Integrability, localisation and bifurcation of an elastic conducting rod in a magnetic field

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- PhD in Hamiltonian dynamical system, post-doc in interaction of ensembles of driven nonlinear oscillators also at UCL (cavitation in tissue), another post-doc solving large scale PDEs (focused ultrasound therapy) at the Institute of Cancer Research

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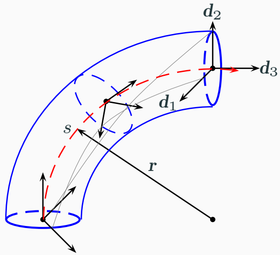
Introduction

- Classical problem: modern approach
- The reduction estimated at a billion dollars over ten years for the international space station alone [1]
- Tethers need to be described as an elastic rod rather wire
- Highly (geometrically) nonlinear

A Cosserat rod is defined by a centreline $\boldsymbol{r}$ and directors $\{\boldsymbol{d}_1, \boldsymbol{d}_2, \boldsymbol{d}_3\}$ by

$$\boldsymbol{r}' = \boldsymbol{v} \quad \text{and} \quad \boldsymbol{d}_i' = \boldsymbol{u} \times \boldsymbol{d}_i.$$

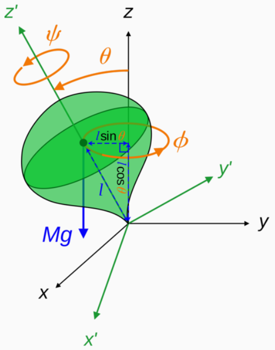
The strains $\boldsymbol{u}$, $\boldsymbol{v}$ relates the configuration to the balance equations through the hyperelastic constitutive relations. In order to separate geometric from material nonlinearities, linear constitutive relations are chosen throughout



$$u_1 = m_1/B_1, \quad u_2 = m_2/B_2, \quad u_3 = m_3/C \quad \text{and} \quad v_3 = 1 + n_3/K.$$

Kinetic Analogy

- Parallels between the motion of a heavy spinning top and the shape of a rod under torque and tension
  - Arc-length plays the role of time
  - Sleeping top - straight rod
  - Precession - helical rod
  - Homoclinic - localised modes
  - Regions of stability reversed
  - Simplification, no shear, no extension, no initial curvature, nonlinear constitutive relations ...



Simplest rod model is the force-free rod. In the spatial frame

$$\boldsymbol{m}' = \boldsymbol{0}.$$

In the director frame

$$\mathsf{m}' = \mathcal{J}(\mathsf{m})\nabla\mathcal{H}(\mathsf{m}) = \mathsf{m} \times \mathsf{u}$$

structure matrix and Hamiltonian given by

$$\mathcal{J} = -\mathcal{J}^T = \hat{\mathsf{m}} = \begin{pmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{pmatrix} \quad \text{and} \quad \mathcal{H} = \frac{1}{2}\mathsf{m} \cdot \mathsf{u}(\mathsf{m}).$$

One-dimensional null-space spanned by the gradient of the Casimir

$$\nabla C_1 = \frac{1}{2}(m_1, m_2, m_3)^T \Rightarrow C_1 = \mathsf{m} \cdot \mathsf{m}.$$

The system is globally superintegrable [2].

The non-canonical system is in fact a Lie-Poisson system

$$\{f, g\}_\mu = \left\langle \mu, \left[\frac{\delta f}{\delta \mu}, \frac{\delta g}{\delta \mu}\right]\right\rangle$$

where $\mu \in \mathfrak{g}^*$ and $f, g : \mathfrak{g}^* \mapsto \mathbb{R}$, with an associated inner product $\langle \cdot, \cdot \rangle$ and Lie bracket on the Lie algebra $[\cdot, \cdot]$.

The inner product associates elements of the Lie algebra with its dual and the Lie bracket acts on the function derivatives of f and g with respect to the field variable, that is $\delta f / \delta \mu : \mathfrak{g}^* \mapsto \mathfrak{g}$.

The Poisson bracket is a Lie-Poisson bracket

$$\{f, g\}_{(\mathrm{m})} = -\underbrace{\mathrm{m} \cdot (\nabla_{\mathrm{m}} f \times \nabla_{\mathrm{m}} g)}_{\text{twist}} .$$

where the Lie bracket is given by the direct sum of elements

$$[\xi, \eta] = \xi \times \eta$$

and the inner product is the dot product.

Corresponds to the Euler top.

For a rod under end tension and moment in the spatial frame

$$\boldsymbol{m}' = \boldsymbol{n} \times \boldsymbol{r}' \quad \text{and} \quad \boldsymbol{n}' = \boldsymbol{0}.$$

In the body frame

$$\begin{pmatrix} \mathsf{m} \\ \mathsf{n} \end{pmatrix}' = \begin{pmatrix} \hat{\mathsf{m}} & \hat{\mathsf{n}} \\ \hat{\mathsf{n}} & 0 \end{pmatrix} \begin{pmatrix} \mathsf{u} \\ \mathsf{v} \end{pmatrix}, \quad \mathcal{H} = \frac{1}{2}\mathsf{u} \cdot \mathsf{m} + \frac{1}{2}(\mathsf{v} - \mathsf{d}_3) \cdot \mathsf{n} + \mathsf{n} \cdot \mathsf{d}_3$$

where $\mathsf{d}_3 = (0, 0, 1)$. System has two Casimirs

$$C_1 = \mathsf{n} \cdot \mathsf{n} \quad \text{and} \quad C_2 = \mathsf{n} \cdot \mathsf{m}.$$

If the principal bending stiffnesses are equal, i.e. isotropic, is integrable.

$$I_1 = \mathsf{m} \cdot \mathsf{d}_3.$$

Other integrable cases: Kovalevskaya and Chaplygin-Goryachev.

The force is fixed in the spatial frame, but rotates in the body frame. The dynamics are generated by a Poisson bracket with a semi-direct product extension

$$\{f, g\}_{(\mathsf{m},\mathsf{n})} = -\mathsf{m} \cdot (\nabla_{\mathsf{m}} f \times \nabla_{\mathsf{m}} g) - \underbrace{\mathsf{n} \cdot (\nabla_{\mathsf{m}} f \times \nabla_{\mathsf{n}} g + \nabla_{\mathsf{n}} f \times \nabla_{\mathsf{m}} g)}_{\text{force}} .$$

which is a Lie-Poisson bracket for

$$[(\xi, u), (\eta, v)] = (\xi \times \eta, \xi \times v - \eta \times u) .$$

Corresponds to a heavy top

Superintegrable solutions are helices with additional integral $\mathsf{m} \cdot \mathsf{m}$.

Integrability

For a conducting rod in a uniform magnetic field, in the spatial frame ($\boldsymbol{B} = B\boldsymbol{e}_3$)

$$\boldsymbol{m}' = \boldsymbol{n} \times \boldsymbol{r}', \quad \boldsymbol{n}' = \underbrace{\lambda \boldsymbol{e}_3 \times \boldsymbol{r}'}_{\text{Lorentz force, } \lambda = IB} \quad \text{and} \quad {\boldsymbol{e}_3}' = \boldsymbol{0}.$$

In the director frame - noncanonical Hamiltonian system

$$\begin{pmatrix} \mathsf{m} \\ \mathsf{n} \\ \mathsf{e}_3 \end{pmatrix}' = \begin{pmatrix} \hat{\mathsf{m}} & \hat{\mathsf{n}} & \hat{\mathsf{e}}_3 \\ \hat{\mathsf{n}} & \lambda \hat{\mathsf{e}}_3 & 0 \\ \hat{\mathsf{e}}_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathsf{u} \\ \mathsf{v} \\ 0 \end{pmatrix}, \quad \mathcal{H} = \frac{1}{2}\mathsf{u} \cdot \mathsf{m} + \frac{1}{2}(\mathsf{v} - \mathsf{d}_3) \cdot \mathsf{n} + \mathsf{d}_3 \cdot \mathsf{n}.$$

The system has a Poisson bracket:

$$
\begin{aligned}
\{f, g\}_{(\mathsf{m},\mathsf{n},\mathsf{e}_3)} = {} & -\mathsf{m}\cdot(\nabla_{\mathsf{m}} f \times \nabla_{\mathsf{m}} g) - \mathsf{n}\cdot(\nabla_{\mathsf{m}} f \times \nabla_{\mathsf{n}} g + \nabla_{\mathsf{n}} f \times \nabla_{\mathsf{m}} g) \\
& - \underbrace{\mathsf{e}_3\cdot(\nabla_{\mathsf{m}} f \times \nabla_{\mathsf{e}_3} g + \nabla_{\mathsf{e}_3} f \times \nabla_{\mathsf{m}} g)}_{\text{evolution of field}} \\
& - \underbrace{\lambda \mathsf{e}_3\cdot(\nabla_{\mathsf{n}} f \times \nabla_{\mathsf{n}} g)}_{\text{effect of field}}
\end{aligned}
$$

extended by semi-direct extension, describing the evolution of the magnetic field in the body, and a cocycle, describing the Lorentz force, called a Liebnitz extension [3]. Lie bracket is

$$
[(\xi, u, w), (\eta, v, x)] = (\xi \times \eta,\ \xi \times v - \eta \times u,\ \xi \times x - \eta \times w - \lambda u \times v).
$$

A nine-dimensional system with three Casimirs

$$C_1 = \mathsf{e}_3 \cdot \mathsf{e}_3, \quad C_2 = \mathsf{e}_3 \cdot \mathsf{n} \quad \text{and} \quad C_3 = \mathsf{n} \cdot \mathsf{n} + 2\lambda \mathsf{m} \cdot \mathsf{e}_3,$$

two additional first integrals when isotropic and inextensible

$$I_1 = \mathsf{m} \cdot \mathsf{d}_3 \quad \text{and} \quad I_2 = \mathsf{n} \cdot \mathsf{m} + \lambda B_1 \mathsf{d}_3 \cdot \mathsf{e}_3$$

and one Hamiltonian

$$\mathcal{H} = \frac{1}{2} \mathsf{u} \cdot \mathsf{m} + \mathsf{d}_3 \cdot \mathsf{n}.$$

Thus system is completely integrable

Conserved quantities have no immediate physical interpretation.

The family of equations can be generated by a Lax pair [4] with a spectral parameter α

$$\frac{\mathrm{d}}{\mathrm{d}s}\Gamma(\alpha) = \left[\Gamma(\alpha), \hat{\mathrm{d}}_3\alpha + \hat{\mathrm{u}}\right],$$

where

$$\Gamma(\alpha) = K\hat{\mathrm{d}}_3\alpha + \Gamma_0 + \Gamma_1\alpha^{-1} + \ldots + \Gamma_n\alpha^{-n} \in \mathfrak{so}(3), \quad n \in \mathbb{N},$$

with

$$\Gamma_0 = \hat{\mathrm{m}}, \quad \Gamma_1 = \hat{\mathrm{n}} \quad \text{and} \quad \Gamma_2 = \lambda\hat{\mathrm{e}}_3 \ldots$$

and conserved quantities

$$I_i = -\frac{1}{4}\text{residue}_{\mu=0}\left(\alpha^{i-1}\text{trace}\left[\Gamma(\alpha)^2\right]\right) \quad \text{for} \quad i = -1, 0, 1, \ldots, n-1,$$
$$C_i = -\frac{1}{4}\text{residue}_{\mu=0}\left(\alpha^{i-1}\text{trace}\left[\Gamma(\alpha)^2\right]\right) \quad \text{for} \quad i = n, n+1, n+2, \ldots, 2n.$$

Thus the governing equations for elastic conducting rod in a non-uniform “twisted” hyper-magnetic field are integrable.

$$\boldsymbol{m}' + \boldsymbol{r}' \times \boldsymbol{n} = \boldsymbol{0}, \quad \boldsymbol{n}' + \boldsymbol{r}' \times \boldsymbol{B} = \boldsymbol{0}, \quad \boldsymbol{B}' + \boldsymbol{r}' \times \boldsymbol{D} = \boldsymbol{0} \quad \text{and} \quad \boldsymbol{D}' = \boldsymbol{0}.$$

Gives $B_x = y$, $B_y = -x$, $B_z = 0$, where (x, y, z) and (B_x, B_y, B_z) are components of $\boldsymbol{r}$ and $\boldsymbol{B}$ relative to the spatial frame $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3\}$, and $\boldsymbol{e}_3$ is chosen to be in the direction of $\boldsymbol{D}$. Thus the system describes a rod in a linearly-varying magnetic field generated by a uniform ‘hypermagnetic’ field $\boldsymbol{D}$. Hamiltonian is as before, structure matrix is

$$\mathcal{J} = \begin{pmatrix} \hat{\mathrm{m}} & \hat{\mathrm{n}} & \hat{\mathrm{B}} & \hat{\mathrm{D}} \\ \hat{\mathrm{n}} & \hat{\mathrm{B}} & \hat{\mathrm{D}} & 0 \\ \hat{\mathrm{B}} & \hat{\mathrm{D}} & 0 & 0 \\ \hat{\mathrm{D}} & 0 & 0 & 0 \end{pmatrix}.$$

Spatial Chaos

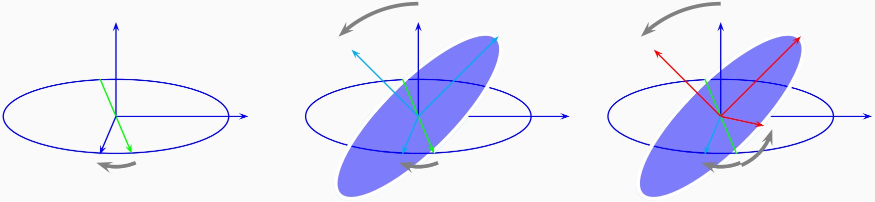


Figure 1: The Euler angles relating two coordinate frames.

Euler angles reduce nine-dimensional non-canonical system to six-dimensional canonical system

$$\mathcal{H}(\theta, p_\theta, \psi, p_\psi) = \frac{p_\theta^2}{2B_1} + \frac{1}{2B_1}\left(\frac{p_\psi - p_\phi \cos\theta}{\sin\theta}\right)^2 + \frac{p_\phi^2}{2C} + C_2 \cos\theta$$

$$+ \boxed{\sin\psi \sin\theta \sqrt{C_3 - C_2^2 - 2\lambda p_\psi}}$$

with two integrals

$$I_1 = p_\phi, \quad I_2 = B_1 \lambda \cos\theta + C_2 p_\psi$$

$$- \sqrt{C_3 - C_2^2 - 2\lambda p_\psi}\left(p_\theta \sin\psi - \cos\psi\left(\frac{p_\phi - p_\psi \cos\theta}{\sin\theta}\right)\right).$$

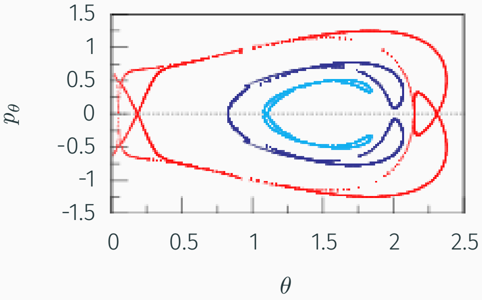
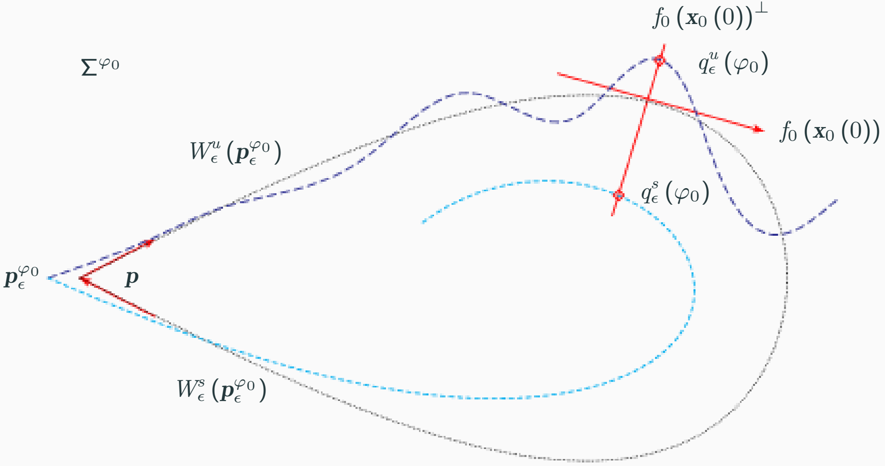


Figure 2: Phase plane at various energy levels at section $\cos\psi = 0.9$.

An approximation to splitting of stable/unstable manifolds



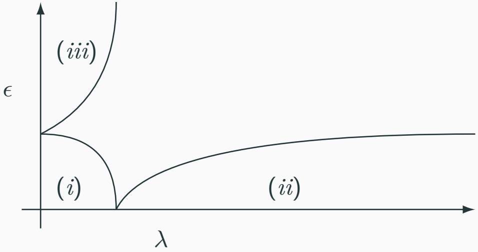
On the scaling

$$2C_1 - C_2^2 = a\delta^2, \quad \lambda = b\delta^2 \quad \text{and} \quad C_2/K = c\delta$$

reduced extensible Hamiltonian takes the form

$$\begin{aligned}\mathcal{H}_\delta\left(\theta, p_\theta, \psi, p_\psi\right) = \mathcal{H}_0\left(\theta, p_\theta\right) + \delta\left(\mathcal{H}_1^\lambda\left(\theta, \psi, p_\psi\right) + \mathcal{H}_1^\epsilon\left(\theta\right)\right) \\ + \delta^2\mathcal{H}_2^{\epsilon\lambda}\left(\theta, \psi, p_\psi\right) + \delta^3\mathcal{H}_3^{\epsilon\lambda^2}\left(\theta, \psi, p_\psi\right) + \mathcal{O}\left(\delta^4\right).\end{aligned}$$

Mel'nikov's method can be applied in three possible cases:



- Perturb Lagrange rod,
- Perturb extensible rod,
- Perturb magnetic rod.

If extensible detailed Mel'nikov analysis shows loss of integrability.

$$
\begin{aligned}
\mathcal{M}_h^{(1)}(\psi_0) &= \int_{-\infty}^{+\infty} f_0 \wedge f_1 \, \mathrm{d}\psi = \int_{-\infty}^{+\infty} \left\{\mathcal{H}_0 + \mathcal{H}_1^{\epsilon}, \mathcal{H}_1^{\lambda}\right\}_{(\theta, p_\theta)} \mathrm{d}s \\
&= 2\sqrt{a - 2bp_\psi} \cos\psi_0 \int_0^{+\infty} p_\theta \cos\bar{\psi} \sin\theta \left(1 + \frac{C_2}{K}\cos\theta\right) \mathrm{d}s \\
&\quad - 2\sqrt{a - 2bp_\psi} \sin\psi_0 \int_0^{+\infty} p_\theta \sin\bar{\psi}\,(1 + \cos\theta)\left(\cos\theta + \frac{C_2}{K}\cos 2\theta\right) \mathrm{d}s.
\end{aligned}
$$

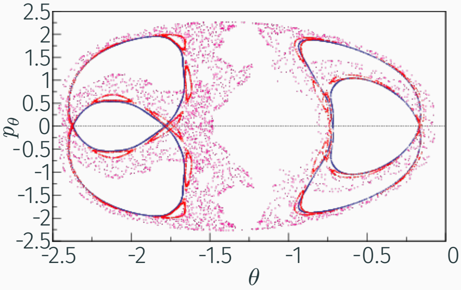


Figure 3: Poincaré sections

For case (ii)

$$\mathcal{M}_h^{(1)}(\psi_0) = \int_{-\infty}^{+\infty} \left\{\mathcal{H}_0, \mathcal{H}_1^{\epsilon} + \mathcal{H}_1^{\lambda}\right\}_{(\theta, p_\theta)} \mathrm{d}s$$

but both perturbations are integrable

$$\int_{-\infty}^{+\infty} \{\mathcal{H}_0, \mathcal{H}_1^{\epsilon}\}_{(\theta, p_\theta)} \mathrm{d}s = 0 \quad \text{and} \quad \int_{-\infty}^{+\infty} \left\{\mathcal{H}_0, \mathcal{H}_1^{\lambda}\right\}_{(\theta, p_\theta)} \mathrm{d}s = 0.$$

Thus second order analysis needs to be performed [5].

$$\begin{aligned}
\mathcal{M}_h^{(2)}(\psi_0) = {} & \frac{1}{2}\int_{\psi_0}^{+\infty} f_0 \wedge D^2 f_0 \left(\mathbf{x}_1^s\right)^2 \mathrm{d}\psi + \frac{1}{2}\int_{-\infty}^{\psi_0} f_0 \wedge D^2 f_0 \left(\mathbf{x}_1^u\right)^2 \mathrm{d}\psi \\
& + \frac{1}{2}\int_{\psi_0}^{+\infty} f_0 \wedge Df_1 \mathbf{x}_1^s \, \mathrm{d}\psi + \frac{1}{2}\int_{-\infty}^{\psi_0} f_0 \wedge Df_1 \mathbf{x}_1^u \, \mathrm{d}\psi \\
& + \int_{-\infty}^{+\infty} f_0(\mathbf{x}_0) \wedge f_2(\mathbf{x}_0, \psi) \, \mathrm{d}\psi.
\end{aligned}$$

- Homoclinic solutions found numerically with angle ψ as 'time' and action $\mathcal{I}$ as Hamiltonian.
- Robust numerical method using bisection method to find first order approximation to flow, $\boldsymbol{x}_1$ computed subject to being bounded and transverse to flow forwards and backwards in time.

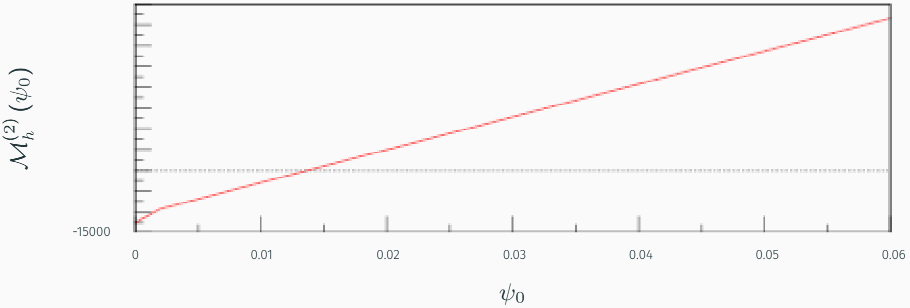


Figure 4: Second order Mel'nikov integral showing existence of simple zero

Bifurcation

Computation

Ordinarily trivial configuration, a straight twisted rod, is about a fixed point, now it is a periodic solution $\boldsymbol{\gamma}(s)$ with period τ.

$$\boldsymbol{\gamma}(s) = (0, 0, 0, 0, 0, 0, 0, 0, \sin(s(1+\nu)/2), \cos(s(1+\nu)/2)).$$

Under the τ-mapping $\boldsymbol{\gamma}(\tau) = \boldsymbol{p}$ the periodic orbit is the fixed point

$$\boldsymbol{p} = \boldsymbol{\gamma}(\tau) = (0, 0, 0, 0, 0, 0, 0, 0, 0, 1).$$

- Computation and continuation of localised solutions needs modification.
- Floquet theory required, monodromy matrix computed.
- No closed form analytical expressions for buckling values.

The system is reversible. Thus two involutions exist

$$
\begin{aligned}
R_1 : (m_1, m_2, m_3, n_1, n_2, n_3, e_{31}, e_{32}, e_{33}) &\mapsto \\
&(-m_1, m_2, m_3, -n_1, n_2, n_3, e_{31}, -e_{32}, e_{33}) \quad \text{and} \\
R_2 : (m_1, m_2, m_3, n_1, n_2, n_3, e_{31}, e_{32}, e_{33}) &\mapsto \\
&(m_1, -m_2, m_3, n_1, -n_2, n_3, e_{31}, -e_{32}, e_{33}) \quad \text{as} \quad s \mapsto -s.
\end{aligned}
$$

R_i-Reversible configurations pass through a symmetric section $\mathcal{S}_i$

$$
\mathcal{S}_i = \mathsf{Fix}\,(R_i) \quad \text{for} \quad \boldsymbol{\gamma}\,(n\tau) = \boldsymbol{p} \quad \text{where} \quad \text{with} \quad \boldsymbol{p} \in \mathcal{S}_i.
$$

Symmetric section is three-dimensional

$$
\mathcal{S}_1 = \left\{ \boldsymbol{x} \in \mathbb{R}^9 \;:\; m_1\,(1) = n_1\,(1) = e_{31}\,(1) = 0 \right\}.
$$

- In order to avoid the polar singularity inherent in the Euler angles when $\theta = 0$ and reduce the dimension of the full system Euler parameters are used to convert quantities from the spatial frame into the director frame.

- The four Euler parameters $q = (q_1, q_2, q_3, q_4)$ are subject to

$$1 = q_1^2 + q_2^2 + q_3^2 + q_4^2.$$

- The ten-dimensional system $\boldsymbol{x} = (\mathrm{m}, \mathrm{n}, q)$ has three Casimirs and a constraint.

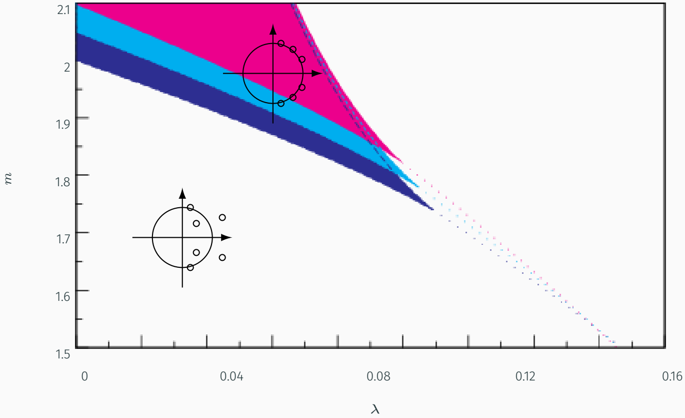
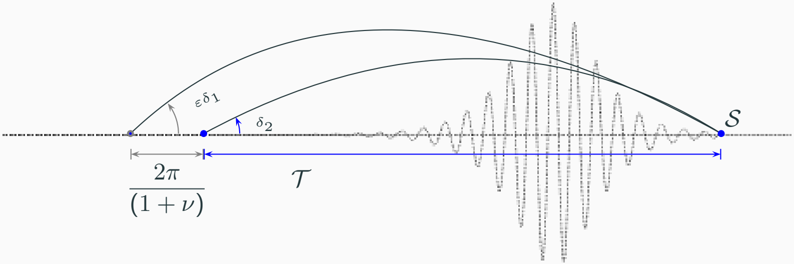


Figure 5: Spectrum of the monodromy matrix when $\nu = 1/3$ with $\epsilon = 0$ (blue), $\epsilon = 0.05$ (cyan) and $\epsilon = 0.1$ (magenta). The coloured regions correspond elliptic periodic orbits, the dashed lines are codimension-one curves at which the multipliers are stationary and reverse direction.

- Shooting from fixed point of a map $\boldsymbol{p}$ over half range into 3D symmetric section $\mathcal{S}$ of a reversibility.
- Three parameter shooting with $(\delta_1, \delta_2, \mathcal{T})$

$$\boldsymbol{x}(0) = \boldsymbol{p} + \varepsilon\delta_1 \left(\boldsymbol{v}_1 \sin\delta_2 + \boldsymbol{v}_2 \cos\delta_2\right)$$

where $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are vectors in the two-dimensional unstable linear subspace of the fixed point $\boldsymbol{p}$.



- Homoclinic solutions are codimension-zero.

Localised solutions are found for isolated branches of the shooting parameters.

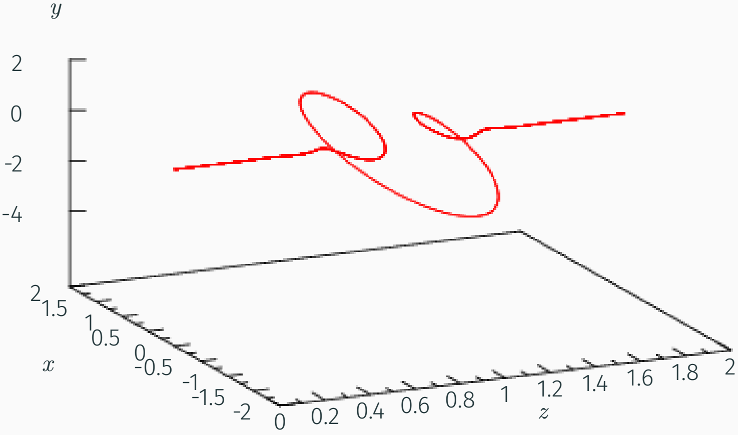


Figure 6: A localised solution

When extensible or anisotropic there are an infinite number of multi-modal solutions comprised of “primary” uni-modal solutions.

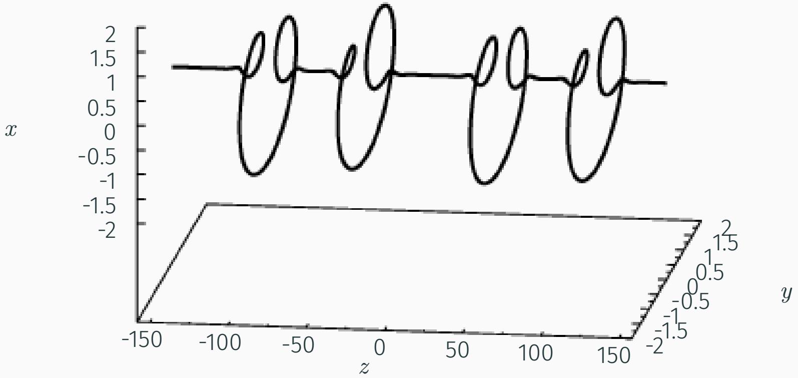


Figure 7: Localised multi-pulse solution

Continuation

With Euler parameters 10D monodromy matrix decouples:

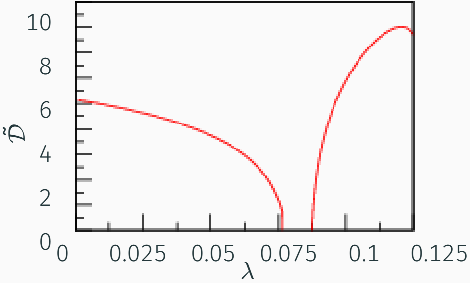
- 4D trivial matrix containing Casimirs and constraint,
- 6D nontrivial matrix contains the 'dynamics'.

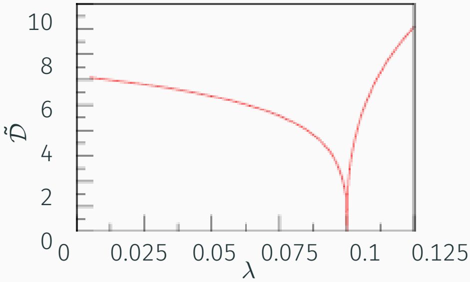
Projection boundary conditions using `auto97` exploiting exponential trichotomies onto two-dimensional stable and centre manifolds of fixed point of periodic orbit [6]

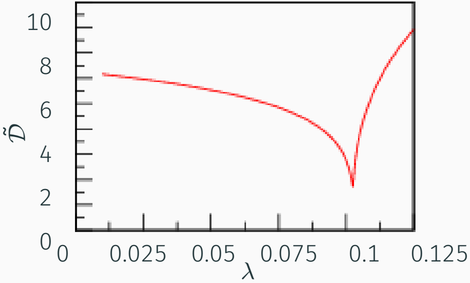
$$(L_{c,s} - \boldsymbol{p})\,\mathbf{x} = \mathbf{0} \quad \text{where} \quad L_{c,s} \in \mathbb{R}^{6\times 2}$$

along with conditions on three-dimensional symmetric section.

- Ill-posed: an extra boundary condition $\Rightarrow$ truncation length $\mathcal{T}$ freed.



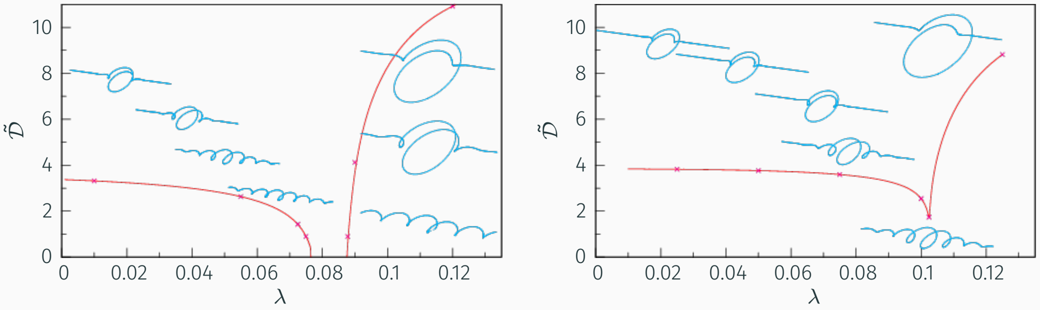




Hamiltonian-Hopf bifurcations at

- two $\lambda_\pm$ when $m > m_c$,
- one λ_c when $m = m_c$,
- zero when $m < m_c$,

critical values.

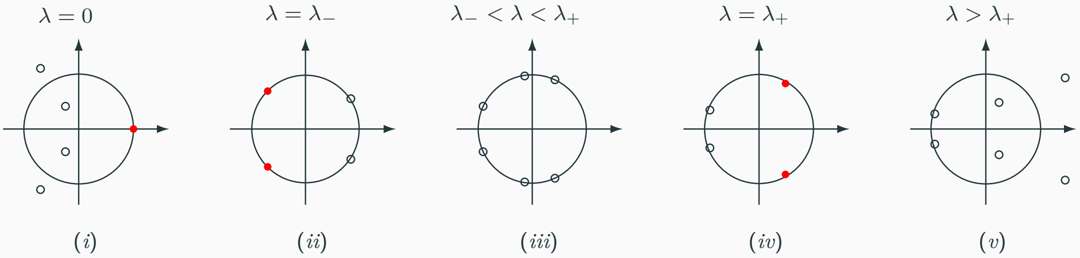


(a) $m = 1.9$

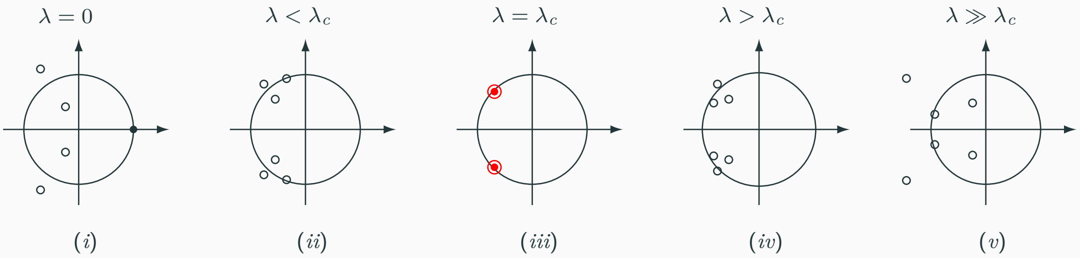
(b) $m = 1.81$

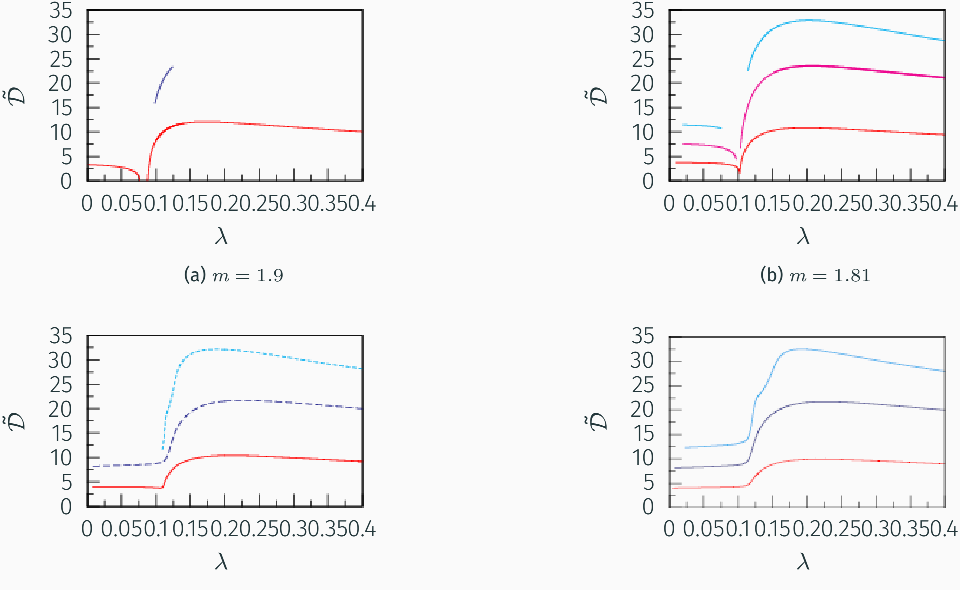
- Hamiltonian-Hopf bifurcation about a periodic solution from right $\lambda = \lambda_{-}$ and left $\lambda = \lambda_{+}$. Localised solutions bifurcate into straight twisted rods.
- Localisation-delocalisation-localisation can occur near codimension-two point.

Behaviour of Floquet multipliers for $m > m_c$



and $m = m_c$.





(a) $m = 1.9$

(b) $m = 1.81$

(c) $m = 1.775$

(d) $m = 1.3985$

Figure 9: Bifurcation diagram for single and multimodal localised configurations

Summary

Conclusions

- System is Hamiltonian, Lie-Poisson and completely integrable
- Physical realisation of abstract system [3]
- System is one member in a family of integrable systems expressed by a Lax pair [4]
- Hidden conditions on integrability: extensibility, shearability ...

Conclusions

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- Physical realisation of abstract system [3]
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- Three parameter shooting, exponential trichotomies
- Hamiltonian-Hopf-Hopf bifurcation
- Buckling due to increasing and decreasing external force
- Sequential merging of limit points for multimodal solutions

Open Problems

- Complete reduction to single degree of freedom system [7] may enable
  - Expression of system as an equivalent oscillator
  - Fixed point solutions: existence of superintegrable solutions
  - Closed form solutions for homoclinic solution: Mel'nikov method in case (iii)
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- Normal form for Hamiltonian-Hopf-Hopf bifurcation in the twistless case

Thankyou for your attention.

Any questions?

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github.com/djps/extensibility
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