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JTMS-MAT-13: Numerical Methods

Exam & Solutions: Thursday 23 May 2024

All questions carry equal marks. Answer 5 questions only. Please only use the booklet provided, clearly stating which questions are to be marked.

All trigonometric values are in radians.

Question 1:

(a) For the function $f(x) = (x + 1/2)\cos(x)$, show that Newton's method is

$$x_{k+1} = x_k - \frac{(x_k + 1/2)\cos(x_k)}{\cos(x_k) - (x_k + 1/2)\sin(x_k)}.$$

(b) From the definition of Newton's method, show that the secant method is given by

$$x_{k+1} = x_k - f(x_k)\frac{x_{k-1} - x_k}{f(x_{k-1}) - f(x_k)}.$$

(c) Compute two iterates of the Newton method with initial guess $x_0 = -1/3$.

(d) Compute two iterates of the secant method with initial guess $x_0 = -1/3$ and $x_1 = -0.4$.

(a) Newton's method is

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)},$$

so for the function given, the derivative is

$$\begin{aligned} f'(x) &= \cos(x) + \left(x + \frac{1}{2}\right)(-\sin(x)) \\ &= \cos(x) - \left(x + \frac{1}{2}\right)\sin(x). \end{aligned}$$

Thus, the formula for Newton iterates is given by

$$x_{k+1} = x_k - \frac{\left(x_k + \frac{1}{2}\right)\cos(x_k)}{\cos(x_k) - \left(x_k + \frac{1}{2}\right)\sin(x_k)}.$$

(b) The secant method uses an approximation of the derivative

$$f'(x_k) \approx \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

which can be substituting into Newton's method, yielding

$$x_{k+1} = x_k - f(x_k)\frac{x_{k-1} - x_k}{f(x_{k-1}) - f(x_k)}.$$

(c) First iteration: x_1, calculate $f(x_0)$:

$$f(x_0) = \left(-\frac{1}{3} + \frac{1}{2}\right)\cos\left(-\frac{1}{3}\right) = \left(\frac{1}{6}\right)\cos\left(-\frac{1}{3}\right)$$

Since $\cos(x)$ is an even function, $\cos(-x) = \cos(x)$:

$$f(x_0) = \frac{1}{6}\cos\left(\frac{1}{3}\right)$$

Calculate $f'(x_0)$:

$$\begin{aligned} f'(x_0) &= \cos\left(-\frac{1}{3}\right) - \left(-\frac{1}{3} + \frac{1}{2}\right)\sin\left(-\frac{1}{3}\right) \\ &= \cos\left(\frac{1}{3}\right) + \frac{1}{6}\sin\left(\frac{1}{3}\right) \end{aligned}$$

Thus, we can compute x_1 as

$$\begin{aligned} x_1 &= x_0 - \frac{f(x_0)}{f'(x_0)} \\ &= -\frac{1}{3} - \frac{\frac{1}{6}\cos\left(\frac{1}{3}\right)}{\cos\left(\frac{1}{3}\right) + \frac{1}{6}\sin\left(\frac{1}{3}\right)} \end{aligned}$$

Numerically evaluating the trigonometric functions, with $\cos\left(\frac{1}{3}\right) \approx 0.87758256$ and $\sin\left(\frac{1}{3}\right) \approx 0.32719470$, thus

$$\begin{aligned} x_1 &\approx -\frac{1}{3} - \frac{0.15749282438578963}{0.9994893957807631} \\ &- 0.490906615301735 \end{aligned}$$

Second iteration for x_2, using the updated $x_1 \approx -0.491$, and calculating $f(x_1)$:

$$\begin{aligned} f(x_1) &= \left(x_1 + \frac{1}{2}\right)\cos(x_1) = (-0.491 + 0.5)\cos(-0.491) \\ &= (0.009)\cos(-0.491) \\ &= 0.009\cos(0.491) \\ &= 0.008019508883621638 \end{aligned}$$

Calculate $f'(x_1)$:

$$\begin{aligned} f'(x_1) &= \cos(x_1) - \left(x_1 + \frac{1}{2}\right)\sin(x_1) \\ &= \cos(-0.491) - (-0.491 + 0.5)\sin(-0.491) \\ &= \cos(0.491) - (0.009)\sin(0.491) \\ &= 0.8861926740400325 \end{aligned}$$

Compute x_2:

$$\begin{aligned} x_2 &= -0.491 - \frac{0.008019508883621638}{0.8861926740400325} \\ &= -0.4999560118026808 \end{aligned}$$

which is 0.500 to three significant figures.

(d) Having computed $f(x_0)$, then with $\cos(0.4) = 0.921$, $f(x_1)$ is given by:

$$\begin{aligned} f(x_1) = (-0.4 + 0.5)\cos(-0.4) & \\ &= 0.1\cos(0.4) \\ &= 0.1 \times 0.921 \\ &= 0.0921 \end{aligned}$$

Substitute $x_0 = -0.33333$, $x_1 = -0.4$, $f(x_0) = 0.15733$, and $f(x_1) = 0.0921$, thus

$$\begin{aligned}
x_2 &= -0.4 - \frac{0.0921(-0.4 + 0.33333)}{0.0921 - 0.15733} \\
&= -0.4 - \frac{0.0921 \times (-0.06667)}{-0.06523} \\
&= -0.4 - \frac{-0.00614}{-0.06523} \\
&= -0.4 - 0.094 \\
&= -0.494
\end{aligned}$$

Using the updated $x_1 = -0.4$ and $x_2 \approx -0.494$, evaluating the function yields

$$\begin{aligned}
f(x_2) &= (-0.494 + 0.5)\cos(-0.494) \\
&= 0.006\cos(0.494) \\
&= 0.006 \times 0.880 \\
&= 0.00528.
\end{aligned}$$

Using the formula for the secant method:

$$\begin{aligned}
x_3 &= -0.494 - \frac{0.00528(-0.494 + 0.4)}{0.00528 - 0.0921} \\
&= -0.494 - \frac{0.00528 \times (-0.094)}{-0.08682} \\
&= -0.494 - \frac{-0.000496}{-0.08682} \\
&= -0.494 - 0.0057 \\
&\approx -0.500.
\end{aligned}$$

Question 2:

Consider the linear ordinary differential equation

$$y''(t) = -4y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1.$$

(a) By converting this 2^{nd} order ordinary differential equation into a system of two coupled first order ODEs, one in $y(t)$ and one in $y'(t)$, show the system can be written as a vector-valued ordinary differential equation in $\vec{v}(t) = (y(t), y'(t))^T$, in the form $f(\vec{v}) = A\vec{v}$ where

$$A = \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix}.$$

(b) Show that the forward Euler method can be written as

$$\vec{u}_{n+1} = (I + hA)\,\vec{u}_n$$

for some approximation $\vec{u} \in \mathbb{R}^2$, and provide the full system for $\vec{u}_{n+1}$ for the ODE presented above.

(c) Show that the backward Euler method yields

$$\vec{u}_{n+1} = \frac{1}{1 + 4h - h^2} \begin{pmatrix} 1 + 4h & +h \\ +h & 1 \end{pmatrix} \vec{u}_n.$$

(d) Calculate approximations $y(0.3)$ and $y'(0.3)$ using forward Euler method with $h = 0.15$.

(e) Calculate $y(0.3)$ and $y'(0.3)$ using backward Euler method with $h = 0.15$.

(a) Splitting yields, $y' = w$, and $v' = -4w + y$, which can be expressed as the linear system

$$\begin{pmatrix} y \\ w \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} y \\ w \end{pmatrix}$$

which, for the vector $\vec{v} = (y, w)^T$, is a linear differential equation $\vec{v}' = f(\vec{v}) = A\vec{v}$, with the matrix A as given.

(b) The forward Euler method is $u_{n+1} = u_n + hf(u_n)$, which for the function $f(\vec{v}_n) = A\vec{v}_n$, where $\vec{v}_n = (y_n, w_n)^T$, yields $\vec{v}_{n+1} = \vec{v}_n + hA\vec{v}_n$, which can be factorized as $\vec{v}_{n+1} = (I + hA)\,\vec{v}_n$.

(c) The backwards Euler scheme is given by $u_{n+1} = u_n + hf(u_{n+1})$, i.e. $\vec{v}_{n+1} = \vec{v}_n + hA\vec{v}_{n+1}$. Thus, $(I - hA)\vec{v}_{n+1} = \vec{v}_n$, so that $\vec{v}_{n+1} = (I - hA)^{-1}\,\vec{v}_n$. The inverse of the matrix is then given by

$$\begin{aligned}
(I - hA)^{-1} &= \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - h \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} \right)^{-1} \\
&= \begin{pmatrix} 1 & -h \\ -h & 1 + 4h \end{pmatrix}^{-1} \\
&= \frac{1}{1 + 4h - h^2} \begin{pmatrix} 1 + 4h & +h \\ +h & 1 \end{pmatrix}.
\end{aligned}$$

(d) With step size $h = 0.15$, then two steps, $\vec{u}_1$ and $\vec{u}_2$, must be computed from the initial data $\vec{u}_0 = (1, 1)$.

Using the matrix derived in the formula given, the solution at $t = 0.15$ is given by

$$
\begin{aligned}
\vec{u}_1 &= (I + hA)\,\vec{u}_0 \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + 0.15 \begin{pmatrix} 0 & 1 \\ 1 & -4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0.15 \\ 1 & 1 - 4 \times 0.15 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0.15 \\ 0.15 & 1 - 0.6 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0.15 \\ 0.15 & 0.4 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 + 0.15 \\ 0.15 + 0.4 \end{pmatrix} = \begin{pmatrix} 1.15 \\ 0.55 \end{pmatrix}
\end{aligned}
$$

Using the matrix and value derived above, the next step is given by

$$
\begin{aligned}
\vec{u}_2 &= (I + hA)\,\vec{u}_1 \\
&= \begin{pmatrix} 1 & 0.15 \\ 0.15 & 0.4 \end{pmatrix} \begin{pmatrix} 1.15 \\ 0.55 \end{pmatrix} \\
&= \begin{pmatrix} 1.15 + 0.55 \times 0.15 \\ 1.15 \times 0.15 + 0.4 \times 0.55 \end{pmatrix} \\
&= \begin{pmatrix} 1.15 + 0.0825 \\ 0.1725 + 0.22 \end{pmatrix} \\
&= \begin{pmatrix} 1.2325 \\ 0.3925 \end{pmatrix}
\end{aligned}
$$

(e) With step size $h = 0.15$, then two steps, $\vec{u}_1$ and $\vec{u}_2$, must be computed from the initial data $\vec{u}_0 = (1, 1)$. Using the matrix derived in the formula given, the solution at $t = 0.15$ is given by

$$
\begin{aligned}
\vec{u}_1 &= \frac{1}{1 + 4 \times 0.15 - 0.15^2} \begin{pmatrix} 1 + 4 \times 0.15 & 0.15 \\ 0.15 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{400}{631} \begin{pmatrix} 1.6 & 0.15 \\ 0.15 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \frac{4}{631} \begin{pmatrix} 160 & 15 \\ 15 & 100 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
&= \begin{pmatrix} (160 + 15) \times 4/631 \\ (100 + 15) \times 4/631 \end{pmatrix} \\
&= \begin{pmatrix} 175 \times 4/631 \\ 115 \times 4/631 \end{pmatrix} \\
&= \begin{pmatrix} 1.109350237717908 \\ 0.7290015847860539 \end{pmatrix}.
\end{aligned}
$$

The second step is then given by

$$
\begin{aligned}
\vec{u}_2 &= \frac{4}{631} \begin{pmatrix} 160 & 15 \\ 15 & 100 \end{pmatrix} \begin{pmatrix} 1.109350237717908 \\ 0.7290015847860539 \end{pmatrix} \\
&= \begin{pmatrix} 1.1944916754780102 \\ 0.5676095850673472 \end{pmatrix}
\end{aligned}
$$

Question 3:

(a) What is meant if a matrix is said to be diagonally dominant?

(b) Give a definition for a positive definite matrix.

(c) What is the range of ω for which the method of successive over relaxation converges for a semi-positive definite matrix?

(d) Show that the matrix

$$A = \begin{pmatrix} 4 & 3 & 0 \\ 3 & 4 & -1 \\ 0 & -1 & 4 \end{pmatrix}$$

has eigenvalues $\lambda = 4$, $4 + \sqrt{10}$ and $4 - \sqrt{10}$, and thus is can be inverted using the method of successive over relaxation.

(a) A matrix is diagonally dominant if, for each row of the matrix, the magnitude of the diagonal element in that row is greater than or equal to the sum of the magnitudes of all the other (non-diagonal) elements in that row.

(b) A positive definite matrix is a symmetric matrix A (i.e., $A = A^T$) with the property that for any non-zero vector $\mathbf{x}$, the quadratic form $\mathbf{x}^T A\mathbf{x} > 0$ is positive. Equivalent definitions are that all eigenvalues of a positive definite matrix are positive or all the leading principal minors (that is, the determinants of the leading principal submatrices) of a positive definite matrix are also positive.

(c) For a semi-positive definite matrix, the optimal range of ω ensuring convergence is: $0 < \omega < 2$.

(b) The characteristic polynomial in λ can be found via

$$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 3 & 0 \\ 3 & 4-\lambda & -1 \\ 0 & -1 & 4-\lambda \end{vmatrix} = 0.$$

Calculating the determinant by expanding along the first row:

$$\det(A - \lambda I) = (4-\lambda)\begin{vmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} - 3\begin{vmatrix} 3 & -1 \\ 0 & 4-\lambda \end{vmatrix} = 0.$$

Calculate the 2×2 determinants:

$$\begin{vmatrix} 4-\lambda & -1 \\ -1 & 4-\lambda \end{vmatrix} = (4-\lambda)(4-\lambda) - (-1)(-1) = (4-\lambda)^2 - 1$$

and

$$\begin{vmatrix} 3 & -1 \\ 0 & 4-\lambda \end{vmatrix} = 3(4-\lambda) - (-1)(0) = 3(4-\lambda).$$

Substitute back:

$$\begin{aligned} \det(A - \lambda I) &= (4-\lambda)((4-\lambda)^2 - 1) - 3 \cdot 3(4-\lambda) \\ &= (4-\lambda)((4-\lambda)^2 - 1) - 9(4-\lambda) \\ &= (4-\lambda)((4-\lambda)^2 - 10). \end{aligned}$$

Set the determinant to zero, $(4-\lambda)((4-\lambda)^2 - 10) = 0$, to solve for the eigenvalues yield $\lambda = 4$, $\lambda = 4 + \sqrt{10}$, and $\lambda = 4 - \sqrt{10}$.

Question 4:

(a) Show, by constructing an cubic polynomial, $q(x)$, which is orthogonal to 1, x and x^2, i.e.

$$\int_{-1}^{1} x^i q(x)\,\mathrm{d}x = 0 \quad \text{for} \quad i = 0, 1, 2$$

that the Gauss nodes for Gaussian quadrature are $x_i^* = -\sqrt{\frac{3}{5}}$, 0, $\sqrt{\frac{3}{5}}$.

(b) Show that the Lagrange polynomials for the function

$$f(x) = 3x\cos(x)$$

which generates the data

i	0	1	2
x_i	$-\pi/4$	0	$\pi/4$
y_i	$-\dfrac{3\pi}{4\sqrt{2}}$	0	$\dfrac{3\pi}{4\sqrt{2}}$

are

$$l_0 = \frac{8}{\pi^2}x\left(x - \pi/4\right), \quad l_1 = -\frac{16}{\pi^2}\left(x^2 - \pi^2/16\right) \quad \text{and} \quad l_2 = \frac{8}{\pi^2}x\left(x + \pi/4\right).$$

(c) Thus, using $A_i = \int_{-1}^{1} l_i(x)\,\mathrm{d}x$, show that the approximation for the integral yields

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{i=0}^{2} A_i f\left(x_i^*\right) = 0.$$

(a) To show that the Gauss nodes for Gaussian quadrature are $x_i^* = -\sqrt{\frac{3}{5}}$, 0, $\sqrt{\frac{3}{5}}$, we need to find a cubic polynomial $q(x)$ which is orthogonal to 1, x, and x^2 over the interval $[-1, 1]$. This means we require:

$$\int_{-1}^{1} q(x)\,\mathrm{d}x = 0 \quad \int_{-1}^{1} xq(x)\,\mathrm{d}x = 0 \quad \text{and} \quad \int_{-1}^{1} x^2 q(x)\,\mathrm{d}x = 0.$$

Let $q(x) = x(x^2 - a)$. We choose this form because it is a cubic polynomial and we will determine the value of a that satisfies the orthogonality conditions. Now, we need to compute the integrals. Firstly, compute $\int_{-1}^{1} q(x)\,\mathrm{d}x$:

$$\int_{-1}^{1} x(x^2 - a)\,\mathrm{d}x = \int_{-1}^{1} (x^3 - ax)\,\mathrm{d}x$$

Since x^3 and x are odd functions, their integrals over symmetric limits around zero are zero, i.e.

$$\int_{-1}^{1} x^3\,\mathrm{d}x = 0 \quad \text{and} \quad \int_{-1}^{1} ax\,\mathrm{d}x = 0$$

Thus,

$$\int_{-1}^{1} (x^3 - ax)\,\mathrm{d}x = 0 - 0 = 0$$

Compute $\int_{-1}^{1} xq(x)\,\mathrm{d}x$:

$$\int_{-1}^{1} x(x(x^2 - a))\,\mathrm{d}x = \int_{-1}^{1} (x^4 - ax^2)\,\mathrm{d}x$$

As

$$\int_{-1}^{1} x^4\,\mathrm{d}x = \frac{2}{5}$$

and

$$\int_{-1}^{1} x^2 \, \mathrm{d}x = \frac{2}{3}$$

Thus we have,

$$\int_{-1}^{1} (x^4 - ax^2) \, \mathrm{d}x = \frac{2}{5} - a \cdot \frac{2}{3} = 0$$

so solving for a yields

$$\frac{2}{5} - a \cdot \frac{2}{3} = 0 \quad \Rightarrow \quad a = \frac{2}{5} \cdot \frac{3}{2} = \frac{3}{5}.$$

Thus, the polynomial $q(x)$ is:

We have constructed the polynomial $q(x)$. Now, we need to find the roots of $q(x) = 0$ to determine the Gauss nodes, i.e. $x(x^2 - \frac{3}{5}) = 0$, which gives $x = 0$ and $x = \pm\sqrt{\frac{3}{5}}$.

(**b**) The Lagrange polynomial for a given set of data points (x_i, y_i) is given by:

$$l_i(x) = \prod_{\substack{0 \le j \le 2 \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

Given data points:

i	0	1	2
x_i	$-\frac{\pi}{4}$	0	$\frac{\pi}{4}$
y_i	$-\frac{3\pi}{4\sqrt{2}}$	0	$\frac{3\pi}{4\sqrt{2}}$

Let's calculate each $l_i(x)$. Firstly $l_0(x)$

$$l_0(x) = \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2}$$

Substitute the values:

$$\begin{aligned} l_0(x) &= \frac{x - 0}{-\frac{\pi}{4} - 0} \cdot \frac{x - \frac{\pi}{4}}{-\frac{\pi}{4} - \frac{\pi}{4}} \\ &= \frac{x}{-\frac{\pi}{4}} \cdot \frac{x - \frac{\pi}{4}}{-\frac{\pi}{2}} \\ &= \frac{x}{-\frac{\pi}{4}} \cdot \frac{x - \frac{\pi}{4}}{-\frac{\pi}{2}} \\ &= \frac{4x}{\pi} \cdot \frac{2\left(x - \frac{\pi}{4}\right)}{\pi} \\ &= \frac{8}{\pi^2} x \left(x - \frac{\pi}{4}\right) \end{aligned}$$

Similarly, for l_1,

$$l_1(x) = \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2}$$

Substitute the values:

$$\begin{aligned}
l_1(x) &= \frac{x - (-\frac{\pi}{4})}{0 - (-\frac{\pi}{4})} \cdot \frac{x - \frac{\pi}{4}}{0 - \frac{\pi}{4}} \\
&= \frac{x + \frac{\pi}{4}}{\frac{\pi}{4}} \cdot \frac{x - \frac{\pi}{4}}{-\frac{\pi}{4}} \\
&= \frac{4(x + \frac{\pi}{4})}{\pi} \cdot \frac{4(x - \frac{\pi}{4})}{-\pi} \\
&= \frac{16(x + \frac{\pi}{4})(x - \frac{\pi}{4})}{-\pi^2} \\
&= -\frac{16}{\pi^2}\left(x^2 - \left(\frac{\pi}{4}\right)^2\right) \\
&= -\frac{16}{\pi^2}\left(x^2 - \frac{\pi^2}{16}\right).
\end{aligned}$$

Finally,

$$l_2(x) = \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1}$$

Substitute the values:

$$\begin{aligned}
l_2(x) &= \frac{x - (-\frac{\pi}{4})}{\frac{\pi}{4} - (-\frac{\pi}{4})} \cdot \frac{x - 0}{\frac{\pi}{4} - 0} \\
&= \frac{x + \frac{\pi}{4}}{\frac{\pi}{2}} \cdot \frac{x}{\frac{\pi}{4}} \\
&= \frac{2\left(x + \frac{\pi}{4}\right)}{\pi} \cdot \frac{4x}{\pi} \\
&= \frac{8}{\pi^2} x \left(x + \frac{\pi}{4}\right)
\end{aligned}$$

Thus, the Lagrange polynomials are:

$$\begin{aligned}
l_0(x) &= \frac{8}{\pi^2} x \left(x - \frac{\pi}{4}\right), \\
l_1(x) &= -\frac{16}{\pi^2}\left(x^2 - \frac{\pi^2}{16}\right) \quad \text{and} \\
l_2(x) &= \frac{8}{\pi^2} x \left(x + \frac{\pi}{4}\right).
\end{aligned}$$

(c) Compute the coefficients A_i, which are the integrals of the Lagrange polynomials over the interval $[-1, 1]$:

$$A_i = \int_{-1}^{1} l_i(x)\,\mathrm{d}x.$$

Firstly, A_0 is given by

$$A_0 = \int_{-1}^{1} \frac{8}{\pi^2} x \left(x - \frac{\pi}{4}\right) \mathrm{d}x = \frac{8}{\pi^2} \int_{-1}^{1} \left(x^2 - \frac{\pi}{4} x\right) \mathrm{d}x$$

Note that since $\int_{-1}^{1} x\,\mathrm{d}x = 0$ (odd function over a symmetric interval), then

$$\begin{aligned}
A_0 &= \frac{8}{\pi^2}\left(\int_{-1}^{1} x^2\,\mathrm{d}x - \frac{\pi}{4}\int_{-1}^{1} x\,\mathrm{d}x\right) \\
&= \frac{8}{\pi^2}\int_{-1}^{1} x^2\,\mathrm{d}x \\
&= \frac{8}{\pi^2} \cdot \frac{2}{3} \\
&= \frac{16}{3\pi^2}.
\end{aligned}$$

Calculating A_1

$$\begin{aligned}
A_1 &= \int_{-1}^{1} -\frac{16}{\pi^2}\left(x^2 - \frac{\pi^2}{16}\right) \mathrm{d}x \\
&= -\frac{16}{\pi^2}\left(\int_{-1}^{1} x^2 \,\mathrm{d}x - \frac{\pi^2}{16}\int_{-1}^{1} 1 \,\mathrm{d}x\right) \\
&= -\frac{16}{\pi^2}\left(\frac{2}{3} - \frac{\pi^2}{16}\cdot 2\right) \\
&= 2 - \frac{32}{3\pi^2}
\end{aligned}$$

Calculating A_2

$$A_2 = \int_{-1}^{1} \frac{8}{\pi^2} x \left(x + \frac{\pi}{4}\right) \mathrm{d}x = \frac{8}{\pi^2}\int_{-1}^{1}\left(x^2 + \frac{\pi}{4}x\right) \mathrm{d}x$$

Since $\int_{-1}^{1} x \,\mathrm{d}x = 0$, (as before, integrating an odd function over a symmetric interval):

$$\begin{aligned}
A_2 &= \frac{8}{\pi^2}\left(\int_{-1}^{1} x^2 \,\mathrm{d}x + \frac{\pi}{4}\int_{-1}^{1} x \,\mathrm{d}x\right) \\
&= \frac{8}{\pi^2}\int_{-1}^{1} x^2 \,\mathrm{d}x \\
&= \frac{8}{\pi^2}\cdot\frac{2}{3} \\
&= \frac{16}{3\pi^2}
\end{aligned}$$

The approximation for the integral is given by:

$$\int_{-1}^{1} f(x) \,\mathrm{d}x \approx \sum_{i=0}^{2} A_i f(x_i)$$

which, given the data:

$$f(x_0) = -\frac{3\pi}{4\sqrt{2}}, \quad f(x_1) = 0, \quad \text{and} \quad f(x_2) = \frac{3\pi}{4\sqrt{2}}$$

yields

$$\begin{aligned}
\int_{-1}^{1} f(x) \,\mathrm{d}x &\approx A_0 f(x_0) + A_1 f(x_1) + A_2 f(x_2) \\
&= \frac{16}{3\pi^2}\left(-\frac{3\pi}{4\sqrt{2}}\right) + \left(2 - \frac{32}{3\pi^2}\right)(0) + \frac{16}{3\pi^2}\left(\frac{3\pi}{4\sqrt{2}}\right).
\end{aligned}$$

Notice that $f(x_0) = -f(x_2)$, so that

$$\begin{aligned}
\int_{-1}^{1} f(x) \,\mathrm{d}x &\approx \frac{16}{3\pi^2}\left(-\frac{3\pi}{4\sqrt{2}} + \frac{3\pi}{4\sqrt{2}}\right) \\
&= 0
\end{aligned}$$

Question 5:

Given the following data:

i	0	1	2
x_i	0	1	3
y_i	1	3	2

Using polynomial interpolation, what is the value of $y(2)$?

- ◯ $y(2) = 3/2$
- ◯ $y(2) = 3/4$
- ✗ $y(2) = 10/3$
- ◯ $y(2) = 11/4$
- ◯ $y(2) = 4/9$
- ◯ $y(2) = 0$

To find the value of $y(2)$ using polynomial interpolation, we can use either the Lagrange interpolation formula or Newton's formula.

The Lagrange interpolation polynomial for the given set of data points (x_i, y_i) is given by:

$$P(x) = \sum_{i=0}^{n} y_i l_i(x)$$

where $l_i(x)$ are the Lagrange basis polynomials defined as:

$$l_i(x) = \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}.$$

Given data points:

i	0	1	2
x_i	0	1	3
y_i	1	3	2

We have $n = 2$, so we need to compute $l_0(x)$, $l_1(x)$ and $l_2(x)$. Calculating $l_0(x)$

$$\begin{aligned}
l_0(x) &= \frac{x - x_1}{x_0 - x_1} \cdot \frac{x - x_2}{x_0 - x_2} \\
&= \frac{x - 1}{0 - 1} \cdot \frac{x - 3}{0 - 3} \\
&= \frac{-(x - 1)}{1} \cdot \frac{-(x - 3)}{3} \\
&= (1 - x) \cdot \left(\frac{3 - x}{3}\right) \\
&= \frac{(1 - x)(3 - x)}{3} \\
&= \frac{3 - x - 3x + x^2}{3} \\
&= \frac{x^2 - 4x + 3}{3}
\end{aligned}$$

Calculating $l_1(x)$

$$\begin{aligned}
l_1(x) &= \frac{x - x_0}{x_1 - x_0} \cdot \frac{x - x_2}{x_1 - x_2} \\
&= \frac{x - 0}{1 - 0} \cdot \frac{x - 3}{1 - 3} \\
&= x \cdot \frac{x - 3}{-2} \\
&= x \cdot \left(\frac{3 - x}{2}\right) \\
&= \frac{x(3 - x)}{2}
\end{aligned}$$

Lastly, calculating $l_2(x)$

$$\begin{aligned}
l_2(x) &= \frac{x - x_0}{x_2 - x_0} \cdot \frac{x - x_1}{x_2 - x_1} \\
&= \frac{x - 0}{3 - 0} \cdot \frac{x - 1}{3 - 1} \\
&= \frac{x}{3} \cdot \frac{x - 1}{2} \\
&= \frac{x(x - 1)}{6}
\end{aligned}$$

Constructing the interpolation polynomial $P(x)$

$$\begin{aligned}
P(x) &= y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) \\
&= 1 \cdot l_0(x) + 3 \cdot l_1(x) + 2 \cdot l_2(x) \\
&= 1 \cdot \frac{x^2 - 4x + 3}{3} + 3 \cdot \frac{3x - x^2}{2} + 2 \cdot \frac{x^2 - x}{6} \\
&= \frac{x^2 - 4x + 3}{3} + \frac{9x - 3x^2}{2} + \frac{2(x^2 - x)}{6} \\
&= \frac{x^2 - 4x + 3}{3} + \frac{9x - 3x^2}{2} + \frac{x^2 - x}{3} \\
&= \frac{(x^2 - 4x + 3) + (x^2 - x)}{3} + \frac{9x - 3x^2}{2} \\
&= \frac{x^2 - 4x + 3 + x^2 - x}{3} + \frac{9x - 3x^2}{2} \\
&= \frac{2x^2 - 5x + 3}{3} + \frac{9x - 3x^2}{2} \\
&= \frac{2x^2 - 5x + 3}{3} + \frac{9x - 3x^2}{2} \\
&= \frac{4x^2 - 10x + 6}{6} + \frac{27x - 9x^2}{6} \\
&= \frac{4x^2 - 10x + 6 + 27x - 9x^2}{6} \\
&= \frac{-5x^2 + 17x + 6}{6} \\
&= -\frac{5}{6}x^2 + \frac{17}{6}x + 1.
\end{aligned}$$

Thus, evaluating $P(x)$ at $x = 2$ yields

$$\begin{aligned}
P(2) &= -\frac{5}{6}(2)^2 + \frac{17}{6}(2) + 1 \\
&= -\frac{5}{6}(4) + \frac{17}{6}(2) + 1 \\
&= -\frac{20}{6} + \frac{34}{6} + 1 \\
&= \frac{34 - 20}{6} + 1 \\
&= \frac{14}{6} + 1 \\
&= \frac{7}{3} + 1 \\
&= \frac{7}{3} + \frac{3}{3} \\
&= \frac{10}{3}
\end{aligned}$$

Question 6:

The differential equation

$$y'(t) = 1 - 4y(t) \quad \text{with} \quad y(0) = 1$$

has the exact solution $y = \frac{1}{4}\left(3e^{-4t} + 1\right)$. From the Runge-Kutta scheme given by the Butcher array

$$\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & & 1 & \\
\hline
 & 1/6 & 1/3 & 1/3 & 1/6
\end{array}$$

where

$$u_{k+1} = u_k + h\sum_{i=1}^{4} b_i f\left(u_k + h\sum_{j=1}^{4} a_{i,j}k_j, t_k + c_i h\right)$$

and using step size $h = 0.1$, what is the $|y(2h) - u_2|$, i.e. the global truncation error after two steps?

- ○ 0.1
- ○ 0.449
- ○ 0
- ○ 0.5
- ○ 0.839
- ☒ 0.164

Using the formula given, the Butcher array yields a Runge-Kutta scheme of the form:

$$\begin{aligned}
k_1 &= f\left(u_n, t_n\right) \\
k_2 &= f\left(u_n + hk_1/2, t_n + h/2\right) \\
k_3 &= f\left(u_n + hk_2/2, t_n + h/2\right) \\
k_4 &= f\left(u_n + hk_3, t_n + h\right)
\end{aligned}$$

and

$$u_{n+1} = u_n + (h/6)\left(k_1 + 2k_2 + 2k_3 + k_4\right),$$

for the function

$$f(u_n) = 1 - 4u_n.$$

Given the initial condition and the step size, to compute two time steps means to compute approximations to $y(0.1)$ and $y(0.2)$. For the first time step, with $u_0 = 1.0$ and $h = 0.1$,

$$k_1 = -3, \quad k_2 = -2.94, \quad k_3 = -2.9412 \quad \text{and} \quad k_4 - 2.882352.$$

Thus $u_1 = 1 - 17.644752/60 = 0.7059208$.

The second evaluation yields

$$k_1 = -2.88236832, \quad k_2 = -2.8247209536, \quad k_3 - 2.825873900928 \quad \text{and} \quad k_4 = -2.76933336396288.$$

Thus $u_2 = 0.423372610116352$

The exact values are given by $y(0.1) = 0.752740034$ and $y(0.2) = 0.58699672$, thus the difference at $t = 0.2$ is given by $|y(0.2) - u_2| = 0.163624109883648$.

Question 7:

Given the integral

$$I = \int_{1/4}^{1/5} \frac{1}{2} + \sin(\pi x)\, \mathrm{d}x$$

what is the error of the approximate integral for the Trapezium rule when using five subintervals?

- ○ 2.30e-5
- ○ 1.44e-6
- ○ 2.11e-4
- ○ 1.67e-3
- ✗ 2.67e-6
- ○ 1.44e-7

Determine the interval width h:

$$a = \frac{1}{4} \quad \text{and} \quad b = \frac{1}{5} \quad \text{with} \quad n = 5$$

then

$$h = \frac{b-a}{n} = \frac{\frac{1}{5} - \frac{1}{4}}{5} = \frac{\frac{4-5}{20}}{5} = \frac{-1}{20 \cdot 5} = -\frac{1}{100}.$$

Calculate the points x_i:

$$x_i = a + ih \quad \text{for} \quad i = 0, 1, 2, 3, 4, 5$$

that is

$$x_0 = \frac{1}{4}, \quad x_1 = \frac{1}{4} - \frac{1}{100}, \quad x_2 = \frac{1}{4} - \frac{2}{100}, \quad x_3 = \frac{1}{4} - \frac{3}{100}, \quad x_4 = \frac{1}{4} - \frac{4}{100}, \quad x_5 = \frac{1}{5}.$$

Evaluate the function $f(x) = \frac{1}{2} + \sin(\pi x)$ at these points:

$$\begin{aligned}
f(x_0) &= \frac{1}{2} + \sin(\pi \cdot 0.25) = \frac{1}{2} + \sin\left(\frac{\pi}{4}\right) = \frac{1}{2} + \frac{\sqrt{2}}{2} = 1.2071067811865475 \\
f(x_1) &= \frac{1}{2} + \sin(\pi \cdot 0.24) = 1.1845471059286887 \\
f(x_2) &= \frac{1}{2} + \sin(\pi \cdot 0.23) = 1.1613118653236518 \\
f(x_3) &= \frac{1}{2} + \sin(\pi \cdot 0.22) = 1.1374239897486897 \\
f(x_4) &= \frac{1}{2} + \sin(\pi \cdot 0.21) = 1.1129070536529764 \\
f(x_5) &= \frac{1}{2} + \sin(\pi \cdot 0.20) = 1.0877852522924731
\end{aligned}$$

Approximate the integral using the Trapezium rule:

$$I_h = \frac{h}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + 2f(x_4) + f(x_5)\right).$$

Substituting the values of h and $f(x_i)$:

$$I_h = -\frac{1}{200}\left(f(0.25) + 2f(0.24) + 2f(0.23) + 2f(0.22) + 2f(0.21) + f(0.20)\right).$$

Summing the values:

$$\begin{aligned}
I_h &= -\frac{1}{200}\left(1.20710678 + 2(1.18454711) + 2(1.16131187) + 2(1.13742399) + 2(1.11290705) + 1.08778525\right) \\
&= -0.05743636031393516
\end{aligned}$$

The exact integral is given as

$$
\begin{aligned}
I &= \int_{1/4}^{1/5} \frac{1}{2} + \sin \pi x \, \mathrm{d}x \\
&= \left[\frac{x}{2} - \frac{\cos(\pi x)}{\pi} \right]_{1/4}^{1/5} \\
&= \frac{1}{2}\left(\frac{1}{5} - \frac{1}{4}\right) - \frac{1}{\pi}\left(\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{\pi}{4}\right)\right) \\
&= \frac{1}{2}\frac{1}{20} - \frac{1}{\pi}\left(\cos\left(\frac{\pi}{5}\right) - \cos\left(\frac{\pi}{4}\right)\right) \\
&= -0.0574390283609654
\end{aligned}
$$

Thus, the error is given by $|I - I_h|$,

$$|-0.0574390283609654 - -0.05743636031393516| = 2.668047030231213\mathrm{e} - 6.$$

Question 8:

Given the matrix

$$A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix}$$

what is the Cholesky matrix L for the matrix A?

- [x] $\begin{pmatrix} 2 & 0 & 0 \\ 6 & 2 & 0 \\ -8 & 5 & 1 \end{pmatrix}$
- [ ] $\begin{pmatrix} 1 & 2 & 1 \\ 2 & 5 & -4 \\ 3 & 8 & 1 \end{pmatrix}$
- [ ] $\begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}$
- [ ] $\begin{pmatrix} 2 & 1 \\ 2 & 5 \end{pmatrix}$
- [ ] $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 6 \end{pmatrix}$
- [ ] $\begin{pmatrix} 2 & 0 & 0 \\ 2 & 5 & 0 \\ 7 & 9 & 1 \end{pmatrix}$

Initialize L as a lower triangular matrix with unknown elements:

$$L = \begin{pmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}$$

and solve for $L^T L = A$. Thus first compute l_{11}

$$l_{11} = \sqrt{a_{11}} = \sqrt{4} = 2 \quad \Rightarrow \quad L = \begin{pmatrix} 2 & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}.$$

Compute l_{21}:

$$l_{21} = \frac{a_{21}}{l_{11}} = \frac{12}{2} = 6 \quad \Rightarrow \quad L = \begin{pmatrix} 2 & 0 & 0 \\ 6 & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{pmatrix}.$$

Compute l_{31}:

$$l_{31} = \frac{a_{31}}{l_{11}} = \frac{-16}{2} = -8 \quad \Rightarrow \quad L = \begin{pmatrix} 2 & 0 & 0 \\ 6 & l_{22} & 0 \\ -8 & l_{32} & l_{33} \end{pmatrix}.$$

Compute l_{22}:

$$l_{22} = \sqrt{a_{22} - l_{21}^2} = \sqrt{40 - 6^2} = \sqrt{40 - 36} = \sqrt{4} = 2 \quad \Rightarrow \quad L = \begin{pmatrix} 2 & 0 & 0 \\ 6 & 2 & 0 \\ -8 & l_{32} & l_{33} \end{pmatrix}.$$

Compute l_{32}:

$$l_{32} = \frac{a_{32} - l_{31} l_{21}}{l_{22}} = \frac{-38 - (-8 \cdot 6)}{2} = \frac{-38 + 48}{2} = \frac{10}{2} = 5 \quad \Rightarrow \quad L = \begin{pmatrix} 2 & 0 & 0 \\ 6 & 2 & 0 \\ -8 & 5 & l_{33} \end{pmatrix}.$$

Lastly, compute l_{33}:

$$l_{33} = \sqrt{a_{33} - l_{31}^2 - l_{32}^2} = \sqrt{90 - (-8)^2 - 5^2} = \sqrt{90 - 64 - 25} = \sqrt{1} = 1.$$

Thus, the Cholesky matrix L for the given matrix A is:

$$L = \begin{pmatrix} 2 & 0 & 0 \\ 6 & 2 & 0 \\ -8 & 5 & 1 \end{pmatrix}.$$