Reservoir Dynamical System Analysis

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1 General 3D Reservoir System

The reservoir computer dynamical system is

$$\dot{\mathbf{r}}(t) = \gamma\big[-\mathbf{r}(t) + \tanh\big(A\mathbf{r}(t) + W_{\text{in}}\mathbf{u}(t)\big)\big]$$

We will simplify the system to a three dimensional system. Write

$$\mathbf{r}(t) = \begin{bmatrix} x(t) & y(t) & z(t) \end{bmatrix}^T$$

and

$$W_{\text{in}}\mathbf{u}(t) = \begin{bmatrix} a(t) & b(t) & c(t) \end{bmatrix}^T$$

Then the entire system is:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \gamma \begin{bmatrix} -x(t) + \tanh\Big(a_{11}x(t) + a_{12}y(t) + a13z(t) + a(t)\Big) \\ -y(t) + \tanh\Big(a_{21}x(t) + a_{22}y(t) + a23z(t) + b(t)\Big) \\ -z(t) + \tanh\Big(a_{31}x(t) + a_{32}y(t) + a33z(t) + c(t)\Big) \end{bmatrix}$$

Let's investigate a few different adjacency matrices:

1.1 Single Path

Let

$$A = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then our system simplifies to:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \gamma \begin{bmatrix} -x(t) + \tanh\Big(a(t)\Big) \\ -y(t) + \tanh\Big(x(t) + b(t)\Big) \\ -z(t) + \tanh\Big(y(t) + c(t)\Big) \end{bmatrix}$$

We solve for the location of the fixed point:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \gamma \begin{bmatrix} -x(t) + \tanh\Big(a(t)\Big) \\ -y(t) + \tanh\Big(x(t) + b(t)\Big) \\ -z(t) + \tanh\Big(y(t) + c(t)\Big) \end{bmatrix}$$

$$\begin{bmatrix} x^\star(t) \\ y^\star(t) \\ z^\star(t) \end{bmatrix} = \begin{bmatrix} \tanh\Big(a(t)\Big) \\ \tanh\Big(x^\star(t) + b(t)\Big) \\ \tanh\Big(y^\star(t) + c(t)\Big) \end{bmatrix}$$

Then, substitute in the solution for $x^\star$ into the equation for $y^\star$ and the solution for $y^\star$ into $z^\star$ to obtain:

$$\begin{aligned} x^\star &= \tanh\big(a(t)\big) \\ y^\star &= \tanh\big(\tanh\big(a(t)\big) + b(t)\big) \\ z^\star &= \tanh\big(\tanh\big(\tanh\big(a(t)\big) + b(t)\big) + c(t)\big) \end{aligned}$$

The interpretation of this solution is that the location of the fixed point depends only on the input signals. The first reservoir node state $x(t)$ responds only to the input driving it directly $a(t)$. The second reservoir node state $y(t)$ responds to the input driving it directly, $b(t)$ as well as to $\tanh(a(t))$, the transformed input to $x(t)$. This means that input from $a(t)$ will have an impact on the location of $y^\star$, but it will mapped to the interval $(-1, 1)$. Therefore, if $b(t)$ is a lot larger in magnitude, $a(t)$ be overpowered by $b(t)$.

1.2 Three Node Loop

Let

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Then our system simplifies to:

$$\begin{bmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{bmatrix} = \gamma \begin{bmatrix} -x(t) + \tanh\Big(z(t) + a(t)\Big) \\ -y(t) + \tanh\Big(x(t) + b(t)\Big) \\ -z(t) + \tanh\Big(y(t) + c(t)\Big) \end{bmatrix}$$

We solve for the location of the fixed point:

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \gamma \begin{bmatrix} -x(t) + \tanh\Big(z(t) + a(t)\Big) \\ -y(t) + \tanh\Big(x(t) + b(t)\Big) \\ -z(t) + \tanh\Big(y(t) + c(t)\Big) \end{bmatrix}$$

$$\begin{bmatrix} x^\star \\ y^\star \\ z^\star \end{bmatrix} = \begin{bmatrix} \tanh\Big(z^\star + a(t)\Big) \\ \tanh\Big(x^\star + b(t)\Big) \\ \tanh\Big(y^\star + c(t)\Big) \end{bmatrix}$$

Then, substitute in the solution for $x^\star$ into the equation for $y^\star$ and the solution for $y^\star$ into $z^\star$ to obtain:

$$x^\star = \tanh\Bigg(\tanh\Bigg(\tanh\Big(x^\star + b(t)\Big) + c(t)\Bigg) + a(t)\Bigg)$$

$$y^\star = \tanh\big(\tanh\Big(\tanh\Big(y^\star + c(t)\Big) + a(t)\Big) + b(t)\big)$$

$$z^\star = \tanh\big(\tanh\Big(\tanh\Big(z^\star + a(t)\Big) + b(t)\Big) + c(t)\big)$$

References

[1] Tanaka, Gouhei, et. al. "Recent advances in physical reservoir computing: A review", *Neural Networks*, vol. 115. (2019).