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JTMS-MAT-13: Numerical Methods

Exam & Solutions: Friday 22 August 2025

All questions carry equal marks. Only five questions will be marked. Please use the booklet provided, clearly indicating in the inside cover which questions are to be marked.

Note that all trigonometric values should be expressed in radians.

Question 1:

(a) What three operations form the elementary row operations used in Gaussian elimination? [3]

(b) Given $a = 2$, find the solution to the linear system $A\vec{x} = \vec{b}$ with

$$A = \begin{pmatrix} 1 & 4 & 5 \\ 3 & -1 & 3 \\ 0 & a & 1 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}.$$

[7]

(c) For the matrix A, when $a = 2$, what rows should be swapped in order to perform partial pivoting on the first column? [3]

(d) Show that when $a = \dfrac{13}{12}$ the linear system has no unique solution. [7]

(a) The three operations are:

- Row swapping: interchanging two rows of the matrix
- Row multiplication: multiplying all elements of a row by a non-zero scalar
- Row addition: adding a multiple of one row to another row

(b) The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 3 & -1 & 3 & 2 \\ 0 & 2 & 1 & 5 \end{array}\right).$$

First, eliminate the 3 in the second row of the first column by $R_2 - 3R_1 \rightarrow R_2$. Thus

$$\left(\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 0 & -13 & -12 & -1 \\ 0 & 2 & 1 & 5 \end{array}\right)$$

Next, eliminate the 2 in the third row, second column using $R_3 + \frac{2}{13}R_2 \rightarrow R_3$, then

$$\left(\begin{array}{ccc|c} 1 & 4 & 5 & 1 \\ 0 & -13 & -12 & -1 \\ 0 & 0 & -\frac{11}{13} & \frac{63}{13} \end{array}\right).$$

Now perform back substitution

$$-\frac{11}{13}z = \frac{63}{13} \Rightarrow z = -\frac{63}{11}.$$

Thus

$$-13y - 12\left(-\frac{63}{11}\right) = -1 \Rightarrow -13y + \frac{756}{11} = -1 \Rightarrow y = \frac{59}{11}.$$

Substitute y and z into the first row:

$$x + 4\left(\frac{59}{11}\right) + 5\left(-\frac{63}{11}\right) = 1 \Rightarrow x = \frac{90}{11}.$$

The solution to the system is $\vec{x} = \left(\frac{90}{11}, \frac{59}{11}, -\frac{63}{11}\right)^T$.

(c) The first and the second rows should be swapped.

(d) The system has no unique value when the determinant of the matrix is zero. Expanding along the first row, the determinant of A is given by:

$$\det(A) = 1\left((-1 \times 1) - 3a\right) - 4\left(3 \times 1 - (3 \times 0)\right) + 5\left(3a - (-1 \times 0)\right).$$

This simplifies to:

$$\det(A) = -1 - 3a - 12 + 15a = 12a - 13.$$

Thus, when $a = 13/12$ then the determinant if zero, hence no unique solutions. Equivalently, show that the rows or columns are not linearly independent.

Question 2:

Consider the integral

$$I = \int_0^1 e^{2x+1}\,\mathrm{d}x.$$

Let the integral be discretized into subintervals of width $h = 1/2^k$, where $k = 1, 2$ and 3 so that n, the number of intervals, is $n = 2, 4$ and 8. The number of points for each discretization is given by $m = n + 1$. Noting that the Trapezium rule is given by

$$I_n = \frac{h}{2}\left(f(x_0) + 2\sum_{i=1}^{m-1} f(x_i) + f(x_m)\right)$$

and Simpson's rule is given by

$$I_n = \frac{h}{3}\left(f(x_0) + 4\sum_{i=1}^{m/2} f(x_{2i-1}) + 2\sum_{i=1}^{m/2-1} f(x_{2i}) + f(x_m)\right).$$

(a) Compute the first three approximations using Trapezium rule. [8]

(b) Compute the first three approximations using Simpson's rule. [8]

(c) For the Trapezium rule, the error is proportional to which power of h? [2]

(d) By which process can Simpson's rule be derived from the Trapezium rule? [2]

(a) For $n = 2$, $h = \frac{1}{2}$ and the subintervals, $x_0 = 0$, $x_1 = \frac{1}{2}$ and $x_2 = 1$. The function evaluated at these points

$$\begin{aligned} f(x_0) &= e^{2(0)+1} = e, \\ f(x_1) &= e^{2(\frac{1}{2})+1} = e^2, \\ f(x_2) &= e^{2(1)+1} = e^3. \end{aligned}$$

The, applying trapezium rule:

$$\begin{aligned} I_2 &= \frac{h}{2}\left(f(x_0) + 2f(x_1) + f(x_2)\right) \\ &= \frac{1}{4}\left(e + 2e^2 + e^3\right) \end{aligned}$$

For $n = 4$, $h = \frac{1}{4}$ and the subintervals, $x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_2 = \frac{3}{4}$ and $x_4 = 1$. Computing the function evaluations and applying trapezium rule yields

$$\begin{aligned} I_4 &= \frac{h}{2}\left(f(x_0) + 2(f(x_1) + f(x_2) + f(x_3)) + f(x_4)\right) \\ &= \frac{1/4}{2}\left(e + 2(e^{\frac{3}{2}} + e^2 + e^{\frac{5}{2}}) + e^3\right) \\ &= \frac{1}{8}\left(e + 2(e^{\frac{3}{2}} + e^2 + e^{\frac{5}{2}}) + e^3\right). \end{aligned}$$

For $n = 8$, the process yields

$$I_8 = \frac{1}{16}\left(e + 2(e^{\frac{5}{4}} + e^{\frac{3}{2}} + e^{\frac{7}{4}} + e^2 + e^{\frac{9}{4}} + e^{\frac{5}{2}} + e^{\frac{11}{4}}) + e^3\right).$$

(b) The widths, evaluation points, and function evaluations are the same for Simpson's rule as for the Trapezium rule, only the formula is different.

For $n = 2$, $h = \frac{1}{2}$ and the subintervals, are $x_0 = 0$, $x_1 = \frac{1}{2}$ and $x_2 = 1$. Note that when applying Simpson's rule, the interior point is x_1 which is odd-numbered, so that

$$\begin{aligned}
I_2 &= \frac{h}{3}\left(f(x_0) + 4f(x_1) + f(x_2)\right) \\
&= \frac{1/2}{3}\left(e + 4e^2 + e^3\right) \\
&= \frac{1}{6}(e + 4e^2 + e^3).
\end{aligned}$$

For $n = 4$, there are five evaluations, with two boundary points (x_0 and x_4) and three interior points: two are odd numbered (x_1 and x_3), one is even numbered (x_2). Applying Simpson's rule:

$$I_4 = \frac{1}{12}\left(e + 4\left(e^{\frac{3}{2}} + e^{\frac{5}{2}}\right) + 2e^2 + e^3\right).$$

For $n = 8$, applying Simpson's rule yields

$$\begin{aligned}
I_8 &= \frac{h}{3}\left(f(x_0) + 4\left(f(x_1) + f(x_3) + f(x_5) + f(x_7)\right) + 2\left(f(x_2) + f(x_4) + f(x_6)\right) + f(x_8)\right) \\
&= \frac{1}{24}\left(e + 4\left(e^{\frac{5}{4}} + e^{\frac{7}{4}} + e^{\frac{9}{4}} + e^{\frac{11}{4}}\right) + 2\left(e^{\frac{3}{2}} + e^2 + e^{\frac{5}{2}}\right) + e^3\right).
\end{aligned}$$

(c) The error in the Trapezium rule is proportional to the square of the subintervals, i.e. $\mathcal{O}\left(h^2\right)$. The error E in the Trapezoidal rule over the interval (a, b) with n subintervals can be expressed as:

$$E = -\frac{(b-a)^3}{12n^2} f''(\xi)$$

for some $\xi \in [a, b]$.

(d) Romberg's method can be used to improve the accuracy of the Trapezium rule, giving Simpson's rule. Equivalently, this is an application of Richardson interpolation. It is a Newton-Coates formula.

Question 3:

(a) What is an upper triangular matrix? [2]

(b) For LU decomposition of a matrix A, the matrix A must be invertible. What condition on the determinant is necessary for a matrix to be invertible? [2]

(c) What condition on the eigenvalues of A is necessary for Cholseky decomposition to be performed? [2]

(d) If the entries of the matrix are real numbers, what addition condition on the matrix A is necessary for Cholseky decomposition to be performed? [2]

(e) Find the lower triangular Cholesky matrix $\tilde{L}$, such that $A = \tilde{L}\tilde{L}^T$, for

$$A = \begin{pmatrix} 4 & 2 & 14 \\ 2 & 17 & -5 \\ 14 & -5 & 83 \end{pmatrix} \quad \text{i.e.} \quad \tilde{L} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

[12]

(a) An upper triangular matrix is a type of square matrix where all the entries below the main diagonal are zero, that is $a_{i,j} = 0$ for all $i > j$.

(b) The determinant of the matrix A must be non-zero.

(c) All eigenvalues of A must be real and positive.

(d) The matrix A must be symmetric.

(e) Consider the matrix $\tilde{L}$, such that $A = \tilde{L}\tilde{L}^T$, where $\tilde{L}$ is given by

$$\tilde{L} = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

Thus, to find the entries of $\tilde{L}$, consider $a_{1,1} = 4$, which is given by $a_{1,1} = l_{1,1}^2 \Rightarrow l_{1,1} = 2$.

Next, consider $a_{1,2} = 2$, which is given by $a_{1,2} = l_{1,1}l_{2,1} \Rightarrow l_{2,1} = 1$.

The last term in the first row of A is given by $a_{3,1} = 14$. This is given by $a_{1,3} = l_{1,1}l_{3,1} \Rightarrow l_{3,1} = 7$.

Now $a_{2,2} = 17$, and is given by $a_{2,2} = l_{2,1}^2 + l_{2,2}^2$, which given $l_{2,1} = 1 \Rightarrow l_{2,2} = 4$.

As $a_{2,3} = -5$, and is given by $a_{2,3} = l_{3,1}l_{2,1} + l_{2,2}l_{3,2}$, yields $-5 = 7 \times 1 + 4 \times l_{3,2} \Rightarrow= l_{3,2} = -3$.

The last unknown term can be found from $a_{33}^2 = 83 = l_{3,1}^2 + l_{3,2}^2 + l_{3,3}^2 = 7^2 + (-3)^2 + l_{3,3}^2 \Rightarrow l_{3,3} = 5$.

$$\tilde{L} = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 4 & 0 \\ 7 & -3 & 5 \end{pmatrix}.$$

Question 4:

Consider the measurement values $p_0 = 5, p_1 = 4$, and $p_2 = 6$ that have been obtained at the nodes $u_0 = 0$, $u_1 = \frac{\pi}{4}$, and $u_2 = \frac{\pi}{2}$.

Let the function $p(u) = a\cos(u) + bu = \sum_{k=0}^{1} \beta_k \varphi_k(u)$ approximate the data in the least squares sense, where $\beta_0 = a$ and $\beta_1 = b$, and $\varphi_0 = \cos(u)$ and $\varphi_1 = u$.

(a) By minimizing

$$\frac{\partial E}{\partial \beta_j} = -2\sum_{i=0}^{2}\left(p_i - \sum_{k=0}^{1}\beta_k\varphi_k(u_i)\right)\varphi_j(u_i),$$

show that the normal equations are given by

$$\begin{aligned} a\sum_{i=0}^{2}\cos^2(u_i) + b\sum_{i=0}^{2}u_i\cos(u_i) &= \sum_{i=0}^{2}p_i\cos(u_i), \\ a\sum_{i=0}^{2}u_i\cos(u_i) + b\sum_{i=0}^{2}u_i^2 &= \sum_{i=0}^{2}p_iu_i. \end{aligned}$$ [6]

(b) Write this as a linear equation or as a pair of simultaneous linear equations and solve for a and b. [7]

(c) Compute the error in the L_2 sense that is minimized [7]

(a) To derive the normal equations, consider error E, for a least squares approximation, defined as:

$$E = \sum_{i=0}^{2}\left(p_i - \sum_{k=0}^{1}\beta_k\varphi_k(u_i)\right)^2.$$

To minimize, compute the partial derivatives of E with respect to a and b, i.e.

$$\frac{\partial E}{\partial \beta_j} = -2\sum_{i=0}^{2}\left(p_i - \sum_{k=0}^{1}\beta_k\varphi_k(u_i)\right)\frac{\partial}{\partial \beta_j}\left(\sum_{k=0}^{1}\beta_k\varphi_k(u_i)\right).$$

The derivative of the inner sum with respect to β_j is simply $\varphi_j(u_i)$:

$$\frac{\partial E}{\partial \beta_j} = -2\sum_{i=0}^{2}\left(p_i - \sum_{k=0}^{1}\beta_k\varphi_k(u_i)\right)\varphi_j(u_i).$$

Setting this derivative to zero for the least squares approximation gives the normal equations:

$$\sum_{i=0}^{2}\left(p_i - \sum_{k=0}^{1}\beta_k\varphi_k(u_i)\right)\varphi_j(u_i) = 0.$$

So, for $j = 0$, i.e. a:

$$\begin{aligned} \sum_{i=0}^{2}(p_i - (a\cos(u_i) + bu_i))\cos(u_i) &= 0 \\ \sum_{i=0}^{2}p_i\cos(u_i) - a\sum_{i=0}^{2}\cos^2(u_i) - b\sum_{i=0}^{2}u_i\cos(u_i) &= 0 \\ a\sum_{i=0}^{2}\cos^2(u_i) + b\sum_{i=0}^{2}u_i\cos(u_i) &= \sum_{i=0}^{2}p_i\cos(u_i). \end{aligned}$$

Similarly, for b:

$$a\sum_{i=0}^{2}u_i\cos(u_i) + b\sum_{i=0}^{2}u_i^2 = \sum_{i=0}^{2}p_iu_i.$$

Thus,

$$\begin{aligned} a\sum_{i=0}^{2}\cos^2(u_i) + b\sum_{i=0}^{2}u_i\cos(u_i) &= \sum_{i=0}^{2}p_i\cos(u_i), \\ a\sum_{i=0}^{2}u_i\cos(u_i) + b\sum_{i=0}^{2}u_i^2 &= \sum_{i=0}^{2}p_iu_i. \end{aligned}$$

(b) To solve the linear system for a and b, first calculate all the required sums:

- $\sum_{i=0}^{2} \cos^2(u_i)$:

$$\cos^2(u_0) + \cos^2(u_1) + \cos^2(u_2) = \cos^2(0) + \cos^2\left(\frac{\pi}{4}\right) + \cos^2\left(\frac{\pi}{2}\right) = 1^2 + \left(\frac{\sqrt{2}}{2}\right)^2 + 0^2 = 1 + \frac{1}{2} + 0 = \frac{3}{2}$$

- $\sum_{i=0}^{2} u_i \cos(u_i)$:

$$0 \cdot 1 + \frac{\pi}{4} \cdot \frac{\sqrt{2}}{2} + \frac{\pi}{2} \cdot 0 = \frac{\pi\sqrt{2}}{8}$$

- $\sum_{i=0}^{2} u_i^2$:

$$0^2 + \left(\frac{\pi}{4}\right)^2 + \left(\frac{\pi}{2}\right)^2 = 0 + \frac{\pi^2}{16} + \frac{\pi^2}{4} = 0 + \frac{\pi^2}{16} + \frac{4\pi^2}{16} = \frac{5\pi^2}{16}$$

- $\sum_{i=0}^{2} p_i \cos(u_i)$:

$$5 \cdot 1 + 4 \cdot \frac{\sqrt{2}}{2} + 6 \cdot 0 = 5 + 2\sqrt{2}$$

- $\sum_{i=0}^{2} p_i u_i$:

$$5 \cdot 0 + 4 \cdot \frac{\pi}{4} + 6 \cdot \frac{\pi}{2} = 0 + 4 \cdot \frac{\pi}{4} + 6 \cdot \frac{\pi}{2} = \pi + 3\pi = 4\pi$$

Substituting all these values into the normal equations yields

$$\begin{aligned} a \cdot \frac{3}{2} + b \cdot \frac{\pi\sqrt{2}}{8} &= 5 + 2\sqrt{2} \\ a \cdot \frac{\pi\sqrt{2}}{8} + b \cdot \frac{5\pi^2}{16} &= 4\pi \end{aligned}$$

which as a matrix equation, $A\vec{x} = \vec{b}$, has

$$\vec{b} = \begin{pmatrix} 5 + 2\sqrt{2} \\ 4\pi \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} \frac{3}{2} & \frac{\pi\sqrt{2}}{8} \\ \frac{\pi\sqrt{2}}{8} & \frac{5\pi^2}{16} \end{pmatrix} \quad \Rightarrow \quad A^{-1} = \begin{pmatrix} \frac{5}{7} & -\frac{2\sqrt{2}}{7\pi} \\ -\frac{2\sqrt{2}}{7\pi} & \frac{24}{7\pi^2} \end{pmatrix}$$

After some algebraic manipulation and solving for a and b yields

$$\begin{aligned} a &= \frac{5}{7}\left(5 + 2\sqrt{2}\right) - 8\sqrt{2}/7 \\ &= \left(25 + 2\sqrt{2}\right)/7 = 3.975489589249456 \\ b &= -2\sqrt{2}/(7\pi) \cdot (5 + 2\sqrt{2}) + 4\pi \cdot 24/(7\pi^2) \\ &= (96 - 4 \cdot 2 - 10\sqrt{2})/(7\pi) \\ &= \left(88 - 10\sqrt{2}\right)/(7\pi) = 3.358526914769721 \end{aligned}$$

(c) To compute the L_2 error that is minimized in the least squares sense, we need to evaluate the error function E defined as:

$$\begin{aligned} E &= \sum_{i=0}^{2} (p_i - p(u_i))^2 \\ &= \sum_{i=0}^{2} (p_i - a\cos(u_i) - bu_i)^2 \\ &= (p_0 - a\cos(u_0) - bu_0)^2 + (p_1 - a\cos(u_1) - bu_1)^2 + (p_2 - a\cos(u_2) - bu_2)^2 \\ &= (5 - a\cos(0))^2 + \left(4 - a\cos\left(\frac{\pi}{4}\right) - b\frac{\pi}{4}\right)^2 + \left(6 - a\cos\left(\frac{\pi}{2}\right) - b\frac{\pi}{2}\right)^2 \\ &= (5 - a)^2 + \left(4 - \frac{1}{2}\left(a\sqrt{2} + b\frac{\pi}{2}\right)\right)^2 + \left(6 - b\frac{\pi}{2}\right)^2 \end{aligned}$$

where $p(u) = a\cos(u) + bu$ is the approximated function using the coefficients a and b obtained from the normal equations then yields

$$\begin{aligned} E &= \left(5 - (25 + 2\sqrt{2})/7\right)^2 + \left(4 - \frac{1}{2}\left((25\sqrt{2} + 4)/7 + 6 - 35\sqrt{2}/7\right)\right)^2 + \left(6 - (88 - 10\sqrt{2})/14\right)^2 \\ &= \left((10 - 2\sqrt{2})/7\right)^2 + \left((4 - 10\sqrt{2}))/7\right)^2 + \left((5\sqrt{2} - 2)/7\right)^2 \\ &= \frac{2}{7}\left(27 - 10\sqrt{2}\right) = 3.673675536076871 \end{aligned}$$

Question 5:
Newton's method is given by in a general form as

$$\vec{x}_{n+1} = \vec{x}_n - J^{-1}(\vec{x}_n)\vec{f}(\vec{x}_n)$$

where J is the Jacobian matrix of $\vec{f}$. What is the second iterate, $\vec{x}_2$, for the function

$$\vec{f}(\vec{x}) = \begin{pmatrix} 2u - v + \frac{1}{9}e^{-u} - 1 \\ -u + 2v + \frac{1}{9}e^{-v} \end{pmatrix}$$

where $\vec{x} = (u, v)^T$ with initial guess $\vec{x}_0 = (1, 1)^T$

- ○ (-0.5972, -0.2556)
- ○ (0.2556, 0.5972)
- ○ (0.5972, -0.2556)
- ○ (-0.5972, 0.2556)
- ☒ (0.5972, 0.2556)
- ○ (0.2556, -0.5972)

Let $\vec{f} = (f_1, f_2)$. The partial derivatives for the Jacobian are given by

$$\frac{\partial f_1}{\partial u} = 2 - \frac{1}{9}e^{-u}$$
$$\frac{\partial f_1}{\partial v} = -1$$
$$\frac{\partial f_2}{\partial u} = -1$$
$$\frac{\partial f_2}{\partial v} = 2 - \frac{1}{9}e^{-v}$$

So that

$$J = \begin{pmatrix} 2 - \frac{1}{9}e^{-u} & -1 \\ -1 & 2 - \frac{1}{9}e^{-v} \end{pmatrix}.$$

Then the determinant is

$$\det(J) = \left(2 - \frac{1}{9}e^{-u}\right)\left(2 - \frac{1}{9}e^{-v}\right) - (-1)(-1)$$

so the inverse of the Jacobian is given by

$$J^{-1} = \frac{1}{\det(J)}\begin{pmatrix} 2 - \frac{1}{9}e^{-v} & 1 \\ 1 & 2 - \frac{1}{9}e^{-u} \end{pmatrix}.$$

Thus,

$$\vec{x}_1 = \vec{x}_0 - J^{-1}(\vec{x}_0)\vec{f}(\vec{x}_0)$$

where

$$\vec{f}(\vec{x}_0) = \begin{pmatrix} \frac{1}{9}e^{-1} \\ 1 + \frac{1}{9}e^{-1} \end{pmatrix} \quad \text{and} \quad J^{-1}(\vec{x}_0) = \frac{1}{\left(2 - \frac{1}{9}e^{-1}\right)^2 - 1}\begin{pmatrix} 2 - \frac{1}{9}e^{-1} & 1 \\ 1 & 2 - \frac{1}{9}e^{-1} \end{pmatrix}$$

yields

$$\begin{aligned}\vec{x}_1 &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{\left(2 - \frac{1}{9}e^{-1}\right)^2 - 1}\begin{pmatrix} 2 - \frac{1}{9}e^{-1} & 1 \\ 1 & 2 - \frac{1}{9}e^{-1} \end{pmatrix}\begin{pmatrix} \frac{1}{9}e^{-1} \\ 1 + \frac{1}{9}e^{-1} \end{pmatrix} \\ &= \begin{pmatrix} 0.60504264 \\ 0.26710485 \end{pmatrix}\end{aligned}$$

Thus,

$$f\left(\vec{x}_1\right) = \begin{pmatrix} 0.00365278 \\ 0.01423292 \end{pmatrix} \quad \text{and} \quad \det\left(J\left(\vec{x}_1\right)\right) = 2.7136847146624117$$

then the second approximation is given by

$$\begin{aligned} \vec{x}_2 &= \begin{pmatrix} 0.60504264 \\ 0.26710485 \end{pmatrix} - \frac{1}{2.7136847146624117} \begin{pmatrix} 1.91493413 & 1 \\ 1 & 1.93932765 \end{pmatrix} \begin{pmatrix} 0.00365278 \\ 0.01423292 \end{pmatrix} \\ &= \begin{pmatrix} 0.60504264 \\ 0.26710485 \end{pmatrix} - \frac{1}{2.7136847146624117} \begin{pmatrix} 0.02122775 \\ 0.03125508 \end{pmatrix} \\ &= \boxed{\begin{pmatrix} 0.59722016 \\ 0.25558727 \end{pmatrix}} \end{aligned}$$

Question 6:

Given the following data which fits $y = \sin(x)$,

i	0	1	2
x_i	0	$\frac{\pi}{2}$	π
y_i	0	1	0

Using any polynomial interpolation, such as Lagrange polynomials, which are given by

$$p(x) = \sum_{i=0}^{N} y_i \, l_i(x), \quad \text{where} \quad l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{N} \frac{x - x_j}{x_i - x_j},$$

derive a quadratic approximation. What is the co-efficient of the quadratic term?

- ○ $4/\pi$
- ○ $4/\pi^2$
- ○ $2/\pi^2$
- ☒ $-4/\pi^2$
- ○ $3/\pi^2$
- ○ $-1/\pi^2$

The first Lagrange polynomial

$$\begin{aligned} l_0(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\ &= \frac{(x - \frac{\pi}{2})(x - \pi)}{(0 - \frac{\pi}{2})(0 - \pi)} \\ &= \frac{(x - \frac{\pi}{2})(x - \pi)}{-\frac{\pi}{2} \cdot -\pi} \\ &= \frac{2(x - \frac{\pi}{2})(x - \pi)}{\pi^2} \end{aligned}$$

For $l_1(x)$:

$$\begin{aligned} l_1(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\ &= \frac{(x - 0)(x - \pi)}{(\frac{\pi}{2} - 0)(\frac{\pi}{2} - \pi)} \\ &= \frac{x(x - \pi)}{\frac{\pi}{2} \cdot -\frac{\pi}{2}} \\ &= -\frac{4x(x - \pi)}{\pi^2} \end{aligned}$$

For $l_2(x)$

$$\begin{aligned} l_2(x) &= \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} \\ &= \frac{(x - 0)(x - \frac{\pi}{2})}{(\pi - 0)(\pi - \frac{\pi}{2})} \\ &= \frac{x(x - \frac{\pi}{2})}{\pi \cdot \frac{\pi}{2}} \\ &= \frac{2x(x - \frac{\pi}{2})}{\pi^2} \end{aligned}$$

The Lagrange interpolation polynomial $p(x)$ is given by

$$p(x) = y_0\, l_0(x) + y_1\, l_1(x) + y_2\, l_2(x).$$

Hence $p(x) = -\dfrac{4x(x-\pi)}{\pi^2}$, then the co-efficient of x^2 is $\boxed{-4/\pi^2}$.

Question 7:

The ordinary differential equation

$$y'(t) = \left(1 - \frac{y}{K}\right) y \quad \text{with} \quad y(0) = 1$$

has the solution, $y(t) = \dfrac{y_0 e^t}{1 + y_0 e^t / K}$. Let $K = 2$, then using the forward Euler method, i.e.

$$u_{n+1} = u_n + h f(t_n, u_n)$$

with step size $h = 0.1$, show the global truncation error $|y(2h) - u_2|$ after two steps is

- ○ -0.935662
- ○ 0.000288
- ○ 0.0071290
- ☒ 0.341875
- ○ 0.05656
- ○ 0.189544

From the ordinary differential equation, f is given by

$$f = y\,(1 - y/2).$$

Hence the initial evaluation is

$$f(t_0, u_0) = y_0\,(1 - y_0/2) = \frac{1}{2}.$$

Thus, the approximation given by the forward Euler method is

$$\begin{aligned} u_1 &= u_0 + h f(t_0, u_0) \\ &= 1 + 0.1 \times \frac{1}{2} \\ &= 21/20. \end{aligned}$$

Then for the next time step

$$f(t_1, u_1) = (1 - 21/40) \times \frac{21}{20} = 0.49875.$$

So the approximation to the solution is

$$\begin{aligned} u_2 &= u_1 + h f(t_1, u_1) \\ &= 1.05 + 0.1 \times 0.49875 \\ &= 1.05 + 0.049875 \\ &= 1.099875. \end{aligned}$$

The exact solution is given by

$$y(0.2) = \frac{2e^{0.2}}{2 + e^{0.2}} \approx 0.7580$$

Thus the global truncation error is given by

$$\boxed{|y(0.2) - u_2| = 0.341875}$$

Question 8:

The Jacobi scheme solves the linear equation $A\vec{x} = \vec{b}$ iteratively as

$$\vec{x}_{n+1} = \left(I - D^{-1}A\right)\vec{x}_n + D^{-1}\vec{b}$$

where D is the diagonal matrix of A. Find the residual, $|\vec{x}^* - \vec{x}_2|$ between the exact solution, $\vec{x}^*$, and the Jacobi solution after two iterates with initial guess $\vec{x}_0 = (1,1)^T$, where

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \vec{b} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

- [x] 1/4
- [ ] 1/5
- [ ] 2/5
- [ ] 1/2
- [ ] 3/4
- [ ] 0

Exact solution is given by $\vec{x}^* = A^{-1}\vec{b}$, where A^{-1} is given by

$$A^{-1} = \frac{1}{2\times 2 - (-1)\times 1}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}$$

Then

$$\begin{aligned}\vec{x}^* &= \frac{1}{5}\begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\ &= \frac{1}{5}\begin{pmatrix} 1\times 2 + (-1)\times 2 \\ 1\times 1 + 2\times 2 \end{pmatrix} \\ &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}.\end{aligned}$$

The diagonal matrix D is

$$D = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \Rightarrow D^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus

$$\begin{aligned}I - D^{-1}A &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 1/2 \\ -1/2 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}\end{aligned}$$

and

$$
\begin{aligned}
D^{-1}b &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \end{pmatrix} \\
&= \begin{pmatrix} 1/2 \\ 1 \end{pmatrix}.
\end{aligned}
$$

The first iterate is then given by

$$
\begin{aligned}
\vec{x}_1 &= \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -1/2 + 1/2 \\ 1/2 + 1 \end{pmatrix} \\
&= \begin{pmatrix} 0 \\ 3/2 \end{pmatrix}
\end{aligned}
$$

and the second iterate

$$
\begin{aligned}
\vec{x}_2 &= \begin{pmatrix} 0 & -1/2 \\ 1/2 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ 3/2 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} -3/4 + 1/2 \\ 0 + 1 \end{pmatrix} \\
&= \begin{pmatrix} -1/4 \\ 1 \end{pmatrix}
\end{aligned}
$$

Thus the residual error is given by

$$
\boxed{|\vec{x}^* - \vec{x}_2| = 1/4.}
$$