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JTMS-MAT-13: Numerical Methods

Exam & Solutions

All questions carry equal marks. Only five questions will be marked. Please use the booklet provided, clearly indicating in the inside cover which questions are to be marked.

All trigonometric values are in radians.

Question 1:

(a) Given Newton's method in one dimension is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)},$$

using a first-order approximation for the derivative, derive the secant method.

(b) Given a two-dimensional function, $\vec{f} = (f_1, f_2)^T$, which takes two variables as arguments, what is the Jacobian?

(c) For

$$\vec{f} = \begin{pmatrix} x^2 + y^2 - xy \\ y^2 + e^{x+y} \end{pmatrix}$$

show the inverse of the Jacobian is given by

$$J^{-1} = \frac{1}{\det(J)} \begin{pmatrix} 2y + e^{x+y} & x - 2y \\ -e^{x+y} & 2x - y \end{pmatrix}, \quad \text{where} \quad \det(J) = 3e^{x+y}(x-y) + 2y(2x-y).$$

(d) With initial condition $(x_0, y_0)^T = (1, 3)^T$, find the first iterate of the Newton method.

(a) The first order approximation to the derivative $f'(x)$ can be written as

$$f'(x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$$

which, when substituted in Newton's method is

$$x_{n+1} = x_n - f(x_n)\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}$$

(b) The Jacobian matrix is given by

$$J = \begin{pmatrix} \dfrac{\partial f_1}{\partial x} & \dfrac{\partial f_1}{\partial y} \\ \dfrac{\partial f_2}{\partial x} & \dfrac{\partial f_2}{\partial y} \end{pmatrix}$$

(c) As $f_1 = x^2 + y^2 - xy$, so

$$\frac{\partial f_1}{\partial x} = 2x - y$$

and

$$\frac{\partial f_1}{\partial y} = 2y - x.$$

As $f_2 = y^2 + e^{x+y}$, so

$$\frac{\partial f_2}{\partial x} = e^{x+y}$$

and

$$\frac{\partial f_2}{\partial y} = 2y - e^{x+y}.$$

Thus,

$$J = \begin{pmatrix} 2x - y & 2y - x \\ e^{x+y} & 2y - e^{x+y} \end{pmatrix}.$$

The determinant is

$$\begin{aligned} \det(J) &= (2x - y)(2y - e^{x+y}) - (2y - x)e^{x+y} \\ &= 3e^{x+y}(x - y) + 2y(2x - y) \end{aligned}$$

and the inverse is of a 2×2 matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is

$$A^{-1} = \frac{1}{ad - cb} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Hence

$$J^{-1} = \frac{1}{\det(J)} \begin{pmatrix} 2y + e^{x+y} & x - 2y \\ -e^{x+y} & 2x - y \end{pmatrix}$$

(d) Substituting the values into the vector Newton iteration scheme, is

$$\begin{aligned} \vec{x}_1 &= \vec{x}_0 - J^{-1}(\vec{x}_0)\vec{f}(\vec{x}_0) \\ &= \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} - \frac{1}{\det(J)} \begin{pmatrix} 2y_0 + e^{x_0+y_0} & x_0 - 2y_0 \\ -e^{x_0+y_0} & 2x_0 - y_0 \end{pmatrix} \begin{pmatrix} x_0^2 + y_0^2 - x_0 y_0 \\ y_0^2 + e^{x_0+y_0} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{1}{\det(J)} \begin{pmatrix} 2 \times 3 + e^{1+3} & 1 - 2 \times 3 \\ -e^{1+3} & 2 \times 1 - 3 \end{pmatrix} \begin{pmatrix} 1^2 + 3^2 - 1 \times 3 \\ 3^2 + e^{1+3} \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{1}{-6(1 + e^4)} \begin{pmatrix} 6 + e^4 & -5 \\ -e^4 & -1 \end{pmatrix} \begin{pmatrix} 7 \\ 9 + e^4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{1}{6(1 + e^4)} \begin{pmatrix} -3 - 2e^4 \\ -9 - 8e^4 \end{pmatrix} \\ &= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \frac{1}{6(1 + e^4)} \begin{pmatrix} 3 - 2e^4 \\ 9 + 8e^4 \end{pmatrix} \\ &= \begin{pmatrix} 1.31834 \\ 4.33633 \end{pmatrix} \end{aligned}$$

Question 2:

(**a**) What is a singular matrix?

(**b**) Given the matrix

$$A = \begin{pmatrix} 4 & 3 & 1 \\ 3 & 4 & 3 \\ 1 & 3 & 5 \end{pmatrix},$$

show the row echelon form of the matrix can be written as

$$U = \begin{pmatrix} 4 & 3 & 1 \\ 0 & 7 & 9 \\ 0 & 0 & 52 \end{pmatrix}.$$

(**c**) By applying Gaussian elimination, or any other method, find the solution to the linear equation $A\vec{x} = \vec{b}$, where $\vec{b}$ is given by

$$\vec{b} = \begin{pmatrix} 61 \\ 63 \\ 43 \end{pmatrix}.$$

(**d**) If an $n \times n$ matrix is invertible, what is the order of the upper limit for the number of arithmetic operations to yield the inverse for Gaussian elimination?

(**a**) A singular matrix has a determinant which is equal to zero. Other answers are that the matrix does not have full rank, etc.

(**b**) There are many ways of providing equivalent row echelon matrices, depending on the elementary row operations which are performed. The first step is $R_2 \mapsto 4R_2 - 3R_1$, so that

$$\bar{A} = \begin{pmatrix} 4 & 3 & 1 \\ 0 & 7 & 9 \\ 1 & 3 & 5 \end{pmatrix},$$

Next $R_3 \mapsto 4R_3 - R_1$

$$\bar{A} = \begin{pmatrix} 4 & 3 & 1 \\ 0 & 7 & 9 \\ 0 & 9 & 19 \end{pmatrix},$$

Finally, $R_3 \mapsto 7R_3 - 9R_2$ then yields

$$\bar{A} = \begin{pmatrix} 4 & 3 & 1 \\ 0 & 7 & 9 \\ 0 & 0 & 52 \end{pmatrix},$$

Note that if partial pivoting is performed and a row swap performed, the signs change: once for the row swap and once in the row operation.

(**c**) The first step is $R_2 \mapsto 4R_2 - 3R_1$, so that

$$\bar{A} = \left(\begin{array}{ccc|c} 4 & 3 & 1 & 61 \\ 0 & 7 & 9 & 69 \\ 1 & 3 & 5 & 43 \end{array}\right),$$

Next $R_3 \mapsto 4R_3 - R_1$

$$\bar{A} = \left(\begin{array}{ccc|c} 4 & 3 & 1 & 61 \\ 0 & 7 & 9 & 69 \\ 0 & 9 & 19 & 111 \end{array}\right),$$

Finally, $R_3 \mapsto 7R_3 - 9R_2$ then yields

$$\bar{A} = \left(\begin{array}{ccc|c} 4 & 3 & 1 & 61 \\ 0 & 7 & 9 & 69 \\ 0 & 0 & 52 & 156 \end{array}\right),$$

Thus, if $\vec{x} = (x, y, z)^T$, then evaluating row three gives $52z = 156$, so $z = 3$. Substituting this into the second row gives $7y + 27 = 69$, thus $7y = 42$, so that $y = 6$. The last equation is $4x + 18 + 3 = 61$, thus $4x = 40$. Hence

$$x = \begin{pmatrix} 10 \\ 6 \\ 3 \end{pmatrix}.$$

(d) The order of Gaussian elimination is $\mathcal{O}\left(n^3\right)$.

(e) The order of back substitution is $\mathcal{O}\left(n^2\right)$.

Question 3:

(a) Show, by constructing an cubic polynomial, $q(x)$, which is orthogonal to 1, x and x^2, i.e.

$$\int_{-1}^{1} x^i \, q(x) \, \mathrm{d}x = 0 \quad \text{for} \quad i = 0, 1, 2$$

that the Gauss nodes for Gaussian quadrature are $x_i^* = -\sqrt{\frac{3}{5}}, \, 0, \, \sqrt{\frac{3}{5}}$.

(b) Lagrange polynomials are defined as

$$l_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}.$$

Derive the three Lagrange polynomials for the x_i^* given above.

(c) Show that the weights c_i such that

$$c_i \int_{-1}^{1} l_i(x) \, \mathrm{d}x = 1.$$

for each Lagrange polynomial, are given by $c_0 = \frac{9}{5}$, $c_1 = \frac{9}{8}$ and $c_2 = \frac{9}{8}$.

(a) To show that the Gauss nodes for Gaussian quadrature are $x_i^* = -\sqrt{\frac{3}{5}}, \, 0, \, \sqrt{\frac{3}{5}}$, we need to find a cubic polynomial $q(x)$ which is orthogonal to 1, x, and x^2 over the interval $[-1, 1]$. This means we require:

$$\int_{-1}^{1} q(x) \, \mathrm{d}x = 0 \quad \int_{-1}^{1} x q(x) \, \mathrm{d}x = 0 \quad \text{and} \quad \int_{-1}^{1} x^2 q(x) \, \mathrm{d}x = 0.$$

Let $q(x) = x(x^2 - a)$. We choose this form because it is a cubic polynomial and we will determine the value of a that satisfies the orthogonality conditions. Now, we need to compute the integrals. Firstly, compute $\int_{-1}^{1} q(x) \, \mathrm{d}x$:

$$\int_{-1}^{1} x(x^2 - a) \, \mathrm{d}x = \int_{-1}^{1} (x^3 - ax) \, \mathrm{d}x$$

Since x^3 and x are odd functions, their integrals over symmetric limits around zero are zero, i.e.

$$\int_{-1}^{1} x^3 \, \mathrm{d}x = 0 \quad \text{and} \quad \int_{-1}^{1} ax \, \mathrm{d}x = 0$$

Thus,

$$\int_{-1}^{1} (x^3 - ax) \, \mathrm{d}x = 0 - 0 = 0$$

Compute $\int_{-1}^{1} x q(x) \, \mathrm{d}x$:

$$\int_{-1}^{1} x(x(x^2 - a)) \, \mathrm{d}x = \int_{-1}^{1} (x^4 - ax^2) \, \mathrm{d}x$$

As

$$\int_{-1}^{1} x^4 \, \mathrm{d}x = \frac{2}{5}$$

and

$$\int_{-1}^{1} x^2 \, \mathrm{d}x = \frac{2}{3}$$

Thus we have,

$$\int_{-1}^{1} (x^4 - ax^2) \, \mathrm{d}x = \frac{2}{5} - a \cdot \frac{2}{3} = 0$$

so solving for a yields

$$\frac{2}{5} - a \cdot \frac{2}{3} = 0 \quad \Rightarrow \quad a = \frac{2}{5} \cdot \frac{3}{2} = \frac{3}{5}.$$

Thus, the polynomial $q(x)$ is:

We have constructed the polynomial $q(x)$. Now, we need to find the roots of $q(x) = 0$ to determine the Gauss nodes, i.e. $x(x^2 - \frac{3}{5}) = 0$, which gives $x = 0$ and $x = \pm\sqrt{\frac{3}{5}}$.

(**b**) Lagrange polynomials are defined as

$$l_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

Thus for $x_0 = -\sqrt{\dfrac{3}{5}}$, $x_1 = 0$ and $x_2 = \sqrt{\dfrac{3}{5}}$, then

$$\begin{aligned}
l_0(x) &= \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} \\
&= \frac{(x - 0)\left(x - \sqrt{\frac{3}{5}}\right)}{\left(-\sqrt{\frac{3}{5}} - 0\right)\left(-\sqrt{\frac{3}{5}} - \sqrt{\frac{3}{5}}\right)} \\
&= \frac{x\left(x - \sqrt{\frac{3}{5}}\right)}{-\sqrt{\frac{3}{5}}\left(-2\sqrt{\frac{3}{5}}\right)} \\
&= \frac{x\left(x - \sqrt{\frac{3}{5}}\right)}{2\frac{3}{5}} \\
&= \frac{5}{6}x\left(x - \sqrt{\frac{3}{5}}\right).
\end{aligned}$$

For $l_1(x)$, then

$$\begin{aligned}
l_1(x) &= \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} \\
&= \frac{\left(x + \sqrt{\frac{3}{5}}\right)\left(x - \sqrt{\frac{3}{5}}\right)}{\left(0 + \sqrt{\frac{3}{5}}\right)\left(0 - \sqrt{\frac{3}{5}}\right)} \\
&= \frac{\left(x^2 - \frac{3}{5}\right)}{-\frac{3}{5}} \\
&= -\frac{5}{3}\left(x^2 - \frac{3}{5}\right) \\
&= \left(1 - \frac{5}{3}x^2\right).
\end{aligned}$$

Finally, l_2, which is similar to l_0,

$$\begin{aligned}
l_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= \frac{\left(x+\sqrt{\frac{3}{5}}\right)(x-0)}{\left(\sqrt{\frac{3}{5}}+\sqrt{\frac{3}{5}}\right)\left(\sqrt{\frac{3}{5}}-0\right)} \\
&= \frac{x\left(x+\sqrt{\frac{3}{5}}\right)}{\left(2\sqrt{\frac{3}{5}}\right)\sqrt{\frac{3}{5}}} \\
&= \frac{5}{6}x\left(x+\sqrt{\frac{3}{5}}\right).
\end{aligned}$$

(c) Considering $l_0(x) = \frac{5}{6}x\left(x-\sqrt{\frac{3}{5}}\right)$, then

$$\begin{aligned}
\int_{-1}^{1} l_0(x)\,\mathrm{d}x &= \int_{-1}^{1} \frac{5}{6}x\left(x-\sqrt{\frac{3}{5}}\right)\mathrm{d}x \\
&= \frac{5}{6}\int_{-1}^{1} x^2\,\mathrm{d}x - \frac{5}{6}\sqrt{\frac{3}{5}}x\,\mathrm{d}x \\
&= \frac{5}{6}\left[\frac{x^3}{3}\right]_{-1}^{1} - \frac{5}{6}\sqrt{\frac{3}{5}}\left[\frac{x^2}{2}\right]_{-1}^{1} \\
&= \frac{5}{6}\left[\frac{1}{3}-\frac{-1}{3}\right] - \frac{5}{6}\sqrt{\frac{3}{5}}\left[\frac{1}{2}-\frac{1}{2}\right] \\
&= \frac{5}{6}\frac{2}{3} \\
&= \frac{5}{9}.
\end{aligned}$$

Thus, $c_0\int_{-1}^{1} l_0(x)\,\mathrm{d}x = 1 \Rightarrow c_0 = \frac{9}{5}$. Similarly for $l_2(x)$,

$$\begin{aligned}
\int_{-1}^{1} l_2(x)\,\mathrm{d}x &= \int_{-1}^{1} \frac{5}{6}x\left(x+\sqrt{\frac{3}{5}}\right)\mathrm{d}x \\
&= \frac{5}{6}\int_{-1}^{1} x^2\,\mathrm{d}x + \frac{5}{6}\sqrt{\frac{3}{5}}x\,\mathrm{d}x \\
&= \frac{5}{9}.
\end{aligned}$$

Hence, $c_2 = \dfrac{9}{5}$. Finally, for $l_1(x)$

$$\begin{aligned}
\int_{-1}^{1} l_1\left(x\right) \mathrm{d}x &= \int_{-1}^{1} \left(1 - \frac{5}{3}x^2\right) \mathrm{d}x \\
&= \int_{-1}^{1} 1 \,\mathrm{d}x - \frac{5}{3}\int_{-1}^{1} x^2 \,\mathrm{d}x \\
&= \left[x\right]_{-1}^{1} - \frac{5}{3}\left[\frac{x^3}{3}\right]_{-1}^{1} \\
&= 2 - \frac{5}{3}\frac{2}{3} \\
&= \frac{18}{9} - \frac{10}{9} \\
&= \frac{8}{9}.
\end{aligned}$$

Hence, $c_1 = \dfrac{9}{8}$.

Question 4:

(a) What condition on the Butcher array makes a Runge-Kutta scheme implicit?

(b) The differential equation

$$y'(t) = 1 - 2y(t) \quad \text{with} \quad y(0) = 1$$

has the exact solution $y = \dfrac{1}{2}\left(e^{-2t} + 1\right)$. From the Runge-Kutta scheme given by the Butcher array

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ 1 & 0 & 0 & 1 & \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

where

$$u_{k+1} = u_k + h \sum_{i=1}^{4} b_i f\left(u_k + h \sum_{j=1}^{4} a_{i,j} k_j, t_k + c_i h\right) = u_k + h \sum_{i=1}^{4} b_i k_i$$

and using step size $h = 0.1$, show that to compute u_1, the k coefficients are given by

$$k_1 = -1, \quad k_2 = -0.9, \quad k_3 = -0.91 \quad \text{and} \quad k_4 = -0.818.$$

(c) For an s-stage Runge-Kutta scheme, what condition on the coefficients b_i is necessary for stability?

(d) Find the global truncation error after two steps, i.e. $|y(2h) - u_2|$.

(a) A Runge-Kutta scheme is said be to be implicit if the matrix A is lower triangular.

(b) Given $u_0 = 1$ at $x = 0$ and $f(u_n, t_n) = 1 - 2u_n$, then

$$k_1 = f(u_0, x_0) = 1 - 2u_0 = -1$$

and

$$\begin{aligned} k_2 = f\left(u_0 + \frac{h}{2}k_1, x_0 + \frac{h}{2}\right) &= 1 - 2\left(1 + \frac{0.1}{2}(-1)\right) \\ &= 1 - 2\left(1 - \frac{1}{20}\right) \\ &= 1 - 19/10 \\ &= 1 - 1.9 \\ &= -0.9 \end{aligned}$$

and

$$\begin{aligned} k_3 = f\left(u_0 + \frac{h}{2}k_2, x_0 + \frac{h}{2}\right) &= 1 - 2\left(1 + \frac{0.1}{2}(-0.9)\right) \\ &= -0.91 \end{aligned}$$

and lastly,

$$\begin{aligned} k_4 = f\left(u_0 + h\,k_3, x_0 + h\right) &= 1 - 2\left(1 + 0.1(-0.91)\right) \\ &= -0.818 \end{aligned}$$

(c) For a Runge-Kutta scheme to be stable, a necessary condition is that $\sum_{i=1}^{s} b_i = 1$.

(**d**) The first step is given by

$$\begin{aligned} u_1 &= u_0 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \\ &= 1 + \frac{1}{60}\left(-1 - 2 \times 0.9 - 2 \times 0.91 - 0.818\right) \\ &= 0.9093\dot{6} \end{aligned}$$

The second step requires

$$k_1 = f\left(u_1, x_1\right) = 1 - 2u_1 = -0.81873\dot{3}$$

and

$$\begin{aligned} k_2 = f\left(u_1 + \frac{h}{2}k_1, x_1 + \frac{h}{2}\right) &= 1 - 2\left(u_1 + \frac{h}{2}k_1\right) \\ &= 1 - 2\left(0.9093\dot{6} - \frac{0.1}{2} \times 0.81873\dot{3}\right) \\ &= -0.736859998667 \end{aligned}$$

and

$$\begin{aligned} k_3 = f\left(u_1 + \frac{h}{2}k_2, x_1 + \frac{h}{2}\right) &= 1 - 2\left(u_1 + \frac{h}{2}k_2\right) \\ &= 1 - 2\left(0.9093\dot{6} - \frac{0.1}{2} \times 0.736859998667\right) \\ &= -0.7450473334532999 \end{aligned}$$

and lastly,

$$\begin{aligned} k_4 = f\left(u_1 + h\,k_3, x_1 + h\right) &= 1 - 2\left(u_1 + hk_3\right) \\ &= -0.6697238666293399 \end{aligned}$$

thus,

$$\begin{aligned} u_2 &= u_1 + \frac{h}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right) \\ &= 0.9093 + \frac{1}{60}\left(-0.81873\dot{3} - 2 \times 0.736859998667 - 2 \times 0.7450473334532999 - 0.6697238666293399\right) \\ &= 0.835128802265501 \end{aligned}$$

The exact value is $y = \frac{1}{2}\left(e^{-0.4} + 1\right) = 0.8351600230178197$.

Thus, the global truncation error is given by

$$|0.8351600230178197 - 0.835128802265501| = 3.122075231865029e - 05 = 3.122 \times 10^{-5}.$$

Question 5: Given the following data:

i	0	1	2
x_i	0	1	3
y_i	2	4	3

Using polynomial interpolation, such as Lagrange or Newton interpolation, what is the value of $y(2)$?

- ○ 14/3
- ○ 4
- ○ 10/3
- ☒ 13/3
- ○ 43/9
- ○ 41/10

To find the value of $y(2)$ using polynomial interpolation, we can use either the Lagrange interpolation formula or Newton's formula. The Lagrange interpolation polynomial for the given set of data points (x_i, y_i) is given by:

$$P(x) = \sum_{i=0}^{n} y_i l_i(x)$$

where $l_i(x)$ are the Lagrange basis polynomials defined as:

$$l_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}.$$

Given data points:

i	0	1	2
x_i	0	1	3
y_i	2	4	3

We have $n = 2$, so we need to compute $l_0(x)$, $l_1(x)$ and $l_2(x)$. Calculating $l_0(x)$

$$\begin{aligned} l_0(x) &= \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} \\ &= \frac{x - 1}{0 - 1} \cdot \frac{x - 3}{0 - 3} \\ &= \frac{-(x - 1)}{1} \cdot \frac{-(x - 3)}{3} \\ &= (1 - x) \cdot \left(\frac{3 - x}{3}\right) \\ &= \frac{(x - 1)(x - 3)}{3} \\ &= \frac{x^2 - 4x + 3}{3} \end{aligned}$$

Calculating $l_1(x)$

$$\begin{aligned} l_1(x) &= \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} \\ &= \frac{x - 0}{1 - 0} \cdot \frac{x - 3}{1 - 3} \\ &= x \cdot \frac{x - 3}{-2} \\ &= x \cdot \left(\frac{3 - x}{2}\right) \\ &= \frac{x(3 - x)}{2} \end{aligned}$$

Lastly, calculating $l_2(x)$

$$
\begin{aligned}
l_2(x) &= \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} \\
&= \frac{x - 0}{3 - 0} \cdot \frac{x - 1}{3 - 1} \\
&= \frac{x}{3} \cdot \frac{x - 1}{2} \\
&= \frac{x(x - 1)}{6}
\end{aligned}
$$

Constructing the interpolation polynomial $P(x)$

$$
\begin{aligned}
P(x) &= y_0 \cdot l_0(x) + y_1 \cdot l_1(x) + y_2 \cdot l_2(x) \\
&= 2 \cdot l_0(x) + 4 \cdot l_1(x) + 3 \cdot l_2(x) \\
&= \frac{2}{3}\left(x^2 - 4x + 3\right) + 2x\,(3 - x) + \frac{1}{2}x\,(x - 1) \\
&= \left(\frac{2}{3} - 2 + \frac{1}{2}\right)x^2 + \left(-\frac{1}{2} + 6 - \frac{8}{3}\right)x + 2 \\
&= \frac{4 - 12 + 3}{6}x^2 + \frac{36 - 3 - 16}{6}x + 2 \\
&= -\frac{5}{6}x^2 + \frac{17}{6}x + 2.
\end{aligned}
$$

Thus, evaluating $P(x)$ at $x = 2$ yields

$$
\begin{aligned}
P(2) &= -\frac{5}{6}(2)^2 + \frac{17}{6}(2) + 2 \\
&= -\frac{5}{6}(4) + \frac{17}{6}(2) + 2 \\
&= -\frac{20}{6} + \frac{34}{6} + 2 \\
&= \frac{34 - 20}{6} + 2 \\
&= \frac{14}{6} + 2 \\
&= \frac{7}{3} + 2 \\
&= \frac{7}{3} + \frac{6}{3} \\
&= \frac{7 + 6}{3} \\
&= \boxed{\frac{13}{3}}
\end{aligned}
$$

Question 6: Given the integral

$$I = \int_{1/4}^{1/5} \frac{1}{3} + \sin(\pi x)\, \mathrm{d}x$$

what is the error of the approximate integral for the Trapezium rule when using five subintervals?

- ○ 2.30e-5
- ○ 8.00e-6
- ○ 2.11e-4
- ☒ 2.67e-6
- ○ 8.44e-7
- ○ 8.67e-3

Determine the interval width h:

$$a = \frac{1}{4} \quad \text{and} \quad b = \frac{1}{5} \quad \text{with} \quad n = 5$$

then

$$h = \frac{b-a}{n} = \frac{\frac{1}{5} - \frac{1}{4}}{5} = \frac{\frac{4-5}{20}}{5} = \frac{-1}{20 \cdot 5} = -\frac{1}{100}.$$

Calculate the points x_i:

$$x_i = a + ih \quad \text{for} \quad i = 0, 1, 2, 3, 4, 5$$

that is

$$x_0 = \frac{1}{4}, \quad x_1 = \frac{1}{4} - \frac{1}{100}, \quad x_2 = \frac{1}{4} - \frac{2}{100}, \quad x_3 = \frac{1}{4} - \frac{3}{100}, \quad x_4 = \frac{1}{4} - \frac{4}{100}, \quad x_5 = \frac{1}{5}.$$

Evaluate the function $f(x) = \frac{1}{3} + \sin(\pi x)$ at these points:

$$\begin{aligned}
f(x_0) &= \frac{1}{3} + \sin(\pi \cdot 0.25) = \frac{1}{3} + \sin\left(\frac{\pi}{4}\right) = \frac{1}{3} + \frac{\sqrt{2}}{2} = 1.040440114519881 \\
f(x_1) &= \frac{1}{3} + \sin(\pi \cdot 0.24) = 1.017880439262022 \\
f(x_2) &= \frac{1}{3} + \sin(\pi \cdot 0.23) = 0.9946451986569851 \\
f(x_3) &= \frac{1}{3} + \sin(\pi \cdot 0.22) = 0.970757323082023 \\
f(x_4) &= \frac{1}{3} + \sin(\pi \cdot 0.21) = 0.9462403869863099 \\
f(x_5) &= \frac{1}{3} + \sin(\pi \cdot 0.20) = 0.9211185856258064
\end{aligned}$$

Approximate the integral using the Trapezium rule:

$$I_h = \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right).$$

Substituting the values of h and $f(x_i)$:

$$I_h = -\frac{1}{200}\left(f(0.25) + 2f(0.24) + 2f(0.23) + 2f(0.22) + 2f(0.21) + f(0.20)\right).$$

Summing the values:

$$\begin{aligned}
I_h = -\frac{1}{200}\big(&1.040440114519881 + 2 \cdot 1.017880439262022 + 2 \cdot 0.9946451986569851 + 2 \cdot 0.970757323082023 + \\
&2 \cdot 0.9462403869863099 + 0.9211185856258064\big) \\
= -&0.04910302698060184
\end{aligned}$$

The exact integral is given as

$$
\begin{aligned}
I &= \int_{1/4}^{1/5} \frac{1}{3} + \sin \pi x \,\mathrm{d}x \\
&= \left[\frac{x}{3} - \frac{\cos(\pi x)}{\pi} \right]_{1/4}^{1/5} \\
&= \frac{1}{3}\left(\frac{1}{5} - \frac{1}{4}\right) - \frac{1}{\pi}\left(\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{\pi}{4}\right)\right) \\
&= -\frac{1}{3}\frac{1}{20} - \frac{1}{\pi}\left(\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{\pi}{4}\right)\right) \\
&= -0.04910569502763208
\end{aligned}
$$

Thus, the error is given by $|I - I_h|$,

$$|-0.04910569502763208 - -0.04910569502763208| = 2.668047030231213\mathrm{e}-6.$$

which is $\boxed{2.67\mathrm{e}-6.}$

Question 7: Given the matrix

$$A = \begin{pmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{pmatrix}$$

what is the Cholesky matrix L for the matrix A?

- ○ $\begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ -1 & 5 & 1 \end{pmatrix}$
- ○ $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & -4 \\ 3 & 8 & 1 \end{pmatrix}$
- ○ $\begin{pmatrix} 5 & 0 & 0 \\ 2 & 3 & 0 \\ 2 & 2 & 3 \end{pmatrix}$
- ○ $\begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix}$
- ○ $\begin{pmatrix} 5 & 3 & 3 \\ 0 & 3 & 6 \\ 0 & 0 & 3 \end{pmatrix}$
- ✗ $\begin{pmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{pmatrix}$

For the Cholesky factorization, $A = LL^T$. The coefficients of L are given by

$$l_{k,i} = \frac{a_{k,i} - \sum_{j=1}^{i-1} l_{ij} l_{k,j}}{l_{i,i}}$$

and

$$l_{k,k} = \sqrt{a_{k,k} - \sum_{j=1}^{k-1} l_{k,j}^2}.$$

Thus, for the matrix

$$A = \begin{pmatrix} 25 & 15 & -5 \\ 15 & 18 & 0 \\ -5 & 0 & 11 \end{pmatrix}$$

then

$$\begin{aligned} l_{1,1} &= \sqrt{a_{1,1}} \\ &= \sqrt{25} = 5, \end{aligned}$$

and subsequently

$$\begin{aligned} l_{2,1} &= \frac{a_{2,1}}{l_{1,1}} = \frac{15}{5} = 3, \\ l_{2,2} &= \sqrt{a_{2,2} - l_{2,1}^2} \\ &= \sqrt{18 - 3^2} = \sqrt{18 - 9} = \sqrt{9} = 3 \\ l_{3,1} &= \frac{a_{3,1}}{l_{1,1}} = -5/5 = -1, \\ l_{3,2} &= \frac{a_{3,2} - l_{3,1} \times l_{2,1}}{l_{2,2}} \\ &= \frac{0 - (-1) \times 3}{3} = 1 \end{aligned}$$

and finally,

$$l_{3,3} = \sqrt{a_{3,3} - l_{3,1}^2 - l_{3,2}^2} = \sqrt{11 - (-1)^2 - 1^2} = \sqrt{11 - 2} = 3.$$

Thus, the answer is

$$\boxed{L = \begin{pmatrix} 5 & 0 & 0 \\ 3 & 3 & 0 \\ -1 & 1 & 3 \end{pmatrix}.}$$

Question 8: Heun's method for a differential equation can be written in the form:

$$u_{n+1} = u_n + \frac{h}{2}\left(f\left(x_n, u_n\right) + f\left(x_{n+1}, u^*_{n+1}\right)\right) \quad \text{where} \quad u^*_{n+1} = u_n + hf\left(x_n, u_n\right).$$

Show that for the differential equation in $y(x)$ given by

$$\frac{\mathrm{d}y}{\mathrm{d}x} = y - \ln(x+1), \quad \text{with} \quad y_0 = 1,$$

that for $h = 0.2$, the value for u_2 is given by:

○ 1.20177	○ 2.40899
☒ 1.41063	○ 0.98452
○ 1.33333	○ −0.18431

With $f(x,y) = y - \ln(1+x)$, the intermediate step is required first

$$u^*_1 = u_0 + hf\left(x_0, u_0\right)$$

As $f(0, u_1) = 1 - \ln(1) = 1 - 0 = 1$, thus

$$u^*_1 = u_0 + hf(x_0, u_0) = 1 + 0.2 \times f(0,1) = 1 + 0.2 \times (1 - \ln(0+1)) = 1 + 0.2 \times (1 - \ln(1)) = 1 + 0.2 \times (1-0) = 1.2$$

To calculate the first step, consider

$$f(x_1, u^*_1) = f(0.2, 1.2) = 1.2 - \ln(0.2+1) = 1.2 - \ln(1.2)$$

which is

$$f(0.2, 1.2) \approx 1.2 - 0.1823 = 1.0177.$$

Thus u_1 is then given by

$$\begin{aligned} u_1 &= 1 + \frac{0.2}{2}(1 + 1.0177) \\ &= 1 + 0.1 \times 2.0177 = 1 + 0.20177 = 1.20177. \end{aligned}$$

To compute the second term, again compute the intermediate value u^*_2, first noting that

$$f(0.2, 1.20177) = 1.20177 - \ln(0.2+1) = 1.20177 - 0.1823 \approx 1.01947.$$

So that u^*_2 is given by

$$u^*_2 = 1.20177 + 0.2 \times 1.01947 = 1.20177 + 0.203894 = 1.405664.$$

With

$$f\left(x_2, u^*_2\right) = f(0.4, 1.405664) = 1.405664 - \ln(0.4+1) = 1.405664 - \ln(1.4) = 1.405664 - 0.3365 \approx 1.069164,$$

so u_2 can be evaluated as

$$= 1.20177 + 0.1(1.01947 + 1.069164) = 1.20177 + 0.12.088634 \approx 1.20177 + 0.2088634 \approx 1.4106334.$$

The required solution is $\boxed{1.41063.}$