

JTMS-MAT-13: Numerical Methods

Assignment Sheet 6. Released: May 02, 2024

Due: May 14, 2024

Exercise 1 [4+2+2+3 Points]:

- (a) For Gaussian quadrature on $[-1, 1]$ and $n = 3$, show that the four Gauss nodes x_0, \dots, x_3 as the roots of the polynomial $q(x)$ that is orthogonal to $1, x, x^2$ and x^3 are given by

$$x = \pm \sqrt{\frac{15 \pm 2\sqrt{30}}{35}}$$

- (b) To illustrate the full quadrature, determine the first corresponding weight A_0 of the 4 point Gaussian quadrature.
- (c) Based on your results of (a), find the corresponding Legendre polynomial that uses the additional condition of $q(1) = 1$.
- (d) Consider Gauss quadrature with 2 nodes on the interval $[-1, 1]$. Derive the resulting Legendre polynomial $q(x)$ as a solution to the orthogonality condition:

$$\int_{-1}^1 q(x)x^k dx = 0$$

for $k = 0, 1$. Determine the roots of $q(x)$ to arrive at the Gauss nodes. Use your result to approximate the integral:

$$\int_{-\frac{3\pi}{\sqrt{3}}}^{\frac{3\pi}{\sqrt{3}}} \sin(x) \cos^2(x) dx$$

Remember that you need to convert into the appropriate interval. You may give the result either as an exact value or use a calculator to arrive at the final number. What kind of method is this?

Exercise 2 [3+4+4+2+2 Points]:

Consider the linear ordinary differential equation

$$y''(t) = -3y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1.$$

- (a) Convert this 2nd order ordinary differential equation into a system of two coupled first order ODEs, one in $y(t)$ and one in $y'(t)$. Write the system as a vector-valued ordinary differential equation in $\vec{v}(t) = (y(t), y'(t))^T$, in the form $f(\vec{v}) = A\vec{v}$.
- (b) Show that the backward Euler method can be written as

$$\vec{u}_{n+1} = (I - hA)^{-1} \vec{u}_n$$

and provide the full system for \vec{u}_{n+1} for the ODE presented above.

- (c) Noting that $f_n = A\vec{u}_n$, so that $f(\vec{u}_n + hf_n) = A\vec{u}_n + hA^2\vec{u}_n$, show that Heun's method for the case of the vector-valued ODE presented above is

$$\vec{u}_{n+1}(t) = \begin{pmatrix} 1 + \frac{h^2}{2} & h\left(1 - \frac{3}{2}h\right) \\ h\left(1 - \frac{3}{2}h\right) & h(5h - 3) \end{pmatrix} \vec{u}_{n+1}(t).$$

- (d) Calculate $y(0.3)$ and $y'(0.3)$ using backward Euler method with $h = 0.15$.
- (e) Calculate $y(0.3)$ and $y'(0.3)$ using Heun's method with $h = 0.15$.

Exercise 3 [5 Points]:

Consider the ordinary differential equation

$$y'(t) = -y(t) + \cos(t) \quad \text{with} \quad y(0) = 1.$$

- (a) Showing all working, calculate four time steps of the approximate solution using RK4 with step size $h = 0.1$.