2$  and defined by the given Boolean function.

*Proof.*

$$\begin{aligned} |\mathcal{BF}_n| &= |\mathbb{F}_2^{2^n}| \\ &= |\{(a_{\emptyset}, \dots, a_I, \dots) : a_I \in \mathbb{F}_2\}| \\ &= |\mathbb{F}_2[x_0, \dots, x_{n-1}]/(x_0^2 \oplus x_0, \dots, x_{n-1}^2 \oplus x_{n-1})| \end{aligned}$$



Because every Boolean function is determined by at least one polynomial and the size of the polynomial ring equals the size of the set of all Boolean functions, each Boolean function must be uniquely determined by a polynomial in the polynomial ring.  $\square$

### 2.3 The Discrete Fourier Transform

The Fourier transform of a function has many applications in signals analysis. The idea is to express a given signal as a function of frequency to reveal the dominant frequencies in the signal. Today, digital signals have taken over analog. This fact has made the discrete Fourier transform much more applicable to modern communication systems.

The discrete Fourier transforms (DFTs) of  $n$ -variable pseudo Boolean functions are studied here. The following discussion on DFTs will follow [?]. To begin the discussion about DFTs, *characters* are defined.

**Definition 2.4.** [?] A *character*  $\chi$  of a finite abelian group  $G$  is a group homomorphism from  $G$  into the multiplicative group of complex numbers.

For the purposes of this paper, it should be clear that  $\chi_\lambda(x) := (-1)^{\lambda \cdot x}$  where  $\lambda, x \in \mathbb{F}_2^n$  is a *group character* of  $\mathbb{F}_2^n$ . Define the *dual group*  $\hat{\mathbb{F}}_2^n$  to be the set of all characters of  $G$ . The group operation in  $\hat{\mathbb{F}}_2^n$  is pointwise multiplication of functions:

$$\chi\psi(x) = \chi(x)\psi(x)$$

This operation is closed under multiplication. The isomorphism  $\mathbb{F}_2^n \cong \hat{\mathbb{F}}_2^n$  is shown by constructing a one-to-one and onto mapping  $\Upsilon : \mathbb{F}_2^n \rightarrow \hat{\mathbb{F}}_2^n$  where  $\Upsilon(\lambda) := \chi_\lambda$ .

*Proof.* Every character in  $\hat{\mathbb{F}}_2^n$  corresponds to an element of  $\mathbb{F}_2^n$ . Thus,  $|\mathbb{F}_2^n| = |\hat{\mathbb{F}}_2^n|$ . If  $\Upsilon$  is one-to-one, then it must be an isomorphism. Let  $\Upsilon(\lambda_1) = \Upsilon(\lambda_2)$ . Then  $\forall x$

$$\begin{aligned} (-1)^{\lambda_1 \cdot x} &= (-1)^{\lambda_2 \cdot x} \\ &= (-1)^{(\lambda_1 + \lambda_1 + \lambda_2) \cdot x} \\ &= (-1)^{\lambda_1 \cdot x} (-1)^{(\lambda_1 + \lambda_2) \cdot x}. \end{aligned}$$

Finally,  $(\lambda_1 + \lambda_2) \cdot x = 0$  for all  $x \in \mathbb{F}_2^n$ , which implies  $\lambda_1 + \lambda_2 = 0$ . Therefore,  $\lambda_1 = \lambda_2$ .  $\square$

Addition in  $\mathbb{F}_2^n$  corresponds to multiplication in  $\hat{\mathbb{F}}_2^n$ . Define  $\chi_{\lambda_1}\chi_{\lambda_2}(x) = (-1)^{(\lambda_1 + \lambda_2) \cdot x}$ . Then,

$$\chi_{\lambda_1}(x)\chi_{\lambda_2}(x) = (-1)^{\lambda_1 \cdot x}(-1)^{\lambda_2 \cdot x} = (-1)^{(\lambda_1 + \lambda_2) \cdot x} = \chi_{\lambda_1 + \lambda_2}(x).$$

For discrete Fourier transforms, it is useful to consider *pseudo Boolean functions* which have codomain  $\{-1, 1\}$ .

**Definition 2.5.** Any function  $\hat{f} : \mathbb{F}_2^n \rightarrow \{1, -1\}$  such that  $\hat{f}(x) = (-1)^{f(x)}$  for  $f \in \mathcal{BF}$  is called a *pseudo Boolean function*

**Definition 2.6.** The *discrete Fourier transform* or DFT of a Boolean function is defined by

$$\mathcal{F}f(\lambda) = \sum_{x \in \mathbb{F}_2^n} f(x) \chi_\lambda(x) \quad (6)$$

**Lemma 2.1.** The characters of  $\mathbb{F}_2^n$  are functions in  $\hat{\mathcal{BF}}_n = \{\hat{f} : f \in \mathcal{BF}_n \text{ and form an orthonormal basis of that set.}$

*Proof.* The dimension of  $\hat{\mathcal{BF}}_n$  is  $2^n$ , and there are  $2^n$  characters of  $\mathbb{F}_2^n$ .

$$\begin{aligned} \sum_{x \in \mathbb{F}_2^n} \chi_{\lambda_i}(x) \cdot \chi_{\lambda_j}(x) &= \sum_{x \in \mathbb{F}_2^n} (-1)^{(\lambda_i + \lambda_j) \cdot x} \\ &= \begin{cases} 0 & \text{if } i \neq j \\ 2^n & \text{if } i = j. \end{cases} \end{aligned}$$

Therefore, the characters of  $\mathbb{F}_2^n$  form an orthonormal basis of  $\hat{\mathcal{BF}}_n$ .  $\square$

Every pseudo Boolean function can be written as a linear combination of the characters of  $\mathbb{F}_2^n$ . The coefficients in these linear combinations reveal important properties of the functions. Rothaus rewrote the pseudo Boolean function as a linear combination as follows [17].

**Lemma 2.2.**

$$\hat{f}(x) = \frac{1}{2^{n/2}} \sum_{\lambda \in \mathbb{F}_2^n} c(\lambda) \chi_\lambda(x) \quad (7)$$

where  $c(\lambda)$ , the Fourier coefficients of  $\hat{f}(x)$  are given by

$$c(\lambda) = \frac{1}{2^{n/2}} \mathcal{F}\hat{f}(\lambda).$$

As observed by Rothaus [17],  $2^{n/2}c(\lambda)$  is the number of zeros minus the number of ones of the function  $f(x) + \lambda \cdot x$ . The Hamming weight of  $f$  is easily determined using the zero Fourier coefficient  $c(0)$ :

$$\begin{aligned} c(0) &= \frac{1}{2^{n/2}} \sum_{x \in \mathbb{F}_2^n} (-1)^{f(x)} \\ &= \frac{1}{2^{n/2}} ((2^n - wt(f)) - wt(f)) \\ \Rightarrow wt(f) &= 2^{n-1} - 2^{n/2-1}c(0). \end{aligned} \quad (8)$$

**Definition 2.7.** Let  $f \in \mathcal{BF}_n$  and  $\lambda \in \mathbb{F}_2^n$ . Then the *Walsh transform* of  $f$  is defined by:

$$\mathcal{W}_f(\lambda) = \mathcal{F}\hat{f}(\lambda). \quad (9)$$

Clearly  $\mathcal{W}_f(\lambda) = \frac{1}{2^{n/2}}c(\lambda)$ . For large  $|\mathcal{W}_f(\lambda)|$ ,  $f$  is close to an affine function in  $\mathcal{BF}_n$ .

**Example 3.** There are two cases that should be clear to the reader.

1. Let  $f(x) = \lambda \cdot x$ . Then  $\mathcal{W}_f(\lambda) = 2^n$ .
  2. Let  $f(x) \neq \lambda \cdot x \quad \forall x$ . Then  $f(x) = \lambda \cdot x + 1 \quad \forall x$ . So,  $\mathcal{W}_f(\lambda) = -2^n$ .
- In both cases,  $f$  is an affine function.

## 2.4 Bent Functions

Bent functions were originally defined in a paper by Rothaus in the Journal of Combinatorial Theory in 1976 [17]. These functions are useful for cryptographic applications because they are *perfectly nonlinear* which makes them resistant to differential attacks. The original definition of the *bent function* is based on the Fourier coefficients.

**Definition 2.8.** If all of the Fourier coefficients of  $\hat{f}$  are  $\pm 1$  then  $f$  is a *bent function*.

Here is Rothaus's first theorem about bent functions.

**Theorem 2.3.** If  $f$  is a bent function on  $\mathbb{F}_2^n$ , then  $n$  is even,  $n = 2k$ ; the degree of  $f$  is at most  $k$ , except in the case  $k = 1$ .

*Proof.*  $c(\lambda) = \pm 1$ . This implies  $2^{n/2}c(\lambda)$  is an integer. Therefore  $n$  must be even.

Let  $n = 2k$  with  $k > 1$ , and let  $r > k$ . Consider the polynomial  $f(x_1, x_2, \dots, x_r, 0, 0, \dots, 0) = g(x_1, x_2, \dots, x_r)$  (up to this point all numbering has started at 0; it is more convenient in this proof to begin numbering at 1). Then by Equation (7),

$$\hat{g}(x) = \frac{1}{2^{r/2}} \sum_{\lambda_1, \lambda_2, \dots, \lambda_r=0,1} b(\lambda_1, \dots, \lambda_r) \chi_{(\lambda_1, \dots, \lambda_r)}(x)$$

and

$$\hat{f}(x, 0) = \frac{1}{2^{n/2}} \sum_{\lambda_1, \lambda_2, \dots, \lambda_n=0,1} c(\lambda_1, \dots, \lambda_n) \chi_{(\lambda_1, \dots, \lambda_n)}(x, 0).$$

Because  $f(x, 0) = g(x)$  and the uniqueness of the Fourier expansion,  $b$  and  $c$  are related such that

$$b(\lambda_1, \dots, \lambda_r) = \frac{1}{2^{(n-r)/2}} \sum_{\lambda_{r+1}, \dots, \lambda_n=0,1} c(\lambda_1, \dots, \lambda_r, \lambda_{r+1}, \dots, \lambda_n).$$

Then,

$$\begin{aligned} wt(f(x, 0)) &= wt(g(x)) \\ &= 2^{r-1} - 2^{r/2-1}b(0) \\ &= 2^{r-1} - 2^{r-n/2-1} \sum_{\lambda_{r+1}, \dots, \lambda_n=0,1} c(0, \dots, 0, \lambda_{r+1}, \dots, \lambda_n). \end{aligned}$$

There are  $2^{n-r}$  summands in  $\sum c(0, \dots, 0, \lambda_{r+1}, \dots, \lambda_n)$ . Since  $f$  is bent,  $c(\lambda) = \pm 1$ . By rewriting  $1 = -1 + 2$ ,

$$\begin{aligned} \sum c(0, \dots, 0, \lambda_{r+1}, \dots, \lambda_n) &= -2^{n-r} + 2wt(c(0, \dots, 0, \lambda_{r+1}, \dots, \lambda_n)) \\ &= 2(wt(c(0, \dots, 0, \lambda_{r+1}, \dots, \lambda_n)) - 2^{n-r-1}) \end{aligned}$$

Thus,  $wt(g(x))$  is even. This implies that  $g(x)$  is the sum of an even number of atomic Boolean functions. Therefore the coefficient of  $x_1 x_2 \cdots x_r$  in the polynomial representing  $g(x)$  must be 0. This is true for every  $r > k$ , so the degree of  $f$  must not be greater than  $k$ .  $\square$

## 2.5 Constructions of Bent Functions

A simple bent function construction is accomplished by the Boolean functions in the *Maierana-McFarland class*. This is the set  $\mathcal{M}$  which contains all Boolean function on  $\mathbb{F}_2^n = \{(x, y) : x, y \in \mathbb{F}_2^{n/2}\}$ , of the form:

$$f(x, y) = x \cdot \pi(y) \oplus g(y)$$

where  $\pi$  is any permutation on  $\mathbb{F}_2^{n/2}$  and  $g$  any Boolean function on  $\mathbb{F}_2^{n/2}$ .

All functions in the Maierana-McFarland class of Boolean functions are bent.

## 2.6 Monotone Boolean Functions

**Definition 2.9.** Let  $f \in \mathcal{BF}_n$  and  $r, r' \in \mathbb{F}_2^n$  such that  $r_i = 1 \Rightarrow r'_i = 1$  for  $0 \leq i \leq n-1$ . If  $f(r) = 1 \Rightarrow f(r') = 1$ , then  $f$  is a monotone function.

**Theorem 2.4.** Let  $f \in \mathcal{BF}_n$  be monotone and  $x \in \mathbb{F}_2^n$  such that  $f(x) = 1$  and  $\nexists y \in \mathbb{F}_2^n$  such that  $\text{supp}(y) \subsetneq \text{supp}(x)$  where  $f(y) = 1$ . Then,  $f$  is not bent.

*Proof.* Let  $f \in \mathcal{BF}_n$  be monotone and bent, and  $x \in \mathbb{F}_2^n$  such that  $f(x) = 1$  and  $\nexists y \in \mathbb{F}_2^n$  such that  $\text{supp}(y) \subsetneq \text{supp}(x)$  where  $f(y) = 1$ . Then  $0 < |\text{supp}(x)| < n$ . Define  $\lambda \in \mathbb{F}_2^n$  such that  $\text{supp}(x) \subsetneq \text{supp}(\lambda)$ .

$$\begin{aligned} c(\lambda) &= \frac{1}{2^{n/2}} \sum_{u \in \mathbb{F}_2^n} (-1)^{f(u) + u \cdot \lambda} \\ &= \frac{1}{2^{n/2}} \left( \sum_{x_I : \text{supp}(x) \subseteq I} (-1)^{f(x_I) + x_I \cdot \lambda} + \sum_{x_J : \text{supp}(x) \not\subseteq J} (-1)^{f(x_J) + x_J \cdot \lambda} \right) \\ &= \frac{1}{2^{n/2}} \left( \sum_{x_I : \text{supp}(x) \subseteq I} (-1)^{1 + x_I \cdot \lambda} + \sum_{x_J : \text{supp}(x) \not\subseteq J} (-1)^{f(x_J) + 0} \right) \end{aligned}$$

$f(x_I) = 1$  because  $f$  is monotone, so the first sum equals  $\pm 2^{n-|supp(x)|+1} \mp 1$ . The second sum equals  $2wt(f) - 2^{n-|supp(x)|+1} - 2^n$ .

$$c(\lambda) = \begin{cases} \frac{1}{2^{n/2}} (2^{n-|supp(x)|+1} - 1 + 2wt(f) - 2^{n-|supp(x)|+1} - 2^n) & \text{for } |supp(x)| \text{ even} \\ \frac{1}{2^{n/2}} (-2^{n-|supp(x)|+1} + 1 + 2wt(f) - 2^{n-|supp(x)|+1} - 2^n) & \text{for } |supp(x)| \text{ odd} \end{cases} =$$

It follows that  $c(\lambda)$  cannot be an integer. For a Bent function,  $c(\lambda) = \pm 1 \in \mathbb{Z}$ . Therefore,  $f$  is not bent.  $\square$

### 3 $N$ -adic Integers

The goal of this section is to introduce the reader to the  $N$ -adic integer ring and define the  $N$ -adic valuation. The 2-adic valuation will be used to analyze bent sequences.

#### 3.1 $N$ -adic Integer Ring

The notation used in the definition of the  $N$ -adic numbers will follow the same notation used by Borevich and Shafarevich in Chapter 1 of *Number Theory* [1]. In this section, the set of  $N$ -adic integers is shown to be a commutative ring with an identity.

**Definition 3.1.** Let  $N$  be an integer. Then the infinite integer sequence  $(x_n)$  determines an  $N$ -adic integer  $\alpha$ , or  $(x_n) \rightarrow \alpha$ , if

$$x_{i+1} \equiv x_i \pmod{N^{i+1}} \quad \forall i \geq 0. \quad (10)$$

Two sequences  $(x_n)$  and  $(x'_n)$  determine the same  $N$ -adic integer if

$$x_i \equiv x'_i \pmod{N^{i+1}} \quad \forall i \geq 0. \quad (11)$$

The set of all  $N$ -adic integers will be denoted by  $\mathbb{Z}_N$ .

Each integer  $x$  is associated with a  $N$ -adic integer, determined by the sequence  $(x, x, \dots, x, \dots)$ . These integers will be called *rational integers* in the  $N$ -adic integers.

**Example 4.** Let  $(x_n) \rightarrow \alpha \in \mathbb{Z}_3$ . Then the first 5 terms of  $(x_n)$  may look something like:

$$\begin{aligned} (x_n) &= (1, 1 + 2 \cdot 3, 1 + 2 \cdot 3 + 1 \cdot 3^2, \\ &\quad 1 + 2 \cdot 3 + 1 \cdot 3^2, 1 + 2 \cdot 3 + 1 \cdot 3^2 + 1 \cdot 3^4, \dots) \\ &= (1, 7, 16, 16, 97, \dots) \end{aligned}$$

Then equivalent sequences to  $(x_n)$  could begin differently for the first few terms:

$$\begin{aligned} (y_n) &= (4, 25, 16, 178, 583, \dots) \\ (z_n) &= (-2, -47, 232, -308, 97, \dots) \end{aligned}$$

The sequences for  $(y_n)$  and  $(z_n)$  satisfy equation (10) for the first 5 terms, so they could be  $N$ -adic integers up to this point. Also, both are equivalent to  $(x_n)$  according to the equivalence defined in equation (11).

$$\begin{aligned} 1 &\equiv 4 \equiv 2 \pmod{3} \\ 7 &\equiv 25 \equiv -47 \pmod{3^2} \\ 16 &\equiv 16 \equiv 232 \pmod{3^3} \\ 16 &\equiv 178 \equiv -308 \pmod{3^4} \\ 97 &\equiv 583 \equiv 97 \pmod{3^5} \end{aligned}$$

Therefore  $(x_n), (y_n), (z_n) \rightarrow \alpha$ .

Because there are infinitely many sequence representations for any  $N$ -adic integer, it is useful to define a canonical sequence to be used when writing  $N$ -adic integers as sequences.

**Definition 3.2.** For a given  $N$ -adic integer  $\alpha$ , a given sequence  $(a_n)$  with the properties:

- i.  $(a_n) \rightarrow \alpha$
- ii.  $(a_n) = (a_0, a_0 + a_1 \cdot N, \dots, a_0 + \dots + a_i \cdot N^i, \dots) : 0 \leq a_i < N \quad \forall i \geq 0$

will be called *canonical*. The sequence  $(a_0, a_1, a_2, \dots, a_i, \dots)$  is the *digit representation* of  $\alpha$ . When  $N \leq 10$ , the digits are usually written adjacent to one another.

**Example 5.** In Example 4, the sequence  $(x_n)$  was a canonical sequence that determined the  $N$ -adic integer  $\alpha$ . A few more examples of canonical sequences determining 7-adic integers are given here:

$\beta = 3164\dots$ , then the canonical sequence  $(b_n) \rightarrow \beta$  is

$$\begin{aligned} (b_n) &= (3, 3 + 1 \cdot 7, 3 + 1 \cdot 7 + 6 \cdot 7^2, 3 + 1 \cdot 7 + 6 \cdot 7^2 + 4 \cdot 7^3, \dots) \\ &= (3, 10, 304, 1676, \dots) \end{aligned}$$

$\gamma = 0164\dots$ , then the canonical sequence  $(c_n) \rightarrow \gamma$  is

$$\begin{aligned} (c_n) &= (0, 1 \cdot 7, 1 \cdot 7 + 6 \cdot 7^2, 1 \cdot 7 + 6 \cdot 7^2 + 4 \cdot 7^3, \dots) \\ &= (0, 7, 301, 1673, \dots) \end{aligned}$$

$\delta = 5031\dots$ , then the canonical sequence  $(d_n) \rightarrow \delta$  is

$$\begin{aligned} (d_n) &= (5, 5, 5 + 3 \cdot 7^2, 5 + 3 \cdot 7^2 + 1 \cdot 7^3, \dots) \\ &= (5, 5, 152, 495, \dots). \end{aligned}$$

**Definition 3.3.** Addition and multiplication in  $\mathbb{Z}_N$  are done term by term. Let  $\alpha, \beta \in \mathbb{Z}_N$  and  $(x_n) \rightarrow \alpha, (y_n) \rightarrow \beta$ . Then,

$$\begin{aligned}(x_n) + (y_n) &:= (x_0 + y_0, x_1 + y_1, \dots) \rightarrow \alpha + \beta \\ (x_n) \cdot (y_n) &:= (x_0 \cdot y_0, x_1 \cdot y_1, \dots) \rightarrow \alpha \cdot \beta\end{aligned}$$

Define  $(0, 0, 0, \dots) \rightarrow 0 \in \mathbb{Z}_N$  and  $(1, 1, 1, \dots) \rightarrow 1 \in \mathbb{Z}_N$

**Lemma 3.1.** For  $\alpha \in \mathbb{Z}_N$ ,  $\alpha + 0 = 0 + \alpha = \alpha$  and  $1 \cdot \alpha = \alpha \cdot 1 = \alpha$ .

*Proof.* Let  $(x_n) \rightarrow \alpha \in \mathbb{Z}_N$ .

$$\begin{aligned}(x_n) + (0, 0, \dots) &= (x_0 + 0, x_1 + 0, \dots, x_i + 0, \dots) \\ &= (x_0, \dots, x_i, \dots). \\ (0, 0, \dots) + (x_n) &= (0 + x_0, 0 + x_1, \dots, 0 + x_i, \dots) \\ &= (x_0, \dots, x_i, \dots).\end{aligned}$$

$(x_n) = (x_n) + (0, 0, \dots) = (0, 0, \dots) + (x_n)$  implies  $\alpha = \alpha + 0 = 0 + \alpha$ . Therefore, the additive identity in  $\mathbb{Z}_N$  is 0.

$$\begin{aligned}(x_n) \cdot (1, 1, \dots) &= (x_0 \cdot 1, x_1 \cdot 1, \dots, x_i \cdot 1, \dots) \\ &= (x_0, \dots, x_i, \dots). \\ (1, 1, \dots) \cdot (x_n) &= (1 \cdot x_0, 1 \cdot x_1, \dots, 1 \cdot x_i, \dots) \\ &= (x_0, \dots, x_i, \dots).\end{aligned}$$

$(x_n) = (x_n) \cdot (1, 1, \dots) = (1, 1, \dots) \cdot (x_n)$  implies  $\alpha = \alpha \cdot 1 = 1 \cdot \alpha$ . Therefore, the multiplicative identity in  $\mathbb{Z}_N$  is 1.  $\square$

Finally, this section defines a *ring* and proves that  $\mathbb{Z}_N$  is a *commutative ring with an identity*.

**Definition 3.4.** A *ring* is a set  $R$  with two binary operations defined on it. These are usually called addition denoted by  $+$ , and multiplication denoted by  $\cdot$  or juxtaposition, satisfying the following six axioms:

1. Addition is commutative:  $a + b = b + a \quad \forall a, b \in R$ .
2. Addition is associative:  $a + (b + c) = (a + b) + c \quad \forall a, b, c \in R$ .
3. There exists an additive identity, denoted by 0, such that  $a + 0 = a \quad \forall a \in R$ .
4.  $\forall a \in R$  there exists an additive inverse, denoted by  $-a$ , such that  $a + (-a) = 0$ .
5. Multiplication is associative:  $a(bc) = (ab)c \quad \forall a, b, c \in R$

6. Multiplication is left and right distributive over addition:

$$\begin{aligned} a(b + c) &= ab + ac \\ (b + c)a &= ba + ca \end{aligned}$$

If it is also true that

7. Multiplication is commutative:  $ab = ba \quad \forall a, b \in R$ , then  $R$  is a *commutative ring*.

Further if

8. There exists a multiplicative identity denoted by 1 such that  $a \cdot 1 = a$  and  $1 \cdot a = a \quad \forall a \in R$ , then  $R$  is a *ring with an identity*.

If  $R$  satisfies all eight properties, then  $R$  is a *commutative ring with an identity*.

**Theorem 3.1.**  $\mathbb{Z}_N$  is a commutative ring with an identity.

*Proof.* Let  $(x_n), (y_n), (z_n)$  determine  $\alpha, \beta, \gamma \in \mathbb{Z}_N$  respectively. Then

1. *Commutativity of Addition*

$$\begin{aligned} (x_n) + (y_n) &= (x_0 + y_0, \dots, x_i + y_i, \dots) \\ &= (y_0 + x_0, \dots, y_i + x_i, \dots) \\ &= (y_n) + (x_n). \end{aligned}$$

$(x_n) + (y_n) \rightarrow \alpha + \beta$  and  $(x_n) + (y_n) = (y_n) + (x_n) \rightarrow \beta + \alpha$ . Therefore, by Definition 3.1,  $\alpha + \beta = \beta + \alpha$ .

2. *Associativity of Addition*

$$\begin{aligned} (x_n) + ((y_n) + (z_n)) &= (x_n) + (y_0 + z_0, \dots, y_i + z_i, \dots) \\ &= (x_0 + (y_0 + z_0), \dots, x_i + (y_i + z_i), \dots) \\ &= ((x_0 + y_0) + z_0, \dots, (x_i + y_i) + z_i, \dots) \\ &= (x_0 + y_0, \dots, x_i + y_i, \dots) + (z_n) \\ &= ((x_n) + (y_n)) + (z_n). \end{aligned}$$

Therefore,  $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ .

3. *Existence of the Additive Identity*

By Lemma 3.1, 0 is the additive identity.

4. *Existence of Additive Inverses*

Define  $-(x_n) = (p - x_0, p^2 - x_1, \dots, p^{i+1} - x_i, \dots) \rightarrow -\alpha$ . Then

$$\begin{aligned} (x_n) + (-(x_n)) &= (x_0 + p - x_0, x_1 + p^2 - x_1, \dots, x_i + p^{i+1} - x_i, \dots) \\ &= (p, p^2, \dots, p^{i+1}, \dots) \\ &\equiv (0, 0, \dots) \\ &= 0. \end{aligned}$$



Therefore,  $\alpha + (-\alpha) = 0$ .

5. *Associativity of Multiplication*

$$\begin{aligned}
(x_n)((y_n)(z_n)) &= (x_n)(y_0 z_0, \dots, y_i z_i, \dots) \\
&= (x_0(y_0 z_0), \dots, x_i(y_i z_i), \dots) \\
&= ((x_0 y_0) z_0, \dots, (x_i y_i) z_i, \dots) \\
&= (x_0 y_0, \dots, x_i y_i, \dots)(z_n) \\
&= ((x_n)(y_n))(z_n).
\end{aligned}$$

Therefore,  $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ .

7. *Commutativity of Multiplication*

$$\begin{aligned}
(x_n)(y_n) &= (x_0 y_0, \dots, x_i y_i, \dots) \\
&= (y_0 x_0, \dots, y_i x_i, \dots) \\
&= (y_n)(x_n).
\end{aligned}$$

Therefore,  $\alpha\beta = \beta\alpha$ .

6. *Left and right distributivity of multiplication over addition*

$$\begin{aligned}
(x_n)((y_n) + (z_n)) &= (x_n)(y_0 + z_0, \dots, y_i + z_i, \dots) \\
&= (x_0(y_0 + z_0), \dots, x_i(y_i + z_i), \dots) \\
&= (x_0 y_0 + x_0 z_0, \dots, x_i y_i + x_i z_i, \dots) \\
&= (x_n)(y_n) + (x_n)(z_n).
\end{aligned}$$

By commutativity of multiplication,

$$\begin{aligned}
((y_n) + (z_n))(x_n) &= (x_n)((y_n) + (z_n)) \\
&= (x_n)(y_n) + (x_n)(z_n) \\
&= (y_n)(x_n) + (z_n)(x_n).
\end{aligned}$$

Therefore,  $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$  and  $(\beta + \gamma)\alpha = \beta\alpha + \gamma\alpha$ .

8. *Existence of a multiplicative identity*

By Lemma 3.1, 1 is the multiplicative identity.

Properties 1 through 8 from Definition 3.4 are satisfied, so  $\mathbb{Z}_N$  is a commutative ring with an identity.

□

**Theorem 3.2.** *An  $N$ -adic integer  $\alpha$ , which is determined by a sequence  $(x_n)$ , is a unit if and only if  $x_0$  is relatively prime to  $N$ .*

*Proof.* Let  $\alpha$  be a unit. Then there is an  $N$ -adic integer  $\beta$  such that  $\alpha\beta = 1$ . If  $\beta$  is determined by the sequence  $(y_n)$ , then

$$x_i y_i \equiv 1 \pmod{N^{i+1}} \quad \forall i \geq 0. \quad (12)$$

In particular,  $x_0 y_0 \equiv 1 \pmod{N}$ , hence  $x_0$  is relatively prime to  $N$ .

Conversely, let  $x_0$  be relatively prime to  $N$ . Then  $x_0 \not\equiv 0 \pmod{N}$ . From Equation (10)

$$\begin{aligned} x_1 &\equiv x_0 \pmod{N} \\ &\vdots \\ x_{i+1} &\equiv x_i \pmod{N^i}. \end{aligned}$$

Working backward,  $x_{i+1} \equiv x_i \equiv \dots \equiv x_1 \equiv x_0 \pmod{N}$ . Thus,  $x_i$  is relatively prime to  $N \forall i \geq 0$ , which implies  $x_i$  is relatively prime to  $N^{i+1}$ . Consequently,  $\forall i \geq 0 \exists y_i$  such that  $x_i y_i \equiv 1 \pmod{N^{i+1}}$ . Since  $x_{i+1} \equiv x_i \pmod{N^i}$  and  $x_{i+1} y_{i+1} \equiv x_i y_i \pmod{N^i}$ . Then,  $y_{i+1} \equiv y_i \pmod{N^i}$ . Therefore the sequence  $(y_n)$  determines some  $N$ -adic integer  $\beta$ . Because  $x_i y_i \equiv 1 \pmod{N^{i+1}} \forall i \geq 0$ ,  $\alpha\beta = 1$ . This means  $\alpha$  is a unit.  $\square$

From this theorem it follows that a rational integer  $a \in \mathbb{Z}_N$  is a unit if and only if  $a$  is relatively prime to  $N$ . If  $a$  is invertible in  $\mathbb{Z}_N$ , then for any rational integer  $b \in \mathbb{Z}_N$ ,  $b/a = a^{-1}b \in \mathbb{Z}_N$ .

### 3.2 $N$ -adic Valuation and Norm

The 2-adic valuation will be used to analyze the Bent sequences presented later. The general definitions for the  $N$ -adic valuation and norm are given here.

**Definition 3.5.** Let  $\alpha = (a_n) \in \mathbb{Z}_N \setminus (0)$ . If  $m$  is the smallest number in  $\mathbb{N}$  such that  $a_m \not\equiv 0 \pmod{p}^{m+1}$ , then the  $N$ -adic valuation of  $\alpha$  is  $m$ , or  $\log_N(\alpha) = m$ . If  $\alpha = 0$ , then  $\log_N(\alpha) = \infty$ .

**Definition 3.6.** If  $\alpha \in \mathbb{Z}_N$ , then the  $N$ -adic norm of  $\alpha$  is  $\|\alpha\|_N = N^{-m}$  where  $m = \log_N(\alpha)$ .

This paper will be careful not to confuse  $\log_N$  in  $\mathbb{Z}_N$  with  $\log_N$  in  $\mathbb{R}$ .

**Example 6.** Let the  $\alpha = 00001011011 \dots \in \mathbb{Z}_N$ . Then  $\log_2(\alpha) = 4$ .

## 4 Constructing Sequences Determining $\frac{b}{a}$ in $\mathbb{Z}_N$

For any rational number  $b/a$ ,  $a$  relatively prime to  $N$ , there exists a sequence  $(x_n) \rightarrow b/a \in \mathbb{Z}_N$ . At this point, it is worth using the digit representation for integers in  $\mathbb{Z}_N$ . So  $(x_n) = \{x_0, x_0 + x_1 N, \dots, x_0 + \dots + x_i N^i, \dots\}$  and

$b/a = x_0x_1 \dots x_i \dots$ . Rather than finding  $a^{-1} \pmod{N^{i+1}}$  to determine each  $x_i$ , it is not too difficult for every  $i$  to find  $\sum_{k=0}^i x_k N^k$  such that

$$b \equiv a \sum_{k=0}^i x_k N^k \pmod{N^{i+1}}. \quad (13)$$

Then,

$$x_i = \frac{\sum_{k=0}^i x_k N^k - \sum_{k=0}^{i-1} x_k N^k}{N^i}. \quad (14)$$

Nearly all of the digits for any rational number in  $\mathbb{Z}_N$  can also be found using powers of  $N^{-1}$ , which is much simpler to analyze than the brute force search for the digits mentioned above.

**Theorem 4.1.** *Let  $u_0, q, N \in \mathbb{Z}$ , where  $q$  is relatively prime to  $N$ ,  $|u_0| < q$ , and  $q = -q_0 + \sum_{i=1}^r q_i N^i$  for  $0 \leq q_i < N$ . Define  $\alpha = u_0/q \in \mathbb{Z}_N$  such that  $\alpha = \sum_{i=0}^{\infty} a_i N^i$  for  $0 \leq a_i < N$ . Also, define  $u_k \in \mathbb{Z}$  such that  $u_k/q = \sum_{i=k}^{\infty} a_i N^{i-k} \in \mathbb{Z}_N$  and  $\gamma \equiv N^{-1} \pmod{q}$ . Then, there exist  $u_k$  for every  $k \geq 0$  such that*

$$a_k \equiv q^{-1} u_k \pmod{N}. \quad (15)$$

*If  $-q < u_0 < 0$ , then  $u_k \in \{-q, \dots, -1\}$  for  $k \geq 0$ . Otherwise, for  $k > \lfloor \log_N(q) \rfloor = r$ ,  $u_k \in \{-q, \dots, -1\}$ .*

*Let  $\omega \in \{-q, \dots, -1\}$  such that  $\omega \equiv \gamma^k u_0 \pmod{q}$ . Then for  $k > \lfloor \log_N(q) \rfloor = r$ , or if  $-q < u_0 < 0$ , then  $k \geq 0$ ,*

$$a_k \equiv q^{-1} \omega \pmod{N}. \quad (16)$$

*Proof.* Write  $u_0/q$  in terms of  $u_k$ .

$$\begin{aligned} \frac{u_0}{q} &= a_0 + N \frac{u_1}{q} = a_0 + a_1 N + N^2 \frac{u_2}{q} = \dots \\ &= \sum_{i=0}^{k-1} a_i N^i + N^k \frac{u_k}{q} \quad \forall k \geq 1. \end{aligned} \quad (17)$$

Rewrite (17) to be

$$N^k u_k = u_0 - q \left( \sum_{i=0}^{k-1} a_i N^i \right) \quad \forall k \geq 1 \quad (18)$$

Then  $|u_0| < q$  and  $0 \leq a_i < p$  from the assumptions and equation (18). These imply for all  $k \geq 1$ ,  $|u_0| = |q \sum_{i=0}^{k-1} a_i N^i + N^k u_k| < q$ . Then,

$$-q \left( \frac{1 + \sum_{i=0}^{k-1} a_i N^i}{N^k} \right) < u_k < q \left( \frac{1 - \sum_{i=0}^{k-1} a_i N^i}{N^k} \right).$$

$u_k$  may only be greater than zero when  $\frac{1 - \sum_{i=0}^{k-1} a_i N^i}{N^k}$  is greater than zero. This only occurs when the sequence  $(a_0, \dots, a_j) = (0, \dots, 0)$  for  $j \geq 0$ . Such a sequence occurs if and only if  $u_0 \geq 0$  and  $u_0 \equiv 0 \pmod{N^i}$  for  $0 \leq i \leq j$ ,  $j \geq 0$ . This is clear from the construction of  $N$ -adic sequences for rational numbers. Therefore  $u_k$  may only be greater than zero if  $u_0 \geq 0$  and  $u_0 \equiv 0 \pmod{N^i}$  for  $0 \leq i \leq j$ ,  $j \geq 0$ . The lower bound for  $u_k$  is greater than  $-q$ . This is clear because  $\frac{1 + \sum_{i=0}^{k-1} a_i N^i}{N^k} \leq 1$ . Therefore,

$$-q < u_k < 0 \quad \text{for } -q < u_0 < 0.$$

If  $0 \leq u_0 < q$ , then the upper bound remains unchanged.

$$-q < u_k < q \left( \frac{1 - \sum_{i=0}^{k-1} a_i N^i}{N^k} \right) \quad \text{for } 0 \leq u_0 < q$$

There is still work to be done on the upper bound.

$$\begin{aligned} 0 &\leq \sum_{i=0}^{k-1} a_i N^i < N^k \text{ for } k \geq 1 \\ \Rightarrow -q &\left( \sum_{i=0}^{k-1} a_i N^i \right) \leq 0 \\ \Rightarrow u_0 - q &\left( \sum_{i=0}^{k-1} a_i N^i \right) < q \\ \Rightarrow N^k u_k &< q \\ \Rightarrow u_k &< \frac{q}{N^k}. \end{aligned}$$

For  $k > \lfloor \log_N(q) \rfloor = r$ ,  $|q/N^k| < 1$ . Therefore,  $-q < u_k < 0$  for  $0 \leq u_0 < q$  and  $k > r$ . Further lowering the upperbound, if  $u_k = 0$ , then  $u_0/q = \sum_{i=0}^{k-1} a_i N^i + 0$ . This implies  $u_0/q$  is a rational integer, which is not true. Noting finally that  $u_k$  must be an integer. If  $|u_0| < q$  and  $u_0 < 0$ , or  $|u_0| < q$ ,  $u_0 \geq 0$ , and  $k > \lfloor \log_N(q) \rfloor = r$ , then

$$u_k \in \{-q, \dots, -1\}.$$

It has now been shown, for certain restrictions,  $u_k$  belongs to a specific set of representatives for the residue classes of  $\mathbb{Z}/(q)$ . Define  $\gamma \equiv N^{-1} \pmod{q}$ . Reducing Equation (18) modulo  $q$  shows that

$$u_k \equiv \gamma u_{k-1} \pmod{q}. \quad (19)$$

Since this is true for all  $k$  greater than or equal to 1, it is clear that

$$u_k \equiv \gamma^k u_0 \pmod{q}. \quad (20)$$

Reducing (18) modulo  $p$  shows that

$$a_k \equiv q^{-1}u_k \pmod{N}. \quad (21)$$

Define  $\omega \equiv \gamma^k u_0 \pmod{q}$ , and  $\rho \equiv q^{-1} \pmod{N}$ . Finally, if  $|u_0| < q$  and  $u_0 < 0$ , or  $|u_0| < q$ ,  $u_0 \geq 0$ , and  $k > \lfloor \log_N(q) \rfloor = r$ , then

$$a_k \equiv \rho\omega \pmod{N}. \quad (22)$$

□

**Corollary 4.1.** *Let  $0 \leq u_0 < q$ . Define  $j$  to be the greatest integer such that  $u_0 \equiv 0 \pmod{p^j}$ . Then the following are true:*

- i.  $(a_0, \dots, a_{j-1}) = (0, \dots, 0)$
- ii.  $u_k > 0$  for  $k = j$
- iii.  $u_k \not\equiv 0 \pmod{p}$

Theorem 4.1 shows that for  $-q < u_0 < 0$ , there is a sequence of numerators  $\{u_k\}$  directly related to the sequence of digits  $\{a_k\}$  for  $u_0/q \in \mathbb{Z}_N$ . This provides a more powerful tool for the analysis of the sequences generated by AFSRs. The results shown here fill in the gaps of the incomplete proof shown in Theorem 10 of Klapper and Xu's paper [10].

## 5 Boolean Sequences

The interest of this paper is stream ciphers, and there are a few different ways to use bent functions in the implementation of a stream cipher. Sequences generated using bent functions have nice cryptographic properties because of their perfect nonlinearity. These sequences can be generated multiple ways. Two easy examples are a filtering function on a shift register producing an  $m$ -sequence or a shift register which uses  $n$  different shift registers as input into a bent function. These two techniques are discussed by Carlet [2]. Both of these constructions use input vectors from  $\mathbb{F}_2^n$  in a pseudorandom order to generate the sequence. Before scrambling the input in this way, the sequences generated by lexicographic ordering of input vectors is considered.

**Definition 5.1.** Let  $(a_n)$  be a sequence. If  $T$  is the smallest integer such that  $a_i = a_{i+T}$ , then the *minimal period* of  $(a_n)$  is  $T$ .

**Definition 5.2.** Let  $f \in \mathcal{BF}_n$  and  $v_i \in \mathbb{F}_2^n$  such that  $v_i = B^{-1}(i)$  for  $0 \leq i < 2^n$ . Then,

$$\text{seq}(f) = (f(v_0), f(v_1), \dots, f(v_{2^n-1}), f(v_0), \dots) \quad (23)$$

is a *lexicographical Boolean sequence*.

Defined in this way, all lexicographical Boolean sequences have a minimal period at most  $2^n$ . Using the lexicographic ordering, the Boolean sequence generated will be repeated columns of the outputs for the Boolean function read from the truth table of the Boolean function. For example, the lexicographic Boolean sequence of  $f$  in Table 2.1 is

$$(0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 0, 1, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 1, \dots).$$

**Theorem 5.1.** *The lexicographical Boolean sequence of a Bent function has a period exactly  $2^n$ .*

*Proof.* For  $x_i, \lambda_i \in \mathbb{F}_2$  and  $0 \leq i \leq n-2$ , define  $(x, 1) = (x_0, \dots, x_{n-2}, 1)$ ,  $(x, 0) = (x_0, \dots, x_{n-2}, 0)$ , and  $(\lambda, 1) = (\lambda_0, \dots, \lambda_{n-2}, 1)$ . Suppose  $f \in \mathcal{BF}_n$  and  $\text{seq}(f)$  has a period  $T = 2^j < 2^n$ . Then,  $f(x, 0) = f(x, 1)$ .

$$\begin{aligned} c(\lambda, 1) &= \frac{1}{2^{n/2}} \left( \sum_{x \in \mathbb{F}_2^{n-1}} (-1)^{f(x,0) + (x,0) \cdot (\lambda,1)} + (-1)^{f(x,1) + (x,1) \cdot (\lambda,1)} \right) \\ &= \frac{1}{2^{n/2}} \left( \sum_{x \in \mathbb{F}_2^{n-1}} (-1)^{f(x,0)} \left( (-1)^{(x,0) \cdot (\lambda,1)} + (-1)^{(x,1) \cdot (\lambda,1)} \right) \right) \\ &= \frac{1}{2^{n/2}} \left( \sum_{x \in \mathbb{F}_2^{n-1}} (-1)^{f(x,0)} \left( (-1)^{0 \cdot 1} + (-1)^{1 \cdot 1} \right) \right) \\ &= \frac{1}{2^{n/2}} \left( \sum_{x \in \mathbb{F}_2^{n-1}} (-1)^{f(x,0)} \cdot 0 \right) \\ &= 0. \end{aligned}$$

One of the Fourier coefficients of  $f$  must equal zero. Thus,  $f$  cannot be a Bent function. Clearly, every lexicographical Boolean sequence has a minimal period at most  $2^n$ . Therefore, if  $g$  is a Bent function, then  $\text{seq}(g)$  has period exactly  $2^n$ .  $\square$

Boolean sequences will be considered as the coefficients of the power series used to define 2-adic integers.

**Definition 5.3.** Let  $f \in \mathcal{BF}_n$  and  $v_i \in \mathbb{F}_2^n$  such that  $v_i = B^{-1}(i)$  for  $0 \leq i < 2^n$ . Then,

$$\alpha_f = (f(v_0), f(v_0) + f(v_1) \cdot 2, \dots, f(v_0) + \dots + f(v_i) \cdot 2^i, \dots) \quad (24)$$

where  $\alpha_f \in \mathbb{Z}_2$  is called the 2-adic expansion of  $f$ .

**Lemma 5.1.** *The digit representation of  $\alpha_f$  is  $\text{seq}(f)$ .*

Recall the Maiorana-McFarland class of Boolean functions, and consider the subset of these functions where  $g(y) = 0$ . Then the following theorem is true.

**Theorem 5.2.**  $\log_2(\alpha_{x \cdot \pi(y)}) = 2^{n/2} + 2^{\bar{\pi}(y_0)}$

## 6 Finite State Machines

It is interesting to consider FCSRs in the context of general finite state machines. In 1967, Solomon W. Golomb's *Shift Register Sequences* [?] established a definition of finite state machines and shift registers used in much of the literature today. This section will begin by defining the finite state machine according to Golomb and move toward the definition of generalized shift registers. In the next section, FCSRs will be defined with Golomb's style in mind.

**Definition 6.1.** A *finite state machine* consists of a finite collection of *states*  $K$ , sequentially accepts a sequence of *inputs* from a finite set  $A$ , and produces a sequence of *outputs* from a finite set  $B$ . Moreover, there is an *output function*  $\mu$  which computes the present output as a fixed function of present input and present state, and a *next state function*  $\delta$  which computes the next states as a fixed function of present input and present state. In a more mathematical manner,  $\mu$  and  $\delta$  are defined such that

$$\mu : K \times A \rightarrow B \quad \mu(k_n, a_n) = b_n \quad (25)$$

$$\delta : K \times A \rightarrow K \quad \delta(k_n, a_n) = k_{n+1} \quad (26)$$

Golomb presents two important theorems about these machines. Both deal with the periodicity of any finite state machine, which include FCSRs. The theorems and proofs are presented here.

**Theorem 6.1.** *If the input sequence to a finite state machine is eventually periodic, then the output sequence is eventually periodic.*

*Proof.* Let  $p$  be the period of the inputs once the machine becomes periodic at time  $t$ . Then, for  $h > 0$  and  $c > t$ ,  $a_c = a_{c+hp}$ . Similar to the proof of Theorem ??, using the fact that  $K$  is finite, there must be  $r > s > t$  such that, for some  $h > 0$ ,

$$k_{r+1} = \delta(k_r, a_r) = \delta(k_s, a_{r+hp}) = k_{s+1}.$$

It should also be clear that  $a_{r+i} = a_{r+i+hp}$  for  $h > 0$ . So by induction,  $\forall i > 0$

$$k_{r+i+1} = \delta(k_{r+i}, a_{r+i}) = \delta(k_{s+i}, a_{r+i+hp}) = k_{s+i+1}$$

Finally, this proves  $b_{r+i+1} = b_{s+i+1}$ . Thus, the eventual period of this machine is  $r - s$ .  $\square$

The next object defined is called an  $N$ -ary  $n$ -stage machine. It can be used to represent any finite state machine. It is also a natural generalization of shift registers, so thinking of finite state machines in the context of  $N$ -ary  $n$ -stage machines will make the transition to talking about shift registers much smoother.

**Definition 6.2.** Choose  $n, m, r \in \mathbb{N}$ . An  $N$ -ary  $n$ -stage machine consists of the following:

1.  $D = \{0, \dots, N - 1\}$ . This set contains the  $N$ -ary *digits* of the machine.

2.  $K = \{\sum_{i=0}^n x_i N^i : x_i \in D\}$ . This set contains the  $N$ -ary states of the machine.
3.  $A = \{\sum_{i=0}^m y_i N^i : y_i \in D\}$ . This set contains the  $N$ -ary inputs of the machine.
4.  $B = \{\sum_{i=0}^r z_i N^i : z_i \in D\}$ . This set contains the  $N$ -ary outputs of the machine.
5.  $F = \{f_i(x_0, \dots, x_n, y_0, \dots, y_m) : 0 \leq i < n\}$ . This set contains the  $N$ -ary next state functions of the machine.
6.  $G = \{g_i(x_0, \dots, x_n, y_0, \dots, y_m) : 0 \leq i < r\}$ . This set contains the  $N$ -ary output functions of the machine.

The next state and output are determined from the current state and input by the following equations:

$$x_i^* = f_i(x_0, \dots, x_n, y_0, \dots, y_m) \quad 0 \leq i < n \quad (27)$$

$$z_i = g_i(x_0, \dots, x_n, y_0, \dots, y_m) \quad 0 \leq i < r \quad (28)$$

## 7 Feedback with Carry Shift Registers

A *feedback with carry shift register* is a feedback shift register which uses a linear combination each state to a description of  $N$ -ary feedback with carry shift registers is defined here. The definition of the register follows the one given in Andrew Klapper's book [6].

**Definition 7.1.** Let  $q_0, q_1, \dots, q_m \in \mathbb{Z}/(p)$  for  $p \in \mathbb{Z}$  and assume that  $q_0 \not\equiv 0 \pmod{p}$ . An *algebraic feedback shift register* (or *AFSR*) over  $(\mathbb{Z}, p, S)$  of length  $m$  with *multipliers* or *taps*  $q_0, q_1, \dots, q_m$  is a discrete state machine whose states are collections

$$(a_0, a_1, \dots, a_{m-1}; z) \text{ where } a_i \in S \text{ and } z \in \mathbb{Z}$$

consisting of cell contents  $a_i$  and memory  $z$ . The state changes according to the following rules:

1. Compute

$$\sigma = \sum_{i=1}^m q_i a_{m-i} + z.$$

2. Find  $a_m \in S$  such that  $-q_0 a_m \equiv \sigma \pmod{p}$ . That is  $a_m \equiv -q_0^{-1} \sigma \pmod{p}$ .
3. Replace  $(a_0, \dots, a_{m-1})$  by  $(a_1, \dots, a_m)$  and replace  $z$  by  $\sigma(\text{div } p) = (\sigma + q_0 a_m)/p$ .



## 8 Xu's Rational Approximation Algorithm

In Andrew Klapper's book, he roughly describes Xu's rational approximation algorithm for  $\pi$ -adic sequences in any ring  $R$ . A description of Xu's Algorithm is presented here in the context of the ring  $\mathbb{Z}_p$ , where  $p$  does not have to be prime.

The rational approximation problem is presented here: Given the eventually periodic digit representation of  $\frac{a}{b} \in \mathbb{Z}_p$ , find  $a$  and  $b$ .

If there are no constraints on  $a$  and  $b$ , then the entire sequence must be known to solve the problem. Specifically, a single algebraic feedback shift register can not produce all  $\frac{a}{b} \in \mathbb{Z}_p$ .

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