Attack Design for Continuous and Discrete Systems

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Constructing Continuous Time $\Delta(s)$

To design such an attack for continuous time systems we follow atack design methods from MIT Open Courseware. [**?**]

Proof

Assume that $G(s)$ is stable. In order to create a feasable attack we need to create $\Delta(s)$ satifying:

- $\Delta(s)$ is a viable transfer function (Matrix of rational functions)
- Stable (poles are negative)
- $||\Delta(j\omega_0)G(j\omega_0)||_2 = 1$ for some $j\omega_0$

Fix $j\omega_0$. We construct $\Delta(j\omega_0)$ by taking the singular value decompostion of $G(j\omega_0) = U\Sigma V^*$ where $*$ denotes conjugate transpose. Set

$$\Delta(j\omega_0) = V \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & 0 & \\ & & \ddots \end{bmatrix} U^*$$

Thus,

$$\begin{aligned}
||\Delta(j\omega_0)G(j\omega_0)||_2 &= ||V \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & 0 & \\ & & \ddots \end{bmatrix} U^*U \begin{bmatrix} \sigma_1 & & \\ & \sigma_2 & \\ & & \ddots \end{bmatrix} V^*||_2 \\
&= ||V \begin{bmatrix} 1 & & \\ & 0 & \\ & & \ddots \end{bmatrix} V^*||_2 \\
&= 1
\end{aligned}$$

Thus, at $s = j\omega_0$, $\Delta(j\omega_0)$ will destabilize $G(j\omega_0)$.

We know the value of $\Delta(s)$ at $s = j\omega_0$ but we need to define $\Delta(s)$ for all s. We have that

$$\Delta(j\omega_0) = V \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & 0 & \\ & & \ddots \end{bmatrix} U^* = \frac{1}{\sigma_1} v_1 u_1^* = \frac{1}{\sigma_1} \begin{bmatrix} r_1 e^{j\theta_1} \\ \vdots \\ r_n e^{j\theta_n} \end{bmatrix} \begin{bmatrix} \rho_1 e^{j\phi_1} \cdots \rho_n e^{j\phi_n} \end{bmatrix}$$

when the entries of v_1 and u_1^* are written in polar form. Define

$$\Delta(s) = \frac{1}{\sigma_1} \begin{bmatrix} r_1 f_1(s) \\ \vdots \\ r_n f_n(s) \end{bmatrix} \begin{bmatrix} \rho_1 g_1(s) \cdots \rho_n g_n(s) \end{bmatrix}$$

with appropriate functions $f_i(s)$ and $g_i(s)$.

Continuous Time Entry Functions

Each $f_i(s)$ and $g_i(s)$ must have the following properties:

1. $f_i(j\omega_0) = e^{j\theta_i}$ and $g_i(j\omega_0) = e^{j\phi_i}$

2. The poles of f_i and g_i are negative
3. Each f_i and g_i is unitary
4. Each f_i and g_i is a rational polynomial

Therefore, if appropriate functions are used, $\Delta(s)$ will be a minimal destabilizing attack. An obvious choice is

$$f_i(s) = e^{j\theta_i}$$

a constant function. The function has no poles, is unitary and is a polynomial of degree zero. Another option is the all-pass filter:

$$f_i(s) = \pm\frac{s-\alpha_i}{s+\alpha_i}$$

By adjusting the sign of the function, we can solve for a positive real α_i such that $\frac{j\omega_0-\alpha_i}{j\omega_0+\alpha_i} = e^{j\theta_i}$ for any $\theta_i \in [0,\pi]$ and $-\frac{j\omega_0-\alpha_i}{j\omega_0+\alpha_i} = e^{j\theta_i}$ for any $\theta_i \in [0,-\pi]$.

Finding the appropriate α_i parameter is accomplished by solving the equation

$$\frac{j\omega_0-\alpha_i}{j\omega_0+\alpha_i} = e^{j\theta_i}$$

$$\alpha_i = j\omega_0\frac{1-e^{\theta_i}}{1+e^{\theta_i}}$$

By adjusting the sign of the function, we can solve for a positive α_i, ensuring that the function's pole is negative. Furthermore, observe that for any ω and any α,

$$||\frac{j\omega-\alpha}{j\omega+\alpha}|| = 1$$

Thus the function is unitary and satifies all requirements for creating a viable $\Delta(s)$.

$\blacksquare$

These two examples of functions work to destabilize G, but other functions may work as well. Additionally, our SVD based construction of $\Delta(j\omega_0)$ is not the only contruction that works. For example,

$$\Delta(j\omega_0) = \frac{G^*(j\omega_0)}{||G(j\omega_0)||^2}$$

also works.

Proof

Let $G(s)$ be given. Fix $j\omega_0$. From spectral theory, have that given a complex valued matrix A, the maximum eigenvalue of A^*A is equal to $||A||^2$. Futhermore, $||A^*A|| = ||A||^2$. We apply this to our attack design by setting

$$\Delta(j\omega_0) = \frac{1}{||G(j\omega_0)||^2} G^*(j\omega_0)$$

This produces

$$||\Delta(j\omega_0)G(j\omega_0)|| = \frac{1}{||G(j\omega_0)||^2} ||G^*(j\omega_0)G(j\omega_0)|| = 1$$

with

$$||\Delta(j\omega_0)|| = \frac{1}{||G(j\omega_0)||}\,.$$

Then an appropriate $\Delta(s)$ will destabilize $G(s)$ at $s = j\omega_0$.

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