Policy Iteration and Approximations

Rollout and approximate policy iteration

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Policy iteration

Policy iteration solves the $\min_\mu J_\mu$ *by solving a sequence of single period problems* $\mu_{k+1} \in \operatorname{argmin}_\mu T_\mu J_{\mu_k}$

- For $k = 0, 1, 2 \cdots$
  1. Policy evaluation:
  $$J_{\mu_k} = g_{\mu_k} + \alpha P_{\mu_k} J_{\mu_k}$$
  2. Policy improvement: $\mu_{k+1} \in G(J_{\mu_k}) = \{\mu : T_\mu J_{\mu_k} = T J_{\mu_k}\}$, i.e.
  $$\mu_{k+1}(s) \in \operatorname*{argmin}_{u \in U(s)} g_{\mu_k}(s,u) + \alpha \sum_{s' \in \mathcal{S}} P_{s,s'}(u) J_{\mu_{k+1}}(s') \qquad \forall s \in \mathcal{S}$$

Policy iteration with Q functions

Define the state-action cost-to-go function

$$Q_\mu(s,u) = g(s,u) + \alpha \sum_{s'} P_{ss'}(u) J_\mu(s')$$

This satisfies the Bellman equation:

$$Q_\mu(s,u) = g(s,u) + \alpha \sum_{s'} P_{ss'}(u)) Q_\mu(s', \mu(s'))$$

Approximate PI:

- For $k = 0, 1, 2 \cdots$
  1. Policy evaluation:

     $$Q_{\mu_k}(s,u) = g(s,u) + \alpha \sum_{s'} P_{ss'}(u) Q_{\mu_k}(s', \mu_k(s')) \qquad \forall s, u$$

  2. Policy improvement:

     $$\mu_{k+1}(s) \in \operatorname*{argmin}_{u \in U(s)} Q_{\mu_k}(s,u) \qquad \forall s \in \mathcal{S}$$

Policy improvement property

Each step of policy iteration produces and improved policy, and the improvement is strict until an optimal policy is reached:

$$J_{\mu_{k+1}} = J_{\mu_k} \iff J_{\mu_k} = TJ_{\mu_k} \iff J_{\mu_k} = J^*.$$

Lemma $J_{\mu_{k+1}} \preceq TJ_{\mu_k} \preceq J_{\mu_k}$

Proof.
$J_{\mu_k} = T_{\mu_k} J_{\mu_k} \succeq TJ_{\mu_k} = T_{\mu_{k+1}} J_{\mu_k} \succeq T^2_{\mu_{k+1}} J_{\mu_k} \succeq \cdots \succeq J_{\mu_{k+1}}.$ $\square$

Classic convergence analysis of policy iteration

- Assume the state space is finite.
- As long as the current policy is suboptimal, policy iteration produces a strictly better policy.
- Since there are only finitely many policies, *policy iteration will reach an optimal policy within a finite number of iterations*.

A simple convergence rate

Policy iteration is at least as fast as value iteration.

Lemma: $\|J_{\mu_k} - J^*\|_\infty \leq \alpha^k \|J_{\mu_0} - J^*\|_\infty$.

Proof.
Since $J_{\mu_k} \preceq TJ_{\mu_{k-1}}$.

$$J_{\mu_k} - J^* \preceq TJ_{\mu_{k-1}} - J^* = TJ_{\mu_{k-1}} - TJ^*$$

Since both sides are non-negative, taking the max-norm gives

$$\|J_{\mu_k} - J^*\|_\infty \leq \|TJ_{\mu_{k-1}} - TJ^*\|_\infty \leq \alpha \|J_{\mu_{k-1}} - J^*\|.$$

The result follows by induction. $\square$

Policy iteration iteration vs. value iteration

- Each iteration of policy iteration is more costly than value iteration
  - It requires evaluating the cost-to-go function J_{μ_k} (e.g. by solving a linear system, with cost $O(|\mathcal{S}|^3)$.
- Practical experience suggests policy iteration often converges in very few iterations.
  - But this is not necessarily true. Your homework includes a "bad example" for PI where it requires as many iterations as states.
  - We'll look at a bit of theory suggesting why it sometimes converges in few iterations.

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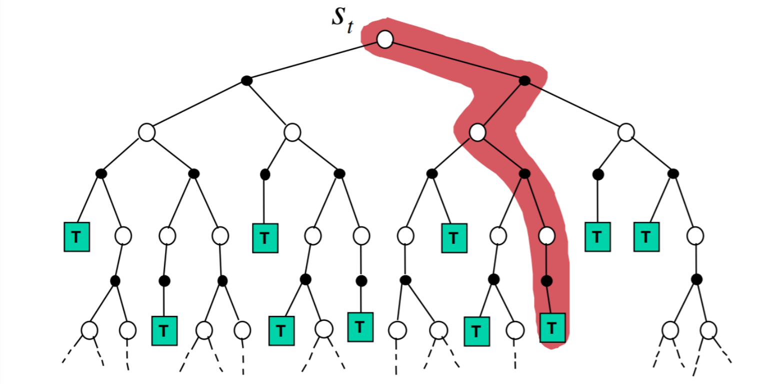
A success story

Rollout

Choose which action to take at the current state by lookahead.



Rollout (one-period lookahead)

- We have a base policy $\bar{\mu}$.
- At time k, in state s_k, we select

$$u_k \in \operatorname*{argmin}_u Q_{\bar{\mu}}(s_k, u)$$

- But we do this without storing the function Q_μ. How?
  - We can evaluate each control u by simulating many trajectories in which we first apply u and apply $\bar{\mu}$ thereafter:

$$Q_\mu(s,u) = g(s,u) + \alpha \sum_{s' \in \mathcal{S}} P_{ss'}(u) J_\mu(s')$$
$$= \mathbb{E}\left[\sum_{k=0}^{\infty} \alpha^k g(s_k, u_k) : u_0 = u, s_0 = s,\ u_k = \bar{\mu}(s_k)\ k > 0\right]$$

- This is done at decision-time, once s_k is observed.
  - No need to compute a full policy iteration update. Just lookahead in the current subproblem.

Policy improvement property

Rollout can only improve the base policy.

- A single period Rollout is a policy iteration update
  - If at decision time, we apply single period rollout to the base policy $\bar{\mu}$, then our decision policy is the policy iteration update $\mu^{+} \in G(J_{\bar{\mu}})$.
- It follows that $J_{\mu^{+}} \preceq TJ_{\bar{\mu}} \preceq J_{\bar{\mu}}$.

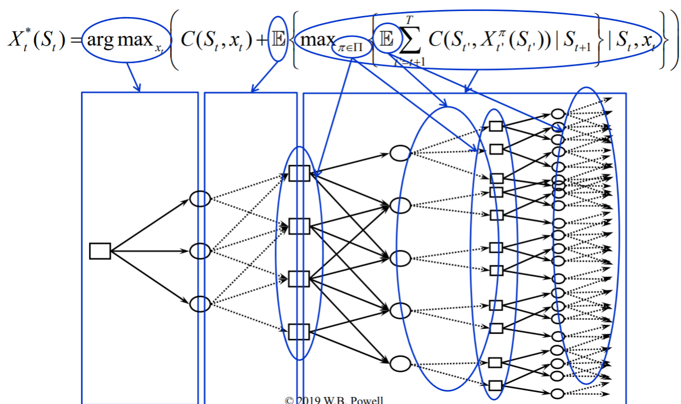
Multi-period rollout

At time k apply the control that is optimal under m-step lookahead:

$$u_k = \operatorname*{argmin}_{u \in U(s_k)} g(s_k, u) + \alpha \sum_{s'} P_{s_k, s'}(u) T^m J_{\bar{\mu}}(s')$$

$$= \operatorname*{argmin}_{u \in U(s_k)} g(s_k, u) + \alpha \sum_{s'} P_{s_k, s'}(u) \min_{\mu_1, \cdots \mu_m} T_{\mu_1} \cdots T_{\mu_m} J_{\bar{\mu}}(s')$$

Picture from Warren Powell. . . uses different notation. . .



Multi-period rollout

Multi-period rollout involves nested single period rollouts.
It's feasible when the ("branching factor") number of actions and successor states is not too large.

View 1-period rollout as an approximation to $\min_u Q_{\bar{\mu}}(s, u)$

We could in principle querry this algorithm many times.

Single Period Rollout

Input: $\bar{\mu}$, current state s

Approximate $\hat{Q}(s, u) \approx Q_\mu(s, u)$ for all $u \in U(S)$ by simulation.

Return: $\mathrm{argmin}_u \hat{Q}(s, u)$, $\min_u \hat{Q}(s, u)$

Computing a two-period rollout

At time k we want to apply the control

$$
\begin{aligned}
u_k &= \operatorname*{argmin}_{u \in U(s_k)} g(s_k, u) + \alpha \sum_{s'} P_{s_k, s'}(u) \min_{\mu} T_{\mu} J_{\bar{\mu}}(s') \\
&= \operatorname*{argmin}_{u \in U(s_k)} g(s_k, u) + \alpha \sum_{s'} P_{s_k, s'}(u) \min_{u; \in U(s')} Q_{\bar{\mu}}(s', u)
\end{aligned}
$$

Simulation based approximation

1. For each $u \in U(s_k)$. Draw N successor states $s_u^1, \cdots s_u^N$
2. For each *unique* successor state sampled s' approximate $\min_{u'} \hat{Q}_{\bar{\mu}}(s', u')$ by single-period rollout.
3. Find $\operatorname{argmin}_{u \in U(s_k)} N^{-1} \sum_{i=1}^{N} \left[g(s_k, u) + \alpha \min_{u'} \hat{Q}_{\bar{\mu}}(s_u^i, u') \right]$

Multi-period policy improvement property

At time k apply the control that is optimal under $m+1$-step lookahead:

$$\mu_R(s_k) = \operatorname*{argmin}_{u \in U(s_k)} g(s_k, u) + \alpha \sum_{s'} P_{s_k, s'}(u) T^m J_{\bar{\mu}}(s')$$

The rollout policy μ_R satisfies $\mu_R \in G(T^m J_{\bar{\mu}})$, i.e

$$T_{\mu_R}(T^m J_{\bar{\mu}}) = T(T^m J_{\bar{\mu}})$$

Akin to the analysis of policy iteration, one can show:

$$J_{\mu_R} \preceq T^{m+1} J^{\bar{\mu}} \preceq J^{\bar{\mu}}$$

and

$$\|J_{\mu_R} - J^*\|_\infty \leq \alpha^{m+1} \|J_{\bar{\mu}} - J^*\|_\infty.$$

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Making this work in practice (1): Truncated rollouts with cost-to-go approximations

It is common to use K-period simulation with terminal values given by a cost-to-go approximation.

Given approximate cost-to-go function $\hat{J}_\theta \approx J_{\bar{\mu}}$,

$$
\begin{aligned}
Q_\mu(s,u) &= g(s,u) + \alpha \sum_{s' \in \mathcal{S}} P_{ss'}(u) J_\mu(s') \\
&= \mathbb{E}\left[\sum_{k=0}^{\infty} \alpha^k g(s_k,u_k) : u_0 = u, s_0 = s,\ u_k = \bar{\mu}(s_k)\ k > 0\right] \\
&\approx \mathbb{E}\left[\sum_{k=0}^{K-1} \alpha^k g(s_k,u_k) + \alpha^K \hat{J}_\theta(s_K) \mid u_0 = u, s_0 = s,\ u_k = \bar{\mu}(s_k)\right]
\end{aligned}
$$

We can approximate the final expectation by simulation.

Making this work in practice (2:) Planning with an approximate model

As an example, consider the nolinear continuous control problem:

$$\begin{aligned} \min \quad & \mathbb{E} \sum_{k=0}^{\infty} \alpha^k \left[s_k^\top Q s_K + u_k^\top R u_k \right] \\ \text{subject to} \quad & s_{k+1} = f(x_k, u_k, w_k) \\ & u_k \in U(s_k) \end{aligned}$$

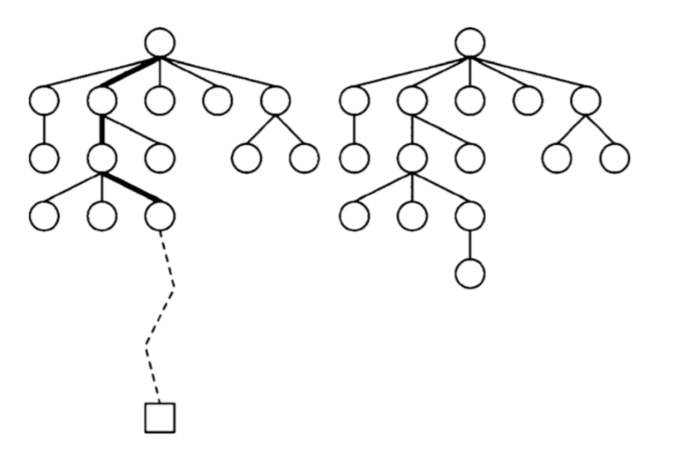
<u>**Model predictive control:**</u> To compute u_k given current state s_k

- Create a locally linear, certainty equivalent (i.e. no w_k) model: $s_{t+1} \approx A s_t + B u_t \quad \text{for } s_t \approx s_k$.
- Return the u_k by solving for a sequence of controls:

$$\begin{aligned} \min_{u_k, \cdots u_M} \quad & \mathbb{E} \sum_{t=k}^{N+k} \alpha^t \left[s_t^\top Q s_t + u_t^\top R u_t \right] \\ \text{subject to} \quad & s_{t+1} = A s_t + B u_t,\ \ s_t = s_k,\ \ s_{k+N} = 0,\ \ u_t \in U(s_t) \end{aligned}$$

Making this work in practice (3:) Selective search depth

Monte Carlo Tree Search



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Policy iteration is faster than value iteration? Connection with Newton's method

PI is Newton's method applied to solving Bellman's equation.

Define the gap in Bellman's equation:

$$B(J) = J - TJ$$

Assuming differentiability, Newton's method is

$$J_{k+1} = J_k - [\nabla B(J_k)]^{-1} B(J_k)$$

Newton's method converges quadratically. That is, for J_k sufficiently close to J^*, we should have

$$\|J_{k+1} - J^*\| \leq C \|J_k - J^*\|^2.$$

Asymptotically faster than the linear rate under value iteration.

Connection with Newton's iteration (a reminder)

Recall this variational form of Bellman's equation:

Lemma: For any $J \in \mathbb{R}^n$ and policy μ,

$$J - J_\mu = (I - \alpha P_\mu)^{-1}(J - T_\mu J).$$

Connection with Newton's iteration (continued)

Fix J and suppose there is a unique greedy policy $G(J) = \{\mu\}$.

For some sufficiently small ϵ

$$\begin{aligned} \|J' - J\| \leq \epsilon &\implies G(J') = \{\mu\} \\ &\implies B(J') = J' - T_\mu J' = (I - \alpha P_\mu) J' + g_\mu. \end{aligned}$$

We find

$$\nabla B(J) = (I - \alpha P_\mu)$$

A Newton step to J produces J_μ:

$$\begin{aligned} J - [\nabla B(J)]^{-1} B(J) &= J - (I - \alpha P_\mu)^{-1} (J - T_\mu J) \\ &= J - (J - J_\mu) \\ &= J_\mu \end{aligned}$$

Hence $\{J_{\mu_k}\}$ is the sequence produced by Newton's method with initial iterate J_{μ_0}.

When is policy iteration fast? . . . an attempt

Each policy iteration step makes enormous progress if that policy visits states with frequency similar to an optimal policy.

Let $u = (1/|\mathcal{S}|, \cdots, 1/|\mathcal{S}|)$ be the uniform distribution.
Recall the discounted state occupancy measure

$$d_\infty^\mu = (1-\alpha)u\left(I - \alpha P_\mu\right)^{-1} = (1-\alpha)\sum_{t=0}^{\infty} \alpha^t u P_\mu^t$$

Define the distribution shift constant

$$c_k = \left\| \frac{d_\infty^{\mu^*}}{d_\infty^{\mu_{k+1}}} \right\|_\infty = \max_s \frac{d_\infty^{\mu^*}(s)}{d_\infty^{\mu_{k+1}}(s)}$$

Proposition: $c_k \leq |\mathcal{S}|/(1-\alpha)$ and

$$\left\| J_{\mu_{k+1}} - J^* \right\|_1 \preceq \left(1 - c_k^{-1}\right) \left\| J_{\mu_k} - J^* \right\|_1$$

When is policy iteration fast?...an attempt (2)

A more precisely quantification of policy improvement.

Lemma: $J_{\mu_k} - J_{\mu_{k+1}} = (I - \alpha P_{\mu_{k+1}})^{-1} (J_{\mu_k} - TJ_{\mu_k})$

Proof.
Apply the variational Bellman eq w/ $J \equiv J_{\mu_k}$ and $\mu \equiv \mu_{k+1}$:

$$\begin{aligned} J_{\mu_k} - J_{\mu_{k+1}} &= (I - \alpha P_{\mu_{k+1}})^{-1} (J_{\mu_k} - T_{\mu_{k+1}} J_{\mu_k}) \\ &= (I - \alpha P_{\mu_{k+1}})^{-1} (J_{\mu_k} - TJ_{\mu_k}) \end{aligned}$$

$\square$

Heuristically at least, this is suggestive of much faster convergence than value iteration:

- $J_{\mu_k} - TJ_{\mu_k}$ is on the order of $\alpha(J_{\mu_k} - J^*)$.
- $(I - \alpha P_{\mu_k})^{-1}$ is on the order of $(1 - \alpha)^{-1}$ if $(I - \alpha P_{\mu_k})^{-1}$ is fairly uniform.

When is policy iteration fast?... an attempt (3)

Lemma: $J_{\mu_k} - J_{\mu_{k+1}} = (I - \alpha P_{\mu_{k+1}})^{-1} (J_{\mu_k} - TJ_{\mu_k})$

Lemma: $J_{\mu_k} - J^* \preceq (I - \alpha P_{\mu^*})^{-1} (J_{\mu_k} - TJ_{\mu_k})$

Proof.

$$\begin{aligned} J_{\mu_k} - J_{\mu^*} &= (I - \alpha P_{\mu^*})^{-1} (J_{\mu_k} - T_{\mu^*} J_{\mu_k}) \\ &\preceq (I - \alpha P_{\mu^*})^{-1} (J_{\mu_k} - TJ_{\mu_k}) . \end{aligned}$$

$\square$

When is policy iteration fast?...an attempt (4)

Lemma: $J_{\mu_k} - J_{\mu_{k+1}} = (I - \alpha P_{\mu_{k+1}})^{-1} (J_{\mu_k} - TJ_{\mu_k})$

Lemma: $J_{\mu_k} - J^* \preceq (I - \alpha P_{\mu^*})^{-1} (J_{\mu_k} - TJ_{\mu_k})$

Proof of Proposition: The definition of c_k gives

$$c_k u(I - \alpha P_{\mu_{k+1}})^{-1} \succeq u(I - \alpha P_{\mu^*})^{-1}.$$

Let $e = (1, 1, \cdots, 1)$ be a column vector of 1's.

$$\begin{aligned}
\|J_{\mu_{k+1}} - J^*\|_1 &= e^\top (J_{\mu_{k+1}} - J^*) \\
&= e^\top \left[J_{\mu_k} - J^* - (I - \alpha P_{\mu_{k+1}})^{-1} (J_{\mu_k} - TJ_{\mu_k}) \right] \\
&\preceq e^\top [J_{\mu_k} - J^*] - c_k^{-1} e^\top \left[(I - \alpha P_{\mu^*})^{-1} (J_{\mu_k} - TJ_{\mu_k}) \right] \\
&\preceq \left(1 - c_k^{-1}\right) e^\top (J_{\mu_k} - J^*) \\
&= \left(1 - c_k^{-1}\right) \|J_{\mu_k} - J^*\|_1
\end{aligned}$$

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Policy iteration reminder

- For $k = 0, 1, 2 \cdots$
  1. Evaluate the current policy:
  $$J_{\mu_k} = g_{\mu_k} + \alpha P_{\mu_k} J_{\mu_k}$$
  2. Policy improvement: $\mu_{k+1} \in G(J_{\mu_k}) = \{\mu : T_\mu J_{\mu_k} = TJ_{\mu_k}\}$:
  $$\mu_{k+1}(s) \in \operatorname*{argmin}_{u \in U(s)} g_{\mu_k}(s, u) + \alpha \sum_{s' \in \mathcal{S}} P_{s,s'}(u) J_{\mu_{k+1}}(s') \qquad \forall s \in \mathcal{S}$$

Approximate policy iteration

- For $k = 0, 1, 2 \cdots$
  1. Approximate the cost-to-go under the current policy $J_{\theta_k} \approx J_{\mu_k}$.
  2. Policy improvement: $\mu_{k+1} \in G(J_{\theta_k}) = \{\mu : T_\mu J_{\mu_k} = TJ_{\mu_k}\}$:

$$\mu_{k+1}(s) \in \operatorname*{argmin}_{u \in U(s)} g_{\mu_k}(s, u) + \alpha \sum_{s' \in \mathcal{S}} P_{s,s'}(u) J_{\theta_k}(s') \qquad \forall s \in \mathcal{S}$$

Approximate Policy iteration with Q functions

Q functions are typically used instead, because it is easy to apply the one-step policy improvement with respect to a Q estimate.

Define the state-action cost-to-go function

$$Q_\mu(s,u) = g_\mu(u) + \alpha \sum_{s'} P_{ss'}(u) Q_\mu(s', \mu(s'))$$

The policy $\operatorname{argmin}_{u \in U(s)} Q_\mu(s,a)$ is a policy iteration update to μ.

Approximate PI:

- For $k = 0, 1, 2 \cdots$
  1. Approximate the cost-to-go under μ_k: $Q_{\theta_k} \approx Q_{\mu_k}$
  2. Solve for an improved policy

$$\mu_{k+1}(s) \in \operatorname*{argmin}_{u \in U(s)} Q_{\theta_k}(s,u) \qquad \forall s \in \mathcal{S}$$

Q_{μ_k} can be approximated by either TD or Monte Carlo methods.

Monte-carlo Q-function approximation

Suppose $\mu_i(s) = \operatorname{argmin}_{u \in U(s)} Q_{\theta_i}(s,u)$. We want to solve

$$\theta_{i+1} = \min_\theta \| Q_\theta - Q_{\mu_i} \|^2_{2,\nu} = \min_\theta \mathbb{E}_{(s,u)\sim\nu} \left[\left(Q_\theta(s,u) - Q_{\mu_i}(s,u) \right)^2 \right].$$

Regression based approximation to Q_μ by simulation μ

For simulation replications $m = 1, 2, \cdots, M$

- Sample $(s_0^m, u_0^m) \sim \nu$ and horizon $H_m \sim \mathrm{Geom}(1-\alpha)$.
- Apply u_0 and observe the next state s_1^m.
- For $k = 1, 2, \cdots, H_m$
  - Apply control $u_k^m = \operatorname{argmin}_u Q_{\theta_i}(s_k^m, u)$ and observe s_{k+1}^m.
- Set $G^m = \sum_{k=0}^{H_m} \alpha^k g_\mu(s_k^m)$

Solve $\min_\theta \frac{1}{M} \sum_{m=1}^{M} \left(Q_\theta(s_0^m, u_0^m) - G^m \right)^2$.

Analysis of approximate PI (J_{θ_i} version)

Lemma: Let $\epsilon = \max_{i \leq k} \|J_{\theta_i} - J_{\mu_i}\|_\infty$. Then

$$\|J_{\mu_k} - J^*\|_\infty \leq \alpha^k \|J_{\mu_0} - J^*\|_\infty + \frac{\alpha\epsilon}{(1-\alpha)^2}$$

Proof $J_{\theta_i} \preceq J_{\mu_i} + \epsilon e$. By the definition of μ_{i+1} and the monotonicity/constant shift properties

$$T_{\mu_{i+1}} J_{\theta_i} = T J_{\theta_i} \preceq T\left(J_{\mu_i} + \epsilon e\right) = T J_{\mu_i} + \alpha\epsilon e.$$

Applying $T_{\mu_{i+1}}$ repeatedly to each side gives

$$J_{\mu_{i+1}} \preceq T J_{\mu_i} + \frac{\alpha\epsilon}{1-\alpha}$$

Subtracting J^* and the LHS and TJ^* on the RHS gives

$$\|J_{\mu_{i+1}} - J^*\|_\infty \leq \|TJ_{\mu_i} - TJ^*\|_\infty + \frac{\epsilon\alpha}{1-\alpha} \leq \alpha\|J_{\mu_i} - J^*\|_\infty + \frac{\epsilon\alpha}{1-\alpha}$$

We now apply this recursively.

Policy chattering in approximate policy iteration

Approximate policy iteration often does not converge. Instead, it cycles endlessly between a set of policies.

I will illustrate the issue in a simple case with on policy sampling

Example: Suppose we have 1 state with 2 actions. So we write $Q(u) = Q^*(s, u)$ for $u = 1, 2$,

We approximate $Q^*(u) \approx Q_\theta(u) = \Phi(u)\theta$, $\theta \in \mathbb{R}$.

Suppose, $\alpha = 0$, $Q^* = (g(1), g(2)) = (1, -1)$ and $\Phi = (2, 1)$.

- If $\theta > 0$, we think we should play action 1.
- If $\theta < 0$, we think we should play action 2.

Policy chattering continued

Here we consider on-policy sampling, i.e. ν is determined by simulating the current policy.

API with on-policy sampling

Input: $\theta_0,\ u_0 = \arg\max_u(\Phi\theta_0)(a)$

For $k = 0, 1, 2, \ldots$

- Evaluate policy: $\theta_{k+1} = \arg\min_\theta \left((\Phi\theta)(u_k) - Q(u_k)\right)^2$
- Policy improvement: $u_{k+1} = \arg\min_u(\Phi\theta_k)(u)$

So what happens with $(\theta_0 = -\frac{1}{2})$?

$$\theta_0 = \frac{1}{2} \Rightarrow Q_{\theta_0} = (1, \frac{1}{2}) \Rightarrow u_0 = 2$$

$$\Phi(2)\theta_1 = Q(2) \Rightarrow \theta_1 = -1 \Rightarrow u_1 = \arg\min_u Q_{\theta_1}(u) = 1$$

$$\Phi(1)\theta_2 = Q(1) \Rightarrow \theta_2 = \frac{1}{2} \Rightarrow u_2 = \arg\min_u Q_{\theta_2}(u) = 2$$

Policies cycle endlessly.