Eigenvector Transfer proof

Theorem

(Eigenvectors of Specialized Graphs) Let $G = (V, E, \omega)$ be a graph and $B \subseteq V$ a base.
(i) If $(\lambda, \mathbf{x})$ is an eigenpair of the graph G and $\lambda \notin \sigma(G|_{\bar{B}})$ then there is an eigenpair $(\lambda, \mathbf{y})$ of $\mathcal{S}_B(G)$ such that $\mathbf{x}_B = \mathbf{y}_B$.
Furthermore, suppose G is strongly connected with positive edge weights and let $\mathbf{u}$ be a leading eigenvector of G. Also, let Z be a strongly connected component of $\beta \in \mathcal{B}_S(G)$. Then there is a leading eigenvector $\mathbf{v}$ of $\mathcal{S}_B(G)$ such that the following hold.
(ii) For all $Z_i \in \mathcal{C}(Z)$ the eigenvector restriction

$$\mathbf{v}_{Z_i} = T(\beta, Z)\mathbf{v}_B.$$

Hence, if $Z_i, Z_j \in \mathcal{C}(Z)$ have the same incoming branch then $\mathbf{v}_{Z_i} = \mathbf{v}_{Z_j}$.
(iii) For $Z_i \in \mathcal{C}(Z)$ let $\cup_{k=1}^{\ell}\{Z_k\}$ be the copies of Z that have the same outgoing branch as Z_i. Then

$$\mathbf{u}_Z = \sum_{k=1}^{\ell} \mathbf{v}_{Z_k} = \sum_{k=1}^{\ell} T(\beta, Z)\mathbf{v}_B.$$

Proof of (ii)

Let Z be a strongly connected component of $G|_{\overline{B}}$ and let Z_i be a copy of Z in $\mathcal{S}_B(G)$. Then there exists a component branch β corresponding to Z_i, of the form $\beta = \{v_j, e_0, C_1, ..., C_k, e_k, Z, C_{k+1}, ..., C_n\}$. Then, because specialization isolates each component branch, we can write the adjacency matrix A of $\mathcal{S}_B(G)$ in the form,

$$A = \begin{bmatrix} \mathcal{A}(In(\beta, Z)) & W \\ Y & X \end{bmatrix} = \begin{bmatrix} \underline{B} & & & & & & W \\ Y_0 & \underline{C_1} & & & & & \\ & Y_1 & \underline{C_2} & & & & \\ & & \ddots & \ddots & & & \\ & & & Y_{k-1} & \underline{C_k} & & \\ & & & & Y_k & \underline{Z_i} & \\ & & & & & Y_{k+1} & X \end{bmatrix}.$$

Here, $\underline{B} = \mathcal{A}(G|_B)$, $\underline{Z_i} = \mathcal{A}(Z_i)$ and for $1 \leq j \leq k$, $\underline{C_j} = \mathcal{A}(C_j)$. For $0 \leq j \leq k+1$, the matrix Y_j is a single entry matrix corresponding to the edge $e_j \in \beta$.

Suppose $(\lambda, \mathbf{v})$ is an eigenpair of G, i.e. $A\mathbf{v} = \lambda\mathbf{v}$, such that $\lambda \notin \sigma(G|_{\bar{B}})$. As $\sigma(G|_{\bar{B}}) = \cup_{j=1}^{m}\sigma(C_i)$ where $C_1, C_2, \ldots, C_m$ are the strongly connected components of $G|_{\bar{B}}$ then λ is not an eigenvalue of any strongly connected component of

β. We may conformally partition $\mathbf{v}$ into $\mathbf{v} = [\mathbf{v}_B, \mathbf{v}_{C_1}, \cdots \mathbf{v}_{C_k}, \mathbf{v}_{Z_i}, \mathbf{v}_X]^T$ so that entries in each piece correspond with the appropriate sub-matrix of A. Then, applying the eigenvector equation produces,

$$A\mathbf{v} = \begin{bmatrix} \underline{B} & & & & & & W \\ Y_0 & \underline{C_1} & & & & & \\ & Y_1 & \underline{C_2} & & & & \\ & & \ddots & \ddots & & & \\ & & & Y_{k-1} & \underline{C_k} & & \\ & & & & Y_k & \underline{Z_i} & \\ & & & & & Y_{k+1} & X \end{bmatrix} \begin{bmatrix} \mathbf{v}_B \\ \mathbf{v}_{C_1} \\ \mathbf{v}_{C_2} \\ \vdots \\ \mathbf{v}_{C_k} \\ \mathbf{v}_{Z_i} \\ \mathbf{v}_X \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{v}_B \\ \mathbf{v}_{C_1} \\ \mathbf{v}_{C_2} \\ \vdots \\ \mathbf{v}_{C_k} \\ \mathbf{v}_{Z_i} \\ \mathbf{v}_X \end{bmatrix} = \lambda\mathbf{v}.$$

We solve for $\mathbf{v}_Z$ in terms of $\mathbf{v}_B$. Multiplying $\mathbf{v}$ by the appropriate block row of A gives $Y_k\mathbf{v}_{C_k} + \underline{Z_i}$, implying that

$$\mathbf{v}_{Z_i} = \lambda\mathbf{v}_{Z_i}.$$
$$\mathbf{v}_{Z_i} = (\lambda I - \underline{Z_i})^{-1} Y_k \mathbf{v}_{C_k} \tag{1}$$

In a similar manner for $2 \leq i \leq k$, we may solve for $\mathbf{v}_{C_i}$ in terms of $\mathbf{v}_{C_{i-1}}$ producing,

$$\mathbf{v}_{C_i} = (\lambda I - \underline{C_i})^{-1} Y_{i-1} \mathbf{v}_{C_{i-1}}.$$

Combining this with equation **(6)** gives,

$$\mathbf{v}_{Z_i} = (\lambda I - \underline{Z_i})^{-1} Y_k (\lambda I - \underline{C_k})^{-1} Y_{k-1} \cdots Y_2 (\lambda I - \underline{C_2})^{-1} Y_1 \mathbf{v}_{C_1}.$$

We solve for $\mathbf{v}_{C_1}$ in a similar manner and find that

$$\mathbf{v}_{C_1} = (\lambda I - \underline{C_1})^{-1} Y_0 \mathbf{v}_B.$$

Then,

$$\mathbf{v}_{Z_i} = (\lambda I - \underline{Z_i})^{-1} Y_k (\lambda I - \underline{C_k})^{-1} Y_{k-1} \cdots Y_1 (\lambda I - \underline{C_1})^{-1} Y_0 \mathbf{v}_B.$$

Since Y_0,...,Y_k, C_1 ... C_k and Z_i are all the appropriate submatrixes of $\mathcal{A}(\ In(\beta, Z_i)\ )$, by definition of the eigenvector transfer matrix,

$$\mathbf{v}_{Z_i} = (\lambda I - \underline{Z_i})^{-1} Y_k (\lambda I - \underline{C_k})^{-1} Y_{k-1} \cdots Y_1 (\lambda I - \underline{C_1})^{-1} Y_0 \mathbf{v}_B = T(\beta, Z_i, \lambda)\mathbf{v}_B.$$

Note that each inverse in this equation exists since $\lambda \notin \sigma(Z_i)$ and $\lambda \notin \sigma(C_j)$ for $j = 1, 2, \ldots, k$.

Definition (Partial component branch)

Given a graph $H = (V, E, \omega)$ that is not strongly connected, let $S = \{S_1, S_2, ..., S_k\}$ be the strongly connected components of H.

If there exist edges $e_1 \cdots e_{k-1}$ such that, e_j is an edge from S_j to S_{j+1} for $1 \le j \le k-1$, we call the ordered set $\alpha = \{S_1, e_1, S_2, ..., S_k\}$ a partial component branch from S_1 to S_k in H.

We let $\mathcal{P}(S_1, S_k)$ denote the set of all partial component branches from S_1 to S_k in H. We furthermore define $\mathcal{P}(S_i, S_i)$ to be the set containing only $\alpha = \{S_i\}$.

Definition (Partial Eigenvector Transfer Matrix)

Let $H = (V, E, \omega)$ be a graph that is not strongly connected and let S and T be strongly connected components of H. For $\alpha = \{S, e_0, C_1, e_1, ...C_m, e_m T\} \in \mathcal{P}(S, T)$, let the adjacency matrix of α be

$$\begin{bmatrix} \underline{S} & & & & \\ Y_0 & \underline{C_1} & & & \\ & Y_1 & \underline{C_2} & & \\ & & \ddots & \ddots & \\ & & & Y_m & \underline{T} \end{bmatrix}$$

Where $\underline{S} = \mathcal{A}(S)$, $\underline{T} = \mathcal{A}(T)$ and $\underline{C_i} = \mathcal{A}(C_i)$ for $1 \le i \le m$. We call the matrix

$$P(\alpha, \lambda) = (\lambda I - T)^{-1} Y_m (\lambda I - C_m)^{-1} Y_{m-1} \cdots Y_m (\lambda I - T)^{-1}$$

the partial eigenvector transfer matrix of α, where λ is a spectral parameter.

Lemma

Assume G $= (V, E, \omega)$ is not strongly connected with strongly connected components S_1, S_2,...,S_k and

$$A = \mathcal{A}(G) = \begin{bmatrix} \underline{S_1} & & & \\ Y_{21} & \underline{S_2} & & \\ \vdots & & \ddots & \\ Y_{k1} & \dots & Y_{kk-1} & \underline{S_k} \end{bmatrix}$$

where $\underline{S_i} = \mathcal{A}(S_i)$ for $1 \le i \le k$. If $\lambda \notin \sigma(G)$ then,

$$(\lambda I - A)^{-1} = \begin{bmatrix} X_{11} & & & \\ X_{21} & X_{22} & & \\ \vdots & \vdots & \ddots & \\ X_{kl} & X_{k2} & \cdots & X_{kk} \end{bmatrix}$$

where

$$X_{ij} = \sum_{\alpha \in \mathcal{P}(S_j, S_i)} P(\alpha, \lambda).$$

Proof

Since A is block lower triangular, $(\rho I - A)$ and $(\rho I - A)^{-1}$ are also block lower triangular. Since we can write,

$$(\rho I - A) = \begin{bmatrix} (\rho I - S_1) & & & \\ -Y_{21} & (\rho I - S_2) & & \\ \vdots & & \ddots & \\ -Y_{k1} & \dots & -Y_{kk-1} & (\rho I - S_k) \end{bmatrix}$$

we can write $(\rho I - A)^{-1}$ as,

$$(\rho I - A)^{-1} = \begin{bmatrix} X_{11} & & & \\ X_{21} & X_{22} & & \\ \vdots & \vdots & \ddots & \\ X_{kl} & X_{k2} & \cdots & X_{kk} \end{bmatrix}$$

where X_{ii} has the same dimensions as S_i for $1 \leq i \leq k$ and X_{ij} has the same dimensions as Y_{ij} when $1 \leq j < i \leq k$. Then it must be the case that,

$$\begin{bmatrix} (\rho I - S_1) & & & \\ -Y_{21} & (\rho I - S_2) & & \\ \vdots & & \ddots & \\ -Y_{k1} & \dots & -Y_{kk-1} & (\rho I - S_k) \end{bmatrix} \begin{bmatrix} X_{11} & & & \\ X_{21} & X_{22} & & \\ \vdots & \vdots & \ddots & \\ X_{kl} & X_{k2} & \cdots & X_{kk} \end{bmatrix} = \begin{bmatrix} I_1 & & & \\ & I_2 & & \\ & & \ddots & \\ & & & I_k \end{bmatrix}$$

Where each I_i is the identity matrix with the same dimensions as S_i. Let $j \in \{1, 2, ...,k\}$. Then,

$$\begin{bmatrix} (\rho I - S_1) & & & \\ -Y_{21} & (\rho I - S_2) & & \\ \vdots & & \ddots & \\ -Y_{k1} & \dots & -Y_{kk-1} & (\rho I - S_k) \end{bmatrix} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ X_{jj} \\ X_{(j+1)j} \\ \vdots \\ X_{kj} \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ I_j \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

Multiplying the jth row of $(\rho I - A)$ by the jth column of $(\rho I - A)^{-1}$ produces

$$(\rho I - S_j)X_{jj} = I_j$$

$$X_{jj} = (\rho I - S_j)^{-1} = P(\{S_j\}, \lambda) \tag{2}$$

Since $\mathcal{P}(S_j, S_j)$ only contains $\{S_j\}$ by definition, it follows that,

$$X_{jj} = \sum_{\alpha \in \mathcal{P}(S_j, S_j)} P(\alpha, \lambda)$$

We will show by induction that

$$X_{ij} = \sum_{\alpha \in \mathcal{P}(S_j, S_i)} P(\alpha, \lambda)$$

when $i > j$. As a base case, consider $X_{(j+1)j}$. By multiplying row $(j+1)$ of $(\rho I - A)$ by the jth column of $(\rho I - A)^{-1}$, we obtain the equations:

$$-Y_{(j+1)j}(\rho I - S_j)^{-1} + (\rho I - S_{(j+1)})X_{(j+1)j} = 0$$

$$X_{(j+1)j} = (\rho I - S_{(j+1)})^{-1} Y_{(j+1)j} (\rho I - S_j)^{-1} \tag{3}$$

Let m be the number of nonzero entries in $Y_{(j+1)j}$ then we can write,

$$Y_{(j+1)j} = \sum_{k=1}^{m} Y_{(j+1)j}^{(k)}$$

where each $Y_{(j+1)j}^{(k)}$ has one non zero entry and that entry is equal to a non-zero entry of $Y_{(j+1)j}$. Thus,

$$X_{(j+1)j} = (\rho I - S_{j+1})^{-1} \sum_{k=1}^{m} Y_{(j+1)j}^{(k)} (\rho I - S_j)^{-1}$$

$$X_{(j+1)j} = \sum_{k=1}^{m} (\rho I - S_{j+1})^{-1} Y_{(j+1)j}^{(k)} (\rho I - S_j)^{-1}$$

For each k, the matrix

$$\begin{bmatrix} S_j & \\ Y_{(j+1)j}^{(k)} & S_{j+1} \end{bmatrix}$$

is an adjacency matrix for $\alpha^{(k)} = \{S_j, e^{(k)}, S_{j+1}\}$ where $e^{(k)}$ is an edge from S_j to S_{j+1}. Then $\alpha^{(k)} \in \mathcal{P}(S_j, S_{j+1})$ and

$$X_{(j+1)j} = \sum_{k=1}^{m} P(\alpha^{(k)})$$

Clearly $\cup_{k=1}^{n}\{\alpha^{(k)}\} \subset \mathcal{P}(S_j, S_{j+1})$. We assert that $\cup_{k=1}^{n}\{\alpha^{(k)}\} = \mathcal{P}(S_j, S_{j+1})$. Let $\alpha \in \mathcal{P}(S_j, S_{j+1})$. Then $\alpha = \{S_j, e, S_{j+1}\}$ because if α contained any other strongly connected component S_l, it would imply that a path exists from S_j to S_l to S_{j+1} and because of the structure of A, it must be the case that $l < j$ or $j+1 < l$. If an edge existed from S_j to S_l to S_{j+1} the matrix A would have an entry above the diagonal. This is a contradiction. Thus,

$$X_{(j+1)j} = \sum_{\alpha \in In(S_j, S_{j+1})} P(\alpha)$$

By induction hypothesis assume that when $i < n$,

$$X_{(j+i)j} = \sum_{\alpha \in In(S_j, S_{j+i})} P(\alpha).$$

Consider $X_{(j+n)j}$. By multiplying the $j+n$th row of $(\rho I - A)$ by the jth column of $(\rho I - A)^{-1}$ we obtain the equations,

$$-Y_{(j+n)j}(\rho I - S_j)^{-1} - Y_{(j+n)(j+1)}X_{(j+1)j} \cdots - Y_{(j+n)(j+n-1)}X_{(j+n-1)j} + (\rho I - S_{j+n})X_{(j+n)j} = 0$$

$$X_{(j+n)j} = (\rho I - S_{j+n})^{-1}Y_{(j+n)j}(\rho I - S_j)^{-1} + \sum_{i=1}^{n-1}(\rho I - S_{j+n})^{-1}Y_{(j+n)(j+i)}X_{(j+i)j} \tag{4}$$

As shown in the base case, the first term can be broken up into a sum of partial eigenvector transfer matrices . Let m_0 be the number of non-zero entries in $Y_{(j+n)j}$. If we define $Y^{(k)}_{(j+n)j}$ so that each $Y^{(k)}_{(j+1)j}$ has a single nonzero entry that is equal to a distinct non-zero entry of $Y_{(j+1)j}$ and

$$Y_{(j+n)j} = \sum_{k=1}^{m_0} Y^{(k)}_{(j+n)j}$$

then,

$$(\rho I - S_{j+n})^{-1}Y_{(j+n)j}(\rho I - S_j)^{-1} = \sum_{k=1}^{m_0}(\rho I - S_{j+n})^{-1}Y^{(k)}_{(j+n)j}(\rho I - S_j)^{-1}.$$

For each $1 \leq k \leq m_0$, $(\rho I - S_{j+n})^{-1}Y^{(k)}_{(j+n)j}(\rho I - S_j)^{-1}$ is the partial eigenvector transfer matrix for a distinct partial component branch α in $\mathcal{P}(S_{j+n}, S_j)$ that does not contain any strongly connected components except S_{j+n} and S_j. Let D_0 denote the set of all such branches. By definition of an adjacency matrix, D_0 contains one branch for each non zero entry in $Y_{(j+n)j}$. Thus,

$$(\rho I - S_{j+n})^{-1}Y_{(j+n)j}(\rho I - S_j)^{-1} = \sum_{\beta \in D_0} P(\beta, \lambda) \tag{5}$$

We consider the other terms in the sum (3). Let $1 \leq i \leq n-1$. By the induction hypothesis,

$$(\rho I - S_{j+n})^{-1} Y_{(j+n)(j+i)} X_{(j+i)j} = \sum_{\alpha \in \mathcal{P}(S_j, S_{j+i})} (\rho I - S_{j+n})^{-1} Y_{(j+n)(j+i)} P(\alpha, \lambda)$$

Let m_i represent the number of nonzero entries in $Y_{(j+n)j+i}$. As before we write $Y_{(j+n)j+i}$ as a sum of m_i single entry matrices. $Y_{(j+n)j+i} = \sum_{k=1}^{m_i} Y^{(k)}_{(j+n)(j+i)}$. Then,

$$(\rho I - S_{j+n})^{-1} Y_{(j+n)(j+i)} X_{(j+i)j} = \sum_{\alpha \in \mathcal{P}(S_j, S_{j+i})} \sum_{k=1}^{m_i} (\rho I - S_{j+n})^{-1} Y^{(k)}_{(j+n)(j+i)} P(\alpha, \lambda)$$

It is clear that for each $\alpha \in \mathcal{P}(S_j, S_{j+i})$ and k, the term

$$(\rho I - S_{j+n})^{-1} Y^{(k)}_{(j+n)(j+i)} P(\alpha, \lambda)$$

is a partial eigenvector transfer matrix for some incoming branch $\gamma \in \mathcal{P}(S_j, S_j + n)$, because the matrix $Y^{(k)}_{(j+n)(j+i)}$ is non zero if and only if an edge exists from S_{j+i} to S_{j+n}. If $\mathcal{P}(S_j, S_{j+i})$ is non empty, there is a branch from S_j to S_{j+i}, implying that there must be a branch from S_j to S_{j+n} with partial eigenvector transfer matrix $(\rho I - S_{j+n})^{-1} Y^{(k)}_{(j+n)(j+i)} P(\alpha)$.

What we see here is that for a given i, the term

$$(\rho I - S_{j+n})^{-1} Y_{(j+n)(j+i)} X_{(j+i)j}$$

is equal to the sum of all centrality transfer matrices for the branches in $\mathcal{P}(S_j, S_{j+n})$ that pass through S_{j+i} immediately before reaching S_{j+n}. Let D_i denote the set of all such branches. Then,

$$(\rho I - S_{j+n})^{-1} Y_{(j+n)(j+i)} X_{(j+i)j} = \sum_{\gamma \in D_i} P(\gamma, \lambda) \tag{6}$$

Putting (3), (4), and (5) together gives,

$$X_{(j+n)j} = \sum_{\beta \in D_0} P(\beta, \lambda) + \sum_{i=1}^{n-1} \sum_{\gamma \in D_i} P(\gamma, \lambda)$$

Let $D = \cup_{i=0}^{n-1} D_i$. Then

$$X_{(j+n)j} = \sum_{\alpha \in D} P(\alpha, \lambda)$$

Clearly, $D \subset \mathcal{P}(S_j, S_{j+n})$. We assert that $D = \mathcal{P}(S_j, S_{j+n})$. Let $\alpha \in \mathcal{P}(S_j, S_{j+n})$. If α has only two components, then $\alpha = \{S_j, e, S_{j+n}\}$ for some edge e and $\alpha \in D_0 \subset D$ by definition of D_0. If α has more then two components, then it has a second to last component, S_{j+i} where $1 \leq i \leq n-1$. By definition of D_i, $\alpha \in D_i$. Thus,

$$X_{j+n,j} = \sum_{\alpha \in \mathcal{P}(S_j, S_{j+n})} P(\alpha, \lambda)$$

This concludes the proof.

(iii) For $Z_i \in \mathcal{C}(Z)$ let $\cup_{k=1}^{\ell}\{Z_k\}$ be the copies of Z that have the same outgoing branch as Z_i. Then

$$\mathbf{u}_Z = \sum_{k=1}^{\ell} \mathbf{v}_{Z_k} = \sum_{k=1}^{\ell} T(\beta, Z)\mathbf{v}_B.$$

Proof of (iii)

Since each Z_k is a copy of Z in $\mathcal{S}_B(G)$, each Z_k must correspond to a unique component branch. Let $D = \{\beta_1, ..., \beta_\ell\}$ be the set of such branches indexed so that β_k is the unique branch corresponding to Z_k for $1 \leq k \leq \ell$. Because each Z_k has the same outgoing branch, it follows that $In(\beta_j, Z) \neq In(\beta_k, Z)$ when $j \neq k$. Otherwise, there would exist $k \neq j$ such that $\beta_k = \beta_j$ which contradicts uniqueness of each β_k.

If $In(\beta_k, Z) = \{v_i, e_0, C_1, e_1, ..., e_{n-1}, C_m, e_m, Z\}$ define $F_k = \{C_1, C_2, ...C_m\}$ to be the set of all strongly connected components of $G|_{\overline{B}}$ in $In(\beta_k, Z)$. We let $F = \cup_{k=1}^{\ell} F_k$. Thus F is the set of all strongly connected components of $G|_B(Z)$ that appear before Z in some incoming branch of D.

We may order $F = \{C_1, C_2, ..., C_n,\}$ such that if $1 \leq i < j \leq n$ there are no paths from C_j to C_i. If no such ordering existed, it would imply for some $1 \leq i < j \leq n$, paths exist both from C_i to C_j and from C_j back to C_i. This implies that C_i and C_j must be part of the same strongly connected component which is a contradiction.

Thus we can write the adjacency matrix, A, of G in the form

$$A = \begin{bmatrix} \underline{B} & W_{BT} & W_{BZ} & W_{BX} \\ Y_{LB} & L & & \\ Y_{ZB} & Y_{ZL} & \underline{Z} & \\ Y_{XB} & Y_{XL} & Y_{XZ} & X \end{bmatrix}$$

where L is of the form,

$$L = \begin{bmatrix} \underline{C_1} & & & \\ Y_{21} & \underline{C_2} & & \\ \vdots & & \ddots & \\ Y_{k1} & \dots & Y_{kk-1} & \underline{C_n} \end{bmatrix}$$

and $\underline{B} = \mathcal{A}(G|_B)$, $\underline{Z} = \mathcal{A}(Z)$, $\underline{C_i} = \mathcal{A}(C_i)$ for $1 \le i \le m$. **Since, λ is an eigenvalue of G,** there is a vector $\mathbf{u}$ such that $A\mathbf{u} = \lambda\mathbf{u}$. We may partition $\mathbf{u}$ into $\mathbf{u} = [\mathbf{u}_B, \mathbf{u}_L, \mathbf{u}_Z, \mathbf{v}_X]^T$ so that the number of entries in each sub-vector corresponds with the size of the appropriate sub-matrix of A

We apply the eigenvector equation to solve for $\mathbf{u}_Z$.

$$\begin{bmatrix} \underline{B} & W_{BT} & W_{BZ} & W_{BX} \\ Y_{LB} & L & & \\ Y_{ZB} & Y_{ZL} & \underline{Z} & \\ Y_{XB} & Y_{XL} & Y_{XZ} & X \end{bmatrix} \begin{bmatrix} \mathbf{u}_B \\ \mathbf{u}_L \\ \mathbf{u}_Z \\ \mathbf{u}_X \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{u}_B \\ \mathbf{u}_L \\ \mathbf{u}_Z \\ \mathbf{u}_X \end{bmatrix}$$

We have that,

$$Y_{ZB}\mathbf{v}_B + Y_{ZL}\mathbf{u}_L + \underline{Z}\mathbf{u}_Z = \lambda\mathbf{u}_Z$$
$$\mathbf{u}_Z = (\lambda I - \underline{Z})^{-1} Y_{ZB}\mathbf{u}_B + (\lambda I - \underline{Z})^{-1} Y_{ZL}\mathbf{u}_L.$$

Solving for $\mathbf{u}_L$ produces,

$$\mathbf{u}_L = (\lambda I - L)^{-1} Y_{LB}\mathbf{u}_B.$$

Thus,

$$\mathbf{u}_Z = \big((\lambda I - \underline{Z})^{-1} Y_{ZB} + (\lambda I - \underline{Z})^{-1} Y_{ZL} (\lambda I - L)^{-1} Y_{LB} \big) \mathbf{u}_B.$$

We now show that,

$$\mathbf{u}_Z = \big((\lambda I - \underline{Z})^{-1} Y_{ZB} + (\lambda I - \underline{Z})^{-1} Y_{ZL} (\lambda I - L)^{-1} Y_{LB} \big) \mathbf{u}_B = \sum_{k=1}^{\ell} T(\beta_k, Z, \lambda)\mathbf{u}_B.$$

by showing that,

$$(\lambda I - \underline{Z})^{-1} Y_{ZB} + (\lambda I - \underline{Z})^{-1} Y_{ZL} (\lambda I - L)^{-1} Y_{LB} = \sum_{k=1}^{\ell} T(\beta_k, Z, \lambda). \quad (7)$$

First, we consider $(\lambda I - \underline{Z})^{-1} Y_{ZB}$. Since every nonzero entry in Y_{ZB} corresponds to an edge from $G|_B$ to Z we let n_0 equal the number of non-zero entries in Y_{ZB} and write,

$$Y_{ZB} = \sum_{r=1}^{n_o} Y_{ZB}^{(r)}$$

where each $Y_{ZB}^{(r)}$ has exactly one non zero entry and that entry is equal to a non-zero entry of Y_{ZB}. Then,

$$(\lambda I - \underline{Z})^{-1} Y_{ZB} = \sum_{r=1}^{n_o} (\lambda I - \underline{Z})^{-1} Y_{ZB}^{(r)}.$$

We fix $r \in \{1, ..., n_0\}$ and consider $(\lambda I - \underline{Z})^{-1} Y_{ZB}^{(r)}$. The matrix $Y_{ZB}^{(r)}$ corresponds to exactly one edge from $G|_B$ to Z. Then there must exist $\beta_r \in D$ such that $In(\beta_r, Z) = \{v, e, Z\}$ where $v \in B$ and e corresponds with the non zero entry in $Y_{ZB}^{(r)}$.

Then,

$$(\lambda I - \underline{Z})^{-1} Y_{ZB}^{(r)} = T(\beta_r, Z, \lambda).$$

Furthermore, β_r must be the only branch in D that contains e. If not, must be another $\beta_s \in D$ such that $In(\beta_s, Z) = \{v, e, Z\} = In(\beta_r, Z)$. Since β_r and β_s are in D, they have the same outgoing branch and it must be the case that $\beta_r = \beta_s$. This contradicts uniqueness of each β in D.

Thus, each $Y_{ZB}^{(r)}$ corresponds with exactly one $\beta_r \in D$ such that i$In(\beta_r, Z)$ that has no strongly connected components except Z. Then

$$\mathcal{A}(In(\beta_r, Z)) = \begin{bmatrix} \underline{B} & \\ Y_{ZB}^{(r)} & \underline{Z} \end{bmatrix}$$

and $(\lambda I - \underline{Z})^{-1} Y_{ZB}^{(r)}$ must the eigenvector transfer matrix of β_r with respect to Z.

Consider the set $D_0 = \{\beta_1, ... \beta_{n_0}\}$ of component branches corresponding to $\{Y_{ZB}^{(1)}, ..., Y_{ZB}^{(n_0)}\}$.

Thus

$$(\lambda I - \underline{Z})^{-1} Y_{ZB} = \sum_{r=1}^{n_o} (\lambda I - \underline{Z})^{-1} Y_{ZB}^{(r)} = \sum_{\beta \in D_0} T(\beta, Z, \lambda). \tag{8}$$

Note that for all $\beta \in D_0$, $In(\beta, Z)$ has no strongly connected components except Z. We assert that D_0 is the set of all branches in D that satisfy this property. If $\beta \in D$ and $In(\beta, Z) = \{v, e, Z\}$, then by definition of $\mathcal{A}(G)$, e must correspond to a non zero entry in Y_{ZB} and therefore β corresponds to $Y_{ZB}^{(r)}$ for some $1 \leq r \leq n_0$. Then $\beta \in D_0$ and D_0 is the set of all $\beta \in D$ where $In(\beta, Z)$ contains no strongly connected components except Z.

Next we consider $(\lambda I - \underline{Z})^{-1} Y_{ZL} (\lambda I - L)^{-1} Y_{ZB}$ from **equation (7)** . Once again, we write $Y_{ZL} = \sum_{s=1}^{n_1} Y_{ZL}^{(s)}$ and $Y_{LB} = \sum_{t=1}^{n_2} Y_{LB}^{(t)}$ as the sum of their non zero entries.

By definition of an adjacency matrix, it must be the case that the set $\{Y_{ZL}^{(s)}\}_{s=1}^{n_1}$ is in a bijective correspondence with the edges from components in $\{C_1, ... C_n\}$ to Z and the set $\{Y_{LB}^{(t)}\}_{t=1}^{n_2}$ is in a bijective correspondence with the edges from $G|_B$ to components in $\{C_1, ... C_n\}$. Thus we may let $\{f_s\}_{s=1}^{n_1}$ be the set of edges corresponding with $\{Y_{ZL}^{(s)}\}_{s=1}^{n_1}$ and $\{g_s\}_{t=1}^{n_1}$ be the set of edges corresponding with $\{Y_{LB}^{(s)}\}_{t=1}^{n_1}$.

We have,

$$(\lambda I - \underline{Z})^{-1}Y_{ZL}(\lambda I - L)^{-1}Y_{LB} = \sum_{s=1}^{n_1}\sum_{t=1}^{n_2}(\lambda I - \underline{Z})^{-1}Y_{ZL}^{(s)}(\lambda I - L)^{-1}Y_{LB}^{(t)}$$

Fix, $s \in \{1, ..., n_1\}$ and $t \in \{1, ...n_2\}$ and consider, $Y_{ZL}^{(s)}(\lambda I - L)^{-1}Y_{LB}^{(t)}$. By definition, $Y_{LB}^{(s)}$ represents an edge from $G|_B$ to some $C_i \in F$ and $Y_{ZL}^{(s)}$ represents an edge from some $C_j \in F$ to Z. Since there are no paths from C_j to C_i when $i > j$, it must be the case that $i \leq j$. This gives us information about the location of the non-zero entries in $Y_{LB}^{(s)}$ and $Y_{ZL}^{(s)}$. It must be the case that

$$\begin{bmatrix} Y_{LB}^{(t)} & L \\ & Y_{ZL}^{(s)} \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 \\ 0 \\ Y^{(t)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} & \begin{bmatrix} \underline{C_1} & & & & & & \\ \vdots & \ddots & & & & & \\ Y_{i1} & & \underline{C_i} & & & & \\ \vdots & & & \ddots & & & \\ Y_{j1} & & & & \underline{C_j} & & \\ \vdots & & & & & \ddots & \\ Y_{k1} & \dots & Y_{ki} & \dots & Y_{kj} & \dots & \underline{C_n} \end{bmatrix} \\ & \begin{bmatrix} 0 & \dots & \dots & 0 & Y^{(s)} & 0 & 0 \end{bmatrix} \end{bmatrix}$$

For some $Y^{(t)}$ and $Y^{(s)}$ that contain a single non-zero entry. This is because $Y_{LB}^{(t)}$ and $Y_{ZL}^{(s)}$ represent an edge from $G|_B$ to a component C_i and an edge from a component C_j to Z respectively. Thus we conclude that all entries in $Y_{LB}^{(t)}$ and $Y_{ZL}^{(s)}$ that correspond to edges to or from components besides C_i and C_j respectively must be zero.

By lemma,

$$(\lambda I - L)^{-1} = \begin{bmatrix} X_{11} & & & \\ X_{21} & X_{22} & & \\ \vdots & \vdots & \ddots & \\ X_{kl} & X_{k2} & \cdots & X_{kk} \end{bmatrix}$$

where

$$X_{ij} = \sum_{\alpha \in \mathcal{P}(C_j, C_i)} P(\alpha, \lambda).$$

Using this fact we simplify the expression $Y_{ZL}^{(s)}(\lambda I - L)^{-1}Y_{LB}^{(t)}$ to

$$\begin{bmatrix} 0 & \dots & 0 & Y^{(s)} & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} X_{11} & & & & & & \\ \vdots & \ddots & & & & & \\ X_{i1} & & X_{ii} & & & & \\ \vdots & & \vdots & \ddots & & & \\ X_{j1} & & X_{ji} & & X_{jj} & & \\ \vdots & & \vdots & & & \ddots & \\ X_{k1} & \dots & X_{ki} & \dots & X_{kj} & \dots & X_{nn} \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ Y^{(t)} \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

demonstrating that,

$$Y_{ZL}^{(s)}(\lambda I - L)^{-1}Y_{LB}^{(t)} = Y^{(s)}X_{ji}Y^{(t)} = \sum_{\alpha \in \mathcal{P}(C_i, C_j)} Y^{(s)}P(\alpha, \lambda)Y^{(t)}$$

.

Thus, for each s and t the term,

$$(\lambda I - \underline{Z})^{-1}Y_{ZL}^{(s)}(\lambda I - L)^{-1}Y_{LB}^{(t)} = \sum_{\alpha \in \mathcal{P}(C_{i_t}, C_{j_s})} (\lambda I - Z)^{-1}Y^{(s)}P(\alpha, \lambda)Y^{(t)}$$

where $j_s, i_t \in \{1, ..., n\}$. We now show that for each $\alpha \in \mathcal{P}(C_{i_t}, C_{j_s})$

$$(\lambda I - \underline{Z})^{-1}Y^{(s)}P(\alpha, \lambda)Y^{(t)} = T(\beta, Z, \lambda)$$

where $\beta \in \mathcal{B}_B(G)$. Let $\alpha \in \mathcal{P}(C_{i_t}, C_{j_s})$. Then $\alpha = \{C_{\alpha_0}, e_1, C_{\alpha_1}, e_2, ...C_{\alpha_{N-1}}, e_N C_{\alpha_N}\}$ where $C_{\alpha_0} = C_{i_t}$, $C_{\alpha_N} = C_{j_s}$ and for each $1 \leq k \leq N$, $C_{\alpha_k} \in F$. Because $Y^{(s)}$ and $Y^{(t)}$ correspond to unique edges from C_{j_s} to Z and from $G|_B$ to C_{i_t} respectively, there is a unique component branch $\beta \in \mathcal{B}_B(G)$ such that

$$\beta = \{v, g_t, C_{\alpha_0}, e_1, C_{\alpha_1}, e_2, ...C_{\alpha_{N-1}}, e_N, C_{\alpha_N}, f_s, Z, ...u\}$$

for some $v, u \in B$ and where $g_t, f_s \in E$ are edges corresponding to $Y^{(t)}$ and $Y^{(s)}$ respectively as defined previously. Then,

$$T(\beta, Z, \lambda) = (\lambda I - \underline{Z})^{-1}Y^{(s)}(\lambda I - \underline{C_{\alpha_0}})^{-1}Y_1(\lambda I - \underline{C_{\alpha_1}})^{-1}Y_2...Y_N(\lambda I - \underline{C_{\alpha_N}})^{-1}Y^{(t)}$$

$$T(\beta, Z, \lambda) = (\lambda I - \underline{Z})^{-1}Y^{(s)}P(\alpha, \lambda)Y^{(t)}$$

because, $P(\alpha, \lambda) = (\lambda I - \underline{C_{\alpha_0}})^{-1}Y_1(\lambda I - \underline{C_{\alpha_1}})^{-1}Y_2...Y_N(\lambda I - \underline{C_{\alpha_N}})^{-1}$.

Thus we see that given s, t and α, $(\lambda I - \underline{Z})^{-1}Y^{(s)}P(\alpha, \lambda)Y^{(t)}$ is equal to the eigenvector transfer matrix for some branch β that contains Z where the first edge in β is g_t and the edge before Z is f_s. Let D_{st} represent the set of all $\beta \in \mathcal{B}_B(G)$ where the first edge is g_t and the edge leading to Z is f_s. We have already shown that for each $\alpha \in \mathcal{P}(C_i, C_j)$, there exists a $\beta \in D_{st}$ such that

$$(\lambda I - \underline{Z})^{-1}Y^{(s)}P(\alpha, \lambda)Y^{(t)} = T(\beta, Z, \lambda).$$

We now show that for each $\beta \in D_{st}$ there exists and $\alpha \in \mathcal{P}(C_i, C_j)$ such that the above equation is true. Let $\beta \in D_{st}$. Then $\beta = \{v, g_t, C_{\beta_0}, e_1...e_M, C_{\beta_M}, f_s, Z...u\}$ where for $1 \leq k \leq M$, $e_k \in E$ and $C_{\beta_k} \in F$ when $0 \leq k \leq M$. By definition of a component branch, each e_k is and edge from $C_{\beta_{k-1}}$ to C_{β_k} when $1 \leq k \leq M$. This implies that there is a partial component branch from C_{β_1} to C_{β_M} of the form, $\alpha = \{C_{\beta_0}, e_1...e_M, C_{\beta_M}\}$. Clearly $\alpha \in \mathcal{P}(C_{\beta_1}, C_{\beta_M})$. Then the adjacency matrix for α is of the form,

$$\mathcal{A}(\alpha) = \begin{bmatrix} \underline{C_{\beta_0}} & & & \\ Y_1 & \underline{C_{\beta_1}} & & \\ & \ddots & \ddots & \\ & & Y_m & \underline{C_{\beta_M}} \end{bmatrix}$$

and $P(\alpha, \lambda) = (\lambda I - \underline{C_{\beta_M}})^{-1} Y_M ... Y_1 (\lambda I - \underline{C_{\beta_0}})^{-1}$. Since the components and edges in α are in β, the adjacency matrix of $In(\beta, Z)$ can be written as,

$$\mathcal{A}(In(\beta, Z)) = \begin{bmatrix} B & & & & & & \\ Y^{(t)} & \underline{C_{\beta_0}} & & & & & \\ & Y_1 & \underline{C_{\beta_1}} & & & & \\ & & \ddots & \ddots & & & \\ & & & Y_m & \underline{C_{\beta_M}} & & \\ & & & & Y^{(s)} & Z & \end{bmatrix}$$

. Thus, there exists an α such that $T(\beta, Z, \lambda) = (\lambda I - Z)^{-1} Y^{(s)} P(\alpha, \lambda) Y^{(t)}$.

Since the sets $\mathcal{P}(C_{\beta_1}, C_{\beta_M})$ and D_{st} are in bijective correspondence, and for each $\alpha \in \mathcal{P}(C_{i_t}, C_{j_s})$ there is a $\beta \in D_{st}$ such that

$$(\lambda I - \underline{Z})^{-1} Y^{(s)} P(\alpha, \lambda) Y^{(t)} = T(\beta, Z, \lambda)$$

such that we may substitute each $(\lambda I - \underline{Z})^{-1} Y^{(s)} P(\alpha, \lambda) Y^{(t)}$ in the sum

$$\sum_{\alpha \in \mathcal{P}(C_i, C_j)} (\lambda I - \underline{Z})^{-1} Y^{(s)} P(\alpha, \lambda) Y^{(t)}$$

for the corresponding $T(\beta, Z, \lambda)$, producing

$$\sum_{\alpha \in \mathcal{P}(C_{i_t}, C_{j_s})} (\lambda I - \underline{Z})^{-1} Y^{(s)} P(\alpha, \lambda) Y^{(t)} = \sum_{\beta \in D_{st}} T(\beta, Z, \lambda).$$

We have now shown that

$$\begin{aligned}(\lambda I - \underline{Z})^{-1} Y_{ZL} (\lambda I - L)^{-1} Y_{LB} &= \sum_{s=1}^{n_1} \sum_{t=1}^{n_2} (\lambda I - \underline{Z})^{-1} Y_{ZL}^{(s)} (\lambda I - L)^{-1} Y_{LB}^{(t)} \\ &= \sum_{s=1}^{n_1} \sum_{t=1}^{n_2} \sum_{\alpha \in \mathcal{P}(C_{i_t}, C_{j_s})} (\lambda I - \underline{Z})^{-1} Y^{(s)} P(\alpha, \lambda) Y^{(t)} \\ &= \sum_{s=1}^{n_1} \sum_{t=1}^{n_2} \sum_{\beta \in D_{st}} T(\beta, Z, \lambda).\end{aligned}$$

Since

$$\mathbf{u}_Z = \left((\lambda I - \underline{Z})^{-1} Y_{ZB} + (\lambda I - \underline{Z})^{-1} Y_{ZL} (\lambda I - L)^{-1} Y_{LB} \right) \mathbf{u}_B,$$

by **equation (8)** we have:

$$\mathbf{u}_Z = \left(\sum_{\beta \in D_0} T(\beta, Z, \lambda) + \sum_{s=1}^{n_1} \sum_{t=1}^{n_2} \sum_{\beta \in D_{st}} T(\beta, Z, \lambda) \right) \mathbf{u}_B.$$

Finally we show that

$$D = D_0 \cup \left(\bigcup_{s=1}^{n_1} \bigcup_{t=1}^{n_2} D_{st} \right).$$

By definition, $D_0 \subset D$ and $D_{st} \subset D$ for all $1 \leq s \leq n1, 1 \leq t \leq n_2$. Then $D_0 \cup (\bigcup_{s=1}^{n_1} \bigcup_{t=1}^{n_2} D_{st}) \subset D$. Let $\beta \in D$. Then $In(\beta, Z)$ either contains a strongly connected component besides Z, or it does not. If it does not, $\beta \in D_0$ as shown previously. If it does, then the first edge in β must connect a node in $G|_B$ to a node in some component $C_i \in F$ and therefore correspond to f_t for some $1 \leq t \leq n_2$. Additionally the edge that appears prior to Z must correspond to g_s for some $1 \leq s \leq n_1$. Thus $\beta \in D_0 \cup (\bigcup_{s=1}^{n_1} \bigcup_{t=1}^{n_2} D_{st})$ and $D \subset D_0 \cup (\bigcup_{s=1}^{n_1} \bigcup_{t=1}^{n_2} D_{st})$. This produces

$$\mathbf{u}_Z = \sum_{\beta \in D} T(\beta, Z, \lambda) \mathbf{u}_B = \sum_{k=1}^{\ell} T(\beta_k, Z, \lambda) \mathbf{u}_B.$$

By (ii), $T(\beta_k, Z, \lambda) \mathbf{u}_B = \mathbf{v}_{Z_k}$ and

$$\mathbf{u}_Z = \sum_{k=1}^{\ell} \mathbf{v}_{Z_k}$$