Jacobs University Bremen

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Spring Semester 2022

CA-MATH-804: Numerical Analysis

Assignment Sheet 4. Due: February 23, 2022

Exercise 1 [5 Points]: Let $I_n(f) = \sum_{k=0}^{n} \alpha_k f(x_k)$ be a Lagrange quadrature formula on $n+1$ nodes. Compute the degree of exactness r for the formula

$$I_4(f) = \frac{1}{4}\left[f(-1) + 3f\left(-\frac{1}{3}\right) + 3f\left(\frac{1}{3}\right) + f(1)\right].$$

Exercise 2 [5 Points]: Let $I_w(f) = \int_0^1 w(x)f(x)\,\mathrm{d}x$ with $w(x) = \sqrt{x}$ and let $Q(f) = \alpha f(x_1)$ be a quadrature formula approximating I_w. Find α and x_1 such that Q has maximum degree of exactness r.

Exercise 3 [5 Points]: Prove that $G^k(x_j) = hG(x_j, x_k)$, where G is the Green's function for the problem

$$u'' = f \quad \text{in}\,(0,1)\,, u(x) = 0 \quad \text{on}\,\{0,1\}\,,$$

and G^k its corresponding discrete counterpart.

Exercise 4 [3+3 Points]: Consider the matrix $A_{\text{fd}} = h^{-2}\text{tridiag}(-1, 2, -1)$ which appears in the finite difference discretization of the second derivative.

a) Show that A_{fd} is symmetric and positive definite

b) Show that A_{fd} is an M-Matrix, i.e. $a_{ij} \leq 0$ for $i \neq j$ and all the entries of its inverse are nonnegative.

Exercise 5 [3+3 Points]: Consider an equidistant grid with nodes x_i and grid-width h and a real valued function f with sufficient smoothness. Using Taylor series expansion show that

a) $\left|f'(x_i) - D_i^- f(x_i)\right| = \frac{h}{2}\left|f''(\xi)\right|$ for some $\xi \in (x_{i-1}, x_i)$,

b) $\left|f''(x_i) - D_i^{\pm} f(x_i)\right| = \frac{h^2}{24}\left|f''''(\xi_1) + f''''(\xi_2)\right|$ for some $\xi_1 \in (x_{i-1}, x_i)$, $\xi_2 \in (x_i, x_{i+1})$.

Exercise 6 [4 Points]: Let $E_0(f)$ and $E_1(f)$ be the quadrature errors of the midpoint ans the trapezoidal formula respectively. Prove that $E_1(f) \approx 2\|E_0(f)\|$.

Exercise 7 [4 Points]: An alternative approach for the construction of the Lagrange interpolation polynomial Π_n involves directly enforcing the interpolation constraints on Π_n and then computing the coefficients a_i. This produces a system of simultaneous linear equations which can be written as the linear system $Xa = y$, where the coefficients of the matrix X are given by $X_{ij} = x_i^{j-1}$ where the points x_i are the interpolation points with $i = 0, \ldots n$. Prove that the matrix X is invertible if the nodes are distinct.