# MT441P - Assignment 5

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1. How many distinguishable colourings of the regular tetrahedron are there, if you may use 5 colours?

**Solution:** We have that $|\Omega| = 5^4$.

- There are 8 rotations by $^{2\pi}/_{3}$ about an axis passing through a vertex and the centre of the opposite face.
- There are 3 rotations by $\pi$ about an axis through 2 opposite edges.

Then we get that the number of distinct colourings is

$$\frac{1}{12}\left(5^4 + 11\left(5^2\right)\right)$$

2. How many distinguishable colourings of the cube are there, if you may use 10 colours?

**Solution:** We have that $|\Omega| = 10^6$.

- There are 6 rotations around an edge by $\pi$.
- There are 8 rotations around a vertex by $\pm^{2\pi}/_{3}$.
- There are 6 rotations around a face by $\pm^{\pi}/_{2}$.
- There are 3 rotations around a face by $\pi$.

Then we get that the number of distinct colourings is

$$\frac{1}{24}\left(10^6 + 12\left(10^3\right) + 11\left(10^2\right)\right)$$

3. How many distinguishable colourings of the octahedron are there, if you may use 8 colours?

**Solution:** We have that $|\Omega| = 8^8$.

- There are 6 rotations around a vertex by $\pi/2$.
- There are 6 rotations around an edge by $\pi$.
- There are 3 rotations around a vertex by $\pi$.
- There are 8 rotations around a face by $\pm 2\pi/3$.

Then we get the number of distinct colourings is

$$\frac{1}{24}\left(8^8 + 6\left(8^2\right) + 17\left(8^4\right)\right)$$

4. How many distinguishable colourings of the dodecahedron are there, if you may use 4 colours?

**Solution:** We have that $|\Omega| = 4^{12}$.

- 4 rotations (by multiples of $2\pi/4$) about centres of 6 pairs of opposite faces.
- 1 rotation (by $\pi$) about centres of 15 pairs of opposite edges.
- 2 rotations (by $\pm 2\pi/3$) about 10 pairs of opposite vertices.

Then we get that the number of distinct colourings is

$$\frac{1}{60}\left(4^{12} + 24\left(4^4\right) + 20\left(4^4\right) + 15\left(4^6\right)\right)$$

5. How many distinguishable colourings of a roulette wheel are there, with 20 identical slots, using 6 colours?

**Solution:** We know that for a roulette wheel with $n$ segments and $k$ colours the number of distinct colourings is:

$$\frac{1}{n}\sum_{\substack{d=1 \\ d|n}}^{n} \phi(d) k^{n/d}$$

Then for $n = 20$ and $k = 6$, we get

$$\begin{aligned}\#\text{orbits} &= \frac{1}{20}\left(\phi(1)6^{20} + \phi(2)6^{10} + \phi(4)6^{5} + \phi(5)6^{4} + \phi(10)6^{2} + \phi(20)6^{1}\right) \\ &= \frac{1}{20}\left(6^{20} + 6^{10} + 2\left(6^{5}\right) + 4\left(6^{4}\right) + 4\left(6^{2}\right) + 8\left(6^{1}\right)\right)\end{aligned}$$

6. How many distinguishable necklaces can you make (with no clasp), using 9 beads, where each can be any of 10 colours?

**Solution:** We know that for $n$ beads, $n$ odd, and $k$ colours the number of orbits is:

$$\frac{1}{2n}\left(\sum_{\substack{d=1 \\ d|n}}^{n} \phi(d)k^{n/d} + n\left(k^{\frac{n+1}{2}}\right)\right)$$

Then for $n = 9$ and $k = 10$, we get

$$\begin{aligned}\#\text{orbits} &= \frac{1}{18}\left(\phi(1)10^{9} + \phi(3)10^{3} + \phi(9)10^{1} + 9\left(10^{5}\right)\right) \\ &= \frac{1}{18}\left(10^{9} + 2\left(10^{3}\right) + 6\left(10^{1}\right) + 9\left(10^{5}\right)\right)\end{aligned}$$