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CTMS-MAT-13: Numerical Methods

Assignment Sheet 6. Released: 30 April 2025
Due: 20 May 2025

Exercise 1 [4+2+2+3 Points]:

(a) For Gaussian quadrature on $[-1, 1]$ and $n = 3$, show that the four Gauss nodes $x_0, \ldots, x_3$ as the roots of the polynomial $q(x)$ that is orthogonal to $1, x, x^2$ and x^3 are given by

$$x = \pm\sqrt{\frac{15 \pm 2\sqrt{30}}{35}}.$$

(b) To illustrate the full quadrature, determine the first corresponding weight A_0 of the 4 point Gaussian quadrature.

(c) Based on your results of **(a)**, find the corresponding Legendre polynomial that uses the additional condition of $q(1) = 1$.

(d) Consider Gauss quadrature with 2 nodes on the interval $[-1, 1]$. Derive the resulting Legendre polynomial $q(x)$ as a solution to the orthogonality condition:

$$\int_{-1}^{1} q\left(x\right) x^k \,\mathrm{d}x = 0$$

for $k = 0, 1$. Determine the roots of $q(x)$ to arrive at the Gauss nodes. Use your result to approximate the integral:

$$\int_{-\frac{3\pi}{\sqrt{3}}}^{\frac{3\pi}{\sqrt{3}}} \sin^2(x)\cos(x)\,\mathrm{d}x.$$

Remember to convert into the appropriate interval. You may give the result either as an exact value or use a calculator to arrive at the final number.

Exercise 2 [3+4+4+2+2 Points]:

Consider the linear ordinary differential equation

$$y''(t) = -\alpha y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1.$$

(a) Convert this second-order ordinary differential equation into a system of two coupled first-order ODEs, one in $y(t)$ and one in $y'(t)$. Write the system as a vector-valued ordinary differential equation for $\vec{v}(t) = \left(y(t), y'(t)\right)^T$, in the form $f\left(\vec{v}\right) = A\vec{v}$.

(b) Show that the backward Euler method can be written as

$$\vec{u}_{n+1} = \left(I - hA\right)^{-1}\vec{u}_n$$

and provide the full system for $\vec{u}_{n+1}$ for the ODE presented above.

(c) Noting that $f_n = A\vec{u}_n$, so that $f(\vec{v}_n + hf_n) = A\vec{v}_n + hA^2\vec{v}_n$, show that Heun's method for the case of the vector-valued ODE given above is

$$\vec{v}_{n+1}(t) = \begin{pmatrix} 1 + \dfrac{h^2}{2} & h\left(1 - \alpha\dfrac{h}{2}\right) \\ h\left(1 - \alpha\dfrac{h}{2}\right) & 1 - h\left(\alpha - \dfrac{h}{2}\left(1 + \alpha^2\right)\right) \end{pmatrix} \vec{v}_n(t).$$

(d) With $\alpha = 3$, calculate $y(0.2)$ and $y'(0.2)$ using the backward Euler method with $h = 0.15$.

(e) With $\alpha = 3$, calculate $y(0.2)$ and $y'(0.2)$ using Heun's method with $h = 0.1$.

Exercise 3 [7 Points]:

Consider the ordinary differential equation

$$y'(t) = -y(t) + \ln(t + 1) \quad \text{with} \quad y(0) = 1.$$

Showing all working, calculate four time steps of the approximate solution using the fourth-order Runge-Kutta method with step size $h = 0.1$.