MECH1010 VECTORS I – Solutions

1. a) $\hat{\mathbf{a}} = \begin{pmatrix} \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \\ \frac{1}{\sqrt{14}} \end{pmatrix}$ it should not be $\frac{1}{\sqrt{14}}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

   b) $3\mathbf{b} - \mathbf{c} = 3\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 2 \\ 5 \end{pmatrix}$

   c) $\mathbf{a} \bullet \mathbf{b} = \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix} = (3\times -1) + (2\times 0) + (1\times 2) = -1$

   d) $\cos\theta = \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-1}{\sqrt{14}\sqrt{5}} \quad \Rightarrow \theta \approx 96.86°$

2. Two points on the line l are (0 0 0) and (1 2 1), hence an equation for the line is:-

$$\mathbf{l} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix} = \lambda\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

   Three points on the plane Π are (1 0 –1), (0 –2 0) and (0 0 4), hence an equation for the plane is:-

$$\Pi = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \mu\left\{\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right\} + \nu\left\{\begin{pmatrix} 0 \\ -2 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix}\right\} = \begin{pmatrix} 0 \\ 0 \\ 4 \end{pmatrix} + \mu\begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} + \nu\begin{pmatrix} 0 \\ -2 \\ -4 \end{pmatrix}$$

   When the plane and line intersect the values of x, y and z must be the same for both, ie $\mathbf{\Pi} = \mathbf{l}$. This gives three equations, one each for the x, y and z components:-

$$\lambda = -\mu \qquad [1]$$
$$2\lambda = -2\mu - 2\nu \qquad [2]$$
$$\lambda = 4 + \mu - 4\nu \qquad [3]$$

   From [1] and [2] $\nu = 0$, from [1] and [3] $\lambda = 2$ and $\mu = -2$.
   Substitute value of $\lambda = 2$ into equation of line to find point of intersection (2 4 2) and check that this is correct by substituting values of $\mu = -2$ and $\nu = 0$ into equation of plane, it should give same point.

3. (Work done) = (force) x (distance moved in the direction of force)

   Let $\mathbf{F}$ be the force vector, then $|\mathbf{F}| = 6\text{N}$ Also $\mathbf{F} = \lambda\mathbf{d}$, hence $\lambda = \frac{|\mathbf{F}|}{|\mathbf{d}|} = \frac{6}{\sqrt{6}} = \sqrt{6}$

   Distance moved in the direction of the force is $|\mathbf{b}|\cos\theta$ where $\mathbf{b} = \begin{pmatrix} -1 \\ -3 \\ -5 \end{pmatrix}$ note moving *to* the origin from (1, 3, 5),

   and θ is the angle between direction of motion and the direction of action of the force

   Thus work done $= |\mathbf{F}||\mathbf{b}|\cos\theta$

$$= \sqrt{6}(\mathbf{d} \bullet \mathbf{b})$$
$$= \sqrt{6}(-1 - 3 + 10)$$
$$= 6\sqrt{6}\ \text{Nm} \approx 14.7\ \text{Nm}$$

4. $\left|\vec{MP}\right| = \left|\vec{OP}\right|\cos\theta$

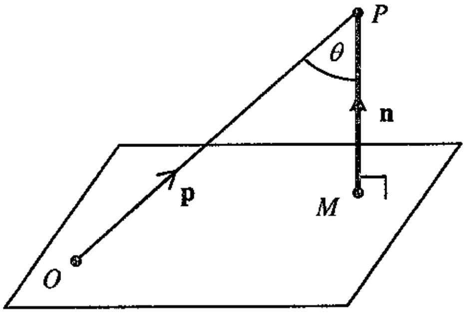
$$= \hat{\mathbf{n}} \bullet \mathbf{p}$$

$$= \frac{\mathbf{n} \bullet \mathbf{p}}{|\mathbf{n}|}$$

$$= \frac{8-4+20}{\sqrt{4+4+16}}$$

$$= \frac{24}{\sqrt{24}}$$

$$= 2\sqrt{6} \approx 4.90$$



5.

a) $\mathbf{a}\times\mathbf{c} = \begin{pmatrix}3\\2\\1\end{pmatrix}\times\begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{pmatrix}(2\times1)-(1\times-2)\\-\{(3\times1)-(1\times2)\}\\(3\times-2)-(2\times2)\end{pmatrix} = \begin{pmatrix}4\\-1\\-10\end{pmatrix}$

b) $\mathbf{b}\times\mathbf{c} = \begin{pmatrix}-1\\0\\2\end{pmatrix}\times\begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{pmatrix}(0\times1)-(2\times-2)\\-\{(-1\times1)-(2\times2)\}\\(-1\times-2)-(0\times2)\end{pmatrix} = \begin{pmatrix}4\\5\\2\end{pmatrix}$

c) The area of the triangle is given by $\frac{1}{2}|\mathbf{a}\times\mathbf{c}| = \frac{1}{2}\left|\begin{pmatrix}4\\-1\\-10\end{pmatrix}\right| = \frac{1}{2}\sqrt{117} \approx 5.408\text{ units}^2$ using the result from part (a)

d) The vector form is $\mathbf{r} = \mathbf{d} + \lambda\mathbf{a} + \mu\mathbf{b}$
To find the Cartesian form the normal vector $\mathbf{n}$ to the plane is required:-

$$\mathbf{n} = \mathbf{a}\times\mathbf{b} = \begin{pmatrix}3\\2\\1\end{pmatrix}\times\begin{pmatrix}-1\\0\\2\end{pmatrix} = \begin{pmatrix}4\\-7\\2\end{pmatrix}$$

Equation has form $4x-7y+2z+D=0$. The value of D can be found as it is known that the plane passes through the point P at (-3 0 5):

$$4(-3)-7(0)+2(5)+D=0$$

This implies that $D = 2$, hence the Cartesian equation of the plane is; $4x-7y+2z+2=0$

6. Angle between planes is the same as the angle between their normal vectors
$\hat{\mathbf{n}}_1 \bullet \hat{\mathbf{n}}_2 = \cos\theta$

$$\theta = \cos^{-1}\left\{\frac{2-2+12}{3\times\sqrt{41}}\right\}$$

$$= \cos^{-1}\left\{\frac{4}{\sqrt{41}}\right\} \approx 51.34°$$

A vector parallel to the line of intersection is give by $\mathbf{n}_1\times\mathbf{n}_2 = \begin{pmatrix}1\\-2\\-2\end{pmatrix}\times\begin{pmatrix}2\\1\\-6\end{pmatrix} = \begin{pmatrix}14\\2\\5\end{pmatrix}$

A point on the line can be found by letting $z = 0$, this reduces Π_1 and Π_2 to $x-2y=10$ and $2x+y=0$
Hence $x = 2$ and $y = -4$.

$\mathbf{r} = \begin{pmatrix}2\\-4\\0\end{pmatrix} + \lambda\begin{pmatrix}14\\2\\5\end{pmatrix}$ or similar depending upon choice of point on line..

An alternative is to find two points on the line (e.g. let $z = 0$ and then let $z = 1$) and fit a line through the points.