

JTMS-MAT-13: Numerical Methods

Exam: Saturday 24 August 2024

All questions carry equal marks. Answer 5 questions only. Please use the booklet provided, clearly stating which questions are to be marked.

Note that all trigonometric values should be expressed in radians.

Question 1:

- (a) State a condition which means a square matrix will not be invertible.
- (b) Given the matrix

$$A = \begin{pmatrix} 4 & 12 & -16 \\ 12 & 40 & -38 \\ -16 & -38 & 90 \end{pmatrix},$$

use Gaussian elimination to show the row echelon form of the matrix A as

$$U = \begin{pmatrix} 4 & 12 & -16 \\ 0 & 4 & 10 \\ 0 & 0 & 1 \end{pmatrix}.$$

- (c) By applying Gaussian elimination, or any other method, show the solution to the linear equation $A\vec{x} = \vec{b}$, where \vec{b} is given by

$$\vec{b} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \quad \text{is} \quad \vec{x} = \begin{pmatrix} 110.25 \\ -24 \\ 9.5 \end{pmatrix}.$$

- (d) If an $n \times n$ matrix is invertible, what is the order of the upper limit for the number of arithmetic operations to yield the inverse for Gaussian elimination?

Question 2

- (a) Find the Jacobian matrix $J = J(x, y)$ for the vector-valued function

$$f(x, y) = \begin{pmatrix} 4x^2 - 20x + \frac{1}{4}y^2 - 8 \\ \frac{1}{2}xy^2 + 2x - 5y + 8 \end{pmatrix}.$$

- (b) Show the inverse of the Jacobian matrix is

$$J^{-1} = \frac{1}{|J|} \begin{pmatrix} xy - 5 & -\frac{1}{2}y \\ -\frac{1}{2}y^2 - 2 & 4(2x - 5) \end{pmatrix} \quad \text{where} \quad |J| = 8x^2y - 40x - 20xy + 100 - \frac{1}{4}y^3 - y.$$

- (c) Let $\vec{u}_n = (x_n, y_n)^T$. Then, using Newton's method, $\vec{u}_{n+1} = \vec{u}_n - J^{-1}(\vec{u}_n)f(\vec{u}_n)$, with an initial guess $\vec{u}_0 = (0, 0)^T$, show that the first iteration of Newton's method is $(-0.4, 1.44)^T$.

Question 3: Consider the integral

$$I = \int_1^2 f(x) dx = \int_1^2 \frac{dx}{x} = \ln(2) = 0.6931471805599453$$

(a) Given the Trapezium rule,

$$I_n = \frac{h}{2} \left(f(x_0) + f(x_n) + 2 \sum_{i=1}^{n-1} f(x_i) \right)$$

where $h = (b - a)/n$ and n is the number of intervals, show that the approximations to the integral for $n = 2^k$ where $k = 0, 1$ and 2 are

k	n	I_n
0	1	0.75
1	2	0.70833333
2	4	0.69702381

(b) Noting that the Trapezium rule has error behaviour

$$I = I_n + a_1 h^2 + a_2 h^4 + \dots$$

for some constants a , and considering the difference between the errors of the Trapezium rule for h and $h/2$, derive the Romberg formula

$$R_k^1 = \frac{1}{3} (4R_k^0 - R_{k-1}^0)$$

where $R_0^0 = I_1$, $R_1^0 = I_2$ etc.

(c) Using the values from the Trapezium rule for $I_k = R_k^0$, show that $R_2^1 = 0.693253$.

Question 4:

(a) For the numerical solutions of ordinary differential equations, $y' = f(y, t)$, what is an explicit method?

(b) Using a Taylor expansion, show the forward Euler scheme for a first-order differential equation is

$$y_{n+1} = y_n + hf(y_n, t_n)$$

where y_n denotes the solution at $t_n = t_0 + nh$ for a time step h and initial condition $y(t_0)$.

(c) For the second-order linear ordinary differential equation

$$y''(t) = -4y'(t) + y(t) \quad \text{with} \quad y(0) = 1 \quad \text{and} \quad y'(0) = 1$$

show, by considering the backward difference approximation $\nabla \bar{u}(t_{n+1}) \approx (\bar{u}_{n+1} - \bar{u}_n)/h$, where $\bar{u} = (y, y')$ and \bar{u}_n denotes the solution at $t_n = t_0 + nh$, that the backward Euler scheme yields

$$\bar{u}_{n+1} = \frac{1}{1 + 4h - h^2} \begin{pmatrix} 1 + 4h & -h \\ -h & 1 \end{pmatrix} \bar{u}_n.$$

(d) Compute the first two steps of the backward Euler scheme for the system given in (c), i.e. $\bar{u}_0 = (1, 1)$ with $h = 0.1$.

Question 8: The differential equation

$$y'(t) = 1 - 3y(t) \quad \text{with} \quad y(0) = 1$$

has the exact solution $y = \frac{1}{3}(2e^{-3t} + 1)$. From the explicit Runge-Kutta scheme given by the Butcher array

$$\begin{array}{c|ccc} 0 & & & \\ \frac{1}{2} & \frac{1}{2} & & \\ \frac{1}{2} & 0 & \frac{1}{2} & \\ 1 & 0 & 0 & 1 \\ \hline & 1/6 & 1/3 & 1/3 & 1/6 \end{array}$$

where

$$u_{k+1} = u_k + h \sum_{i=1}^4 b_i f \left(u_k + h \sum_{j=1}^4 a_{i,j} k_j, t_k + c_i h \right)$$

and using step size $h = 0.1$, what is the $|y(2h) - u_2|$, i.e. the global truncation error after two steps?

☐ 0.002

☐ 0.058

☐ 0.370

☐ 0.000115

☐ 0.965

☐ 0.000019