MT451P - Assignment 2

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1. (a) Find a unit speed parametrisation of the generalised helix, where $a, b > 0$:

$$\alpha(t) = (a\cos t, a\sin t, bt)$$

Solution: We have that
$$\alpha'(u) = (-a\sin u, a\cos u, b)$$

and

$$|\alpha'(u)| = \sqrt{(-a\sin u)^2 + (a\cos u)^2 + b^2)}$$
$$= \sqrt{a^2\sin^2 u + a^2\cos^2 u + b^2}$$
$$= \sqrt{a^2(\sin^2 u + \cos^2 u) + b^2}$$
$$= \sqrt{a^2 + b^2}$$

Suppose that the original function was defined on the interval $[x, y]$. We now get that

$$s(t) = \int_x^t |\alpha(u)|\ \mathrm{d}u = \int_x^t \sqrt{a^2 + b^2}\ \mathrm{d}u = t\sqrt{a^2 + b^2} - x\sqrt{a^2 + b^2}$$

and the inverse of this function is

$$t(s) = x + \frac{s}{\sqrt{a^2 + b^2}}$$

So the unit speed parameterisation is

$$\tilde{\alpha}(s) = \alpha(t(s)) = \left(a\cos\left(x + \frac{s}{\sqrt{a^2 + b^2}}\right), a\sin\left(x + \frac{s}{\sqrt{a^2 + b^2}}\right), \frac{bs}{\sqrt{a^2 + b^2}} + bx\right)$$

and the function is defined as $\tilde{\alpha} : [0, s(y)] \to \mathbb{R}^3$

(b) Compute the acceleration vector to this unit speed parameterisation.

Solution: We have that

$$\tilde{\alpha}(s) = \left(a \cos\left(x + \frac{s}{\sqrt{a^2 + b^2}} \right), a \sin\left(x + \frac{s}{\sqrt{a^2 + b^2}} \right), \frac{bs}{\sqrt{a^2 + b^2}} + bx \right)$$
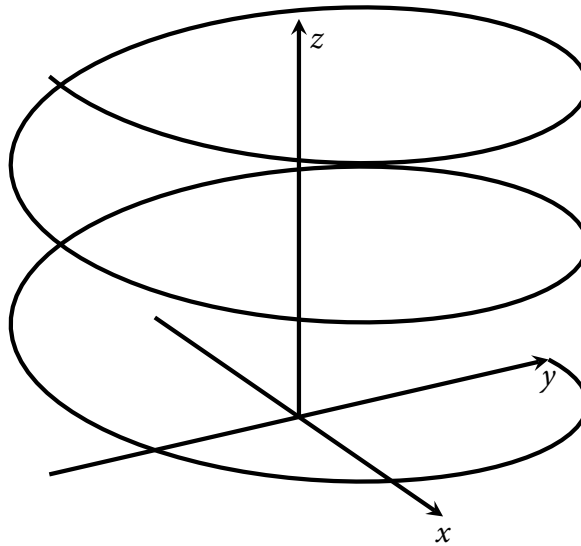
The tangent vector field is equal to

$$\tilde{\alpha}'(s) = \left(\frac{-a \sin\left(x + \frac{s}{\sqrt{a^2+b^2}} \right)}{\sqrt{a^2 + b^2}}, \frac{a \cos\left(x + \frac{s}{\sqrt{a^2+b^2}} \right)}{\sqrt{a^2 + b^2}}, \frac{b}{\sqrt{a^2 + b^2}} \right)$$

The acceleration vector field is equal to

$$\tilde{\alpha}''(s) = \left(\frac{-a \cos\left(x + \frac{s}{\sqrt{a^2+b^2}} \right)}{a^2 + b^2}, \frac{-a \sin\left(x + \frac{s}{\sqrt{a^2+b^2}} \right)}{a^2 + b^2}, 0 \right)$$

(c) Sketch a picture showing the curve and its unit tangent and acceleration vector fields.

Solution:

2. (a) Show that the curve given by

$$\beta(s) = \left(\frac{(1+s)^{3/2}}{3}, \frac{(1-s)^{3/2}}{3}, \frac{s}{\sqrt{2}} \right)$$

defined on $s \in (-1, 1)$ is unit speed.

Solution: We have that

$$\beta'(s) = \left(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}} \right)$$

Then

$$|\beta'(s)| = \sqrt{\left(\frac{\sqrt{1+s}}{2} \right)^2 + \left(-\frac{\sqrt{1-s}}{2} \right)^2 + \left(\frac{1}{\sqrt{2}} \right)^2}$$
$$= \frac{1+s+1-s+2}{4}$$
$$= 1$$

Therefore γ is unit speed.

(b) Compute its Frenet frame.

Solution: From above we have that

$$T(s) = \beta'(s)$$
$$= \left(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}} \right)$$

Then we have that

$$T'(s) = \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0 \right)$$

and

$$|T'(s)| = \sqrt{\left(\frac{1}{4\sqrt{1+s}}\right)^2 + \left(\frac{1}{4\sqrt{1-s}}\right)^2}$$

$$= \sqrt{\frac{1}{16+16s} + \frac{1}{16-16s}}$$

$$= \sqrt{\frac{16-16s+16s+16}{(16-16s)(16+16s)}}$$

$$= \frac{1}{\sqrt{8-8s^2}}$$

$$= \frac{1}{2\sqrt{2(1-s^2)}}$$

So we get that

$$N(s) = \frac{T'(s)}{|T'(s)|}$$

$$= \left(\frac{1}{4\sqrt{1+s}}, \frac{1}{4\sqrt{1-s}}, 0\right) \cdot 2\sqrt{2(1-s^2)}$$

$$= \left(\frac{\sqrt{2(1-s^2)}}{2\sqrt{1+s}}, \frac{\sqrt{2(1-s^2)}}{2\sqrt{1-s}}, 0\right)$$

$$= \left(\frac{\sqrt{-(1-s)}}{\sqrt{2}}, \frac{\sqrt{-(1+s)}}{\sqrt{2}}, 0\right)$$

We know that

$$B(s) = T(s) \times N(s)$$

$$= \left(\frac{\sqrt{1+s}}{2}, -\frac{\sqrt{1-s}}{2}, \frac{1}{\sqrt{2}}\right) \times \left(\frac{\sqrt{-(1-s)}}{\sqrt{2}}, \frac{\sqrt{-(1+s)}}{\sqrt{2}}, 0\right)$$

$$= \left(0, 0, \frac{\sqrt{(-1-s)(-s+1)} - \sqrt{(-1+s)(s+1)}}{4\sqrt{2(1-s^2)}}\right)$$

3. (a) Show that the curve given by

$$\gamma(s) = \left(\frac{4}{5}\cos s, 1 - \sin s, -\frac{3}{5}\cos s\right)$$

determines a circle.

Solution: We have that

$$T(s) = \gamma'(s)$$
$$= \left(-\frac{4}{5}\sin s, -\cos s, \frac{3}{5}\sin s\right)$$

Then we have that

$$T'(s) = \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right)$$

and

$$|T'(s)| = \sqrt{\left(-\frac{4}{5}\cos s\right)^2 + (\sin s)^2 + \left(\frac{3}{5}\cos s\right)^2}$$
$$= \sqrt{\cos^2 s + \sin^2 s}$$
$$= 1$$
$$= \kappa(s)$$

So we get that $N(s) = T'(s)$

We know that

$$B(s) = T(s) \times N(s)$$
$$= \left(-\frac{4}{5}\sin s, -\cos s, \frac{3}{5}\sin s\right) \times \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right)$$
$$= \left(\frac{-3}{5}, 0, \frac{-4}{5}\right)$$

and

$$B'(s) = (0, 0, 0)$$

For the equation $B'(s) = -\tau(s)N(s)$ to hold it must be the case that $\tau = 0$.

Since $\tau = 0$ and $\kappa = 1$ this curves must determine a circle.

(b) Find it's centre and radius.

Solution: We know that
$$r = \frac{1}{\kappa} = 1$$

and

$$c = \gamma + \frac{1}{\kappa}N$$
$$= \left(\frac{4}{5}\cos s, 1 - \sin s, -\frac{3}{5}\cos s\right) + \left(-\frac{4}{5}\cos s, \sin s, \frac{3}{5}\cos s\right)$$
$$= (0, 1, 0)$$

4. Consider a unit speed curve α, with $\kappa > 0$ and $\tau \neq 0$, which lies on the sphere centered at $c \in \mathbb{R}$ of radius r.

(a) Show that
$$\alpha - c = -\rho N - \rho' \sigma B$$

where $\rho = \frac{1}{\kappa}$ and $\sigma = \frac{1}{\tau}$.

Solution: We have that
$$(\alpha - c)^2 = \alpha^2 - 2c\alpha + c^2 = r^2$$

Differentiating both sides w.r.t to s gives us that

$$2\alpha'\alpha - 2c\alpha' = 0$$
$$2\alpha'(\alpha - c) = 0$$
$$\alpha'(\alpha - c) = 0$$
$$T(\alpha - c) = 0 \qquad\qquad \text{as } T = \alpha'$$

Differentiating the above again w.r.t s, we get that

$$T'\alpha + T\alpha' - cT' = 0$$
$$T'(\alpha - c) + T^2 = 0$$

Substituting $T' = \kappa N$, we get

$$\kappa N(\alpha - c) + 1 = 0$$

Differentiating again w.r.t s,

$$\kappa' N(\alpha - c) + \kappa N'(\alpha - c) + \kappa N\alpha' = 0$$
$$\kappa' N(\alpha - c) + \kappa N'(\alpha - c) = 0 \qquad\qquad \text{as } N\alpha' = NT = 0$$
$$-\frac{\kappa'}{\kappa} + \kappa N'(\alpha - c) = 0 \qquad\qquad \text{as } N(\alpha - c) = -\frac{1}{\kappa}$$

Substituting $N' = -\kappa T + \tau B$

$$-\frac{\kappa'}{\kappa} - \kappa^2 T(\alpha - c) + \kappa\tau B(\alpha - c) = 0$$
$$-\frac{\kappa'}{\kappa} + \kappa\tau B(\alpha - c) = 0 \qquad\qquad \text{as } T(\alpha - c) = 0$$
$$B(\alpha - c) = \frac{k'}{k^2 \tau}$$
$$= \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}$$

We now have the following equations:

$$T(\alpha - c) = 0$$
$$N(\alpha - c) = -\frac{1}{\kappa}$$
$$B(\alpha - c) = \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}$$

These 3 equations express the components of the radial vector $\alpha - c$ in the orthonormal frame T, N, B, which gives us that

$$\alpha - c = -\frac{1}{\kappa}N + \left(\frac{1}{\kappa}\right)' \frac{1}{\tau}B$$
$$= -\rho N - \rho'\sigma B$$

(b) Deduce a formula for r in terms of ρ and σ.

Solution: From the part above we know that

$$r^2 = (\alpha - c)^2$$
$$= (-\rho N - \rho'\sigma B)^2$$

Since $\|N\| = \|B\| = 1$ and $NB = 0$, the above equation reduces to

$$
\begin{aligned}
r^2 &= \frac{1}{\kappa^2} + \left(\left(\frac{1}{\kappa}\right)' \frac{1}{\tau}\right)^2 \\
&= \frac{1}{\kappa^2} + \left(-\frac{\kappa'}{\kappa^2}\frac{1}{\tau^2}\right) \\
&= \frac{\kappa'^2 + \kappa^2\tau^2}{\kappa^4\tau^2} \\
r &= \frac{1}{\kappa^2\tau}\sqrt{\kappa'^2 + \kappa^2\tau^2}
\end{aligned}
$$

(c) For an arbitrary unit speed curve with $\kappa > 0$ and $\tau \neq 0$, specify conditions on ρ and σ which would guarantee that α lies on a sphere of radius r.

Solution: We assert that the condition

$$
\frac{\tau}{\kappa} \equiv \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\kappa'}{\kappa^2\tau}\right) = -\frac{\mathrm{d}}{\mathrm{d}s}(\rho'\sigma)
$$

is sufficient to guarantee this property. Define

$$
c(s) := \alpha(s) + \rho(s)N(s) + \rho'(s)\sigma(s)B(s) \tag{1}
$$

$$
r^2(s) := \rho(s)^2 + \rho'(s)^2\sigma(s)^2 \tag{2}
$$

These will define the centre and the square of the radius of the circle, respectively, provided they are both constant. Differentiating (1) with respect to s, we obtain

$$
\begin{aligned}
c' &= \alpha' + \rho'N + \rho N' + \frac{\mathrm{d}}{\mathrm{d}s}(\rho'\sigma)B + (\rho'\sigma)B' \\
&= T + \rho'N + \rho(-\kappa T + \tau B) + \frac{\mathrm{d}}{\mathrm{d}s}(\rho'\sigma)B + (\rho'\sigma)(-\tau N) \\
&= \left(\frac{\rho}{\sigma} + \frac{\mathrm{d}}{\mathrm{d}s}(\rho'\sigma)\right)B = \left(\frac{\tau}{\kappa} - \frac{\tau}{\kappa}\right)B \equiv 0
\end{aligned}
$$

And thus c is constant, as required. Likewise, differentiating (2) with respect to s, we also obtain

$$
\begin{aligned}
r^{2\prime} &= \frac{\mathrm{d}}{\mathrm{d}s}(\rho^2 + (\rho'\sigma)^2) = 2\rho\rho' + 2\rho'\sigma\frac{\mathrm{d}}{\mathrm{d}s}(\rho'\sigma) \\
&= 2\rho\rho' - 2\rho'\sigma\frac{\rho}{\sigma} = 2\rho\rho' - 2\rho\rho' = 0
\end{aligned}
$$

So r^2, and thus r, is constant, as required. Overall, our initial assertion is sufficient to guarantee that α lie on a sphere with centre and radius as defined.