MT451P - Assignment 2

Dheeraj Putta
15329966

1. (a) Find a unit speed parametrisation of the generalised helix, where $a, b > 0$:

$$\alpha(t) = (a\cos t, a\sin t, bt)$$

Solution: We have that

$$\alpha'(u) = (-a\sin u, a\cos u, b)$$

and

$$\begin{aligned}|\alpha'(u)| &= \sqrt{(-a\sin u)^2 + (a\cos u)^2 + b^2)} \\ &= \sqrt{a^2\sin^2 u + a^2\cos^2 u + b^2} \\ &= \sqrt{a^2(\sin^2 u + \cos^2 u) + b^2} \\ &= \sqrt{a^2 + b^2}\end{aligned}$$

Suppose that the original function was defined on the interval $[x, y]$. We now get that

$$s(t) = \int_x^t |\alpha(u)| \, \mathrm{d}u = \int_x^t \sqrt{a^2 + b^2} \, \mathrm{d}u = t\sqrt{a^2 + b^2} - x\sqrt{a^2 + b^2}$$

and the inverse of this function is

$$t(s) = x + \frac{s}{\sqrt{a^2 + b^2}}$$

So the unit speed parameterisation is

$$\tilde{\alpha}(s) = \alpha(t(s)) = \left(a\cos\left(x + \frac{s}{\sqrt{a^2 + b^2}}\right), a\sin\left(x + \frac{s}{\sqrt{a^2 + b^2}}\right), \frac{bs}{\sqrt{a^2 + b^2}} + bx\right)$$

and the function is defined as $\tilde{\alpha} : [0, s(y)] \to \mathbb{R}^3$

(b) Compute the acceleration vector to this unit speed parameterisation.

Solution: We have that

$$\tilde{\alpha}(s) = \left(a\cos\left(x + \frac{s}{\sqrt{a^2+b^2}}\right), a\sin\left(x + \frac{s}{\sqrt{a^2+b^2}}\right), \frac{bs}{\sqrt{a^2+b^2}} + bx\right)$$

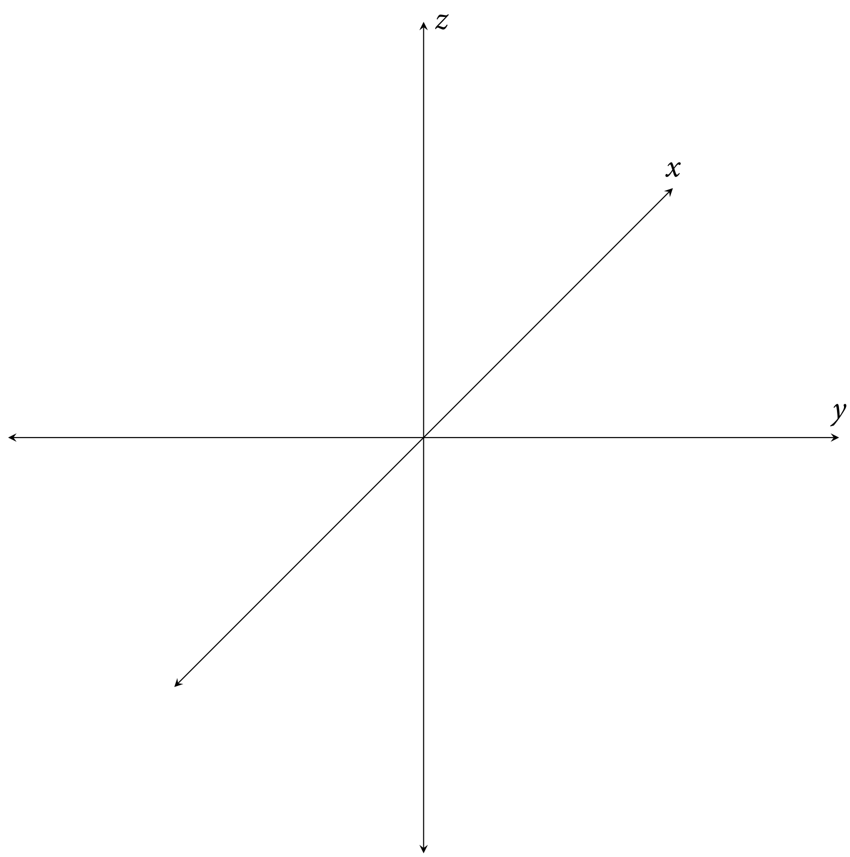
The tangent vector field is equal to

$$\tilde{\alpha}'(s) = \left(\frac{-a\sin\left(x + \frac{s}{\sqrt{a^2+b^2}}\right)}{\sqrt{a^2+b^2}}, \frac{a\cos\left(x + \frac{s}{\sqrt{a^2+b^2}}\right)}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}}\right)$$

The acceleration vector field is equal to

$$\tilde{\alpha}''(s) = \left(\frac{-a\cos\left(x + \frac{s}{\sqrt{a^2+b^2}}\right)}{a^2+b^2}, \frac{-a\sin\left(x + \frac{s}{\sqrt{a^2+b^2}}\right)}{a^2+b^2}, 0\right)$$

(c) Sketch a picture showing the curve and its unit tangent and acceleration vector fields.



2. (a) Show that the curve given by
$$\beta(s) = \left(\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}}\right)$$
defined on $s \in (-1,1)$ is unit speed.

Solution: We have that
$$\beta'(s) = \left(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}}\right)$$
Then
$$\begin{aligned}|\beta'(s)| &= \sqrt{\left(\frac{\sqrt{1+s}}{2}\right)^2 + \left(-\frac{\sqrt{1-s}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \\ &= \frac{1+s+1-s+2}{4} \\ &= 1\end{aligned}$$
Therefore γ is unit speed.

(b) Compute its Frenet frame.

Solution: From above we have that
$$\begin{aligned}T(s) &= \beta'(s) \\ &= \left(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}}\right)\end{aligned}$$
Then we have that
$$T'(s) = \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0\right)$$

and

$$\begin{aligned}
|T'(s)| &= \sqrt{\left(\frac{1}{4\sqrt{1+s}}\right)^2 + \left(\frac{1}{4\sqrt{1-s}}\right)^2} \\
&= \sqrt{\frac{1}{16+16s} + \frac{1}{16-16s}} \\
&= \sqrt{\frac{16-16s+16s+16}{(16-16s)(16+16s)}} \\
&= \frac{1}{\sqrt{8-8s^2}} \\
&= \frac{1}{2\sqrt{2(1-s^2)}}
\end{aligned}$$

So we get that

$$\begin{aligned}
N(s) &= \frac{T'(s)}{|T'(s)|} \\
&= \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0\right) \cdot 2\sqrt{2(1-s^2)} \\
&= \left(\frac{\sqrt{2(1-s^2)}}{2\sqrt{1+s}}, \frac{\sqrt{2(1-s^2)}}{2\sqrt{1-s}}, 0\right) \\
&= \left(\frac{\sqrt{-(1-s)}}{\sqrt{2}}, \frac{\sqrt{-(1+s)}}{\sqrt{2}}, 0\right)
\end{aligned}$$

We know that

$$\begin{aligned}
B(s) &= T(s) \times N(s) \\
&= \left(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}}\right) \times \left(\frac{\sqrt{-(1-s)}}{\sqrt{2}}, \frac{\sqrt{-(1+s)}}{\sqrt{2}}, 0\right) \\
&= \left(0, 0, \frac{\sqrt{(-1-s)(-s+1)} - \sqrt{(-1+s)(s+1)}}{4\sqrt{2(1-s^2)}}\right)
\end{aligned}$$

3. (a) Show that the curve given by
$$\gamma(s) = \left(\frac{4}{5}\cos s, 1 - \sin s, -\frac{3}{5}\cos s\right)$$
determines a circle.

Solution: We have that

$$\begin{aligned} T(s) &= \gamma'(s) \\ &= \left(-\frac{4}{5}\sin s, -\cos s, \frac{3}{5}\sin s\right) \end{aligned}$$

Then we have that

$$T'(s) = \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right)$$

and

$$\begin{aligned} |T'(s)| &= \sqrt{\left(-\frac{4}{5}\cos s\right)^2 + (\sin s)^2 + \left(\frac{3}{5}\cos s\right)^2} \\ &= \sqrt{\cos^2 s + \sin^2 s} \\ &= 1 \\ &= \kappa(s) \end{aligned}$$

So we get that $N(s) = T'(s)$

We know that

$$\begin{aligned} B(s) &= T(s) \times N(s) \\ &= \left(-\frac{4}{5}\sin s, -\cos s, \frac{3}{5}\sin s\right) \times \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right) \\ &= \left(\frac{-3}{5}, 0, \frac{-4}{5}\right) \end{aligned}$$

and

$$B'(s) = (0, 0, 0)$$

For the equation $B'(s) = -\tau(s)N(s)$ to hold it must be the case that $\tau = 0$.

Since $\tau = 0$ and $\kappa = 1$ this curves must determine a circle.

(b) Find it's centre and radius.

Solution: We know that
$$r = \frac{1}{\kappa} = 1$$
and
$$\begin{aligned} c &= \gamma + \frac{1}{\kappa}N \\ &= \left(\frac{4}{5}\cos s, 1 - \sin s, -\frac{3}{5}\cos s\right) + \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right) \\ &= (0, 1, 0) \end{aligned}$$

4. Consider a unit speed curve α, with $\kappa > 0$ and $\tau \neq 0$, which lies on the sphere centered at $c \in \mathbb{R}$ of radius r.

(a) Show that
$$\alpha - c = -\rho N - \rho' \sigma B$$
where $\rho = \frac{1}{\kappa}$ and $\sigma = \frac{1}{\tau}$.

Solution: We have that
$$(\alpha - c)^2 = \alpha^2 - 2c\alpha + c^2 = r^2$$
Differentiating both sides w.r.t to s gives us that
$$\begin{aligned} 2\alpha'\alpha - 2c\alpha' &= 0 \\ 2\alpha'(\alpha - c) &= 0 \\ \alpha'(\alpha - c) &= 0 \\ T(\alpha - c) &= 0 && \text{as } T = \alpha' \end{aligned}$$
Differentiating the above again w.r.t s, we get that
$$\begin{aligned} T'\alpha + T\alpha' - cT' &= 0 \\ T'(\alpha - c) + T^2 &= 0 \end{aligned}$$
Substituting $T' = \kappa N$, we get
$$\kappa N(\alpha - c) + 1 = 0$$

Differentiating again w.r.t s,

$$
\begin{aligned}
\kappa' N(\alpha - c) + \kappa N'(\alpha - c) + \kappa N \alpha' &= 0 \\
\kappa' N(\alpha - c) + \kappa N'(\alpha - c) &= 0 && \text{as } N\alpha' = NT = 0 \\
-\frac{\kappa'}{\kappa} + \kappa N'(\alpha - c) &= 0 && \text{as } N(\alpha - c) = -\frac{1}{\kappa}
\end{aligned}
$$

Substituting $N' = -\kappa T + \tau B$

$$
\begin{aligned}
-\frac{\kappa'}{\kappa} - \kappa^2 T(\alpha - c) + \kappa\tau B(\alpha - c) &= 0 \\
-\frac{\kappa'}{\kappa} + \kappa\tau B(\alpha - c) &= 0 && \text{as } T(\alpha - c) = 0 \\
B(\alpha - c) &= \frac{k'}{k^2\tau} \\
&= \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}
\end{aligned}
$$

We now have the following equations:

$$
\begin{aligned}
T(\alpha - c) &= 0 \\
N(\alpha - c) &= -\frac{1}{\kappa} \\
B(\alpha - c) &= \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}
\end{aligned}
$$

These 3 equations express the components of the radial vector $\alpha - c$ in the orthonormal frame T, N, B, which gives us that

$$
\begin{aligned}
\alpha - c &= -\frac{1}{\kappa} N + \left(\frac{1}{\kappa}\right)' \frac{1}{\tau} B \\
&= -\rho N - \rho' \sigma B
\end{aligned}
$$

(b) Deduce a formula for r in terms of ρ and σ.

Solution: From the part above we know that

$$
\begin{aligned}
r^2 &= (\alpha - c)^2 \\
&= (-\rho N - \rho' \sigma B)^2
\end{aligned}
$$

Since $\|N\| = \|B\| = 1$ and $NB = 0$, the above equation reduces to

$$\begin{aligned} r^2 &= \frac{1}{\kappa^2} + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)^2 \\ &= \frac{1}{\kappa^2} + \left(-\frac{\kappa'}{\kappa^2}\frac{1}{\tau^2}\right) \\ &= \frac{\kappa'^2 + \kappa^2\tau^2}{\kappa^4\tau^2} \\ r &= \frac{1}{\kappa^2\tau}\sqrt{\kappa'^2 + \kappa^2\tau^2} \end{aligned}$$

(c) For an arbitrary unit speed curve with $\kappa > 0$ and $\tau \neq 0$, specify conditions on ρ and σ which would guarantee that α lies on a sphere of radius r.

Solution: We assert that the condition

$$\frac{\tau}{\kappa} \equiv \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\kappa'}{\kappa^2\tau}\right) = -\frac{\mathrm{d}}{\mathrm{d}s}(\rho'\sigma)$$

is sufficient to guarantee this property. Define

$$c(s) \coloneqq \alpha(s) + \rho(s)N(s) + \rho'(s)\sigma(s)B(s) \tag{1}$$
$$r^2(s) \coloneqq \rho(s)^2 + \rho'(s)^2\sigma(s)^2 \tag{2}$$

These will define the centre and the square of the radius of the circle, respectively, provided they are both constant. Differentiating (1) with respect to s, we obtain

$$\begin{aligned} c' &= \alpha' + \rho'N + \rho N' + \frac{\mathrm{d}}{\mathrm{d}s}(\rho'\sigma)B + (\rho'\sigma)B' \\ &= T + \rho'N + \rho(-\kappa T + \tau B) + \frac{\mathrm{d}}{\mathrm{d}s}(\rho'\sigma)B + (\rho'\sigma)(-\tau N) \\ &= \left(\frac{\rho}{\sigma} + \frac{\mathrm{d}}{\mathrm{d}s}(\rho'\sigma)\right)B = \left(\frac{\tau}{\kappa} - \frac{\tau}{\kappa}\right)B \equiv 0 \end{aligned}$$

And thus c is constant, as required. Likewise, differentiating (2) with respect to s, we also obtain

$$\begin{aligned} r^{2\prime} &= \frac{\mathrm{d}}{\mathrm{d}s}(\rho^2 + (\rho'\sigma)^2) = 2\rho\rho' + 2\rho'\sigma\frac{\mathrm{d}}{\mathrm{d}s}(\rho'\sigma) \\ &= 2\rho\rho' - 2\rho'\sigma\frac{\rho}{\sigma} = 2\rho\rho' - 2\rho\rho' = 0 \end{aligned}$$

So r^2, and thus r, is constant, as required. Overall, our initial assertion is sufficient to guarantee that α lie on a sphere with centre and radius as defined.