Jacobs University Bremen | Spring Semester 2022

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# CA-MATH-804: Numerical Analysis

## Assignment Sheet 2. Due: February 23, 2022

**Exercise 1 [5 Points]**: Assuming $p, q = 1, 2, \infty, F$ recover the following table of equivalence constants $c_{pq}$ such that $\forall A \in \mathbb{R}^{n\times n}$ we have $\|A\|_p \le c_{pq} \|A\|_q$

| $c_{pq}$ | $q = 1$ | $q = 2$ | $q = \infty$ | $q = F$ |
|---|---|---|---|---|
| $p = 1$ | $1$ | $\sqrt{n}$ | $n$ | $\sqrt{n}$ |
| $p = 2$ | $\sqrt{n}$ | $1$ | $\sqrt{n}$ | $1$ |
| $p = \infty$ | $n$ | $\sqrt{n}$ | $1$ | $\sqrt{n}$ |
| $p = F$ | $\sqrt{n}$ | $\sqrt{n}$ | $\sqrt{n}$ | $1$ |

**Exercise 2 [5 Points]**: For any square matrix $A \in \mathbb{R}^{n\times n}$, prove the following relations

**a)** $\dfrac{1}{n} K_2(A) \le K_1(A) \le n K_2(A)$,

**b)** $\dfrac{1}{n} K_\infty(A) \le K_2(A) \le n K_\infty(A)$,

**c)** $\dfrac{1}{n^2} K_1(A) \le K_\infty(A) \le n^2 K_1(A)$,

where $K_p(A) = \|A\|_p \left\|A^{-1}\right\|_p$. These relations show that if a matrix is ill-conditioned in a certain norm, it remains so even in another norm, up to a scaling factor.

**Exercise 3 [5 Points]**: Prove the following claims:

**a)** If $A \in \mathbb{R}^{n\times n}$ fulfils one of the following criteria:

1. strict row-sum criterion (strict diagonal dominance) $\sum\limits_{\substack{i=1\\ i\neq j}}^{n} |a_{ij}| < |a_{jj}|$ for all $j$ with $1 \le j \le n$.
2. strict column-sum criterion $\sum\limits_{\substack{i=1\\ i\neq j}}^{n} |a_{ji}| < |a_{jj}|$ for all $j$ with $1 \le j \le n$

then the Jacobi method converges for any initial guess $x^{(0)}$.

**b)** Let $A \in \mathbb{C}^{n\times n}$, then every eigenvalue $\lambda$ of $A$ fulfils one of the following inequalities

$$|\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}| \,.$$

**Exercise 4 [5 Points]**: A matrix in which the sum of the absolute values of the entries of a row is equal for every row, is called *row equilibrated*.

**a)** Show that every regular matrix $A$ can be transformed into a row equilibrated matrix by multiplication with a regular diagonal matrix $D$.

**b)** Let $A$ and $D$ be as in a). Show that all non-singular diagonal matrices $\tilde{D}$ have

$$K_\infty\left(DA\right) \leq K_\infty\left(\tilde{D}A\right).$$

*Hint*: Let $C = DA$ and find a lower estimate for the condition number of $\tilde{D}D^{-1}C$ in terms of the condition number of $C$.

**Exercise 5 [5 Points]**: Prove Theorem 10 from class, which is repeated below.

**Theorem 10** *For $A$, $\delta A \in \mathbb{R}^{n\times n}$ and $b$, $\delta b \in \mathbb{R}^n$, consider perturbations of the problem $Ax = b$. Assume there exists $\gamma > 0$ such that*

$$\|\delta A\| \leq \gamma \|A\| \quad \textit{and} \quad \|\delta b\| \leq \gamma \|b\|$$

*in suitable norms. Also let $\gamma K(A) < 1$ where $K(A)$ is the condition number of $A$ in the norm used above. Then the perturbation $\delta x$ of the solution fulfils*

$$\frac{\|x\delta x\|}{\|x\|} \leq \frac{1+\gamma K(A)}{1-\gamma K(A)} \quad \textit{and} \quad \frac{\|\delta x\|}{\|x\|} \leq \frac{2\gamma K(A)}{1-\gamma K(A)}.$$

*Hints*: Remember that $(A+\delta A)(x+\delta x) = b + \delta b$. Use compatibility and submultiplicativity of the norms. Use Theorem 7 from class which says something about the invertibilty of $\mathrm{Id} + B$ for matrices $B$.

**Exercise 6 [4 Points]**: Verify that the matrix $B \in \mathbb{R}^{n\times n}$ defined by

$$b_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 1 & \text{if } i < j, \\ 1 & \text{if } i > j, \end{cases}$$

has determinant $\det(B) = 1$ and that $K_\infty(B) = n2^{n-1}$.

**Exercise 7 [2 Points]**: Prove that $K(AB) \leq K(A)\,K(B)$ for any two matrices $A$, $B \in \mathbb{R}^{n\times n}$.