# CA-MATH-804: Numerical Analysis

**Assignment Sheet 3. Due: February 23, 2022**

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**Exercise 1 [5 Points]**: Consider the matrix

$$A = \begin{pmatrix} 1 & \gamma \\ 0 & 1 \end{pmatrix}.$$

**a)** Show that for $\gamma \leq 0$ we have $K_{\infty}(A) = K_1(A) = (1+\gamma)^2$.

**b)** For the linear system $Ax = b$, where $b$ is such that $x = (1-\gamma, 1)^T$ is the solution, find a bound for $\|\delta x\|_{\infty} / \|x\|_{\infty}$ in terms of $\|\delta b\|_{\infty} / \|b\|_{\infty}$ when $\delta b = (\delta_1, \delta_2)^T$. What can you say about the condition of the problem?

**Exercise 2 [5 Points]**: Check that the matrix $A = \mathrm{tridiag}_n(-1, \alpha, -1)$ with $\alpha \in \mathbb{R}$ has eigenvalues given by

$$\lambda_i = \alpha - 2\cos(i\theta), \quad i = 1, \ldots, n,$$

and the corresponding eigenvectors are

$$v_i = \left(\sin(i\theta), \sin(2i\theta), \ldots, \sin(ni\theta)\right)^T,$$

where $\theta = \dfrac{\pi}{n+1}$.

**Exercise 3 [5 Points]**: Consider the linear system $Ax = b$, where

$$A = \begin{pmatrix} 1 & 1 & 1 \\ \alpha & 1 & 1 \\ \beta & \gamma & 1 \end{pmatrix}$$

and $\alpha, \beta, \gamma \in \mathbb{R}$. Define an iterative method for $k \geq 0$ as

$$x^{(k+1)} = U^{-1}\left(b - Lx^{(k)}\right),$$

where

$$U = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} 0 & 0 & 0 \\ \alpha & 0 & 0 \\ \beta & \gamma & 0 \end{pmatrix}.$$

**a)** Find all values of $\alpha, \beta, \gamma$ such that the sequence of iterates $\{x^{(k)}\}$ converges for every initial guess $x^{(0)}$ and every right hand side.

**b)** Give an example for $b$ leading to non-convergence in the case $\alpha = \beta = \gamma = -1$.

**c)** Is the solution always found with at most two iterations if $\alpha = \gamma = 0$?

**Exercise 4 [3 × 2 Points]**: Compute the following derivatives:

**a)** $\dfrac{\mathrm{d}}{\mathrm{d}s} \|x + sp\|_q^2$ for $x, p \in \mathbb{R}$ and $1 \le q < \infty$,

**b)** $\nabla \|u(x)\|_2^2$ where $u : \mathbb{R}^n \to \mathbb{R}^n$ is sufficiently smooth,

**c)** $\Delta \|x\|_2^2 := \operatorname{div}\left(\|x\|_2^2\right) := \nabla \cdot \left(\nabla \|x\|_2^2\right)$ for $x \in \mathbb{R}^n$.

**Exercise 5 [5 Points]**: Given $n+1$ distinct points $x_0, \ldots, x_n \in \mathbb{R}$ consider the functions

$$l_i = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \quad i = 0, \ldots n.$$

Show that these functions form a basis for the polynomials $\mathbb{P}_n$ of degree $\le n$ over $\mathbb{R}$.

**Exercise 6 [6 Points]**: Consider the linear system $Ax = b$ with

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad b = \begin{pmatrix} 3 \\ 5 \end{pmatrix},$$

and the following iterative method

$$x^{(k+1)} = B(\theta)\, x^{(k)} + g(\theta)$$

for $k \ge 0$ and a given $x^{(0)}$, where $\theta \in \mathbb{R}$ and

$$B(\theta) = \frac{1}{4} \begin{pmatrix} 2\theta^2 + 2\theta + 1 & -2\theta^2 + 2\theta + 1 \\ -2\theta^2 + 2\theta + 1 & 2\theta^2 + 2\theta + 1 \end{pmatrix}, \quad g(\theta) = \begin{pmatrix} \frac{1}{2} - \theta \\ \frac{1}{2} - \theta \end{pmatrix}.$$

**a)** Show that the method is consistent for all $\theta \in \mathbb{R}$.

**b)** Show that the method is convergent if and only if $-1 < \theta < \frac{1}{2}$.

**c)** Show that the rate of convergence is maximal for the value $\theta = \frac{1}{2}\left(1 - \sqrt{3}\right)$.