MT451P - Assignment 4

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1. Compute the shape operator and Gaussian and mean curvatures for:

   (a) The round sphere of radius r.

Solution: Let $\Sigma = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = r^2\}$ and $p \in \Sigma$. We have that

$$N = -\frac{1}{r}(x, y, z)$$

Then define $\gamma(t) = p + t\vec{v}$ for $\vec{v} \in T_p\Sigma$ such that $\vec{v} = \gamma'(0)$. Then we get that

$$\begin{aligned} S(\vec{v}) = \nabla_{\vec{v}} N &= \frac{d}{dt}\bigg|_{t=0} N(\gamma(t)) \\ &= \frac{d}{dt}\bigg|_{t=0} \left(\frac{1}{r}(p_1 + tv_1, p_2 + tv_2, p_3 + tv_3)\right) \\ &= \frac{1}{r}(v_1, v_2, v_3) = \frac{\vec{v}}{r} \end{aligned}$$

and

$$\text{Matrix } S = \begin{pmatrix} -\frac{1}{r} & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}$$

From this we get that

$$\begin{aligned} K(p) &= \det(S) = \frac{1}{r^2} \\ H(p) &= \text{Trace}(S) = -\frac{2}{r} \end{aligned}$$

   (b) The round cylinder of radius r.

Solution: Let Σ = round cylinder of radius r. Then $T_p\Sigma$ is spanned by the basis vectors $\vec{e}_1$ and $\vec{e}_2$. If we apply the shape operator to both of these vectors we would get that

$$S(\vec{e}_1) = \vec{0}$$

as the cylinder is centred around the x-axis and it would not have any extrinsic curvature.

$$S(\vec{e}_2) = -\frac{\vec{e}_2}{r}$$

For $\vec{v} \in T_p\Sigma$,

$$\begin{aligned} S(\vec{v}) &= S(v_1\vec{e}_1, v_2\vec{e}_2) \\ &= v_1(\vec{0}) + v_2\left(-\frac{\vec{e}_2}{r}\right) \\ &= -\frac{v_2\vec{e}_2}{r} \end{aligned}$$

and

$$\text{Matrix } S = \begin{pmatrix} 0 & 0 \\ 0 & -\frac{1}{r} \end{pmatrix}$$

From this we get that

$$\begin{aligned} K(p) &= \det(S) = 0 \\ H(p) &= \text{Trace}(S) = -\frac{1}{r} \end{aligned}$$

(c) Any flat plane in $\mathbb{R}^3$.

Solution: Let Σ be any flat plane in $\mathbb{R}^3$. Since the shape operator is a measure of curvature and since flat planes have no curvature this must mean that $S = \vec{0}$. This implies that both K and H are both zero as well.

2. Assume that the surface Σ is described locally near p as the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$, with $p = (0, 0, 0)$. Further suppose that f satisfies

$$f_x(0, 0) = f_y(0, 0) = 0$$

Then prove that the shape operator, S, of Σ at $p = (0, 0, 0)$, has matrix (in the standard basis) of the following form:

$$\text{Matrix } S = \begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix}$$

Solution: A local parametrization around around p is given by

$$\varphi : (u, v) \to (u, v, f(u, v))$$

Then the normal vector field is given by

$$N = \frac{\varphi_u \times \varphi_v}{|\varphi_u \times \varphi_v|} = \frac{(-f_u, -f_v, 1)}{\sqrt{1 + f_u^2 + f_v^2}}$$

Define $\alpha = \sqrt{1 + f_u^2 + f_v^2}$. Then

$$\nabla_{\vec{e}_1} N = \left(\frac{\partial}{\partial u}\left(\frac{-f_u}{\alpha}\right), \frac{\partial}{\partial u}\left(\frac{-f_v}{\alpha}\right), \frac{\partial}{\partial u}\left(\frac{1}{\alpha}\right) \right)$$
$$\nabla_{\vec{e}_2} N = \left(\frac{\partial}{\partial v}\left(\frac{-f_u}{\alpha}\right), \frac{\partial}{\partial v}\left(\frac{-f_v}{\alpha}\right), \frac{\partial}{\partial v}\left(\frac{1}{\alpha}\right) \right)$$

The derivative will be similar for all of them so we will only work out 1 of them.

$$\left.\frac{\partial}{\partial u}\right|_{p=0} \left(\frac{-f_u}{\alpha}\right) = \left.\frac{-\alpha f_{uu} + \alpha' f_u}{\alpha^2}\right|_{p=0} = -f_{uu}(0,0)$$

Finally we get that

$$-\nabla_{v_1\vec{e}_1 + v_2\vec{e}_2} N = v_1\left(f_{uu}(0,0), f_{uv}(0,0), 0\right) + v_2\left(f_{vu}(0,0), f_{vv}(0,0), 0\right)$$

which can be represented as

$$\begin{bmatrix} f_{xx}(0,0) & f_{xy}(0,0) \\ f_{yx}(0,0) & f_{yy}(0,0) \end{bmatrix}$$

3. In each case assume that the surface Σ is described locally near p as the graph of the function $f : \mathbb{R}^2 \to \mathbb{R}$, with $p = (0,0,0)$. Compute the shape operator, S, at p and then the Gaussian and mean curvatures of Σ at p in each case.

   (a) $f(x, y) = x^2 + 3y^2$.

Solution: First we check if f satisfies the condition mentioned in Q2. We have that $f_x = 2x$ and $f_y = 6y$, so $f_x(0,0) = 0 = f_y(0,0)$. Then from Q2 we get that

$$\text{Matrix } S = \begin{bmatrix} 2 & 0 \\ 0 & 6 \end{bmatrix}$$

and

$$K(p) = \det(S) = 12$$
$$H(p) = \text{Trace}(S) = 8$$

(b) $f(x, y) = x^2 - y^2$.

Solution: First we check if f satisfies the condition mentioned in Q2. We have that $f_x = 2x$ and $f_y = -2y$, so $f_x(0,0) = 0 = f_y(0,0)$. Then from Q2 we get that

$$\text{Matrix } S = \begin{bmatrix} 2 & 0 \\ 0 & -2 \end{bmatrix}$$

and

$$K(p) = \det(S) = 4$$
$$H(p) = \text{Trace}(S) = 0$$

(c) $f(x, y) = xy$.

Solution: First we check if f satisfies the condition mentioned in Q2. We have that $f_x = y$ and $f_y = x$, so $f_x(0,0) = 0 = f_y(0,0)$. Then from Q2 we get that

$$\text{Matrix } S = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$K(p) = \det(S) = -1$$
$$H(p) = \text{Trace}(S) = 0$$

(d) $f(x, y) = x^3 + y^2$.

Solution: First we check if f satisfies the condition mentioned in Q2. We have that $f_x = 3x^2$ and $f_y = 2y$, so $f_x(0,0) = 0 = f_y(0,0)$. Then from Q2 we get that

$$\text{Matrix } S = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$K(p) = \det(S) = 0$$
$$H(p) = \text{Trace}(S) = 2$$

(e) $f(x,y) = x\left(x + y\sqrt{3}\right)\left(x - y\sqrt{3}\right)$.

Solution: First we check if f satisfies the condition mentioned in Q2. We have that $f_x = 3x^2 + 3y^2$ and $f_y = 6xy$, so $f_x(0,0) = 0 = f_y(0,0)$. Then from Q2 we get that

$$\text{Matrix } S = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$K(p) = \det(S) = 0$$
$$H(p) = \text{Trace}(S) = 0$$

(f) $f(x,y) = y^2$.

Solution: First we check if f satisfies the condition mentioned in Q2. We have that $f_x = 0$ and $f_y = 2y$, so $f_x(0,0) = 0 = f_y(0,0)$. Then from Q2 we get that

$$\text{Matrix } S = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

and

$$K(p) = \det(S) = 0$$
$$H(p) = \text{Trace}(S) = 2$$

4. Suppose M is the cylindrical surface given by the equation $x^2 + y^2 = 100$. Show that the curve

$$\alpha(t) = (10\cos(2t+1), 10\sin(2t+1), 2t+1)$$

is a geodesic on M.

Solution: Consider the following parametrization $\varphi : \mathbb{R}^2 \to M$ given by

$$\varphi(u,v) = (10\cos(2u+1), 10\sin(2u+1), v)$$

Note that

$$\begin{aligned}
\varphi_u &= \left(-20\sin\left(2u+1\right), 20\cos\left(2u+1\right), 0\right) \\
\varphi_v &= \left(0, 0, 1\right) \\
\varphi_u \times \varphi_v &= \left(20\cos\left(2u+1\right), 20\sin\left(2u+1\right), 0\right) \\
\left|\varphi_u \times \varphi_v\right| &= 20\sqrt{\cos^2\left(2u+1\right) + \sin^2\left(2u+1\right)} = 20
\end{aligned}$$

Using the above we get that

$$N = \frac{\varphi_u \times \varphi_v}{\left|\varphi_u \times \varphi_v\right|} = \left(\cos\left(2u+1\right), \sin\left(2u+1\right), 0\right)$$

Observe that $\alpha''(t) = -40 \cdot \left(\cos\left(2t+1\right), \sin\left(2t+1\right), 0\right)$ is a scalar multiple of N, and is hence orthogonal to the surface M. Therefore α is a geodesic.

5. Show that the unit speed geodesics on a round sphere take the form of segments if great circles.

Solution: Let $\alpha : T \to S^2(r)$ be a unit speed geodesic. We know that $|\alpha''(t)| > 0$ as geodesics have constant speed. We may assume W.L.O.G that $N_\alpha = N$. Then we get

$$\begin{aligned}
S(T_\alpha) = \nabla_{T_\alpha} N = \nabla_{T_\alpha} N_\alpha \\
&= -N'_\alpha \\
&= -\left[\kappa T_\alpha + \tau B_\alpha\right]
\end{aligned}$$

We also know from Q1 that

$$S(T_\alpha) = -\frac{T_\alpha}{r}$$

For these two equations to be equal it must be that

$$\kappa = \frac{1}{r}$$

and

$$\tau = 0$$

We have already shown that for a curve with no torsion and $\kappa = {}^1\!/\!_r$ that it is a circle with radius r. The only circles with radius r are the great circles on the surface.