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Examples and principles of mathematical modelling in medicine: Exercises

The rationale of these exercises is to introduce many of the current clinical challenges in image-guided therapies. Many of the exercises are open problem. As such solutions are not be provided.

Microwave Ablation

Exercise 1: At up to 2.45 GHz, assume that steady state solutions and derive the curl-curl form of Maxwell's equations.

Exercise 2: Show that the discrete Maxwell equations on a Yee-cell maintain the divergence relations exactly.

Exercise 3: Write the curl-curl form of Maxwell's equations for vector fields of the form

$$E = \left(E^x_{0,0,0}, E^x_{0,0,1}, \ldots E^x_{0,0,n_z}, E^x_{0,1,0}, \ldots E^z_{n_x,n_y,n_z} \right)$$

and ordered by position in space

$$E = \left(E^x_{0,0,0}, E^y_{0,0,0}, E^z_{0,0,0}, E^x_{0,0,1} \ldots E^z_{n_x,n_y,n_z} \right).$$

Exercise 4: With the left and right preconditioners, K_1 and K_2 which are used to solve the linear system $Ax = b$, via the transformed system $\tilde{A}\tilde{x} = \tilde{b}$, with $\tilde{A} = K_1^{-1} A K_2^{-1}$, $\tilde{x} = K_2 x$ and $\tilde{b} = K_1^{-1} b$, formulate the preconditioned BiCGStab(ℓ) algorithm, based one the unpreconditioned system in Algorithm 1. An open problem is which other algorithms, such as IDR(s)Stab(ℓ), could out perform the stretched coordinate left- and right-preconditioned BiCGStab?

Cryoablation

Exercise 1: Sketch an method to ascertain what is the best update time for the heat equation when considering changes in material properties? Assume that the number of iterations of an iterative solver is proportional to an matrix norm $\left\| \delta A_n \right\|_X$, where $\delta A_n = A_n - A_{n-1}$, where A is the discretized operator for the heat equation, which is dependent on the material properties $A\left(\boldsymbol{\nu}\right)$.

Exercise 2: Construct a scheme to account for contraction and expansion in tissue as the temperature of the material changes, by coupling the thermal field to a deformation field. Consider how to update the thermal field deformation field. What boundary conditions can be imposed on the deformation field?

Exercise 3: Construct a homogenization scheme which considers the internal and external components of each cell to derive effective thermal conductivity.

Exercise 4: If both the in the segmentation of a frozen region in a 2D image, and the 0°C isothermal can be characterised as ellipses with uncertain foci and semi-major and minor axis.

Ultrasound

Exercise 1: Assume that the surface of a ultrasound transducer is characterised by a function $f\left(x, y\right)$. Given the (hyperbolic) form of the acoustic wave equation in image space (x, y, z), in terms of the variables $\boldsymbol{q} = \left(\boldsymbol{\rho}, \boldsymbol{\rho u}, \boldsymbol{\rho v}, \boldsymbol{\rho w}\right)$, where ρ is the density and u, v and w are the components of the velocity vector,

$$\boldsymbol{q}_t + \boldsymbol{f}\left(\boldsymbol{q}\right)_x + \boldsymbol{g}\left(\boldsymbol{q}\right)_y + \boldsymbol{h}\left(\boldsymbol{q}\right)_z = 0$$

Algorithm 1 Unpreconditioned BiCGStab(ℓ) algorithm for a complex matrix. Note that the inner product is the conjugate dot-product, i.e. $\langle \boldsymbol{x}, \boldsymbol{y} \rangle = \boldsymbol{x}^* \cdot \boldsymbol{y}$. paper

$r_0 = (b - Ax_0)$
Choose an arbitrary vector $\hat{r}_0$ such that $(\hat{r}_0, r_0) \neq 0$, e.g., $\hat{r}_0 = r_0$
$\rho_{\text{old}} = \alpha = \omega_0 = 1$
$v_0 = p_0 = \hat{u}_0 = \mathbf{0}$
while $|r_{j+1}| > \epsilon$ and $i < N$ **do**
  $\rho_{\text{old}} = -\omega \rho_{\text{old}}$
  for $j = 0, \dots l-1$ **do** ▷ (Start of BiCG part)
    $\rho_{\text{new}} = \langle r_j, \hat{r}_0 \rangle$
    $\beta = (\rho_{\text{new}} / \rho_{\text{old}})$
    $\rho_{\text{old}} = \rho_{\text{new}}$
    for $k = 0, \dots, j$ **do**
      $\hat{u}_k = \hat{r}_k - \beta \hat{u}_k$
    end for
    $\hat{u}_{j+1} = A\hat{u}_j$
    $\gamma = \langle \hat{u}_{j+1}, \hat{r}_0 \rangle$
    $\alpha = \rho_{\text{new}} / \gamma$
    for $k = 0, \dots, j$ **do**
      $\hat{r}_k = \hat{r}_k - \alpha \hat{u}_{k+1}$
    end for
    $\hat{r}_{j+1} = A\hat{r}_j$
    $x = x + \alpha x$
  end for
  for $j = 1, \dots, l$ **do** ▷ (Start of polynomial part)
    for $k = 1, \dots, j-1$ **do**
      $\tau_{k,j} = \langle \hat{r}_j, \hat{r}_k \rangle / \sigma_k$
      $\hat{r}_j = \hat{r}_j - \tau_{k,j} \hat{u}_k$
    end for
    $\sigma_j = \langle \hat{r}_j, \hat{r}_j \rangle$
    $\gamma' = \langle \hat{r}_0, \hat{r}_j \rangle / \sigma_j$
  end for
  $\gamma_l = \gamma'_l$
  $\omega = \gamma_l$
  for $j = l-1, \dots, 1$ **do**
    $\gamma_j = \gamma'_j - \sum_{k=j+1}^{l} \tau_{j,k} \gamma_k$
  end for
  for $j = 1, \dots, l-1$ **do**
    $\gamma''_j = \gamma_{j+1} + \sum_{k=j+1}^{l-1} \tau_{j,k} \gamma_{k+1}$
  end for
  $x = x + \gamma_l \hat{r}_0$ ▷ (Updates)
  $\hat{r}_0 = \hat{r}_0 - \gamma'_l \hat{r}_l$
  $\hat{u}_0 = \hat{u}_0 - \gamma_l \hat{u}_l$
  for $j = 1, \dots, l-1$ **do**
    $\hat{u}_0 = \hat{u}_0 - \gamma_j \hat{u}_j$
    $x = x + \gamma''_l \hat{r}_j$
    $\hat{r}_0 = \hat{r}_0 - \gamma'_j \hat{r}_j$
  end for
  $i = i + 1$
end while

which can be written as

$$\frac{\partial}{\partial t}\begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho w \end{pmatrix} = \frac{\partial}{\partial x}\begin{pmatrix} \rho u \\ p + \rho u^2 \\ \rho u v \\ \rho u w \end{pmatrix} + \frac{\partial}{\partial y}\begin{pmatrix} \rho v \\ \rho u v \\ p + \rho v^2 \\ \rho v w \end{pmatrix} + \frac{\partial}{\partial z}\begin{pmatrix} \rho w \\ \rho u w \\ \rho v w \\ p + \rho w^2 \end{pmatrix} = \mathbf{0}$$

with a simple equation of state $p = \rho c^2$, write the governing equation in terms of curvilinear coordinates (ξ_1, ξ_2, ξ_3), where the transformation between the two frames can be expressed as

$$x = x(\xi_1, \xi_2, \xi_3), \tag{1}$$
$$y = y(\xi_1, \xi_2, \xi_3), \tag{2}$$
$$z = z(\xi_1, \xi_2, \xi_3). \tag{3}$$

This formulation enable the input condition to be well characterised on the plane ξ_3, avoiding potential staircasing artefacts.

Hint: See the doi: 10.1088/0256-307X/30/7/074302 for a formulation in oblate spheriod coordinates.

Exercise 2: Show that for the one-dimensional pseudo-heat equation

$$\frac{\partial T}{\partial x} = \kappa \frac{\partial^2 T}{\partial t^2},$$

a solution of the form

$$T = C e^{\frac{1}{2\kappa} \int_0^t v(x,\tau)\,\mathrm{d}\tau}$$

yields the (acoustic) Burgers equation written in terms of the fluid velocity v,

$$\frac{\partial v}{\partial x} - v\frac{\partial v}{\partial t} - \kappa \frac{\partial^2 v}{\partial t^2} = 0.$$

Solve the nonlinear acoustic equation with initial condition $v\,(0, t) = \sin\,(t)$.

Hint: See the doi:10.1063/1.1309185

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Exercise 3: Short pulsed ultrasound may be described by a set of Wavelet basis functions. Write Burgers equation in terms of a wavelet basis functions and seek a solution.