MT451P - Assignment 3

Dheeraj Putta
15329966

1. Making use of the Implicit Function Theorem, derive the Regular Level Set Theorem.

Solution: Let $f : \mathbb{R}^2 \times \mathbb{R} \leftarrow \mathbb{R}$ be a smooth function. Let $c \in \operatorname{Im} f$ and define $\Sigma \coloneqq f^{-1}(c) \subseteq \mathbb{R}^3$. Since c is a regular value, we have that Σ contains no critical points of f. For $p \in \Sigma$, we have that

$$Df_p = \left[\frac{\partial f}{\partial x_1}(p), \frac{\partial f}{\partial x_2}(p), \frac{\partial f}{\partial x_3}(p)\right]$$

has a non-zero component. Then by the Implicit Function Theorem, $\exists U \subseteq \mathbb{R}^2$, $V \subseteq \mathbb{R}$ with $p \in U \times V$, and a smooth function $g : U \leftarrow V$, such that $(u_1, u_2, g(u_1, u_2)) \in \Sigma$, $\forall u_1, u_2 \in U$. Then define

$$\tilde{x} : U \leftarrow R^3 : (u_1, u_2) \to (u_1, u_2, g(u_1, u_2))$$

Since g is smooth this must mean that $\tilde{x}$ is smooth as well. Also, since $\tilde{x}(u_1, u_2)$ is uniquely determined by (u_1, u_2), $\tilde{x}$ is injective. Both Σ and g vary continuously, which means that $g(p_1, p_2) = p_3$ and $(p_1, p_2, p_3) \in \operatorname{Im} \tilde{x}$. By the Implicit Function Theorem we get that $\tilde{x}(u_1, u_2) \in \Sigma$, $\forall\ (u_1, u_2) \in U$, so $\operatorname{Im} \tilde{x} \subseteq \Sigma$. We also know that,

$$\left[\frac{\partial \tilde{x}_1}{\partial u_1}, \frac{\partial \tilde{x}_2}{\partial u_1} \frac{\partial \tilde{x}_3}{\partial u_1}\right] \times \left[\frac{\partial \tilde{x}_1}{\partial u_2}, \frac{\partial \tilde{x}_2}{\partial u_2} \frac{\partial \tilde{x}_3}{\partial u_2}\right] = [1, 0, g_{u_1}] \times [0, 1, g_{u_2}] = [-g_{u_1}, g_{u_2}, 1] \neq 0$$

that is, $\tilde{x}_{u_1} \times \tilde{x}_{u_2}$ is non-zero on all of U, which means that $\tilde{x}$ is regular. Since $\tilde{x}$ satisfies the conditions for being a co-ordinate patch containing p, and as this hold $\forall\ p \in \Sigma$, Σ is a surface.

2. Which of the following subsets of $\mathbb{R}^3$ are surfaces. Provide a brief justification for your answer in each case.

   (a) The solution set for the equation
$$\frac{1}{3}z^3 - z = \frac{1}{2}x^2 - \frac{1}{2}y^2$$

Solution: Define
$$f : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \to \frac{1}{2}x^2 - \frac{1}{2}y^2 - \frac{1}{3}z^3 + z$$
Then
$$\nabla f = \left[x, y, 1 - z^2\right]$$
The only points where $\nabla f = 0$ are $(0, 0, \pm 1)$ but since
$$f(0, 0, \pm 1) = \pm 1 \pm \frac{1}{3} \neq 0$$
these points are not in $f^{-1}(0)$. Therefore, 0 is a regular value of f and by the Regular Level Set Theorem, the set $f^{-1}(0)$ is a surface.

   (b) The sphere $S^2 \subset \mathbb{R}^3$ consisting of all points in $\mathbb{R}^3$ whose distance from the origin is 1.

Solution: Define
$$f : \mathbb{R}^3 \to \mathbb{R} : (x, y, z) \to (x^2 + y^2 + z^2)$$
Then
$$\nabla f = [2x, 2y, 2z]$$
The only point where $\nabla f = 0$ is at $(0, 0, 0)$ but since
$$f(0, 0, 0) = 0 \neq 1$$
it is not in $f^{-1}(1)$. Therefore 1 is a regular value of f and by the Regular Level Set Theorem, this set is a surface.

   (c) The set of points $(x, y, f(x, y))$ where $f : \mathbb{R}^2 \to \mathbb{R}$ is a smooth function.

Solution: Define
$$\tilde{x} : \mathbb{R}^2 \to \mathbb{R}^3 : (x, y) \to (x, y, f(x, y))$$
This is a smooth function as f is a smooth function. It is also an injective function as $\tilde{x}$ is uniquely determined by (x, y). Then we get that
$$\left[\frac{\partial \tilde{x}_1}{\partial x}, \frac{\partial \tilde{x}_2}{\partial x} \frac{\partial \tilde{x}_3}{\partial x}\right] \times \left[\frac{\partial \tilde{x}_1}{\partial y}, \frac{\partial \tilde{x}_2}{\partial y} \frac{\partial \tilde{x}_3}{\partial y}\right] = [1, 0, f_x] \times [0, 1, f_y] = [-f_x, f_y, 1] \neq \vec{0}$$

Therefore $\tilde{x}_x \times \tilde{x}_y$ is non-zero everywhere and so it is regular. This means that $\tilde{x}$ is a co-ordinate patch whose domain is all of $\mathbb{R}^2$, meaning it suffices for all inputs and its image is the set we seek. Theorem that set is a surface.

(d) The subset of $\mathbb{R}^3$ obtained upon revolution about the z-axis of the circle given by the equation

$$(x - R)^2 + z^2 = r^2$$

where $R, r > 0$ are constants.

Solution: We can define this subset by using cylindrical co-ordinates. Define

$$f : \mathbb{R}^3 \to \mathbb{R} : (d, \theta, z) \to (d - R)^2 + z^2 = d^2 - 2Rd + R^2 + z^2$$

Then we get that

$$\nabla f = [0, 2d - 2R, 2z]$$

This is only zero points of the form $(R, \theta, 0)$ for $\theta \in [0, 2\pi)$. But $f(R, \theta, 0) = (R - R)^2 + 0^2 = 0 \neq r^2$ as $r > 0$. This means that r^2 is a regular value of f. Since $f^{-1}(r^2)$ contains none of the critical points, by the Level Set Theorem, it is a surface.

(e) The set of all $(x, y, z) \in \mathbb{R}^3$ which satisfy the equation $x^2 + z^2 = y^2$.

Solution: The set described y this equation is the double cone connected at the origin. Suppose this set is a surface. This means there is a co-ordinate patch $\tilde{x}$ which maps from an open area of $\mathbb{R}^2$ to an open neighbourhood of the origin in $\mathbb{R}^3$.

Since $\tilde{x}$ is a homeomorphism this would mean that it would preserve the connectedness property of open sets in $\mathbb{R}^2$. However removing the origin in $\mathbb{R}^3$ would leave us with two disjoint cones. Therefore such an $\tilde{x}$ cannot exist and as such the set described above is not be a surface.

(f) The union of the sets A and B where

$$A = \{(x, y, 0) : x^2 + y^2 < 1\} \quad \text{and} \quad B = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$$

Solution: The union of these two sets is the sphere of radius 1 with a disk inside on the xy-plane. The area where these two sets intersect is the unit circle on the xy-plane. An open neighbourhood on the unit circle will not be locally euclidean, so it must be that $A \cup B$ is not a surface.

(g) The union of the sets A and B where

$$A = \{(x, y, 0) : x^2 + y^2 < 1\} \quad \text{and} \quad B = \{(x, 0, z) : x^2 + z^2 < 1\}$$

Solution: This set describes the disks on the xy and xz plane which intersect on the x plane. If we consider an open neighbourhood around $(1, 0, 0)$ with that point removed then it is not connected. Then by the same reason as in (e), this set cannot be a surface.

(h) The intersection of the sets A and B where

$$A = \{(x, y, z) : 3x^2 + 7y^2 < 1\} \quad \text{and} \quad B = \{(x, y, 0) : x, y > 0\}$$

Solution: The intersection of these two sets is the set $C = \{(x, y, 0) : 3x^2 + 7y^2 < 1 \text{ and } x, y > 0\}$, which is the positive quadrant of an ellipse without boundary. Since this set has no critical points and is an open region of $\mathbb{R}^2$, it is a surface.