CA-MATH-804: Numerical Analysis

Assignment Sheet 6. Due: February 23, 2022

Exercise 1 [5 x 4* Points]: Consider the bending of a clamped beam subject to a transversal force f, which is described by the boundary value problem

$$\begin{aligned} u''''(x) &= f(x) \quad \text{in} \quad (0,1), \\ u(0) &= u(1) = 0, \\ u'(0) &= u'(1) = 0. \end{aligned}$$

a) Show that under certain conditions this problem is equivalent to the following variational (weak) problem:

$$(u'', v'') = (f, v) \quad \forall v \in W$$

where

$$W = \{ v : (0,1) \to \mathbb{R} \,\|\, v \,\text{and}\, v' \,\text{are continuous}, v'' \,\text{is piecewise continuous} \\ \text{and}\, v(0) = v(1) = v'(0) = v'(1) = 0 \}.$$

b) For an interval $I = [a, b]$ define

$$P_3(I) = \{ v : I \to \mathbb{R} \,\|\, v \,\text{is a polynomial of degree} \leq 3, \text{ i.e.} \\ v(x) = a_3 x^3 + a_2 x^2 + a_1 x + a_0 \,\text{for}\, a_i \in \mathbb{R} \}.$$

Show that $v \in P_3(I)$ is uniquely defined by the values $v(a)$, $v'(a)$, $v(b)$, $v'(b)$ and determine the corresponding basis functions $b_i(x)$ such that

$$v(x) = v(a) b_0(x) + v'(a) b_1(x) + v(b) b_2(x) + v'(b) b_3(x).$$

c) Starting from b) use a uniform partitioning of $(0,1)$ to construct a finite dimensional subspace W_h of W consisting of piecewise cubic functions. Specify suitable parameters to describe the functions in W_h and determine the corresponding basis functions of W_h. What is the dimension of the resulting finite element space W_h?

d) Formulate a finite element method for the problem based on the space W_h. Find the corresponding system of equations.

e) Determine the finite element solution in the case of two intervals and $f = 1$. Compare with the exact solution.

Exercise 2 [5 Points]: Prove that $G^k(x_j) = hG(x_i, x_j)$, where $G(x, y)$ is the Green's function for the one-dimensional Laplacian with Dirichlet boundary conditions, i.e.

$$\begin{aligned} -u''(x) = f(x) \quad \text{in} \quad (0,1), \\ u(0) = u(1) = 0 \end{aligned} \tag{1}$$

and $G^k(x_i)$ is the corresponding discrete counterpart with h the uniform step-size.

Exercise 3 [5 Points]: For the differential equation $-u''(x) + ku' + u = f$, find a value for k such that $a(u, v) = 0$ but $v \neq 0$ for some $v \in H^1(0, 1)$. Note: that the Hilbert space $H^1(0, 1)$ contains functions on the unit interval which have a weak derivative, however, for which no boundary values are prescribed.

Exercise 4 [5 Points]: Let $a(\cdot, \cdot)$ be the inner product for a Hilbert space V and let $F : V \to \mathbb{R}$ be a linear mapping. Prove that the following two statements are equivalent:

1. $u \in V$ satisfies $a(u, v) = F(v)$ for all $v \in V$.
2. u minimizes $\frac{1}{2}a(u, v) - F(v)$ over V.

i.e. show that the existence of one implies the existence of the other.

Hint: Consider a similar theorem which we had regarding the gradient method for solving linear systems.

Exercise 5 [5 Points]: Compare the linear systems which result from the finite difference and finite element approximations for the numerical solution of equation 1, assuming that f is a constant, that both methods using equidistant nodes on $[0, 1]$ and that the finite element method uses piecewise linear basis functions. For the integration of the right-hand side of the finite element method, you can use a midpoint quadrature rule.

Exercise 6 [5 Points]: Let a be a bilinear form on $H_0^1(0, 1)$ of the form:

$$a(u, v) = \int_0^1 (u'v' + u'v + uv)\, \mathrm{d}x$$

Show that

$$a(v, v) = \int_0^1 \left((v')^2 + v^2\right) \mathrm{d}x$$

for all $v \in H_0^1(0, 1)$.