Definition Issues

The definitions in your paper are definitely more elegant than mine, and make the theorems a lot nicer. However, the issue I'm running into is the one I've been having a difficult time with for months. It is this:

The main result I need from my lemma is that

$$X_{ij} = \sum_{\alpha \in In(S_j, S_i)} P(\alpha).$$

In other words, the sub diagonal entries of $\rho I - G)^{-1}$ are the sum of centrality transfer matrices from strongly connected component S_j to S_i.

This is a way of explaining how centrality moves from component to component and in my main result, I will work backward through the components to the base and connect invariant base centralities to the centralities in the components.

A subtle detail is that later, I need alpha to be a branch from the **base** to a strongly connected component, so $In(.\,,.)$ needs to work from base to s.c. component as well as s.c. component to s.c. component.

To do this I need a definition of an incoming branch that works for incoming branches between strongly connected components and incoming branches from the base to a strongly connected component.

It's really tricky and I haven't figured out how to do it perfectly. The way I was thinking of doing it was by using a really general definition of a component branch that could begin at a subgraph and end at a component (or at the same subgraph). Then I could define centrality transfer matrixes for both base to component (because the base is a subgraph) and component to component.

I liked your way of connecting the definition to component branches, and I thought that maybe we could make a definition for *component subbranches* , that could be any sub sequence of a component branch that begins and ends with a subgraph.

The problem with that idea is that my lemma is not connected to specialization, and doesn't include the notion of a base, so component branches don't seem to fit in. I could also drop centrality transfer matrixes from the proof, but I'm worried that it would get a lot messier than it already is.

I included my definitions below so that you can see what I tried to do.

Definition (Incoming branch)

Given a graph $G = (V, E, \omega)$, a set $B \subset V$ and a strongly connected component Z of $G|_{\overline{B}}$, let $S_1, S_2 \cdots S_k$ be strongly connected components of $G|_{\overline{B}}$. Let H represent the subgraph $G|_B$.

If there exist edges $e_0 \cdots e_m$ such that,

(i) e_0 is an edge from H to S_1

(ii) e_j is an edge from S_j to S_{j+1} for $1 \leq j \leq k-1$

(iii) e_k is an edge from S_k to Z

The we call the ordered set $\alpha = \{H,\ e_0,\ S_1,\ e_1,\ S_2, ..., S_k,\ e_k,\ Z\}$ an incoming branch from B to Z. We let $In(H, Z)$ denote the set of all incoming branches from H to Z in G.

Definition (Subgraph generated by an ingoing branch)

Let $G = (V, E, \omega)$ with $B \subset V$ and Z a strongly connected component of $G|_{\overline{B}}$. If $\alpha = \{B,\ e_0,\ S_1,\ e_1,\ S_2, ..., S_k,\ e_k,\ Z\} \in In(B, Z)$, then the subgraph generated by $alpha$ is the graph consisting of all all strongly connected components and edges in α together with $G|_B$

Definition (Centrality Transfer Matrix)

Let $G = (V, E, \omega)$ have a centrality vector and associated spectral radius ρ, let $B \subset V$ and let Z be a strongly connected component of $G|_B$. If $\alpha = \{B,\ e_0,$ $S_1,\ e1,\ S_2, ..., S_k,\ e_k,\ Z\ \}$ is an incoming path from B to Z then

$$P(\alpha) = (\rho I - Z)^{-1} Y_k (\rho I - S_k)^{-1} Y_{k-1} \cdots (\rho I - S_1)^{-1} Y_0 (\rho I - B)^{-1}$$

is a centrality transfer matrix of α, if

$$\begin{bmatrix} B & & & & & \\ Y_0 & S_1 & & & & \\ & Y_1 & S_2 & & & \\ & & & \ddots & & \\ & & & Y_{k-1} & S_k & \\ & & & & Y_k & Z \end{bmatrix}$$

is an adjacency matrix for the subgraph generated by α.

Lemma

Assume G $= (V, E, \omega)$ is not strongly connected. Let If $\rho > \max\{|\lambda| : \lambda \in \sigma(G)\}$ then,

$$(\rho I - A)^{-1} = \begin{bmatrix} (\rho I - S_1)^{-1} & & & & \\ X_{21} & (\rho I - S_2)^{-1} & & & \\ X_{31} & X_{32} & (\rho I - S_3)^{-1} & & \\ \vdots & \vdots & \ddots & \ddots & \\ X_{k1} & X_{k2} & & X_{kk-1} & (\rho I - S_k)^{-1} \end{bmatrix}$$

where A is an adjacency matrix of G, S_1, S_2,...,S_k are all adjacency matrices of the strongly connected components of G, and each

$$X_{ij} = \sum_{\alpha \in In(S_j, S_i)} P(\alpha).$$