

MT451P - Assignment 2

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1. (a) Find a unit speed parametrisation of the generalised helix, where $a, b > 0$:

$$\alpha(t) = (a \cos t, a \sin t, bt)$$

Solution: We have that

$$\alpha'(u) = (-a \sin u, a \cos u, b)$$

and

$$\begin{aligned} |\alpha'(u)| &= \sqrt{(-a \sin u)^2 + (a \cos u)^2 + b^2} \\ &= \sqrt{a^2 \sin^2 u + a^2 \cos^2 u + b^2} \\ &= \sqrt{a^2(\sin^2 u + \cos^2 u) + b^2} \\ &= \sqrt{a^2 + b^2} \end{aligned}$$

Suppose that the original function was defined on the interval $[x, y]$. We now get that

$$s(t) = \int_x^t |\alpha(u)| \, du = \int_x^t \sqrt{a^2 + b^2} \, du = t\sqrt{a^2 + b^2} - x\sqrt{a^2 + b^2}$$

and the inverse of this function is

$$t(s) = x + \frac{s}{\sqrt{a^2 + b^2}}$$

So the unit speed parameterisation is

$$\tilde{\alpha}(s) = \alpha(t(s)) = \left(a \cos \left(x + \frac{s}{\sqrt{a^2 + b^2}} \right), a \sin \left(x + \frac{s}{\sqrt{a^2 + b^2}} \right), \frac{bs}{\sqrt{a^2 + b^2}} + bx \right)$$

and the function is defined as $\tilde{\alpha} : [0, s(y)] \rightarrow \mathbb{R}^3$

(b) Compute the acceleration vector to this unit speed parameterisation.

Solution: We have that

$$\tilde{\alpha}(s) = \left(a \cos \left(x + \frac{s}{\sqrt{a^2 + b^2}} \right), a \sin \left(x + \frac{s}{\sqrt{a^2 + b^2}} \right), \frac{bs}{\sqrt{a^2 + b^2}} + bx \right)$$

The tangent vector field is equal to

$$\tilde{\alpha}'(s) = \left(\frac{-a \sin \left(x + \frac{s}{\sqrt{a^2 + b^2}} \right)}{\sqrt{a^2 + b^2}}, \frac{a \cos \left(x + \frac{s}{\sqrt{a^2 + b^2}} \right)}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right)$$

The acceleration vector field is equal to

$$\tilde{\alpha}''(s) = \left(\frac{-a \cos \left(x + \frac{s}{\sqrt{a^2 + b^2}} \right)}{a^2 + b^2}, \frac{-a \sin \left(x + \frac{s}{\sqrt{a^2 + b^2}} \right)}{a^2 + b^2}, 0 \right)$$

(c) Sketch a picture showing the curve and its unit tangent and acceleration vector fields.

2. (a) Show that the curve given by

$$\beta(s) = \left(\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}} \right)$$

defined on $s \in (-1, 1)$ is unit speed.

Solution: We have that

$$\beta'(s) = \left(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}} \right)$$

Then

$$\begin{aligned} |\beta'(s)| &= \sqrt{\left(\frac{\sqrt{1+s}}{2}\right)^2 + \left(-\frac{\sqrt{1-s}}{2}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \\ &= \frac{1+s+1-s+2}{4} \\ &= 1 \end{aligned}$$

Therefore γ is unit speed.

- (b) Compute its Frenet frame.

Solution: From above we have that

$$\begin{aligned} T(s) &= \beta'(s) \\ &= \left(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}} \right) \end{aligned}$$

Then we have that

$$T'(s) = \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0 \right)$$

and

$$\begin{aligned}
 |T'(s)| &= \sqrt{\left(\frac{1}{4\sqrt{1+s}}\right)^2 + \left(\frac{1}{4\sqrt{1-s}}\right)^2} \\
 &= \sqrt{\frac{1}{16+16s} + \frac{1}{16-16s}} \\
 &= \sqrt{\frac{16-16s+16s+16}{(16-16s)(16+16s)}} \\
 &= \frac{1}{\sqrt{8-8s^2}} \\
 &= \frac{1}{2\sqrt{2(1-s^2)}}
 \end{aligned}$$

So we get that

$$\begin{aligned}
 N(s) &= \frac{T'(s)}{|T'(s)|} \\
 &= \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0\right) \cdot 2\sqrt{2(1-s^2)} \\
 &= \left(\frac{\sqrt{2(1-s^2)}}{2\sqrt{1+s}}, \frac{\sqrt{2(1-s^2)}}{2\sqrt{1-s}}, 0\right) \\
 &= \left(\frac{\sqrt{-(1-s)}}{\sqrt{2}}, \frac{\sqrt{-(1+s)}}{\sqrt{2}}, 0\right)
 \end{aligned}$$

We know that

$$\begin{aligned}
 B(s) &= T(s) \times N(s) \\
 &= \left(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}}\right) \times \left(\frac{\sqrt{-(1-s)}}{\sqrt{2}}, \frac{\sqrt{-(1+s)}}{\sqrt{2}}, 0\right) \\
 &= \left(0, 0, \frac{\sqrt{(-1-s)(-s+1)} - \sqrt{(-1+s)(s+1)}}{4\sqrt{2(1-s^2)}}\right)
 \end{aligned}$$

3. (a) Show that the curve given by

$$\gamma(s) = \left(\frac{4}{5} \cos s, 1 - \sin s, -\frac{3}{5} \cos s \right)$$

determines a circle.

Solution: We have that

$$\begin{aligned} T(s) &= \gamma'(s) \\ &= \left(-\frac{4}{5} \sin s, -\cos s, \frac{3}{5} \sin s \right) \end{aligned}$$

Then we have that

$$T'(s) = \left(-\frac{4}{5} \cos s, \sin s, \frac{3}{5} \cos s \right)$$

and

$$\begin{aligned} |T'(s)| &= \sqrt{\left(-\frac{4}{5} \cos s \right)^2 + (\sin s)^2 + \left(\frac{3}{5} \cos s \right)^2} \\ &= \sqrt{\cos^2 s + \sin^2 s} \\ &= 1 \\ &= \kappa(s) \end{aligned}$$

So we get that $N(s) = T'(s)$

We know that

$$\begin{aligned} B(s) &= T(s) \times N(s) \\ &= \left(-\frac{4}{5} \sin s, -\cos s, \frac{3}{5} \sin s \right) \times \left(-\frac{4}{5} \cos s, \sin s, \frac{3}{5} \cos s \right) \\ &= \left(\frac{-3}{5}, 0, \frac{-4}{5} \right) \end{aligned}$$

and

$$B'(s) = (0, 0, 0)$$

For the equation $B'(s) = -\tau(s)N(s)$ to hold it must be the case that $\tau = 0$.

Since $\tau = 0$ and $\kappa = 1$ this curves must determine a circle.

(b) Find its centre and radius.

Solution: We know that

$$r = \frac{1}{\kappa} = 1$$

and

$$\begin{aligned} c &= \gamma + \frac{1}{\kappa}N \\ &= \left(\frac{4}{5} \cos s, 1 - \sin s, -\frac{3}{5} \cos s \right) + \left(-\frac{4}{5} \cos s, \sin s, \frac{3}{5} \cos s \right) \\ &= (0, 1, 0) \end{aligned}$$

4. Consider a unit speed curve α , with $\kappa > 0$ and $\tau \neq 0$, which lies on the sphere centered at $c \in \mathbb{R}$ of radius r .

(a) Show that

$$\alpha - c = -\rho N - \rho' \sigma B$$

where $\rho = \frac{1}{\kappa}$ and $\sigma = \frac{1}{\tau}$.

Solution: We have that

$$(\alpha - c)^2 = \alpha^2 - 2c\alpha + c^2 = r^2$$

Differentiating both sides w.r.t to s gives us that

$$2\alpha'\alpha - 2c\alpha' = 0$$

$$2\alpha'(\alpha - c) = 0$$

$$\alpha'(\alpha - c) = 0$$

$$T(\alpha - c) = 0$$

$$\text{as } T = \alpha'$$

Differentiating the above again w.r.t s , we get that

$$T'\alpha + T\alpha' - cT' = 0$$

$$T'(\alpha - c) + T^2 = 0$$

Substituting $T' = \kappa N$, we get

$$\kappa N(\alpha - c) + 1 = 0$$

Differentiating again w.r.t s ,

$$\kappa'N(\alpha - c) + \kappa N'(\alpha - c) + \kappa N\alpha' = 0$$

$$\kappa'N(\alpha - c) + \kappa N'(\alpha - c) = 0$$

$$\text{as } N\alpha' = NT = 0$$

$$-\frac{\kappa'}{\kappa} + \kappa N'(\alpha - c) = 0$$

$$\text{as } N(\alpha - c) = -\frac{1}{\kappa}$$

Substituting $N' = -\kappa T + \tau B$

$$-\frac{\kappa'}{\kappa} - \kappa^2 T(\alpha - c) + \kappa \tau B(\alpha - c) = 0$$

$$-\frac{\kappa'}{\kappa} + \kappa \tau B(\alpha - c) = 0$$

$$\text{as } T(\alpha - c) = 0$$

$$\begin{aligned} B(\alpha - c) &= \frac{\kappa'}{\kappa^2 \tau} \\ &= \left(\frac{1}{\kappa}\right)' \frac{1}{\tau} \end{aligned}$$

We now have the following equations:

$$T(\alpha - c) = 0$$

$$N(\alpha - c) = -\frac{1}{\kappa}$$

$$B(\alpha - c) = \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}$$

These 3 equations express the components of the radial vector $\alpha - c$ in the orthonormal frame T, N, B , which gives us that

$$\alpha - c = -\frac{1}{\kappa}N - \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B$$

(b) Deduce a formula for r in terms of ρ and σ .

Solution: From the part above we know that

$$\begin{aligned} r^2 &= (\alpha - c)^2 \\ &= (-\rho N - \rho' \sigma B)^2 \end{aligned}$$

Since $\|N\| = \|B\| = 1$ and $NB = 0$, the above equation reduces to

$$\begin{aligned} r^2 &= \frac{1}{\kappa^2} + \left(\left(\frac{1}{\kappa} \right)' \frac{1}{\tau} \right)^2 \\ &= \frac{1}{\kappa^2} + \left(-\frac{\kappa'}{\kappa^2} \frac{1}{\tau^2} \right)^2 \\ &= \frac{\kappa'^2 + \kappa^2 \tau^2}{\kappa^4 \tau^2} \\ r &= \frac{1}{\kappa^2 \tau} \sqrt{\kappa'^2 + \kappa^2 \tau^2} \end{aligned}$$

- (c) For an arbitrary unit speed curve with $\kappa > 0$ and $\tau \neq 0$, specify conditions on ρ and σ which would guarantee that α lies on a sphere of radius r .

Solution: We want to show that α lies on a sphere of radius $r \iff \rho^2 + (\rho'\sigma)^2 = r^2$. First suppose that α lies on a sphere of radius r . Then we have show that

$$\begin{aligned} r^2 &= (-\rho N - \rho'\sigma B)^2 \\ &= \rho^2 + (\rho'\sigma)^2 \end{aligned}$$

Differentiating both sides we get that

$$\begin{aligned} 0 &= 2\rho\rho' + 2(\rho'\sigma)(\rho'' + \rho'\sigma') \\ &= \end{aligned}$$