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JTMS-MAT-13: Numerical Methods

Exam: Saturday 24 August 2024

All questions carry equal marks. Answer 5 questions only. Please use the booklet provided, clearly stating which questions are to be marked.

Note that all trigonometric values should be expressed in radians.

Question 1:

- (a) State a condition which means a square matrix will not be invertible.
- (b) Given the matrix

$$A = \left(\begin{array}{ccc} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{array}\right),$$

use Gaussian elimination to show the row echelon form of the matrix A as

$$U = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 0 & 1 \end{pmatrix}.$$

(c) By applying Gaussian elimination, or any other method, show the solution to the linear equation $A\vec{x} = \vec{b}$, where \vec{b} is given by

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{is} \quad \vec{x} = \begin{pmatrix} 110.25 \\ -24 \\ 9.5 \end{pmatrix}.$$

(d) If an $n \times n$ matrix is invertible, what is the order of the upper limit for the number of arithmetic operations to yield the inverse for Gaussian elimination?

Question 2

(a) Find the Jacobian matrix J = J(x, y) for the vector-valued function

$$f(x,y) = \begin{pmatrix} 4x^2 - 20x + \frac{1}{4}y^2 - 8\\ \frac{1}{2}xy^2 + 2x - 5y + 8 \end{pmatrix}.$$

(b) Show the inverse of the Jacobian matrix is

$$J^{-1} = \frac{1}{|J|} \begin{pmatrix} xy - 5 & -\frac{1}{2}y \\ -\frac{1}{2}y^2 - 2 & 4(2x - 5) \end{pmatrix} \text{ where } |J| = 8x^2y - 40x - 20xy + 100 - \frac{1}{4}y^3 - y.$$

(c) Let $\vec{u}_n = (x_n, y_n)^T$. Then, using Newton's method, $\vec{u}_{n+1} = \vec{u}_n - J^{-1}(\vec{u}_n) f(\vec{u}_n)$, with an initial guess $\vec{u}_0 = (0,0)^T$, show that the first iteration of Newton's method is $(-0.4, 1.44)^T$.

Question 3: Consider the integral

$$I = \int_{1}^{2} f(x) dx = \int_{1}^{2} \frac{dx}{x} = \ln(2) = 0.6931471805599453$$

(a) Given the Trapezium rule,

$$I_n = \frac{h}{2} \left(f(x_0) + f(x_n) + 2 \sum_{i=1}^{n-2} f(x_i) \right)$$

where h = (b-a)/n and n is the number of intervals, show that the approximations to the integral for $n = 2^k$ where k = 0, 1 and 2 are

$$\begin{array}{c|cccc} k & n & I_n \\ \hline 0 & 1 & 0.75 \\ 1 & 2 & 0.70833333 \\ 2 & 4 & 0.69702381 \\ \end{array}$$

(b) Noting that the Trapezium rule has error behaviour

$$I = I_n + a_1 h^2 + a_2 h^4 + \cdots$$

for some constants a, and considering the difference between the errors of the Trapezium rule for h and h/2, derive the Romberg formula

$$R_k^1 = \frac{1}{3} \left(4R_k^0 - R_{k-1}^0 \right)$$

where $R_0^0 = I_1$, $R_1^0 = I_2$ etc.

(c) Using the values from the Trapezium rule for $I_k = R_k^0$, show that $R_2^1 = 0.693253$.

Question 4:

- (a) For the numerical solutions of ordinary differential equations, y' = f(y,t), what is an explicit method?
- (b) Using a Taylor expansion, show the forward Euler scheme for a first-order differential equation is

$$y_{n+1} = y_n + hf(y_n, t_n)$$

where y_n denotes the solution at $t_n = t_0 + nh$ for a time step h and initial condition $y(t_0)$.

(c) For the second-order linear ordinary differential equation

$$y''(t) = -4y'(t) + y(t)$$
 with $y(0) = 1$ and $y'(0) = 1$

show, by considering the backward difference approximation $\nabla \vec{u}(t_{n+1}) \approx (\vec{u}_{n+1} - \vec{u}_n)/h$, where $\vec{u} = (y, y')$ and \vec{u}_n denotes the solution at $t_n = t_0 + nh$, that the backward Euler scheme yields

$$\vec{u}_{n+1} = \frac{1}{1+4h-h^2} \left(\begin{array}{cc} 1+4h & -h \\ -h & 1 \end{array} \right) \vec{u}_n.$$

(d) Compute the first two steps of the backward Euler scheme for the system given in (c), i.e. $\vec{u}_0 = (1,1)$ with h = 0.1.

Question 5: Integrals can be numerically evaluated using the composite Simpson's 1/3 rule, which is given by

$$I = \int_{a}^{b} f(x) dx \approx \frac{h}{3} \left(f(x_0) + 4 \sum_{i=1}^{n/2} f(x_{2i-1}) + 2 \sum_{i=1}^{n/2-1} f(x_{2i}) + f(x_n) \right)$$

where $x_i = a + ih$, with h = (b - a)/n for i = 0, ..., n, where n, the number of subintervals, is even. Given the integral

$$I = \int_0^1 \frac{1}{3} + \cos\left(\pi x\right) \, \mathrm{d}x$$

what is the difference between the exact and the approximate integral using Simpson's rule with six subintervals?

\bigcirc 1/2	$\bigcirc 1/6$
O 0	○ 1/10
\bigcirc 1/3	O 1/7

Question 6: Given the following data:

$$\begin{array}{c|ccccc} i & 0 & 1 & 2 \\ \hline x_i & 0 & 2 & 4 \\ y_i & 2 & 1 & 2 \end{array}$$

Newton interpolation constructs a interpolating polynomial p(x), using the formula $p = \sum_{i=0}^{n} \alpha_i n_i(x)$, where the basis polynomials are defined as

$$n_0(x) = 1$$
, $n_i(x) = (x - x_0)(x - x_1) \cdots (x - x_{i-1})$ for $i \ge 1$

where $p(x_i) = y_i$. Using Newton interpolation, which are the correct collocation matrix Φ and weighting vector $\vec{\alpha}$ such that $\Phi \vec{\alpha} = \vec{y}$ where $\vec{\alpha} = (\alpha_0, \alpha_1, \alpha_2)^T$ and $\vec{y} = (y_0, y_1, y_2)^T$.

$$\bigcirc \begin{pmatrix} 1 & 0 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix} \qquad \bigcirc \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 4 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 0 \\ -1/2 \\ -1/4 \end{pmatrix} \\
\bigcirc \begin{pmatrix} 1 & 1 & 1 \\ 4 & 2 & 0 \\ 8 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ -1 \end{pmatrix} \qquad \bigcirc \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} -2 \\ -1/2 \\ -1/3 \end{pmatrix} \\
\bigcirc \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 \\ -1/2 \\ 1/4 \end{pmatrix} \qquad \bigcirc \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 4 & 8 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Question 7: Using the Jacobi scheme, $\vec{x}_{n+1} = (I - D^{-1}A)\vec{x}_n + D^{-1}b$, where D is the diagonal matrix of A, what is the second iterate, \vec{x}_2 , of the solution to the system Ax = b with

$$A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix}, \quad \vec{x}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \text{ and } \vec{b} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

$$\bigcirc \begin{pmatrix} 0.456 \\ 0.7421 \\ -0.8123 \end{pmatrix} \qquad \bigcirc \begin{pmatrix} 0.6694 \\ 0.2306 \\ 0.5183 \end{pmatrix}$$

$$\bigcirc \begin{pmatrix} 1.7326 \\ -0.6034 \\ 0.6777 \end{pmatrix} \qquad \bigcirc \begin{pmatrix} 0.6789 \\ 0.2481 \\ 0.1111 \end{pmatrix}$$

$$\bigcirc \begin{pmatrix} 0.0013 \\ 0.0299 \\ 3.5217 \end{pmatrix} \qquad \bigcirc \begin{pmatrix} 2.9876 \\ 0.1234 \\ 2.3540 \end{pmatrix}$$

Question 8: The differential equation

$$y'(t) = 1 - 3y(t)$$
 with $y(0) = 1$

has the exact solution $y = \frac{1}{3}(2e^{-3t} + 1)$. From the explicit Runge-Kutta scheme given by the Butcher array

where

$$u_{k+1} = u_k + h \sum_{i=1}^{4} b_i f\left(u_k + h \sum_{j=1}^{4} a_{i,j} k_j, t_k + c_i h\right)$$

and using step size h = 0.1, what is the $|y(2h) - u_2|$, i.e. the global truncation error after two steps?

 \bigcirc 0.002

 $\bigcirc 0.058$

 \bigcirc 0.370

 \bigcirc 0.000115

 \bigcirc 0.965

 \bigcirc 0.000019