Constructor University **Spring Semester 2025**

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CTMS-MAT-13: Numerical Methods

Assignment Sheet 5. Released:
Due:

Exercise 1 [2+3+3+2 Points]:

Given the following system of linear equations:

$$\begin{aligned} 10x_1 - 2x_2 + x_3 &= -1 \\ -6x_1 + 9x_2 + x_3 &= 2 \\ 4x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

(a) Given the starting point $\vec{x}_0 = \vec{0}$, compute the next two iterates $\vec{x}_1$ and $\vec{x}_2$ of the Jacobi method.

(b) Given the starting point $\vec{x}_0 = \vec{0}$, compute the next two iterates $\vec{x}_1$ and $\vec{x}_2$ of the Gauss-Seidel method.

(c) Given the starting point $\vec{x}_0 = \vec{0}$, compute the next two iterates $\vec{x}_1$ and $\vec{x}_2$ of the method of successive over-relaxation with $\omega = 1.25$.

(d) Are the Jacobi and the Gauss-Seidel methods guarenteed to converge?

Exercise 2 [2+3+2+3 Points]:

Let $A = L + D + U$ be a non-singular matrix with L being a lower triangular, D being a diagonal, and U being an upper triangular matrix.

(a) Show that for a linear system of equations $A\vec{x} = \vec{b}$ with solution $\vec{x}$, an iterative solver at iteration step k with an approximation $\vec{x}_k$ has an error $\vec{e}_k = \vec{x} - \vec{x}_k$ which can be written as

$$\vec{e}_k = \left(I - Q^{-1}A\right)^k \vec{e}_0$$

with initial error $\vec{e}_0 = \vec{x} - \vec{x}_0$ for the initial guess $\vec{x}_0$.

(b) Show that the Jacobi iteration for solving $A\vec{x} = \vec{b}$ can be written as

$$\vec{x}_{k+1} = D^{-1}\left(\vec{b} - (L + U)\vec{x}_k\right)$$

(c) Show that the Gauss-Seidel iteration for solving $A\vec{x} = \vec{b}$ can be written as

$$\vec{x}_{k+1} = (D + L)^{-1}\left(\vec{b} - U\vec{x}_k\right)$$

(d) Derive how the successive over-relaxation iteration for solving $A\vec{x} = \vec{b}$ with weight ω can be rewritten in a similar form to the one in **(c)**.

Exercise 3 [2+2+3 Points]:

For the function $f(x) = (x+1)\sin\left(\frac{\pi}{2}(x+1)\right)$, evaluate the integral

$$\int_0^1 f(x)\,\mathrm{d}x$$

using four subintervals for

(a) Trapezium rule,

(b) Simpson's rule and

(c) Compute the error in each case against the exact solution.

Exercise 4 [4+4 Points]:

(a) Show, by applying the Romberg algorithm to the trapezium method, that the next improvement is Simpson's Rule.

(b) For

$$\int_1^2 \frac{1}{x^3}\mathrm{d}x = \frac{3}{8}.$$

Compute for four intervals, the trapezium, Simpson and the next refinement of the Romberg algorithm.