VECTORS II SOLUTIONS

1

General equation of sphere

$$\underset{\sim}{s} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + \underset{\sim}{r} \qquad \text{(A)}$$

where $|\underset{\sim}{r}| = 3$

or $$\underset{\sim}{s} = \begin{pmatrix} 3 \\ -1 \\ 3 \end{pmatrix} + 3\,\hat{\underset{\sim}{r}} \qquad \text{(B)}$$

$\rightarrow$ 2

Equation of line $\underset{\sim}{q} = \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 2 \\ -1 \end{pmatrix}$

let $\underset{\sim}{q} = \underset{\sim}{s}$ (use $3\hat{\underset{\sim}{r}}$ version) and consider components

$$3r_x = -3\mu - 1$$
$$3r_y = 2\mu - 2$$
$$3r_z = -\mu - 2$$

1 each
(Note alternative approach if using eq (A))

but $|\hat{r}| = 1 \Rightarrow r_x^2 + r_y^2 + r_z^2 = 1$ 2

$$9 = (-3\mu - 1)^2 + (2\mu - 2)^2 + (-\mu - 2)^2$$

$$9 = 9\mu^2 + 6\mu + 1 + 4\mu^2 - 8\mu + 4 + \mu^2 + 4\mu + 4$$

$$0 = 14\mu^2 + 2\mu + 0 \qquad 2$$

2

$$\Rightarrow \mu = 0 \text{ or } -1/7$$

Line intersects sphere at $\begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 2\frac{3}{7} \\ -3\frac{2}{7} \\ 1\frac{1}{7} \end{pmatrix}$. 2

$\boxed{13}$

2 cont.

iii) Vector along z axis is $\begin{pmatrix}0\\0\\1\end{pmatrix}$ 1

Line passing through D and parallel to z axis is

$$\underline{r} = \begin{pmatrix}1\\3\\1\end{pmatrix} + \lambda\begin{pmatrix}0\\0\\1\end{pmatrix} \longleftarrow \text{Note, unit vector}$$ 1

Plane containing face ABC is found from normal to plane, use result from ii)

$8x + 5y - 6z = D$ — — — — 1

Passes through point $(0, 1, 2)$ $\Rightarrow D = -7$

$8x + 5y - 6z + 7 = 0$ — — — 1

Point of intersection

$8(1) + 5(3) - 6(1+\lambda) + 7 = 0$

$\Rightarrow \underline{\underline{\lambda = 4}}$ — — — — — 1

Distance of D from plane is 4 as measured along z axis. 1

[Point of intersection is $(1,3,5)$ though this is not required] $\boxed{6}$

iv) Vol. of tetrahedron is $\frac{1}{6}\left(\overrightarrow{AB} \times \overrightarrow{AC} \cdot \overrightarrow{AD}\right)$ 1

(or $\frac{1}{3}$ base area × perp height) $\frac{1}{2}$

$\overrightarrow{AD} = \begin{pmatrix}1\\2\\-1\end{pmatrix}$ — —

$\text{Vol} = \frac{1}{6}\begin{pmatrix}8\\5\\-6\end{pmatrix} \cdot \begin{pmatrix}1\\2\\-1\end{pmatrix}$ $\frac{1}{2}$

$= \frac{1}{6}\left(8 + 10 + 6\right)$ — —

$= \underline{\underline{4}}$ units cubed. 1 $\boxed{3}$

↑ Minus $\frac{1}{2}$ if omitted.

$\boxed{\text{TOTAL } 15}$

VECTORS II

SOLUTIONS

4i

(1) $\dfrac{x-1}{1} = \dfrac{y+6}{-4} = \dfrac{z-3}{-5}$

(2) $\dfrac{1-2x}{2} = \dfrac{y-1}{4} = \dfrac{z}{5}$

re-writing line (2) $\dfrac{x-\frac{1}{2}}{-1} = \dfrac{y-1}{4} = \dfrac{z}{5}.$

Lines are parallel since $\underset{\sim}{U_1} = -\underset{\sim}{U_2} = \begin{pmatrix}1\\-4\\-5\end{pmatrix}$ - - - - - 1

$\underset{\sim}{U_1} = \begin{pmatrix}1\\-4\\-5\end{pmatrix}$ $\quad \underset{\sim}{a_1} = \begin{pmatrix}1\\-6\\3\end{pmatrix}$ $\quad \underset{\sim}{a_2} = \begin{pmatrix}\frac{1}{2}\\1\\0\end{pmatrix}$ $\quad$ ½ Each

Find normal $\underset{\sim}{n}$ to plane containing lines (1) and (2)

$$\underset{\sim}{n} = \underset{\sim}{U_1} \times (\underset{\sim}{a_1} - \underset{\sim}{a_2}) = \begin{pmatrix}1\\-4\\-5\end{pmatrix} \times \begin{pmatrix}\frac{1}{2}\\-7\\3\end{pmatrix}$$

$$= \begin{pmatrix}-47\\-5\frac{1}{2}\\-5\end{pmatrix}$$ — 2

Find vector $\underset{\sim}{d}$ ~~normal to~~ in plane containing (1) and (2), but normal to lines (1) and (2)

$$\underset{\sim}{d} = \underset{\sim}{U_1} \times \underset{\sim}{n} = \begin{pmatrix}1\\-4\\-5\end{pmatrix} \times \begin{pmatrix}-47\\-5\frac{1}{2}\\-5\end{pmatrix}$$

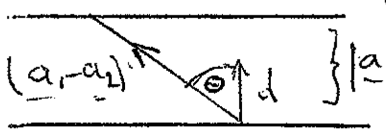
[$\underset{\sim}{d}$ is normal to both (1) and (2)]

$$= \begin{pmatrix}-7\frac{1}{2}\\240\\-193{\cdot}5\end{pmatrix}$$ — 2

$$\text{Distance is } \frac{(\underset{\sim}{a_1}-\underset{\sim}{a_2})\cdot \underset{\sim}{d}}{|d|} = \frac{-3\frac{3}{4} - 1680 - 580{\cdot}5}{308{\cdot}38}$$ $1\frac{1}{2}$

$$= 7{\cdot}34$$ (1)

1



$\}|\underset{\sim}{a_1}-\underset{\sim}{a_2}|\cos\theta$

(2)

$\boxed{10}$

VECTORS II

4 iii)

Line ① $\frac{x}{1} = \frac{y+2}{1} = \frac{z-1}{1/6}$

Line ② $\frac{x+1}{1} = \frac{y}{1/2} = \frac{z}{-1/12}$ or $\underline{r} = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 12 \\ 6 \\ -1 \end{pmatrix}$

$\underline{U}_1 = \begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix}$ $\quad \underline{U}_2 = \begin{pmatrix} 12 \\ 6 \\ -1 \end{pmatrix}$

$\underline{a}_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ $\quad \underline{a}_2 = \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}$ — 2 ($\frac{1}{2}$ Each)

$$\text{Dist} = \frac{(\underline{a}_1 - \underline{a}_2) \cdot (\underline{U}_1 \times \underline{U}_2)}{|\underline{U}_1 \times \underline{U}_2|}$$ 1

$$\underline{U}_1 \times \underline{U}_2 = \begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix} \times \begin{pmatrix} 12 \\ 6 \\ -1 \end{pmatrix} = \begin{pmatrix} -12 \\ 18 \\ -36 \end{pmatrix}$$ 1

$$= 6 \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}.$$

$$\underline{a}_1 - \underline{a}_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

$$\text{Dist} = \frac{-2 - 6 - 6}{\sqrt{4 + 9 + 36}}$$ — 2

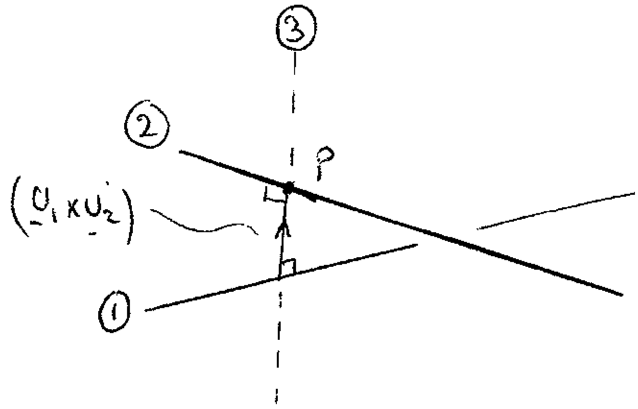
$$= \frac{-14}{7}$$

$$= 2$$

$10 + 6 = 16$

$\boxed{6}$

5) ii Method



Notation refs to Q4 ii

Consider the plane Π containing Line ① and line ③ where line ③ is mutually perpendicular to ① and ②

Equation of line ③ is $\underset{\sim}{P} + \lambda(\underline{U}_1 \times \underline{U}_2)$

$\underline{P}$ is the point where line ② intersects the plane Π

METHOD ②

To find Π. Normal to plane Π $(\underline{U}_1 \times \underline{U}_2) \times \underline{U}_1$

$$= \begin{pmatrix} -12 \\ 18 \\ -36 \end{pmatrix} \times \begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix} \text{ or } \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix} \times \begin{pmatrix} 6 \\ 6 \\ 1 \end{pmatrix} = \begin{pmatrix} +39 \\ -34 \\ -30 \end{pmatrix} \leftarrow ①$$

Point on plane is given by $\underline{a}_1$ $(0, -2, 1)$

$$+39x - 34y - 30z = 38 \leftarrow ①$$

To find $\underline{P}$, look at intersection of ② with Π.

$$39(-1+12\lambda) - 34(0+6\lambda) - 30(0-\lambda) = 38$$

$$\Rightarrow \lambda = \frac{77}{294} = 0.2619\ldots \leftarrow ①$$

$$\underline{P} = \begin{pmatrix} 2.1428 \\ 1.5714 \\ -0.2619 \end{pmatrix} \leftarrow ①$$

$\boxed{6}$

$$\text{Equation of line} = \underline{P} + \mu(\underline{U}_1 \times \underline{U}_2) \approx \begin{pmatrix} 2.1 \\ 1.6 \\ -0.3 \end{pmatrix} + \mu \begin{pmatrix} -2 \\ 3 \\ -6 \end{pmatrix}$$

Q/ $A = \begin{pmatrix} 1 & 3 & -4 \\ 0 & 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 4 \\ -2 & 1 \\ 3 & -1 \end{pmatrix}$, $C = \begin{pmatrix} 1 & 4 \\ 6 & 3 \end{pmatrix}$

(3×2) $\quad$ (2×3) $\quad$ (2×2)

$$AB = \begin{pmatrix} 1 & 3 & -4 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} 0 & 4 \\ -2 & 1 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 1\cdot0 + 3(-2) + (-4)\cdot3 & 1\cdot4 + 3\cdot1 + (-4)(-1) \\ 0\cdot0 + 1(-2) + (-1)\cdot3 & 0\cdot4 + 1\cdot1 + (-1)(-1) \end{pmatrix}$$

$(3\times2)(2\times3)$

$$= \begin{pmatrix} -18 & 3 \\ -5 & 0 \end{pmatrix}$$

$$BC = \begin{pmatrix} 0 & 4 \\ -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 0\cdot1 + 4\cdot6 & 0\cdot4 + 4\cdot3 \\ -2\cdot1 + 1\cdot6 & -2\cdot4 + 1\cdot3 \\ 3\cdot1 - 1\cdot6 & 3\cdot4 - 1\cdot3 \end{pmatrix} = \begin{pmatrix} 24 & 12 \\ 4 & -5 \\ -3 & +9 \end{pmatrix}$$

$(2\times3)(2\times2)$

$$CA = \begin{pmatrix} 1 & 4 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 1 & 3 & -4 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 1\cdot1 + 4\cdot0 & 1\cdot3 + 4\cdot1 & 1\cdot(-4) + (-1)\cdot4 \\ 6\cdot1 + 3\cdot0 & 6\cdot3 + 3\cdot1 & 6\cdot(-4) + 3(-1) \end{pmatrix}$$

$(2\times2)(3\times2)$

$$= \begin{pmatrix} 1 & 7 & -8 \\ 6 & 21 & -27 \end{pmatrix}$$

$$BA = \begin{pmatrix} 0 & 4 \\ -2 & 1 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & -4 \\ 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0\cdot1 + 4\cdot0 & 0\cdot3 + 4\cdot1 & 0(-4) + 4(-1) \\ -2\cdot1 + 1\cdot0 & -2\cdot3 + 1\cdot1 & (-2)\cdot(-4) + (1)(-1) \\ 3\cdot1 + (-1)\cdot0 & 3\cdot3 + (-1)1 & 3\cdot(-4) + (-1)(-1) \end{pmatrix}$$

$(2\times3)(3\times2)$

$$= \begin{pmatrix} 0 & 4 & -4 \\ -2 & -5 & 7 \\ 3 & 8 & -11 \end{pmatrix}$$

$$CC = \begin{pmatrix} 1 & 4 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 6 & 3 \end{pmatrix} = \begin{pmatrix} 1\cdot1 + 1\cdot6 & 1\cdot4 + 4\cdot3 \\ 6\cdot1 + 3\cdot6 & 6\cdot4 + 3\cdot3 \end{pmatrix} = \begin{pmatrix} 7 & 16 \\ 24 & 33 \end{pmatrix}$$

$(2\times2)(2\times2)$

If include transposes $A: (3\times2)$, $A^T: (2\times3)$, $B: (2\times3)$, $B^T: (3\times2)$, $C, C^T = (2\times2)$

AA^T, BB^T, CC^T, A^TA, B^TB, C^TC, A^TB^T, B^TA^T

7/

(i) $\underline{i} \cdot \underline{j} = (1, 0) \cdot (0, 1) = 1 \cdot 0 + 0 \cdot 1 = 0$

Hence orthogonal.

(ii) $\underline{v}_1 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$

(iii) $\hat{\underline{v}}_1 = \dfrac{\underline{v}_1}{|\underline{v}_1|}$ where $|\underline{v}_1| = \sqrt{\cos^2\theta + \sin^2\theta} = 1$

$\Rightarrow \hat{\underline{v}}_1 = \underline{v}_1$

(iv) $\underline{v}_1 \cdot \underline{j} = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \sin\theta$

(v) $\underline{v}_2 = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix}$

$\underline{v}_1 \cdot \underline{v}_2 = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \cdot \begin{pmatrix} -\sin\theta \\ \cos\theta \end{pmatrix} = -\cos\theta\sin\theta + \sin\theta\cos\theta$

$= 0$, Hence orthogonal.