Linear Programming Methods for MDPs

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Quick extensions

The LP Approach to ADP

Deriving the primal LP

Lemma If $J \preceq TJ$ then $J \preceq J^*$.
Proof: $J \preceq TJ \preceq T^2 J \preceq \cdots \preceq T^k J \to J^*$.

J^* is the largest vector satisfying $J \preceq TJ$.
That is, J^* solves the multi-objective problem:

$$\begin{aligned} \text{Maximize} \quad & J(s) \quad \forall s \in \mathcal{S} \\ \text{subject to} \quad & J \preceq TJ \end{aligned}$$

Deriving the primal LP (2)

Introduce state-relevance weights $\nu(s) > 0$ with $\sum_{s \in \mathcal{S}} \nu(s) = 1$

- The specific choice of ν does not affect the solution, but it gives an interpretation of the dual.
- Similarly, that $\sum_{s \in \mathcal{S}} \nu(s) = 1$ helps only in interpreting the dual.

J^* is the unique solution to:

$$\begin{aligned} \text{Maximize} \quad & \nu^\top J \\ \text{subject to} \quad & J \preceq TJ \end{aligned}$$

Deriving the primal LP (3)

The $|\mathcal{S}|$ nonlinear inequalities

$$J(s) \leq TJ(s) = \min_{u \in U(s)} g(s,u) + \alpha \sum_{s' \in \mathcal{S}} p_{ss'}(u) J(s') \qquad \forall s \in \mathcal{S}$$

are equivalent to the $\sum_s |U(s)|$ linear inequalities

$$J(s) \leq g(s,u) + \alpha \sum_{s' \in \mathcal{S}} p_{ss'}(u) J(s') \qquad \forall s \in \mathcal{S},\ u \in U(s).$$

This leads to the final LP formulation of computing J^*:

(Primal) Linear program for computing J^*.

$$\begin{aligned} \text{Maximize} \quad & \sum_{s \in \mathcal{S}} \nu(s) J(s) \\ \text{subject to} \quad & J(s) \leq g(s,u) + \alpha \sum_{s' \in \mathcal{S}} p_{ss'}(u) J(s') \quad \forall s \in \mathcal{S},\ u \in U(s) \end{aligned}$$

Dual LP

(Dual) Linear program

$$\begin{aligned} \text{Minimize} \quad & \frac{1}{1-\alpha} \sum_{s \in \mathcal{S}} \sum_{u \in U(s)} x(s,u) g(s,u) \\ \text{subject to} \quad & \sum_{u \in U(s')} x(s',u) = (1-\alpha)\nu(s') + \alpha \sum_{s \in \mathcal{S}} \sum_{u \in U(s)} x(s,u) P_{ss'}(u) \\ & x(s,u) \geq 0 \end{aligned}$$

You will usually see the from below instead, but I find the term above, where we tranform $x \to (1-\alpha)x$, to be more interpretable.

$$\begin{aligned} \text{Minimize} \quad & \sum_{s \in \mathcal{S}} \sum_{u \in U(s)} x(s,u) g(s,u) \\ \text{subject to} \quad & x(s',u) = \nu(s') + \alpha \sum_{s \in \mathcal{S}} \sum_{u \in U(s)} x(s,u) P_{ss'}(u) \\ & x(s,u) \geq 0 \end{aligned}$$

Interpretation of the dual LP

- Rather than discounting, equivalently think of this as a problem with average cost objective, where in every period there is a fixed probability $1 - \alpha$ that the system "restarts" in a state drawn from ν.
- The variable $x(s, u)$ represents the steady state fraction of the spend playing u in state s.
- The constraint

$$\sum_{u \in U(s')} x(s', u) = (1 - \alpha)\nu(s') + \alpha \sum_{s \in \mathcal{S}} \sum_{u \in U(s)} x(s, u) P_{ss'}(u)$$

  is a balance equation required of a stationary distribution:
  - The LHS is the probability of being in state s'.
  - The RHS is the steady-state "flow" into state s'.

Interpretation of the dual LP (2)

(Dual) Linear program

$$\text{Minimize} \quad \frac{1}{1-\alpha}\sum_{s\in\mathcal{S}}\sum_{u\in U(s)} x(s,u)g(s,u)$$

$$\text{subject to} \quad \sum_{u\in U(s')} x(s',u) = (1-\alpha)\nu(s') + \alpha\sum_{s\in\mathcal{S}}\sum_{u\in U(s)} x(s,u)P_{ss'}(u)$$

$$x(s,u) \geq 0$$

The set of feasible solutions to this LP is precisely the set of all discounted state-occupancy measures

$$x_\mu(s,u) = (1-\alpha)\sum_{t=0}^{\infty}\alpha^t\mathbb{P}_\mu(S_t = s, u_t = u | s_0 \sim \nu)$$

induced by stationary randomized policies μ. The dual LP has the intuitive form of minimizing the geometrically averaged cost incurred among feasible state-action frequencies.

From stationary policies to feasible solutions

Lemma For a stationary (possibly randomized) policy μ, let

$$x_\mu(s,u) = (1-\alpha)\sum_{k=0}^{\infty}\alpha^k \mathbb{P}(s_k = s, u_k = u | s_0 \sim \nu).$$

Then x_μ is a feasible solution.
proof: Define $x_\mu(s) = \sum_{u\in U(s)} x_\mu(s,u)$. Our goal is to show feasibility:

$$\begin{aligned}
x_\mu(s') &= (1-\alpha)\nu(s') + \alpha\sum_{s\in\mathcal{S}}\sum_{u\in U(s)} x_\mu(s,u)P_{ss'}(u) \\
&= (1-\alpha)\nu(s') + \alpha\sum_{s\in\mathcal{S}} x_\mu(s)P_{ss'}(\mu)
\end{aligned}$$

To show this, we will show,

$$\alpha\sum_{s\in\mathcal{S}} x_\mu(s)P_{ss'}(\mu) = x_\mu(s') - (1-\alpha)\nu(s')$$

From stationary policies to feasible solutions (2)

$$\begin{aligned}
\alpha \sum_{s \in \mathcal{S}} x_\mu(s) P_{ss'}(\mu) &= \alpha(1-\alpha) \sum_{s \in \mathcal{S}} \left(\sum_{k=0}^{\infty} \alpha^k \mathbb{P}_\mu(s_k = s | s_0 \sim \nu) \right) P_{ss'}(\mu) \\
&= \alpha(1-\alpha) \sum_{k=0}^{\infty} \alpha^k \left(\sum_{s \in \mathcal{S}} \mathbb{P}_\mu(s_k = s | s_0 \sim \nu) P_{ss'}(\mu) \right) \\
&= \alpha(1-\alpha) \sum_{k=0}^{\infty} \alpha^k \mathbb{P}_\mu(s_{k+1} = s' | s_0 \sim \nu) \\
&= (1-\alpha) \sum_{k=0}^{\infty} \alpha^{k+1} \mathbb{P}_\mu(s_{k+1} = s' | s_0 \sim \nu) \\
&= (1-\alpha) \sum_{k=1}^{\infty} \alpha^k \mathbb{P}_\mu(s_k = s' | s_0 \sim \nu) \\
&= \underbrace{(1-\alpha) \sum_{k=0}^{\infty} \alpha^k \mathbb{P}_\mu(s_k = s' | s_0 \sim \nu)}_{x_\mu(s')} - (1-\alpha)\nu(s').
\end{aligned}$$

From feasible solutions to stationary policies

Lemma: Let $x(s,u)$ be a feasible solution to the dual LP. Let $\mu(s,u) = \frac{x(s,u)}{\sum_{u' \in U(s)} x(s,u')}$. Then,

$$x(s,u) = (1-\alpha)\sum_{k=0}^{\infty} \alpha^k \mathbb{P}_\mu(s_k = s, u_k = u | s_0 \sim \nu) := x_\mu(s,u)$$

Proof: Define $x(s) = \sum_{u \in U(s)} x(s,u)$ and $x_\mu(s) = \sum_{u \in U(s)} x_\mu(s,u)$.

$$\begin{aligned}
x(s') &= (1-\alpha)\nu(s') + \alpha \sum_{s \in \mathcal{S}} \sum_{u \in U(s)} x(s,u) P_{ss'}(u) \\
&= (1-\alpha)\nu(s') + \alpha \sum_{s \in \mathcal{S}} x(s) \sum_{u \in U(s)} \left(\frac{x(s,u)}{\sum_{u' \in U(s)} x(s,u')} \right) P_{ss'}(u) \\
&= (1-\alpha)\nu(s') + \alpha \sum_{s \in \mathcal{S}} x(s) \sum_{u \in U(s)} \mu(s,u) P_{ss'}(u) \\
&= (1-\alpha)\nu(s') + \alpha \sum_{s \in \mathcal{S}} x(s) P_{ss'}(\mu)
\end{aligned}$$

In vector notation, $x^\top = (1-\alpha)\nu^\top + \alpha x^\top P_\mu$, or $x = (1-\alpha)\nu^\top (I - \alpha P_\mu)^{-1} = x_\mu \in \mathbb{R}^{|\mathcal{S}|}$.

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The LP Approach to ADP

Extension: Dual LP for average cost objectives

In a finite state, average cost problem, the goal is to minimize

$$\limsup_{K\to\infty} \frac{1}{K}\mathbb{E}\left[\sum_{k=0}^{K} g(s_k, u_k)\right]$$

For unichain MDPs, where each state is reachable from each other state, the optimal objective value is $\lambda^* = \lim_{\alpha\to 1}(1-\alpha)J^*(s)$.

(Dual) Linear program for average cost, unichain, MDPs

$$\begin{aligned}
\text{Minimize} \quad & \sum_{s\in\mathcal{S}}\sum_{u\in U(s)} x(s,u)g(s,u) \\
\text{subject to} \quad & \sum_{u\in U(s')} x(s',u) = \sum_{s\in\mathcal{S}}\sum_{u\in U(s)} x(s,u)P_{ss'}(u) \\
& \sum_{s\in\mathcal{S}}\sum_{u\in U(s)} x(s,u) = 1 \\
& x(s,u) \geq 0
\end{aligned}$$

Extension: Adding constraints

A central advantage of the LP approach is that it is easy to add constraints on steady-state behavior, as in the following LP. Here c represents an alternative cost measure whose average must lie below a threshold $\bar{c}$.

$$
\begin{aligned}
\text{Minimize} \quad & \sum_{s\in\mathcal{S}} \sum_{u\in U(s)} x(s,u) g(s,u) \\
\text{subject to} \quad & \sum_{u\in U(s')} x(s',u) = \sum_{s\in\mathcal{S}} \sum_{u\in U(s)} x(s,u) P_{ss'}(u) \\
& \sum_{s\in\mathcal{S}} \sum_{u\in U(s)} x(s,u) c(s,u) \le \bar{c} \\
& \sum_{s\in\mathcal{S}} \sum_{u\in U(s)} x(s,u) = 1 \\
& x(s,u) \ge 0
\end{aligned}
$$

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The LP Approach to ADP

The LP Approach to Approximate DP

We can approximate MDP solutions by approximating:

1. The primal LP.
   - Since the decision variable is the cost-to-go function, this is a kind of cost-to-go approximation.
2. The dual LP
   - This is more akin to policy approximation. But to my knowledge, current approaches require approximating the set of feasible occupancy-measures.

We'll look at approach 1 as introduced by Schweitzer and Seidmann [1985] and follow the understanding developed in De Farias and Van Roy [2003, 2004].

Cost-to-go approximation in the primal LP

We face the problem

$$\max\{\nu^\top J : TJ \succeq J\}$$

which can be formulated as an LP. Believing $J^* \approx \Phi\theta$, we introduce the auxiliary problem

$$\begin{aligned} \text{Maximize}_\theta \quad & \nu^\top \Phi\theta \\ \text{subject to} \quad & T\Phi\theta \succeq \Phi\theta \end{aligned}$$

We can write this a an LP with decision variable $\theta \in \mathbb{R}^d$:

$$\begin{aligned} \text{Maximize} \quad & (\nu^\top \Phi)\theta \\ \text{Subject to} \quad & g(s,u) + \alpha \sum_{s' \in \mathcal{S}} p_{ss'}(u) \Phi\theta(s') \geq (\Phi\theta)(s) \quad \forall s \in \mathcal{S},\, u \in U(s). \end{aligned}$$

Constraint sampling

- A first issue is that the LP has $|\mathcal{S}| \times |U|$ constraints, and we have in mind problems where $|\mathcal{S}|$ is enormous.
- De Farias and Van Roy [2004] provide theoretical analysis of a constraint sampling approach, where we impose only a randomly sampled subset of the constraints.
- The intuition is that most constraints are irrelevant, and under some conditions, many provide nearly redundant information. Recall that basic feasible solution is described by $d << |\mathcal{S}|$ active linearly independent constraints.

Approximation guarantee and feature selection

Assume we can exactly solve the problem:

$$\begin{aligned} \mathrm{Maximize}_{\theta} \quad & \nu^{\top}\Phi\theta \\ \text{subject to} \quad & T\Phi\theta \succeq \Phi\theta \end{aligned}$$

In what sense are we approximating J^*?

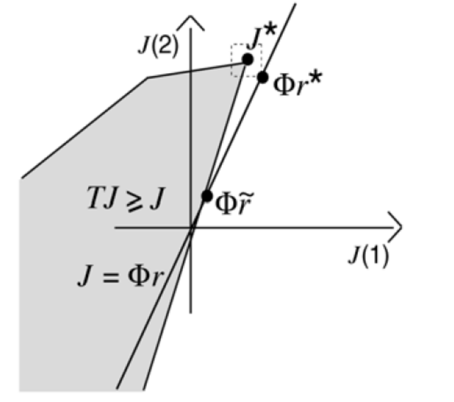
First Lemma: A vector $\tilde{\theta}$ solves the optimization problem above if and only if it solves

$$\begin{aligned} \mathrm{Minimize}_{\theta} \quad & \|J^* - \Phi\theta\|_{1,\nu} \\ \text{subject to} \quad & T\Phi\theta \succeq \Phi\theta \end{aligned}$$

Why? Subtract $\nu^{\top} J^*$ from the objective. The constraint implies $\Phi\theta \preceq J^*$, so $\nu^{\top}(\Phi\theta - J^*) = -\|J^* - \Phi\theta\|_{1,\nu}$.

Quality of the approximation

- In general, we cannot (even nearly) achieve $\min_\theta \|J^* - \Phi\theta\|_{1,\nu}$.
- There may be no *feasible* solution close to the best-approximate cost-to-go function.



A simple error bound and feature selection

Theorem Suppose e is in the span of the columns of Φ. Then, the approximate LP is feasible. If $\tilde{\theta}$ is an optimal solution to the approximate LP,

$$\|J^* - \Phi\tilde{\theta}\|_{1,\nu} \leq \frac{2}{1-\alpha} \min_{\theta} \|J^* - \Phi\theta\|_\infty$$

Proof of feasibility: Let $\Phi\hat{\theta} = e$. For any θ consider a new $\theta_\gamma = \theta - \gamma\hat{\theta}$. We have

$$\Phi\theta_\gamma = \Phi\theta - \gamma e$$

$$T\Phi\theta_\gamma = T(\Phi\theta - \gamma e) = T\Phi\theta - \alpha\gamma e$$

For $\gamma \geq \|\Phi\theta - T\Phi\theta\|_\infty/(1-\alpha)$.

$$T\Phi\theta_\gamma - \Phi\theta_\gamma = (\Phi\theta - T\Phi\theta) - (1-\alpha)\gamma e \succeq 0$$

Proof

Take $\theta^* = \text{argmin}_\theta \|\Phi\theta - J^*\|_\infty$ and $\epsilon = min_\theta \|\Phi\theta - J^*\|_\infty$.

Show

$$\|T\Phi\theta^* - J^*\|_\infty \leq \alpha\|\Phi\theta - J^*\|_\infty = \alpha\epsilon$$

Then,

$$\begin{aligned} T\left(\Phi(\theta^* - \gamma\hat{\theta})\right) &= T\Phi\theta^* - \alpha\gamma e \\ &\geq J^* - \alpha\epsilon e - \alpha\gamma e \\ &\geq \Phi\theta^* - \epsilon e - \alpha\epsilon e - \alpha\gamma e. \quad = \Phi\left(\theta^* - \gamma\hat{\theta}\right) + (1-\alpha)\gamma e \end{aligned}$$

This ensures $\theta^* - \frac{1+\alpha}{1-\alpha}\hat{\theta}$ feasible. The triangle inequality can be used to show

$$\|\Phi\left(\theta^* - \frac{1+\alpha}{1-\alpha}\hat{\theta}\right) - J^*\|_\infty \leq \frac{2}{1-\alpha}\epsilon.$$

Since there exists a feasible solution close to J^* in infinity norm, the optimal solution in $\|\cdot\|_{1,\nu}$ must be at least that close.

Choice of basis and state-relevance weights

What is special about e? It acts a Lyapunov function: $e \succeq 0$ and

$$Te \succeq \kappa e \qquad \text{where } \kappa < 1.$$

The natural choice of state-relevance weights are uniform. Other choices work better in other problems.

Warmup Example: Autonomous Queue

An N state autonomous queue has state dynamics

$$s_{t+1} = \begin{cases} \min\{x_t + 1, N-1\} & \text{with prob. } p \\ \max\{x_t - 1\,, 0\} & \text{otherwise} \end{cases}$$

and no decision-variables. The stationary probabilities are $\pi(x) = \pi(0)\left(\frac{p}{1-p}\right)^x$ for $x = 0, \cdots N-1$. If $V(X) = x^2 + 2/(1-\alpha)$, then

$$TV \preceq \frac{1+\alpha}{2} V$$

De Farias and Van Roy [2003] suggest to choose $\nu = \pi$ and to include the basis vectors $\phi(x) = x^2$ and $\phi(x) = 1$. See the paper for full theory.

Daniela Pucci De Farias and Benjamin Van Roy. The linear programming approach to approximate dynamic programming. *Operations research*, 51(6):850–865, 2003.

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Paul J Schweitzer and Abraham Seidmann. Generalized polynomial approximations in markovian decision processes. *Journal of mathematical analysis and applications*, 110(2):568–582, 1985.