MA 548 Final Project Report

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1 Introduction

For our project, we analyzed the ten highest market capitalized companies in the U.S. These companies were: Apple, Amazon, Alphabet (Google), Facebook, Johnson & Johnson, JP-Morgan Chase, Microsoft, Visa, Exxon Mobil, and, Berkshire Hathaway. Our goal was to create an Exchange-traded fund (ETF) that held these companies. In addition, we wanted to find the fair market price of Asian and European options based off our ETF. Also, We desired to check the projected behavior of the ETF under normal and stressed market conditions so that we could see how they compare during a three month long forecast. There are many ways one can perform simulations, but through the use of variance reduction techniques, one can run much more efficient simulations. Therefore, we used a crude approach and a more sophisticated approach using antithetic variables. We then sought to compare the two methods and show the benefits of using antithetic variables.

2 Assumptions

Simulating stock price behavior requires many assumptions. For example: What should the drift term be? Should we calculate it using the average of the daily returns over a one year, five year, or ten year period? The same questions arise when we try to find an appropriate volatility. Our goal was to project the ETF's behavior over a three month period so we needed the drifts and volatilities for each of the stocks that were a component of the ETF. We obtained the volatilities by finding the three month implied volatilities.

The implied volatility is found by using market option prices and using the Black-Scholes equation to solve for the volatility. To calculate the drift term, we assumed that an average over a three year period was sufficient. For the stressed case, we found the drift using data from late 2018 when interest rates were rising and stocks were falling. In addition, we assumed that our correlation matrix should be obtained using data from the past three years and from the stressed period for the base case and the stressed case respectively. We also assumed that the weighting of each stock in the ETF is fixed and calculated on the basis of market capitalization.

3 Methods

We downloaded three years of historical adjusted closing prices for the various stocks. The daily return for each stock was then calculated using the following formula:

$$u_t = ln(\frac{S_t}{S_{t-1}}) \tag{1}$$

Where "S_t" is the closing price. The market movements are generally correlated, therefore we constructed a correlation matrix to quantify the relationship between the securities. The formula of the correlation between two different securities (X,Y) can be represented as:

$$\rho = \frac{Cov(X,Y)}{\sqrt{Var(X)Var(Y)}} \tag{2}$$

When modeling future price action of the stocks, there are many elements that need to be accounted for. In general, there is a growth component (or a decay component in a stressed period) and a randomness component. The predicted stock price is a function of time, drift, volatility, correlation, and randomness. Therefore, the equation used for each k'th stock was:

$$S_k(t_{i+1}) = S_k(t_i) \cdot e^{(\mu_k - 0.5\sigma_k^2)(t_{i+1}-t_i) + \sqrt{t_{i+1}-t_i}\sum_{j=1}^{10} A_{k,j} Z_{i+1,j}} \tag{3}$$

Where "A" is the lower triangular matrix obtained from Cholesky decomposition i.e.:

$$AA^T = \Sigma \tag{4}$$

Where Σ is the covariance matrix. We can then simulate multiple time dependent paths for each stock. The final prices of the simulations can be averaged to get the expected final price for each stock. The projected value of the ETF could then then be estimated by applying the appropriate weighting. There is obviously variance in the simulated final value of the ETF. Therefore, how many simulations does it take to reach a certain confidence interval? The form of a 100(1-α)% confidence interval as well as relative error is given in the following equations:

$$\bar{X}(n) \pm z_{1-\alpha/2}\sqrt{\frac{S^2(n)}{n}} \tag{5}$$

$$\frac{\mid z_{1-\alpha/2} \mid \sqrt{\frac{S^2(n)}{n}}}{\bar{X}(n)} \tag{6}$$

Using different simulation techniques, the variance can be reduced, thus the number of simulations required to produce a desired confidence interval would be reduced as well. One such variance reduction technique is implemented through the use of antithetic variables. To implement this method, the goal is to create two unbiased estimators that are negatively correlated. Using eq(3) we can simulate an unbiased path. We can also simulate a negativity correlated path by using the same equation except multiplying the same random number by negative one. Using these, we can produce an additional unbiased estimator whose variance is substantially less than the original estimator. Therefore, the two negatively correlated estimators can be formulated as:

$$S1_k(t_{i+1}) = S_k(t_i) \cdot e^{(\mu_k - 0.5\sigma_k^2)(t_{i+1}-t_i) + \sqrt{t_{i+1}-t_i}\sum_{j=1}^{10} A_{k,j} Z_{i+1,j}} \tag{7}$$

$$S2_k(t_{i+1}) = S_k(t_i) \cdot e^{(\mu_k - 0.5\sigma_k^2)(t_{i+1}-t_i) + \sqrt{t_{i+1}-t_i}\sum_{j=1}^{10} A_{k,j} \cdot (-Z_{i+1,j})} \tag{8}$$

The implementation of this method substantially reduces the number of trials required, therefore leading to less run time when running the program. To compare the efficiencies of the two methods, one could find the number of trials required to reach a certain confidence interval using the two methods, then take the ratio of the trial counts.

We constructed different options for our ETF using the base case to calculate the option price. We analyzed calls and puts for Asian and lookback options. The Asian option's payoff is dependent on the average ETF value through the projected period:

$$\bar{S}(T) = \frac{1}{T} \cdot \sum_{0}^{T} S(t) \tag{9}$$

With associated payoffs:

$$Call\ Payoff = max(\bar{S}(T) - K, 0) \tag{10}$$

$$Put\ Payoff = max(K - \bar{S}(T), 0) \tag{11}$$

The payoff of the lookback option is dependent on the extreme values of the asset price. The following are the payoffs for the lookback option:

$$Call\ Payoff = max((S(T) - min(S(t)), 0) \tag{12}$$

$$Put\ Payoff = max(max(S(t)) - S(T), 0) \tag{13}$$

The general procedure for simulating optoin prices is:

(a) simulate the underlying asset price.

(b) calculate the payoff, then discount it.

(c) do this multiple times, then take the average.

4 Data Acquisition

Using eq(2), we computed the correlations. The following is the correlation output where the red values indicate when the correlation is greater than 0.7:

Figure 1: Unstressed Correlation Matrix

	Apple	Amazon	Alphabet	Facebook	JNJ	JPM	Microsoft	Visa	Exxon	Bekshire
Apple	1									
Amazon	0.55885	1								
Alphabet	0.60355	0.70887	1							
Facebook	0.42728	0.55885	0.60918	1						
JNJ	0.25576	0.24114	0.33136	0.18554	1					
JPM	0.34913	0.31452	0.40975	0.26071	0.32001	1				
Microsoft	0.58649	0.69130	0.75864	0.49291	0.35148	0.44434	1			
Visa	0.57178	0.60765	0.69323	0.50162	0.35425	0.45212	0.73035	1		
Exxon	0.30093	0.30343	0.38128	0.25731	0.36843	0.45175	0.35916	0.40293	1	
Bekshire	0.42901	0.37302	0.48103	0.28155	0.44331	0.74412	0.52182	0.54346	0.53851	1

During late last year, the Federal Reserve was raising rates. This caused borrowing costs to increase and therefore caused fears of lower earnings in the future. The stock market fell substantially over this time. We took data from this period to analyze the behavior of the ETF during a stressed period (11/23/2018 - 12/31/2018). The following is the correlation matrix corresponding to stressed market conditions:

Figure 2: Stressed Correlation Matrix

	Apple	Amazon	Alphabet	Facebook	JNJ	JPM	Microsoft	Visa	Exxon	Bekshire
Apple	1									
Amazon	0.93001	1								
Alphabet	0.86745	0.92814	1							
Facebook	0.73340	0.77497	0.73434	1						
JNJ	0.49633	0.51824	0.42308	0.32504	1					
JPM	0.81152	0.81637	0.79642	0.60331	0.36011	1				
Microsoft	0.86694	0.90259	0.87954	0.67774	0.58823	0.76512	1			
Visa	0.89534	0.94322	0.94103	0.72662	0.45917	0.84706	0.91566	1		
Exxon	0.76004	0.71655	0.66996	0.51249	0.45630	0.76626	0.72375	0.72643	1	
Bekshire	0.84442	0.81567	0.82568	0.54504	0.42686	0.92836	0.78335	0.84322	0.81930	1

As can be seen, the stocks are more correlated during this time.

There is associated market risk with each security; to incorporate this into our analysis, we used the three month implied volatilities. These are presented below:

Table 1: Implied Volatilities

Apple	0.225	JPM	0.1894
Amazon	0.261	Microsoft	0.1835
Alphabet	0.191	Visa	0.1512
Facebook	0.311	Exxon	0.155
JNJ	0.117	Berkshire	0.1579

The initial prices used in our model were were obtained on 4/12/19 and their values shown in table 2:

Table 2: Initial Prices

Apple	$198.87	JPM	$111.21
Amazon	$1,843.06	Microsoft	$120.95
Alphabet	$1,217.87	Visa	$159.64
Facebook	$179.10	Exxon	$80.92
JNJ	$135.98	Berkshire	$210.56

To determine the weight of each stock in the ETF, we found each stock's market capital then divided it by the sum of all the company's market capitalization. The market cap and the associated weights for each company are given in tables 3 and 4 respectively. Also, knowing the weights, we found that the initial value of the ETF was $557.92

Table 3: Market Capitalization (Billions $)

Apple	957.01	JPM	369.35
Amazon	919.92	Microsoft	930.79
Alphabet	860.11	Visa	352.58
Facebook	511.15	Exxon	344.87
JNJ	370.21	Berkshire	517.51

Table 4: Weighting

Apple	0.156	JPM	0.060
Amazon	0.150	Microsoft	0.152
Alphabet	0.140	Visa	0.057
Facebook	0.083	Exxon	0.056
JNJ	0.060	Bekshire	0.084

For our base case we found the drift by implementing eq(1) and averaging the results. The same procedure was used for the stressed case during the stressed period.

Table 5: Drift Terms

Company:	Three Year Drift:	Stressed Drift:
Apple	0.0010433	-0.0045583
Amazon	0.0013205	-0.0003912
Alphabet	0.0007373	-0.0000772
Facebook	0.0006009	-0.0011223
JNJ	0.0003121	-0.0035684
JPM	0.0008917	-0.0039084
Microsoft	0.0012754	-0.0006019
Visa	0.0009915	-0.0007449
Exxon	0.0000003	-0.0051502
Berkshire	0.0005541	-0.0012877

The following plot shows the behavior of each stock over the historical period. We can see that they generally move upward (except for Exxon Mobil) and are somewhat correlated.

Figure 3:

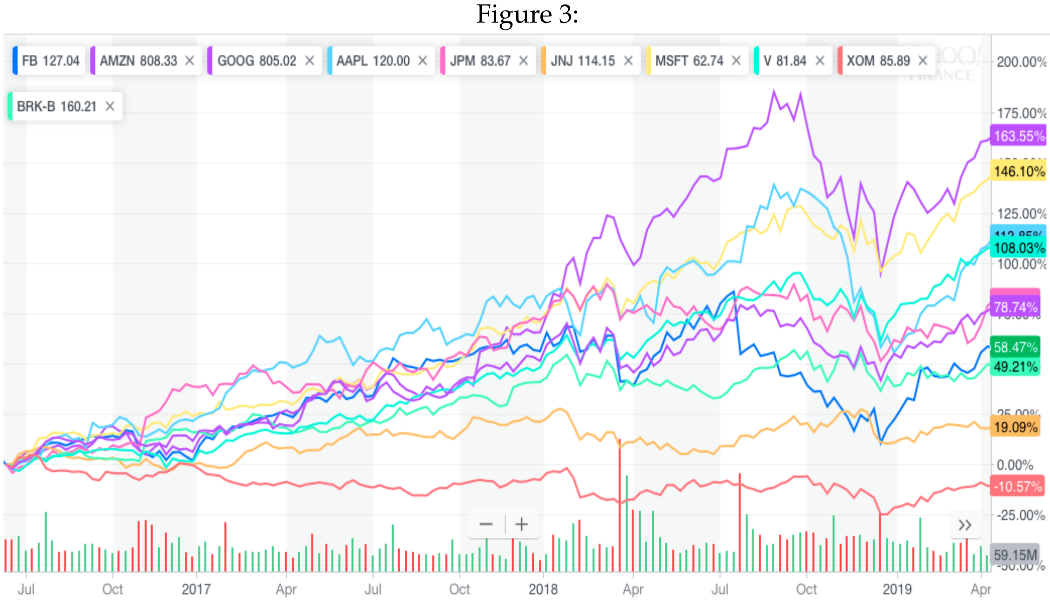
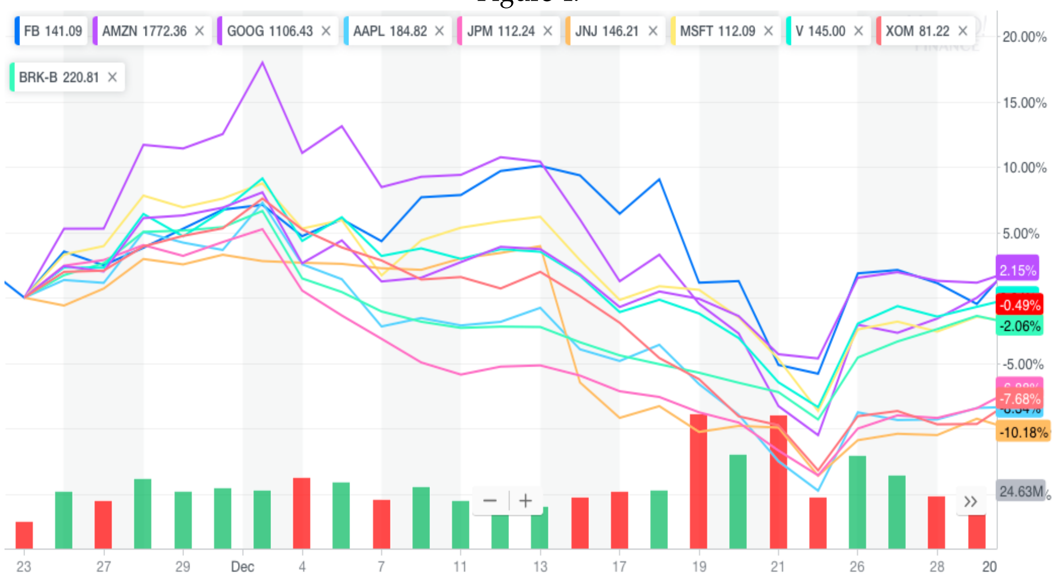


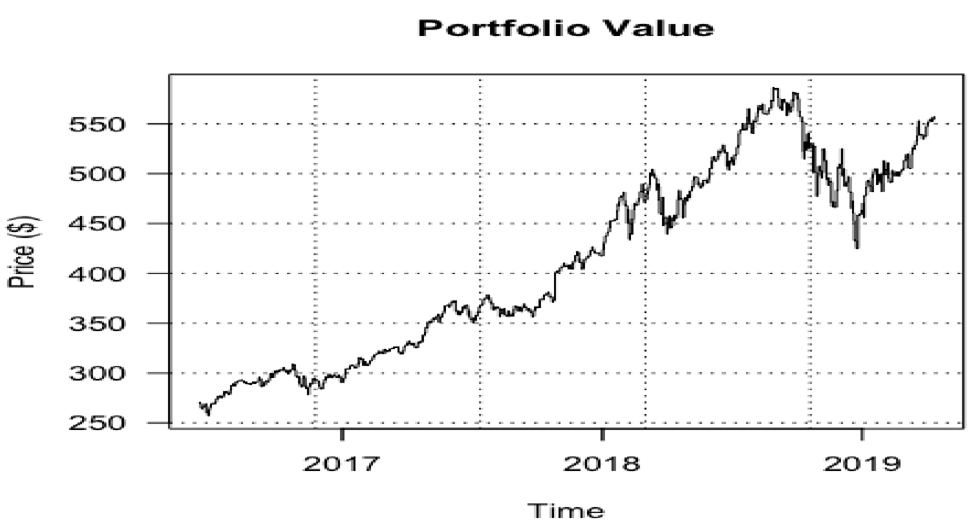
Figure 4 shows the individual stock behavior for the stressed period. The stocks are highly correlated and experience negative growth.

Figure 4:



The ETF historical value is given below.

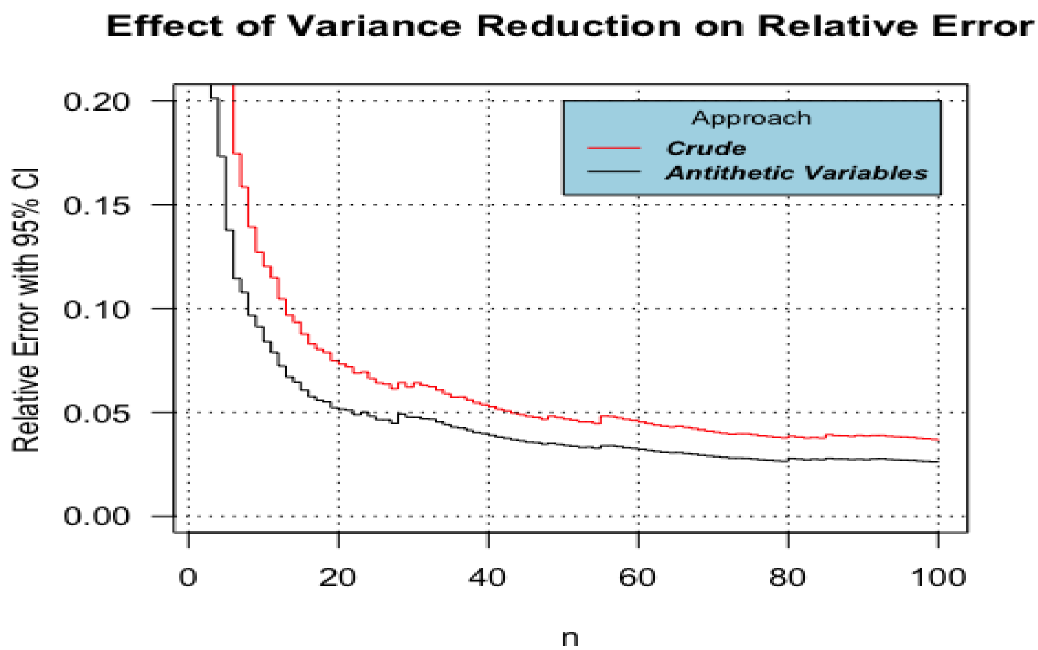
Figure 5:



5 Results

We ran simulations to calculate the final value of the ETF. We wanted to compare the crude Monte Carlo approach to the antithetic method. Figure 6 shows the number of simulations required to reach a given relative error using the different methods. We see that the antithetic approach is far more efficient than the crude approach.

Figure 6:

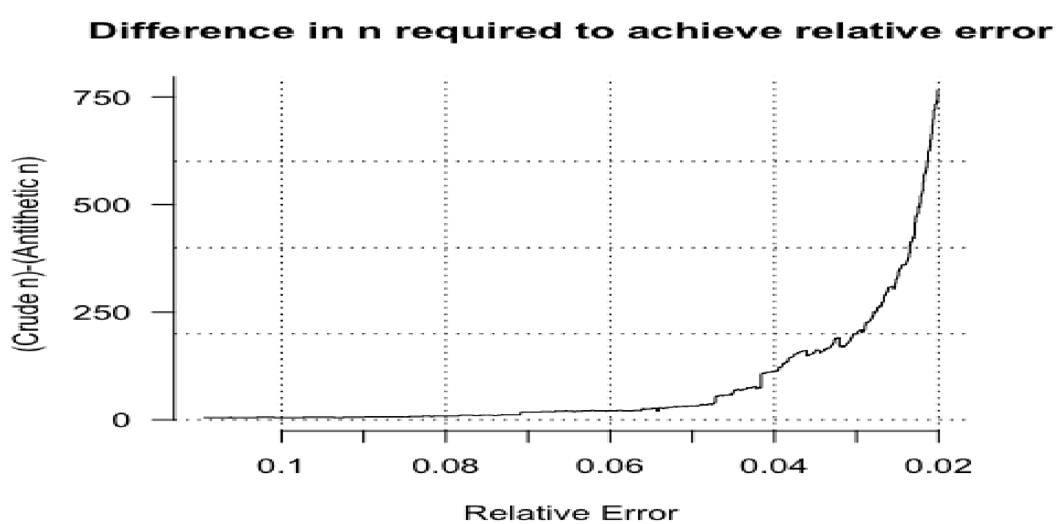


Additional method comparisons are given in figure 7 and 8

Figure 7:

	Relative Error	n Needed with Antithetic Variables	n Needed with Crude Approach	Ratio of n Needed
1	0.10	10	15	1.500000
2	0.09	11	17	1.545455
3	0.08	13	20	1.538462
4	0.07	15	23	1.533333
5	0.06	17	31	1.823529
6	0.05	23	64	2.782609
7	0.04	32	86	2.687500
8	0.03	84	165	1.964286
9	0.02	193	389	2.015544
10	0.01	783	1507	1.924649

Figure 8:



We simulated the ETF value using various cases to see how the portfolio would behave subject to different scenarios. We simulated a ”base” case using the historical data as our inputs. We also estimated the final value of our portfolio assuming the projected period was bearish like late last year. In addition, we used combinations of these scenarios while we were projecting the final ETF value three months into the future. The following gives the projected value of the ETF for each scenario along with a 95% confidence interval. Note the units are in dollars.

Figure 9:

	Lower 95% CI	Mean	Upper 95% CI
Base	591.9639	596.3727	600.7814
1 bad month	579.1194	574.7596	570.3998
2 bad months	558.9838	554.6069	550.2301
3 bad months	539.6503	535.3015	530.9527

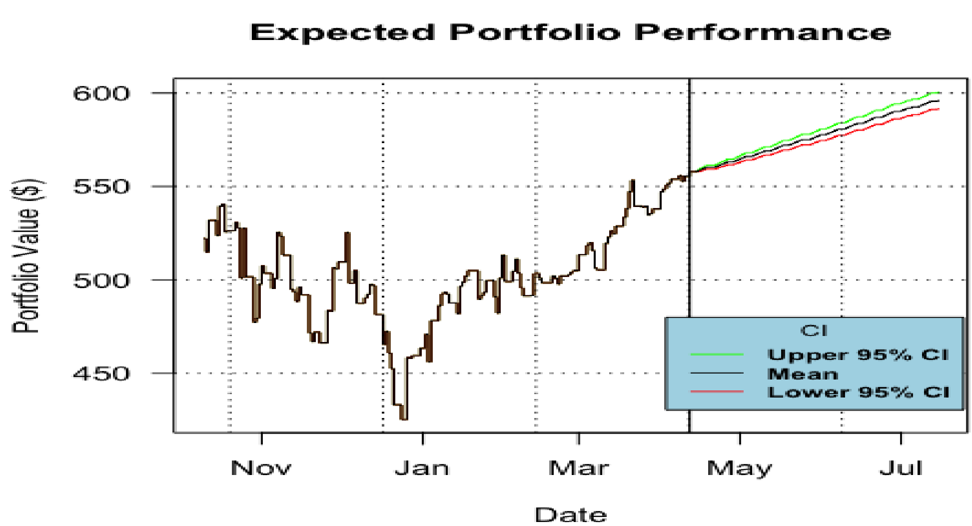
Using the base case scenario, we calculated the price of derivatives for our ETF. We priced Asian calls and puts with a strike price of the value of the portfolio. In addition, we also priced Lookback calls and puts. These prices, along with their associated 95% confidence intervals are presented in figure 10.

Figure 10:

	Lower 95% CI	Mean	Upper 95% CI
Asian call, K=S_0	31.96963	33.58200	35.19436
Asian put, K=S_0	15.32476	16.28617	17.24757
Lookback put	63.85607	65.72702	67.59796
Lookback Call	88.89739	91.88584	94.87429

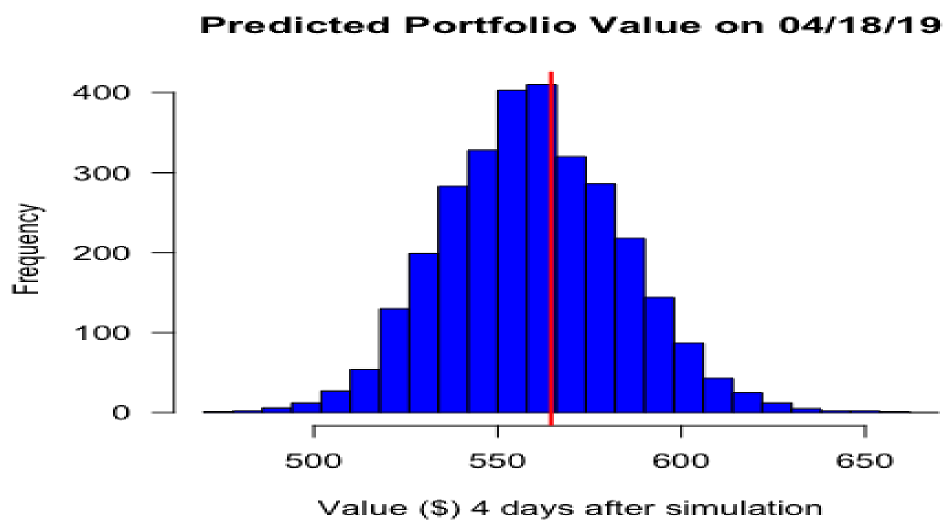
Figure 11 depicts a historical ETF value as well as the projected behavior using the base case.

Figure 11:



To check the predictive power of the model, we waited four days and checked the value of our ETF using real data (red line). We see that the estimated values for this day (calculated four days prior) predicted the value of the ETF quite well.

Figure 12:



The following plots show the expected behavior of the ETF when different scenarios are analyzed.

Figure 13:

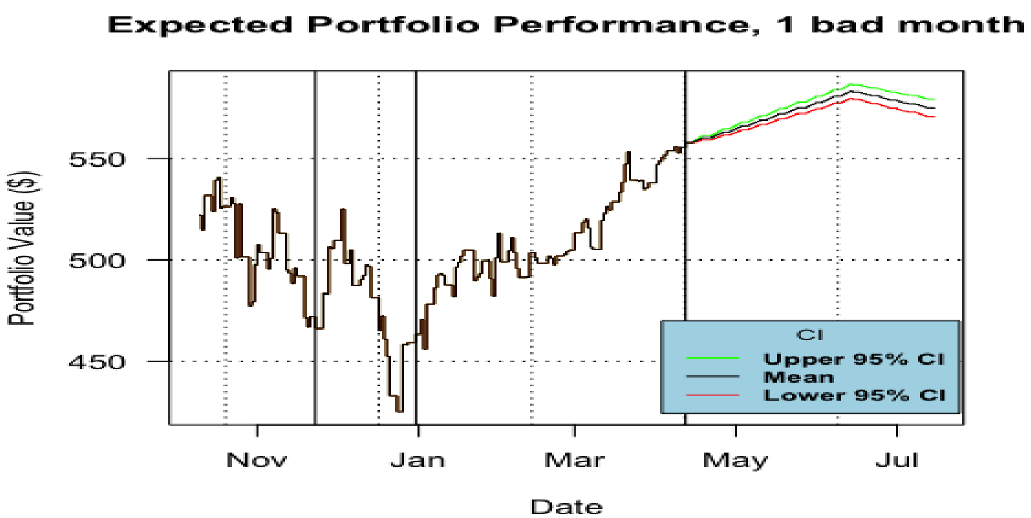


Figure 14:

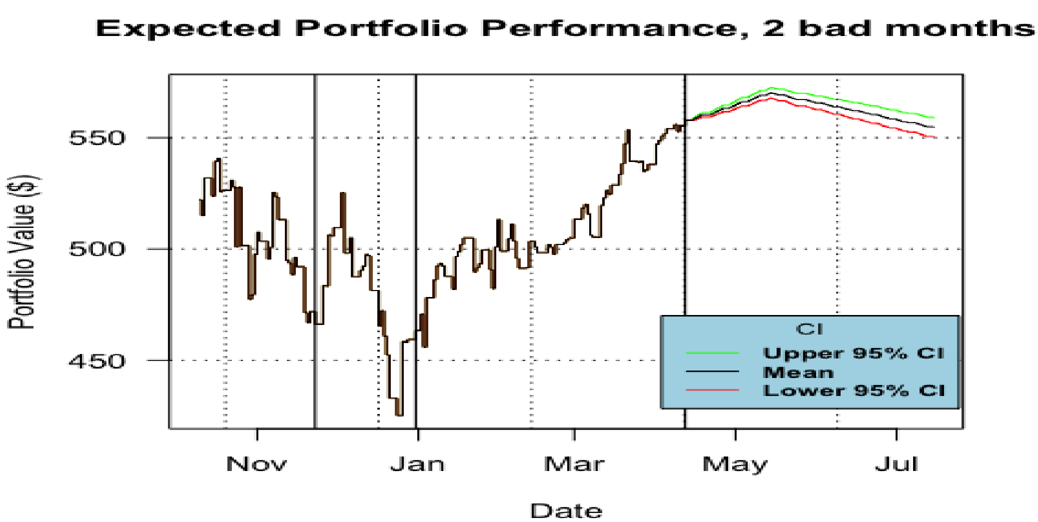
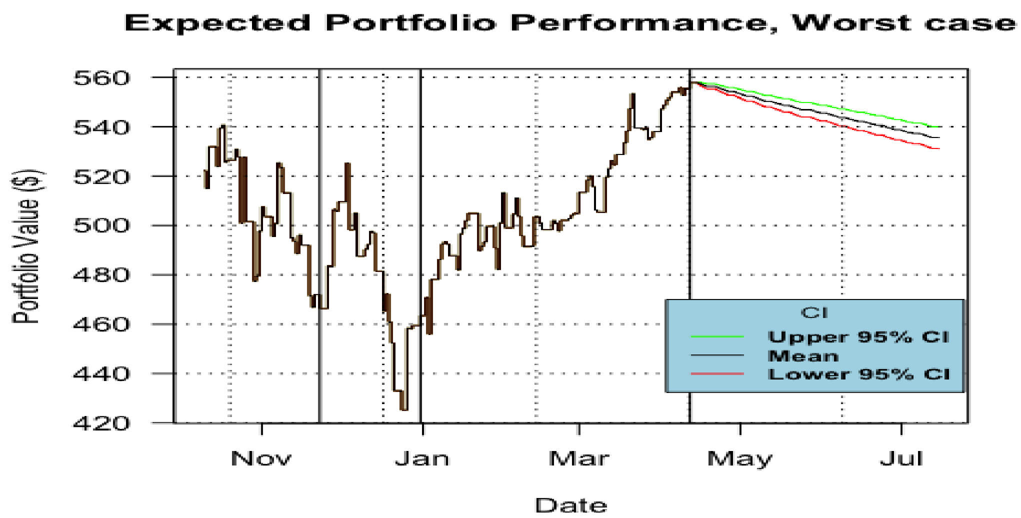


Figure 15:



Expected Portfolio Performance, Worst case
560
540
520
500
480
460
440
420
Portfolio Value ($)
Nov
Jan
Mar
May
Jul
Date
CI
Upper 95% CI
Mean
Lower 95% CI

6 References

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Liu, Yunan. Class notes, North Carolina State University. Print