

JTMS-MAT-13: Numerical Methods

Exam & Solutions: Saturday 24 August 2024

All questions carry equal marks. Answer 5 questions only. Please only use the booklet provided, clearly stating which questions are to be marked.

Note that all trigonometric values should be expressed in radians.

Question 1:

- (a) State a condition which means a square matrix will not be invertible.
- (b) Given the matrix

$$A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix},$$

use Gaussian elimination to show the row echelon form of the matrix A is given by

$$U = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 0 & 1 \end{pmatrix}.$$

- (c) By applying Gaussian elimination, or any other method, show the solution to the linear equation $A\vec{x} = \vec{b}$, where \vec{b} is given by

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \text{ is } \vec{x} = \begin{pmatrix} 110.25 \\ -24 \\ 9.5 \end{pmatrix}.$$

- (d) If an $n \times n$ matrix is invertible, what is the order of the upper limit for the number of arithmetic operations to yield the inverse for Gaussian elimination?

- (a) A matrix is not invertible if the determinant is zero. Equivalent conditions, such as if rank is not full, or rows/columns are not linearly independent are also acceptable.
- (b) To express the matrix in row echelon form, start by eliminating the element in the first column of the second row by subtracting 3 times the first row from the second row:

$$\begin{pmatrix} 4 & 12 & -16 \\ 12 - 3 \cdot 4 & 40 - 3 \cdot 12 & -38 - 3 \cdot (-16) \\ -16 & -38 & 90 \end{pmatrix} = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ -16 & -38 & 90 \end{pmatrix}.$$

Next, eliminate the element in the first column of the third row. To do this, add 4 times the first row to the third row

$$\begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ -16 + 4 \cdot 4 & -38 + 4 \cdot 12 & 90 + 4 \cdot (-16) \end{pmatrix} = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 10 & 26 \end{pmatrix}.$$

Finally, eliminate the element in the second column of the third row by subtracting 2.5 times the second row from the third row

$$\begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 10 - 2.5 \cdot 4 & 26 - 2.5 \cdot 10 \end{pmatrix} = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 0 & 1 \end{pmatrix}.$$

The final upper triangular form of the matrix A is:

$$U = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) Applying the elementary row operations to the righthand side vectors,

$$\left(\begin{array}{ccc|c} 4 & 12 & -16 & 1 \\ 12 & 40 & -38 & 2 \\ -16 & -38 & 90 & 3 \end{array} \right) \mapsto \left(\begin{array}{ccc|c} 4 & 12 & -16 & 1 \\ 0 & 4 & 10 & -1 \\ -16 & -38 & 90 & 3 \end{array} \right) \mapsto \left(\begin{array}{ccc|c} 4 & 12 & -16 & 1 \\ 0 & 4 & 10 & -1 \\ 0 & 10 & 26 & 7 \end{array} \right)$$

and

$$\left(\begin{array}{ccc|c} 4 & 12 & -16 & 1 \\ 0 & 4 & 10 & -1 \\ 0 & 10 & 26 & 7 \end{array} \right) \mapsto \left(\begin{array}{ccc|c} 4 & 12 & -16 & 1 \\ 0 & 4 & 10 & -1 \\ 0 & 0 & 1 & 9.5 \end{array} \right)$$

Thus if the unknown vector $\vec{x} = (x_1, x_2, x_3)$, then $x_3 = 9.5$, so that $4x_2 + 95 = -1 \Rightarrow x_2 = -24$ and finally $4x_1 + 12 \times 24 + 9.5 = 1 \Rightarrow x_1 = 110.25$.

(d) The number of operations is proportional to the cube of the number of rows, i.e. $\mathcal{O}(n^3)$.

Question 2:

(a) Find the Jacobian matrix for the vector-valued function

$$f(x, y) = \begin{pmatrix} 4x^2 - 20x + \frac{1}{4}y^2 - 8 \\ \frac{1}{2}xy^2 + 2x - 5y + 8 \end{pmatrix}.$$

(b) Show the inverse of the Jacobian matrix is

$$J^{-1} = \frac{1}{|J|} \begin{pmatrix} xy - 5 & -\frac{1}{2}y \\ -\frac{1}{2}y^2 - 2 & 4(2x - 5) \end{pmatrix}, \quad \text{where} \quad |J| = 8x^2y - 20xy - 40x + 100 - \frac{1}{4}y^3 - y.$$

(c) Let $\vec{u}_n = (x_n, y_n)^T$. Then, using $\vec{u}_{n+1} = \vec{u}_n - J^{-1}(\vec{u}_n) f(\vec{u}_n)$, with an initial guess $\vec{u}_0 = (0, 0)^T$, show that the first iteration of Newton's method yields $(-0.4, 1.44)^T$.

(a) Compute the partial derivatives:

$$\begin{aligned} \frac{\partial f_1}{\partial x} &= \frac{\partial}{\partial x} \left(4x^2 - 20x + \frac{1}{4}y^2 - 8 \right) \\ &= 8x - 20 = 4(2x - 5) \end{aligned}$$

$$\begin{aligned} \frac{\partial f_1}{\partial y} &= \frac{\partial}{\partial y} \left(4x^2 - 20x + \frac{1}{4}y^2 - 8 \right) \\ &= \frac{1}{2}y \end{aligned}$$

$$\begin{aligned} \frac{\partial f_2}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1}{2}xy^2 + 2x - 5y + 8 \right) \\ &= \frac{1}{2}y^2 + 2 \end{aligned}$$

$$\begin{aligned} \frac{\partial f_2}{\partial y} &= \frac{\partial}{\partial y} \left(\frac{1}{2}xy^2 + 2x - 5y + 8 \right) \\ &= xy - 5 \end{aligned}$$

Thus, the Jacobian matrix J is:

$$J = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{pmatrix} = \begin{pmatrix} 4(2x - 5) & \frac{1}{2}y \\ \frac{1}{2}y^2 + 2 & xy - 5 \end{pmatrix}$$

(b) For the inverse, first compute the determinant:

$$\begin{aligned} \det(J) &= 4(2x - 5)(xy - 5) - \frac{1}{2}y \left(\frac{1}{2}y^2 + 2 \right) \\ &= 4(2x^2y - 5xy - 10x + 25) - \frac{1}{4}y^3 - y \\ &= 8x^2y - 20xy - 40x + 100 - \frac{1}{4}y^3 - y \end{aligned}$$

The formula for the inverse of a 2×2 matrix is given by:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \Rightarrow A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \quad \text{when} \quad \det(A) \neq 0.$$

Thus,

$$J^{-1} = \frac{1}{|J|} \begin{pmatrix} xy - 5 & -\frac{1}{2}y \\ -\frac{1}{2}y^2 - 2 & 4(2x - 5) \end{pmatrix}, \quad \text{where} \quad |J| = 8x^2y - 20xy - 40x + 100 - \frac{1}{4}y^3 - y.$$

[(c) Evaluating the inverse of the Jacobian at the initial guess yields

$$J^{-1}(\vec{u}_0) = \frac{1}{100} \begin{pmatrix} -5 & 0 \\ -2 & -20 \end{pmatrix}.$$

With $\vec{u}_0 = (0, 0)^T$, then

$$f(u_0) = \begin{pmatrix} -8 \\ +8 \end{pmatrix},$$

so

$$\vec{u}_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \frac{1}{100} \begin{pmatrix} -5 & 0 \\ -2 & -20 \end{pmatrix} \begin{pmatrix} -8 \\ +8 \end{pmatrix} = \frac{1}{100} \begin{pmatrix} -40 \\ 144 \end{pmatrix}.$$

Question 3:

Consider the integral

$$I = \int_1^2 f(x) dx = \int_1^2 \frac{dx}{x} = \ln(2) = 0.6931471805599453$$

(a) Given the Trapezium rule,

$$I_n = \frac{h}{2} \left(f(x_0) + f(x_n) + 2 \sum_{i=1}^{n-1} f(x_i) \right)$$

where $h = (b - a) / n$ and n is the number of intervals, show that the approximations to the integral for $n = 2^k$ where $k = 0, 1$ and 2 are

k	n	I_n
0	1	0.75
1	2	0.70833333
2	4	0.69702381

(b) Noting that the Trapezium rule has error behaviour

$$I = I_n + a_1 h^2 + a_2 h^4 + \dots$$

for some constants a , and considering the difference between the errors of the Trapezium rule for h and $h/2$, derive the Romberg formula

$$R_k^1 = \frac{1}{3} (4R_k^0 - R_{k-1}^0)$$

where $R_0^0 = I_1$, $R_1^0 = I_2$ etc.

(c) Using the values from the Trapezium rule for $I_k = R_k^0$, show that $R_2^1 = 0.693253$.

(a) For the first value $h = 1$,

$$R_0^0 = \frac{1}{2} \left(1 + \frac{1}{2} \right) = 0.75.$$

For the second value, $k = 1$, $n = 2$,

$$h = \frac{1-0}{2} = 0.5, \quad x_0 = 1, \quad x_1 = 1.5 \quad \text{and} \quad x_2 = 2.$$

Using the function $f(x) = \frac{1}{x}$

$$f(1) = \frac{1}{1} = 1$$

$$f(1.5) = \frac{1}{1.5} = 2/3$$

$$f(2) = \frac{1}{2} = 0.5$$

Substituting these values into the formula:

$$I_1 = \frac{0.5}{2} (1 + 0.5 + 2 \cdot 2/3) = 0.25 (1.5 + 4/3) = 0.70833$$

For the next value, $k = 2$, $n = 2^2 = 4$, $h = 1/4$

$$\begin{aligned} f(1) &= 1 \\ f(1.25) &= \frac{1}{1.25} = \frac{1}{5/4} = \frac{4}{5} = 0.8 \\ f(1.5) &= 2/3 \\ f(1.75) &= \frac{1}{1.75} = \frac{1}{7/4} = \frac{4}{7} = 0.5714285714285714 \\ f(2) &= 0.5 \end{aligned}$$

Substituting these values into the formula:

$$I_2 = \frac{0.25}{2} \left(1 + 0.5 + 2 \left(\frac{4}{5} + \frac{2}{3} + \frac{4}{7} \right) \right) = 0.69702380952$$

(b) Consider the expansion for I_{i-2} and I_{i-1} , noting that I_{i-1} has twice the steps size.

$$\int_a^b f(x) \, dx = R_{i-1}^0 + a_2 h^2 + a_4 h^4 + \dots$$

and

$$\int_a^b f(x) \, dx = R_i^0 + a_2 \left(\frac{h}{2} \right)^2 + a_4 \left(\frac{h}{2} \right)^4 + \dots$$

then an $\mathcal{O}(h^4)$ approximation is made by multiplying I_{i-1} by four and subtracting I_{i-2} . Let this be denoted by R_1^k . This gives a value which approximates three times the integral, thus the new formula is, as required

$$R_k^1 = \frac{1}{3} (4R_k^0 - R_{k-1}^0).$$

(c) By the formula,

$$R_2^1 = \frac{1}{3} (4R_2^0 - R_1^0) = \frac{1}{3} (4 \cdot 0.69702380952 - 0.70833333333) = 0.69325396825.$$

Question 4:

- (a) For the numerical solutions of ordinary differential equations, $y' = f(y, t)$, what is an explicit method?
- (b) Using a Taylor expansion, show the forward Euler scheme for a first-order differential equation is

$$y_{n+1} = y_n + hf(y_n, t_n)$$

where y_n denotes the solution at $t_n = t_0 + nh$ for a time step h and initial condition $y(t_0)$.

- (c) For the second-order linear ordinary differential equation

$$y''(t) = -4y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1$$

show, by considering the backward difference approximation $\nabla \tilde{u}(t_{n+1}) \approx (\tilde{u}_{n+1} - \tilde{u}_n)/h$, where $\tilde{u} = (y, y')$ and \tilde{u}_n denotes the solution at $t_n = t_0 + nh$, that the backward Euler scheme yields

$$\tilde{u}_{n+1} = \frac{1}{1 + 4h - h^2} \begin{pmatrix} 1 + 4h & -h \\ -h & 1 \end{pmatrix} \tilde{u}_n.$$

- (d) Compute the first two steps of the backward Euler scheme for the system given in (c) with $h = 0.1$.

- (a) The function to be solved only involves the current and past values of the unknown function.
- (b) The Taylor expansion is given by

$$y(t_{n+1}) = y(t_n + h) = y(t_n) + hy'(t_n) + \frac{h^2}{2!}y''(t_n) + \dots$$

where $t_{n+1} = t_n + h$ and h is the step size. For the forward Euler scheme, we will truncate the series after the first-order term

$$y(t_{n+1}) \approx y(t_n) + hy'(t_n)$$

and as $y'(t_n) = f(t_n)$, so

$$y_{n+1} = y_n + hf(t_n, y_n).$$

- (c) Rearranging the expression, yields $\tilde{u}_{n+1} = \tilde{u}_n + h\nabla(\tilde{u}_{n+1})$ The backwards Euler scheme is given by $u_{n+1} = u_n + hf(u_{n+1})$, i.e. $\tilde{v}_{n+1} = \tilde{v}_n + hA\tilde{v}_{n+1}$ as the system is linear and can be written as $\nabla \nabla v = A\tilde{v}$. Thus, $(I - hA)\tilde{v}_{n+1} = \tilde{v}_n$, so that $\tilde{v}_{n+1} = (I - hA)^{-1}\tilde{v}_n$. The inverse of the matrix for the linear system is then given by

$$\begin{aligned} (I - hA)^{-1} &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - h \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} \right)^{-1} \\ &= \begin{pmatrix} 1 & -h \\ -h & 1 + 4h \end{pmatrix}^{-1} \\ &= \frac{1}{1 + 4h - h^2} \begin{pmatrix} 1 + 4h & -h \\ -h & 1 \end{pmatrix}. \end{aligned}$$

- (d) With step size $h = 0.1$, then two steps, \tilde{u}_1 and \tilde{u}_2 , must be computed from the initial data $\tilde{u}_0 = (0, 1)$.

Using the matrix derived in the formula given, the solution at $t = 0.1$ is given by

$$\begin{aligned}
 \vec{u}_1 &= \frac{1}{1 + 4 \times 0.1 - 0.1^2} \begin{pmatrix} 1 + 4 \times 0.1 & -0.1 \\ -0.1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 &= \frac{100}{139} \begin{pmatrix} 1.4 & -0.1 \\ -0.1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 &= \frac{100}{139} \begin{pmatrix} 140 & -10 \\ -10 & 100 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
 &= \frac{100}{139} \begin{pmatrix} 140 & -10 \\ -10 & 100 \end{pmatrix} \\
 &= \begin{pmatrix} -10/139 \\ 100/139 \end{pmatrix} \\
 &= \begin{pmatrix} -0.071942 \\ 0.719424 \end{pmatrix}.
 \end{aligned}$$

The second step is then given by

$$\begin{aligned}
 \vec{u}_2 &= \frac{1}{139} \begin{pmatrix} 140 & -10 \\ -10 & 100 \end{pmatrix} \begin{pmatrix} -10/139 \\ 100/139 \end{pmatrix} \\
 &= \begin{pmatrix} 1.007194 & -0.07194244604316546 \\ -0.07194244604316546 & 0.7194244604316546 \end{pmatrix} \begin{pmatrix} -0.071942 \\ 0.719424 \end{pmatrix} \\
 &= \begin{pmatrix} -0.12421717 \\ 0.52274727 \end{pmatrix}.
 \end{aligned}$$

Question 5:

Integrals can be numerically evaluated using the composite Simpson's 1/3 rule, which is given by

$$I = \int_a^b f(x) dx \approx \frac{h}{3} \left(f(x_0) + 4 \sum_{i=1}^{n/2} f(x_{2i-1}) + 2 \sum_{i=1}^{n/2-1} f(x_{2i}) + f(x_n) \right)$$

where $x_i = a + ih$, with $h = (b - a)/n$ for $i = 0, \dots, n$, where n , the number of subintervals, is even. Given the integral

$$I = \int_0^1 \frac{1}{3} + \cos(\pi x) dx$$

what is the difference between the exact and the approximate integral using Simpson's rule with six subintervals?

☐ 1/2

☐ 1/6

☒ 0

☐ 1/10

☐ 1/3

☐ 1/7

The integral is given by:

$$\begin{aligned} I &= \int_0^1 \frac{1}{3} + \cos(\pi x) dx = \left[\frac{x}{3} - \frac{\sin(\pi x)}{\pi} \right]_0^1 \\ &= \left[\frac{x}{3} \right]_0^1 - \left[\frac{\sin(\pi x)}{\pi} \right]_0^1 \\ &= \frac{1}{3} - 0 - \frac{1}{\pi} (\sin(0) - \sin(\pi)) = \frac{1}{3} \end{aligned}$$

For the approximation, with $a = 0$, $b = 1$, and $n = 6$:

$$\Delta x = \frac{1-0}{6} = \frac{1}{6}.$$

The points x_i are:

$$x_0 = 0, \quad x_1 = \frac{1}{6}, \quad x_2 = \frac{2}{6} = \frac{1}{3}, \quad x_3 = \frac{3}{6} = \frac{1}{2}, \quad x_4 = \frac{4}{6} = \frac{2}{3}, \quad x_5 = \frac{5}{6} \quad \text{and} \quad x_6 = 1.$$

The function values $f(x_i) = \frac{1}{3} + \cos(\pi x_i)$ are:

$$\begin{aligned} f(x_0) &= \frac{1}{3} + \cos(0) = \frac{1}{3} + 1 = \frac{4}{3}, \\ f(x_1) &= \frac{1}{3} + \cos\left(\frac{\pi}{6}\right) = \frac{1}{3} + \frac{\sqrt{3}}{2}, \\ f(x_2) &= \frac{1}{3} + \cos\left(\frac{\pi}{3}\right) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}, \\ f(x_3) &= \frac{1}{3} + \cos\left(\frac{\pi}{2}\right) = \frac{1}{3} + 0 = \frac{1}{3}, \\ f(x_4) &= \frac{1}{3} + \cos\left(\frac{2\pi}{3}\right) = \frac{1}{3} - \frac{1}{2} = -\frac{1}{6}, \\ f(x_5) &= \frac{1}{3} + \cos\left(\frac{5\pi}{6}\right) = \frac{1}{3} - \frac{\sqrt{3}}{2}, \\ f(x_6) &= \frac{1}{3} + \cos(\pi) = \frac{1}{3} - 1 = -\frac{2}{3}. \end{aligned}$$

Now apply Simpson's rule:

$$\begin{aligned} I &\approx \frac{\Delta x}{3} [f(x_0) + 4(f(x_1) + f(x_3) + f(x_5)) + 2(f(x_2) + f(x_4)) + f(x_6)]. \\ &= \frac{1/6}{3} \left[\frac{4}{3} + 4 \left(\frac{1}{3} + \frac{\sqrt{3}}{2} + \frac{1}{3} + \frac{1}{3} - \frac{\sqrt{3}}{2} \right) + 2 \left(\frac{5}{6} - \frac{1}{6} \right) - \frac{2}{3} \right]. \end{aligned}$$

Simplify the sums:

$$\begin{aligned} I &\approx \frac{1}{18} \left[\frac{4}{3} + 4 \left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3} \right) + 2 \left(\frac{4}{6} \right) - \frac{2}{3} \right]. \\ &= \frac{1}{18} \left[\frac{4}{3} + 4 + 2 \left(\frac{2}{3} \right) - \frac{2}{3} \right] \\ &= \frac{1}{18} \left[\frac{4}{3} + \frac{12}{3} + \frac{4}{3} - \frac{2}{3} \right] \\ &= \frac{1}{18} \left[\frac{4 + 12 + 4 - 2}{3} \right] \\ &= \frac{1}{18} \left[\frac{18}{3} \right] \\ &= \frac{1}{3}. \end{aligned}$$

Thus, the difference between the approximation and the exact value is

$$\left| \frac{1}{3} - \frac{1}{3} \right| = 0$$

Question 6:

Given the following data:

i	0	1	2
x_i	0	2	4
y_i	2	1	2

Newton interpolation constructs a interpolating polynomial $p(x)$, using the formula $p = \sum_{i=0}^n \alpha_i n_i(x)$, where the basis polynomials are defined as

$$n_0(x) = 1, \quad n_i(x) = (x - x_0)(x - x_1) \cdots (x - x_{i-1})$$

where $p(x_i) = y_i$. Using Newton interpolation, which are the correct collocation matrix Φ and weighting vector $\vec{\alpha}$ such that $\Phi \vec{\alpha} = \vec{y}$ where $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2)$ and $\vec{y} = (y_0, y_1, y_2)$.

☒ $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}$
☐ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ -1/4 \end{pmatrix}$

☐ $\begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 8 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ -1 \end{pmatrix}$
☐ $\begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} -2 \\ -1/2 \\ -1/3 \end{pmatrix}$

☐ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}$
☐ $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

The Newton interpolating polynomial can be written in the form:

$$p(x) = \alpha_0 + \alpha_1(x - x_0) + \alpha_2(x - x_0)(x - x_1)$$

For the given data points (x_0, y_0) , (x_1, y_1) and (x_2, y_2) , the collocation matrix Φ is constructed by evaluating the basis functions of the Newton polynomial at each x_i , where the basis functions are given by $n_0 = 1$, $n_1 = x - x_0$ and $n_2 = (x - x_0)(x - x_1)$.

- For $x_0 = 0$, $n_0(x_0) = 1$, $n_1(x_0) = 0$ and $n_2(x_0) = 0$.
- For $x_1 = 2$, $n_0(x_1) = 1$, $n_1(x_1) = 2$ and $n_2(x_1) = 0$.
- For $x_2 = 4$, $n_0(x_2) = 1$, $n_1(x_2) = 4$ and $n_2(x_2) = 8$.

The system $\Phi \vec{\alpha} = \vec{y}$ is

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

Therefore, the correct collocation matrix Φ is:

$$\Phi = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}.$$

and

$$\vec{\alpha} = \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix}.$$

So that

$$p = 2 - \frac{x}{2} + \frac{x}{4}(x - 2)$$

Question 7:

Using the Jacobi scheme, $\tilde{x}_{n+1} = (I - D^{-1}A)\tilde{x}_n + D^{-1}b$, where D is the diagonal matrix of A , what is the second iterate, \tilde{x}_2 , of the solution to the system $Ax = b$ with

$$A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix}, \quad \tilde{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

- ☐ $\begin{pmatrix} 0.456 \\ 0.7421 \\ -0.8123 \end{pmatrix}$
☒ $\begin{pmatrix} 0.6694 \\ 0.2306 \\ 0.5183 \end{pmatrix}$
- ☐ $\begin{pmatrix} 1.7326 \\ -0.6034 \\ 0.6777 \end{pmatrix}$
☐ $\begin{pmatrix} 0.6789 \\ 0.2481 \\ 0.1111 \end{pmatrix}$
- ☐ $\begin{pmatrix} 0.0013 \\ 0.0299 \\ 3.5217 \end{pmatrix}$
☐ $\begin{pmatrix} 2.9876 \\ 0.1234 \\ 2.3540 \end{pmatrix}$

First, extract the diagonal matrix D from A :

$$D = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 90 \end{pmatrix}.$$

Next, compute D^{-1} :

$$D^{-1} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{40} & 0 \\ 0 & 0 & \frac{1}{90} \end{pmatrix}.$$

Now, compute the matrix $I - D^{-1}A$:

$$D^{-1}A = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{40} & 0 \\ 0 & 0 & \frac{1}{90} \end{pmatrix} \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix} = \begin{pmatrix} 1 & 3 & -4 \\ 0.3 & 1 & -0.95 \\ -\frac{8}{45} & -\frac{19}{45} & 1 \end{pmatrix}.$$

So

$$I - D^{-1}A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 3 & -4 \\ 0.3 & 1 & -0.95 \\ -\frac{8}{45} & -\frac{19}{45} & 1 \end{pmatrix} = \begin{pmatrix} 0 & -3 & 4 \\ -0.3 & 0 & 0.95 \\ \frac{8}{45} & \frac{19}{45} & 0 \end{pmatrix}.$$

Next, compute $D^{-1}\vec{b}$:

$$D^{-1}\vec{b} = \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ 0 & \frac{1}{40} & 0 \\ 0 & 0 & \frac{1}{90} \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} \\ \frac{1}{40} \\ \frac{1}{90} \end{pmatrix}.$$

Now, iteratively compute \tilde{x}_1 and then \tilde{x}_2

$$\tilde{x}_1 = (I - D^{-1}A)\tilde{x}_0 + D^{-1}\vec{b} = \begin{pmatrix} 0 & -3 & 4 \\ -\frac{3}{10} & 0 & \frac{19}{20} \\ \frac{8}{45} & \frac{19}{45} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} \frac{1}{4} \\ \frac{1}{40} \\ \frac{1}{90} \end{pmatrix}.$$

First, compute the matrix-vector product:

$$\begin{pmatrix} 0 & -3 & 4 \\ -0.3 & 0 & 0.95 \\ \frac{8}{45} & \frac{19}{45} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 - 3 + 4 \\ -0.3 + 0 + 0.95 \\ \frac{8}{45} + \frac{19}{45} \end{pmatrix} = \begin{pmatrix} 1 \\ 0.65 \\ \frac{27}{45} \end{pmatrix}.$$

So

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 0.65 \\ 0.6 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/40 \\ 1/90 \end{pmatrix} = \begin{pmatrix} 1.25 \\ 0.675 \\ 0.6111 \end{pmatrix}.$$

The \vec{x}_2 is given by

$$\vec{x}_2 = (I - D^{-1}A) \vec{x}_1 + D^{-1}\vec{b} = \begin{pmatrix} 0 & -3 & 4 \\ -0.3 & 0 & 0.95 \\ \frac{8}{45} & \frac{19}{45} & 0 \end{pmatrix} \begin{pmatrix} 1.25 \\ 0.675 \\ 0.6111 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/40 \\ 1/90 \end{pmatrix}.$$

Compute the matrix-vector product:

$$\begin{pmatrix} 0 & -3 & 4 \\ -0.3 & 0 & 0.95 \\ \frac{8}{45} & \frac{19}{45} & 0 \end{pmatrix} \begin{pmatrix} 1.25 \\ 0.675 \\ 0.6111 \end{pmatrix} = \begin{pmatrix} 0 - 2.025 + 2.4444 \\ -0.375 + 0 + 0.580545 \\ \frac{10}{45} + \frac{12.825}{45} \end{pmatrix} = \begin{pmatrix} 0.4194 \\ 0.205545 \\ 0.5072 \end{pmatrix}.$$

So:

$$\vec{x}_2 = \begin{pmatrix} 0.4194 \\ 0.205545 \\ 0.5072 \end{pmatrix} + \begin{pmatrix} 1/4 \\ 1/40 \\ 1/90 \end{pmatrix} = \begin{pmatrix} 0.6694 \\ 0.230545 \\ 0.5183 \end{pmatrix}.$$

Therefore, the second iterate \vec{x}_2 is approximately:

$$\vec{x}_2 = \begin{pmatrix} 0.6694 \\ 0.2305 \\ 0.5183 \end{pmatrix}.$$

Question 8:

The differential equation

$$y'(t) = 1 - 3y(t) \quad \text{with} \quad y(0) = 1$$

has the exact solution $y = \frac{1}{3}(2e^{-3t} + 1)$. From the explicit Runge-Kutta scheme given by the Butcher array

0				
$\frac{1}{2}$	$\frac{1}{2}$			
$\frac{1}{2}$	0	$\frac{1}{2}$		
1	0	0	1	
<hr/>				
	1/6	1/3	1/3	1/6

where

$$u_{k+1} = u_k + h \sum_{i=1}^4 b_i f \left(u_k + h \sum_{j=1}^4 a_{i,j} k_j, t_k + c_i h \right)$$

and using step size $h = 0.1$, what is the $|y(2h) - u_2|$, i.e. the global truncation error after two steps?

- ☐ 0.002
 ☐ 0.058
☐ 0.370
 ☐ 0.000115
☐ 0.965
 ☒ 0.000019

Using the formula given, the Butcher array yields a Runge-Kutta scheme of the form:

$$\begin{aligned}
 k_1 &= f(u_n, t_n) \\
 k_2 &= f(u_n + hk_1/2, t_n + h/2) \\
 k_3 &= f(u_n + hk_2/2, t_n + h/2) \\
 k_4 &= f(u_n + hk_3, t_n + h)
 \end{aligned}$$

for the function which is not dependent on time

$$f(u_n) = 1 - 3u_n$$

which generates the next value via

$$u_{n+1} = u_n + (h/6)(k_1 + 2k_2 + 2k_3 + k_4),$$

Given the initial condition and the step size, to compute two time steps means to compute approximations to $y(0.1)$ and $y(0.2)$. For the first time step, with $u_0 = 1.0$ and $h = 0.1$,

$$k_1 = f(u_0) = 1 - 3u_0 = 1 - 3 \cdot 1 = -2$$

Thus the first step can be computed using the values

$$k_1 = -2, \quad k_2 = 1 - 3 \cdot (1 + 0.1 \cdot (-2)/2) = -1.7, \quad k_3 = -1.745 \quad \text{and} \quad k_4 = -1.4765.$$

So that

$$u_1 = 1.0 + (0.1/6)(-2 - 2 \cdot 1.7 - 2 \cdot 1.745 - 1.4765) = 0.827225$$

The second evaluation yields

$$k_1 = 1 - 3 \cdot 0.827225 = 1.481675, \quad k_2 = -1.25942375, \quad k_3 = -1.292760875 \quad \text{and} \quad k_4 = -0.0938467375$$

Thus $u_2 = 0.827225 + (1/60)(7.679892) = 0.6992268$

The exact values are given by $y(0.1) = 0.8272122$ and $y(0.2) = 0.6992078$, thus the difference at $t = 0.2$ is

$$|y(0.2) - u_2| = 1.9 \times 10^{-5}$$