MT451P - Assignment 2

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1. Let $|G| = 144$. Show that G is not simple.

Solution: We know that $|G| = 144 = 2^4 3^2$.
We have that $n_3 = 1, 4$ or 16.

- If $n_3 = 1$ then G is not simple since the 3-Sylow subgroup is normal.
- Suppose $n_3 = 4$. Consider the follwoing map:

$$\varphi\colon G \to \mathrm{Perm}(G/N_G(Q))$$

where Q is a 3-Sylow subgroup. We know that $\varphi \cong S_4$ as $[G : N_G(Q)| = 4$. Since the order of G is less than the order of S_4, we see that we have a homomorphism with a non-trivial kernel. Since $\ker \varphi$ is nontrivial and normal in G, G is not simple.

- Suppose $n_3 = 16$. Now suppose that every pair of 3-Sylow subgroups have trivial intersection. This means that there are $16 \cdot 8$ non-identity elements in the 3-Sylow subgroups and there are 16 elements left to form the 2-Sylow subgroup, which means that there is only one 2-Sylow subgroup. Therefore $n_2 = 1 \implies G$ is not simple.

  Suppose there exists two different Sylow 3-subgroups P and Q such that $|P \cap Q| = 3$. Let $N = N_G(P \cap Q)$ be the normalizer of $P \cap Q$, which is the largest subgroup of G that contains $P \cap Q$ as a normal subgroup. Since $|P_i| = 3^2$, we know that P_i is abelian for all i and therefore $Q \cap P$ is normal in both P and Q. So we have that both $P, Q \subset N \implies PQ \subset N$. We also have that

$$|PQ| = \frac{|P||Q|}{|P \cap Q|} = 27$$

  Hence $|N| > 27$ as 27 does not divide the order of G. We also know that $9 \mid |N|$. So we have that $|N| = 36, 72$ or 144. If $|N| = 144$ then it would mean that $P \cap Q$ is normal in G. If $|N| = 36$ or 72 then $[G\colon N] = 4$ or 2. From a similar argument as above, we have $\varphi\colon G \to \mathrm{Perm}(G/N)$ to get that $\ker \varphi$ is a nontrivial subgroup of G, hence G is not simple.

2. Suppose that $|G| = 60$ and that G has 20 elements of order 3. Show that $G \cong A_5$.

Solution: Since there are 20 elements of order 3, it must be the case that the intersection of each of the subgroups is trivial, so we get that $n_3(3-1) = 20 \implies n_3 = 10$.

Now assume that G is not simple and that it contains a non-trivial proper subgroup H.

- If $3 \mid |H|$, then H contains a 3-Sylow subgroup P of G. Since H is normal in G and every 3-Sylow subgroup is a conjugate of P, H contains all 10 3-Sylow subgroups $\implies |H| \geq 21$. Since $|H| \mid |G|$ and H is proper, $|H| = 30$. This contradicts the fact that every group of order 30 has a unique 3-Sylow subgroup. Therefore $3 \nmid |H|$.
- If $|H| = 10$ or 20, then H has a normal Sylow subgroup which is also normal in G (by Question 9). Thus we may assume, by replacing H by its normal subgroup if necessary, that $|H| = 2, 4$ or 5. Then $|G/H| = 30, 15$ or 12. Which implies that G/H has a normal subgroup K/H of order 3, where K is a normal subgroup of G. Now $|K| = |K/H| \cdot |H| = 5 \cdot |H|$. This implies that $5 \mid |K|$ which contradicts the previous point. Therefore G is simple.

3. Let G be the group of rotations of the Icosahedron. Show that $G \cong A_5$.

Solution: The elements of G are:

- 4 rotations (by multiples of $2\pi/4$) about centres of 6 pairs of opposite faces.
- 1 rotation (by π) about centres of 15 pairs of opposite edges.
- 2 rotations (by $\pm 2\pi/3$) about 10 pairs of opposite vertices.

Since $|G| = 60$ and the number of 3 cycles is 20 we can use the question above to show that $G \cong A_5$.