MT441P - Assignment 4

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2. Let $|G| = 144$. Show that G is not simple.

Solution: We know that $|G| = 144 = 2^4 3^2$.
We have that $n_3 = 1, 4$ or 16.

- If $n_3 = 1$ then G is not simple since the 3-Sylow subgroup is normal.
- Suppose $n_3 = 4$. Consider the following map:

$$\varphi \colon G \to \text{Perm}(G/N_G(Q))$$

where Q is a 3-Sylow subgroup. We know that $\varphi \cong S_4$ as $[G : N_G(Q)| = 4$. Since the order of G is less than the order of S_4, we see that we have a homomorphism with a non-trivial kernel. Since $\ker \varphi$ is nontrivial and normal in G, G is not simple.

- Suppose $n_3 = 16$. Now suppose that every pair of 3-Sylow subgroups have trivial intersection. This means that there are $16 \cdot 8$ non-identity elements in the 3-Sylow subgroups and there are 16 elements left to form the 2-Sylow subgroup, which means that there is only one 2-Sylow subgroup. Therefore $n_2 = 1 \implies G$ is not simple.

  Suppose there exists two different Sylow 3-subgroups P and Q such that $|P \cap Q| = 3$. Let $N = N_G(P \cap Q)$ be the normalizer of $P \cap Q$, which is the largest subgroup of G that contains $P \cap Q$ as a normal subgroup. Since $|P_i| = 3^2$, we know that P_i is abelian for all i and therefore $Q \cap P$ is normal in both P and Q. So we have that both $P, Q \subset N \implies PQ \subset N$. We also have that

$$|PQ| = \frac{|P||Q|}{|P \cap Q|} = 27$$

  Hence $|N| > 27$ as 27 does not divide the order of G. We also know that $9 \mid |N|$. So we have that $|N| = 36, 72$ or 144. If $|N| = 144$ then it would mean that $P \cap Q$ is normal in G. If $|N| = 36$ or 72 then $[G \colon N] = 4$ or 2. From a similar argument as above, we have $\varphi \colon G \to \text{Perm}(G/N)$ to get that $\ker \varphi$ is a nontrivial subgroup of G, hence G is not simple.

5. Suppose that $|G| = 60$ and that G has 20 elements of order 3. Show that $G \cong A_5$.

Solution: Since there are 20 elements of order 3, it must be the case that the intersection of each of the subgroups is trivial, so we get that $n_3(3-1) = 20 \implies n_3 = 10$.

Now assume that G is not simple and that it contains a non-trivial proper subgroup H.

- If $3 \mid |H|$, then H contains a 3-Sylow subgroup P of G. Since H is normal in G and every 3-Sylow subgroup is a conjugate of P, H contains all 10 3-Sylow subgroups $\implies |H| \geq 21$. Since $|H| \mid |G|$ and H is proper, $|H| = 30$. This contradicts the fact that every group of order 30 has a unique 3-Sylow subgroup. Therefore $3 \nmid |H|$.
- If $|H| = 10$ or 20, then H has a normal Sylow subgroup which is also normal in G (by Question 9). Thus we may assume, by replacing H by its normal subgroup if necessary, that $|H| = 2, 4$ or 5. Then $|G/H| = 30, 15$ or 12. Which implies that G/H has a normal subgroup K/H of order 3, where K is a normal subgroup of G. Now $|K| = |K/H| \cdot |H| = 5 \cdot |H|$. This implies that $5 \mid |K|$ which contradicts the previous point. Therefore G is simple.

Since we have already shown that a simple group of order 60 is isomorphic to A_5, we are finished.

6. Let G be the group of rotations of the Icosahedron. Show that $G \cong A_5$.

Solution: The elements of G are:

- 4 rotations (by multiples of $2\pi/4$) about centres of 6 pairs of opposite vertices.
- 1 rotation (by π) about centres of 15 pairs of opposite edges.
- 2 rotations (by $\pm 2\pi/3$) about 10 pairs of opposite faces.

Since $|G| = 60$ and the number of 3 cycles is 20 we can use the question above to show that $G \cong A_5$.

7. Let $|G| = 2^2 \cdot 5 \cdot 7$. Show that G is not simple.

Solution: By Sylow's Theorem n_3 must divide 35 and n_3 has to be one more than a multiple of 3, thus $n_3 \in \{1, 7\}$. Now suppose that G is simple, then $n_3 \neq 1$. Then we have that G is isomorphic to a subgroup of S_7 and that $|G|$ has to divide S_7. But this is false hence we have a contraction. Therefore G is not simple.

8. If $|G| = 2p$, $(p > 2)$ show that G is cyclic or dihedral.

Solution: Choose $a, b \in G$ with $o(a) = 2$ and $o(b) = p$. Let $H = \langle a \rangle$ and $K = \langle b \rangle$. Since every subgroup of index 2 is normal, K is normal in G. Thus $aba^{-1} = b^i$ for some integer i. We also know that $H \cap K = \{1\}$ and hence $G = HK = \langle a, b \rangle$. Now

$$b^{i^2} = (aba^{-1})^i = ab^ia^{-1} = a(aba^{-1})a^{-1} = b$$

as $a^2 = 1$.
Thus $b^{i^2-1} = 1$ and hence $p \mid i^2 - 1$ as $o(b) = p$. Therefore either $p \mid i - 1$ or $p \mid i + 1$.

- If $p \mid i - 1$ then $aba^{-1} = b^i = b$ and so $ab = ba$. Thus G is abelian and $o(ab) = 2p$, so G is cyclic in this case.
- If $p \mid i + 1$ then $aba^{-1} = b^i = b^{-1}$ and so

$$G = \langle a, b \colon a^2 = b^p = 1, aba^{-1} = b^{-1} \rangle \cong D_{2p}$$

Therefore G is either cyclic or dihedral.

9. Suppose that N is a normal subgroup of G, and that P is a p-Sylow subgroup of N. Show that $G = N_G(P)N$.

Solution: Pick $g \in G$. Since $P \subset N$ and $N \trianglelefteq G$, $gPg^{-1} \subset N$. Then by applying Sylows theorem to N, there is an $n \in N$ such that $gPg^{-1} = nPn^{-1}$, so $n^{-1}gPg^{-1}n = P$. That means $n^{-1}g \in N_G(P)$, so $g \in nN_G(P)$. Thus $G = N \cdot N_G(P)$.

10. Suppose that $N \trianglelefteq G$, and that P is a p-Sylow subgroup of N and $P \trianglelefteq N$. Show that $P \trianglelefteq G$.

Solution: We know that $G = N_G(P)N$. However since P is normal in N, we know that $N \leq N_G(P)$. This implies that $N_G(P)N = N_G(P)$. Therefore $G = N_G(P)$. Since the normalizer is the whole group, P is normal in G.

11. Let P be a p-Sylow subgroup of $A_n, n \geq 3$. Show that $N_G(P)$ contains an odd permutation.

Solution: Suppose $G = S_n$.
By Frattini's argument we get that $S_n = A_n N_G(P)$. Since A_n contains only even permutations, the only way to obtain odd permutations is to compose it with odd permutations. Therefore $N_G(P)$ must contain an odd permutation.