Centrality and Specialization

I read through the changes you made and they seem great.

Dallas does use A_{ij} is an edge from v_i to v_j so I'll need to change my proofs.

I looked at the section on dynamic networks and it looks like you were using the A_{ij} is an edge from v_j to v_i, specifically after definition 4.2 in the equation:

$$F_i(\mathbf{x}) = \sum_{j=1}^{n} A_{ij} f_{ij}(x_j), \quad \text{for} \quad i \in N = \{1, 2, \ldots, n\} \tag{1}$$

To me, it looks like you are using A_{ij} to weight the effect x_j has on x_i. Thus it seems to represent the edge weight from v_j to v_i.

This is important because I'm concerned that I need to define eigenvectors for the transpose of the adjacency matrix. When we define the adjacency matrix A the way Dallas did, the dynamics should technically be $F(\mathbf{x}) = A^T\mathbf{x}$ (for example), then the eigenvectors that really matter are the eigenvectors of A^T and my results should be about those. (Hope that made sense...)

Also, I think that we do get a little more than just centrality for free, so why don't we go ahead an include a lambda in the eigenvector transfer matrix to make it more general. Maybe $T_\lambda(\beta, Z)$. I'll think more carefully about this as well as the result for weighted graphs as I rewrite my proof.

I've put together a plan for using the updated definitions to prove my results in the appendix. Let me know what you think.

Theorem

(Eigenvectors of Specialized Graphs) Let $G = (V, E, \omega)$ be a graph and $B \subseteq V$ a base.
(i) If $(\lambda, \mathbf{x})$ is an eigenpair of the graph G and $\lambda \notin \sigma(G|_{\bar{B}})$ then there is an eigenpair $(\lambda, \mathbf{y})$ of $\mathcal{S}_B(G)$ such that $\mathbf{x}_B = \mathbf{y}_B$.
Furthermore, suppose G is strongly connected with positive edge weights and let $\mathbf{u}$ be a leading eigenvector of G. Also, let Z be a strongly connected component of $\beta \in \mathcal{B}_S(G)$. Then there is a leading eigenvector $\mathbf{v}$ of $\mathcal{S}_B(G)$ such that the following hold.
(ii) For all $Z_i \in \mathcal{C}(Z)$ the eigenvector restriction

$$\mathbf{v}_{Z_i} = T(\beta, Z)\mathbf{v}_B.$$

Hence, if $Z_i, Z_j \in \mathcal{C}(Z)$ have the same incoming branch then $\mathbf{v}_{Z_i} = \mathbf{v}_{Z_j}$.
(iii) For $Z_i \in \mathcal{C}(Z)$ let $\cup_{k=1}^{\ell}\{Z_k\}$ be the copies of Z that have the same outgoing

branch as Z_i. Then

$$\mathbf{u}_Z = \sum_{k=1}^{\ell} \mathbf{v}_{Z_k} = \sum_{k=1}^{\ell} T(\beta, Z)\mathbf{v}_B.$$

Proof. (ii) with the new definition this will be really easy

(iii) I'll need the lemma below. I've reworked it to fit with the new definitions □

Definition

Let S_i, S_j be strongly connected components of a graph G. Let S be the set of vertices containing in S_i and S_j. We let

$$B_{in}(S_i\,, S_j) = \{In(\beta, S_j)\,|\,\beta \in \mathcal{B}_{S_i}(G)\}$$

Lemma

Assume $G = (V, E, \omega)$ is not strongly connected and let $\lambda \notin \sigma(G)$. If S_1, S_2,...,S_k are the strongly connected components of G, then there exists an adjacency matrix A for G such that

$$(\rho I - A)^{-1} = \begin{bmatrix} X_{11} & X_{12} & \cdots & X_{1k} \\ & X_{22} & \cdots & X_{2k} \\ & & \ddots & \vdots \\ X_{kk} & & & \end{bmatrix}$$

Where

$$X_{ij} = \sum_{\beta \in B_{in}(S_j, S_i)} T(\beta, S_i)(\rho I - S_j)^{-1}$$

if $i < j$ and

$$X_{ij} = (\rho I - S_j)^{-1}$$

if $i = j$