Global Optimality Guarantees For Policy Gradient Methods

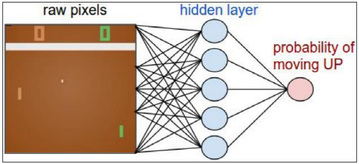
Daniel Russo

April 20, 2020

Columbia University

Table of Contents

Policy Gradient Methods



REINFORCE:

Initialize policy parameters θ arbitrarily

for each episode $\{s_1, a_1, r_2, \cdots, s_{T-1}, a_{T-1}, r_T\} \sim \pi_\theta$ **do**

  for $t = 1$ to $T - 1$ **do**

    $\theta \leftarrow \theta + \alpha \nabla_\theta \log \pi_\theta(s_t, a_t) G_t$

  endfor

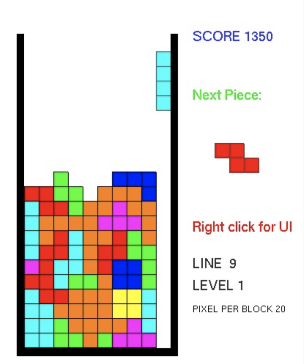
endfor

return θ

https://twitter.com/CovariantAI/status/1232396047948763136

https://youtu.be/kVmp0uGtShk

Policy Search In Tetris



- One approach is a Linear Approximation: $J(s) \approx \phi(s)^\top \theta$.
- $\phi(s)$ encodes features. E.g. column heights, inter-column height differences, max height etc.
- Each (s, a) associated with known successor state s'_a
- Could estimate θ, then play $\text{argmax}_a \, \phi(s'_a)^\top \theta$.
- Often better to directly search over the subclass of policies $\mu_\theta(s, a) \propto \exp(\phi(s'_a)^\top \theta)$

Policy Gradient Methods: Setup

Direct optimization over a policy class by local search.

- Cost-to-go function: $J_\mu(s) = \mathbb{E}_\mu \left[\sum_{t=0}^{\infty} \alpha^t g(s_t, \mu(s_t)) \mid s_0 = s\right]$
- Scalar loss: $\ell(\mu) = (1-\alpha)\mathbb{E}_{s\sim\nu}[J_\mu(s)]$.
    - Assume ν has full support. (*This is essential*!)
- Parameterized policies: $M_\Theta = \{\mu_\theta : \theta \in \Theta\} \subset M$.
    - $\Theta \subset \mathbb{R}^d$ is convex. $M = \{$Markov policies$\}$.
    - Overloaded notation: $\ell(\theta) \equiv \ell(\mu_\theta)$
- (Stochastic) Gradient Descent on $\ell(\cdot)$,

$$\theta_{t+1} = \theta_t - \gamma_t \left(\nabla \ell(\theta_t) + \text{noise}\right) \quad t = 1, 2, \ldots$$

Contrast to policy iteration

Policy gradient methods:

1. Make soft updates to policies
2. Aim to directly minimize a global loss function $\ell(\mu)$ rather than solve the changing surrogate problems $\min_u Q_{\mu_k}(s, u)$.
    - We'll see a connection, though, when we compute the gradients of $\ell(\cdot)$.
3. Approximation is through a policy class rather than a class of cost-to-go functions.
    - There is a connection to approximate PI when considering the set of policies that are (approximately) greedy with respect to a cost-to-go function. (See the tetris example above)
    - Actor-critic methods, which we'll get to later, are a hybrid.

Computing a stochastic gradient via simulation

Justified in the next two slides.

An unbiased estimate of $\nabla_\theta \ell(\mu_\theta)$ for a stochastic policy

Sample $T \sim \mathrm{Geometric}(1-\alpha)$.

Sample $s_0 \sim \nu$

Apply μ_θ for T periods from s_0.

Observe trajectory: $\tau = (s_0, u_0, c_0, \ldots, s_T, u_T, c_T, s_{T+1})$.

Observe total cost: $c(\tau) = c_0 + \cdots + c_T$.

Return $(1-\alpha)c(\tau)\sum_{t=0}^{T} \nabla_\theta \log \mu_\theta(u_t|s_t)$

Many variants and improvements:

- Infintesimal perturbation analysis
- Variance reduction by adding baselines
- Actor critic methods

Score function gradient estimator (Finite horizon case)

Consider a stochastic policy $\mu_\theta(u|s)$
For $\tau = (s_0, u_0, \cdots s_T, u_T)$, set $G(\tau) = \sum_{t=0}^{T} g(s_t, u_t)$
Let $P(\tau; \theta)$ denote its probability under μ_θ.

$$
\begin{aligned}
\nabla_\theta \, \mathbb{E}_\theta \left[\sum_{t=0}^{T} g(s_t, u_t) \right] &= \nabla_\theta \sum_\tau G(\tau) P(\tau; \theta) \\
&= \sum_\tau G(\tau) \nabla_\theta \, P(\tau; \theta) \\
&= \sum_\tau G(\tau) \left(\nabla_\theta \, \log P(\tau; \theta) \right) P(\tau; \theta) \\
&= \mathbb{E}_\theta \left[G(\tau) \left(\nabla_\theta \, \log P(\tau; \theta) \right) \right] \\
&= \mathbb{E}_\theta \left[G(\tau) \left(\nabla_\theta \, \log \prod_{t=0}^{T} \mu_\theta(u_t|s_t) P(s_{t+1}|u_t, s_t) \right) \right] \\
&= \mathbb{E}_\theta \left[G(\tau) \left(\sum_{t=0}^{T} \nabla_\theta \, \log \mu_\theta(u_t|s_t) \right) \right]
\end{aligned}
$$

Score function gradient estimator (infinite horizon case)

Take $T \sim \mathrm{Geom}(1-\alpha)$, so $\mathbb{P}(T \geq t) = \alpha^t$.

$$\begin{aligned}\ell(\theta) = (1-\alpha)\mathbb{E}_\theta\left[\sum_{t=0}^{\infty} \alpha^t g(s_t, u_t)\right] &= (1-\alpha)\mathbb{E}_\theta\left[\sum_{t=0}^{\infty} \mathbb{P}(T \geq t) g(s_t, u_t)\right] \\ &= (1-\alpha)\mathbb{E}_\theta\left[\sum_{t=0}^{T} g(s_t, u_t)\right]\end{aligned}$$

One can show

$$\nabla \ell(\theta) = (1-\alpha)\mathbb{E}_\theta\left[\left(\sum_{t=0}^{T} g(s_t, u_t)\right)\left(\sum_{t=0}^{T} \nabla_\theta \log \mu_\theta(u_t|s_t)\right)\right]$$

You may encounter a slight improvement that recognizes some terms have mean zero and writes

$$\nabla \ell(\theta) = (1-\alpha)\mathbb{E}_\theta\left[\left(\sum_{t=0}^{T} \nabla_\theta \log \mu_\theta(u_t|s_t)\left(\sum_{i=t}^{T} g(s_i, u_i)\right)\right)\right]$$

Challenges with policy gradient methods

This class will cover issue (1). The rest next week.

1. Nonconvexity of the loss function $\ell(\mu)$.

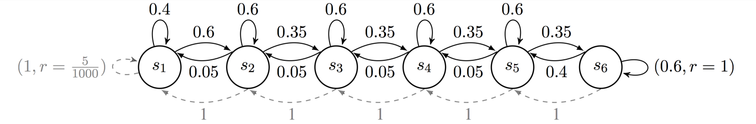
    Due to multi-period objective.

2. Unnatural policy parameterization $\theta \mapsto \mu_\theta$.

    Poor parameterization can lead to vanishing gradients in single period problems. Natural gradient methods are invariant to change of coordinates.

3. Insufficient exploration.

    We rely on the initial distribution $\nu(\cdot)$.



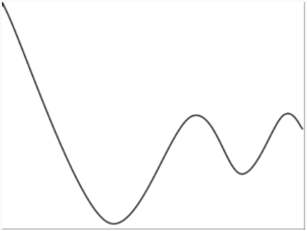
4. Large variance of stochastic gradients.

    Baselines and actor critic methods help.

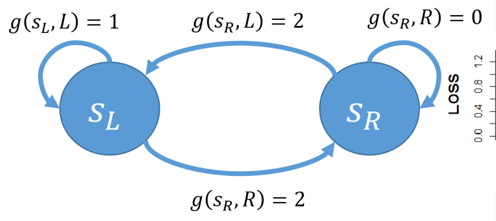
Table of Contents

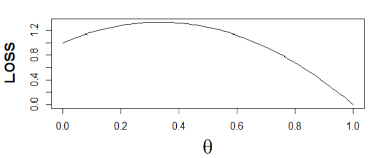
So we've reduced dynamic programming to black-box nonconvex optimization?

- PG is applicable to almost any decision problem.
- **But...** $\mu \mapsto \ell(\mu)$ is nonconvex.
  - Even for classical dynamic programming problems.
- Policy gradient is widely understood to converge toward first-order stationary points (i.e. w/ $\nabla_\theta \ell(\mu_\theta) = 0$).
  - Almost no guarantees on the quality of stationary points.
  - There are bad local minima in simple problems.



A Bad Example For Policy Gradient



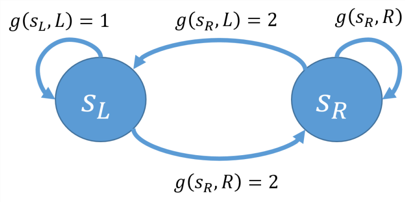


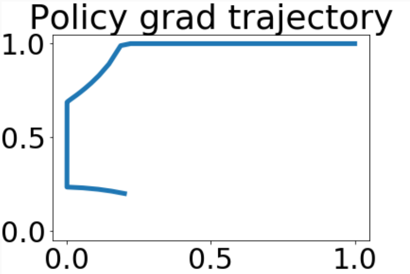
- $\mu_\theta(s_L) = \mu_\theta(S_R) = \theta =$ "Probability of Right."

Policy gradient gets stuck in a bad local minimum

Despite containing the optimal policy, the policy class is not rich enough to allow for a local improvement.

A Good Example For Policy Gradient





- $\mu_\theta(S_L) = \theta_L =$ "Probability of Right from left state."
- $\mu_\theta(S_R) = \theta_R =$ "Probability of Right from right state."

Policy gradient reaches the global optimum

The full policy class is rich enough to allow for local improvement. . . but this is unsatisfying.

Table of Contents

Warmup with the natural special case

- $\mathcal{S} = \{1, \cdots, n\}$ is a finite set.
- $U(s) = U = \Delta^{k-1}$ contains all probability distributions over k deterministic actions.
    - For $u \in U$, $g(s,u) = \sum_{i=1}^{k} g(s,i) u_i$, $P_{ss'}(u) = \sum_{i=1}^{k} u_i P_{ss'}(i)$

Natural parameterization

- $\mu_\theta(s) = \theta_s \in \Delta^{k-1}$ associates each state with a probability distribution over actions.
- Drop the notational dependence on $\theta \ldots$, writing $\mu \in \mathbb{R}^{n \times k}$
- Stationary randomized policies:
  $M = \{\mu \in \mathbb{R}_+^{n \times k} : \sum_{i=1}^{k} \mu_{s,i} = 1 \ \forall s \in \{1, \cdots, n\}\}$.

First order methods

First order methods solve $\min_{\mu \in M} \ell(\mu)$ by iteratively minimizing (regularized) local linear approximations to $\ell(\cdot)$.

Example: Projected Gradient

$$\begin{aligned} \mu_{k+1} &= \Pi_{2,M}\left(\mu_k - \gamma_k \nabla \ell(\mu_k)\right) \\ &= \operatorname*{argmin}_{\mu \in M} \; \ell(\mu_k) + \langle \nabla \ell(\mu_k)\,,\, \mu - \mu_k \rangle + \frac{1}{2\gamma_k} \|\mu - \mu_k\|_2^2 \end{aligned}$$

Projection onto the probability simplex involves a simple soft-thresholding operation.

It is also natural to consider conditional gradient descent or exponentiated gradient descent (... a special case of mirror descent).

Convergence to first order stationary points (1)

Definition: $\ell(\cdot)$ is L–smooth if $\nabla^2 \ell(\mu) \preceq LI$

Informal Proposition (Asymptotic convergence)

If $\ell(\cdot)$ is L–smooth, and stepsizes are set appropriately, then for any limit point μ_∞ of $\{\mu_k\}$.

$$\langle \nabla \ell(\mu_k)\,,\, \mu - \mu_\infty \rangle \geq 0 \qquad \forall \mu \in M$$

Proposition (Convergence rate)

Set $e_k = \min_{\mu \in M} \langle \nabla \ell(\mu_k)\,,\, \mu - \mu_k \rangle$. If $\gamma_k = 1/L$ then,

$$\min_{k \leq K} |e_k| \leq \sqrt{\frac{\ell(\mu_0) - \ell(\mu^*)}{2LR^2K}}$$

where $R = \sup_{\mu,\mu'} \|\mu - \mu'\|_2$.

Similar guarantees apply with stochastic gradient evaluations

Proof of rate to reach approximate stationary point (1)

Descent Lemma

If $\ell(\cdot)$ is L smooth, then $\ell(\bar{\mu}) \le \ell(\mu) + \langle \nabla \ell(\mu)\,,\, \bar{\mu} - \mu \rangle + \frac{L}{2}\|\bar{\mu} - \mu\|_2^2$

For $\gamma_k \le \frac{1}{L}$, projected gradient descent *minimizes a quadratic upper bound* on $\ell(\cdot)$. We find

$$\ell(\mu_{k+1}) \le \min_{\mu \in M}\; \ell(\mu_k) + \langle \nabla \ell(\mu_k)\,,\, \mu - \mu_k \rangle + \frac{1}{2\gamma_k}\|\mu - \mu_k\|_2^2$$

Take $\mu^+ = \operatorname{argmin}_{\mu \in M} \langle \nabla \ell(\mu_k)\,,\, \mu - \mu_k \rangle$ and pick $\gamma_k = 1/L$. Minimizing only over the line segment connecting μ_k, μ^+ gives

$$\begin{aligned}
\ell(\mu_{k+1}) &\le \min_{t \in [0,1]} \ell(\mu_k) + t\langle \nabla \ell(\mu_k)\,,\, \mu^+ - \mu_k \rangle + \frac{Lt^2}{2}\left\|\mu^+ - \mu_k\right\|_2^2 \\
&\le \min_{t \in [0,1]} \ell(\mu_k) + t\langle \nabla \ell(\mu_k)\,,\, \mu^+ - \mu_k \rangle + \frac{t^2}{L2}R^2 \\
&= \ell(\mu_k) - \frac{1}{2LR^2}\left(\min_{\mu \in M}\langle \nabla \ell(\mu_k)\,,\, \mu - \mu_k \rangle\right)^2
\end{aligned}$$

Proof of rate to reach approximate stationary point (2)

Set $e_k = \min_{\mu \in M} \langle \nabla \ell(\mu_k)\,,\, \mu - \mu_k \rangle$ to be the distance from stationarity. We showed

$$\min_{k \leq K} e_k^2 \leq \frac{1}{K} \sum_{k=1}^{K} e_k^2 \leq \frac{2LR^2}{K} \sum_{k=1}^{K} \left(\ell(\mu_{k+1}) - \ell(\mu_k) \right) \leq \frac{\ell(\mu_0) - \ell(\mu^*)}{2LR^2 K}$$

Or

$$\min_{k \leq K} |e_k| \leq \sqrt{\frac{\ell(\mu_0) - \ell(\mu^*)}{2LR^2 K}}$$

Despite nonconvexity, there are no suboptimal stationary points for policy gradient with a full policy class

If $\nu(s) > 0$ for all s the RHS is zero if and only if $J_\mu = TJ_\mu$.

Lemma: If μ^+ is a policy iteration update to μ, then

$$\frac{d}{d\gamma}\ell(\mu + \gamma(\mu^+ - \mu))\bigg|_{\gamma=0} \leq -\|J_\mu - TJ_\mu\|_{1,\nu}$$

Proof: Set $\mu_\gamma = \mu + \gamma(\mu^+ - \mu)$. One can show

$$T_{\mu_\gamma} J_\mu = (1-\gamma) T_\mu J_\mu + \gamma T_{\mu^+} J_\mu = J_\mu - \gamma\,(TJ_\mu - J_\mu) \preceq J_\mu$$

Monotonicity implies $J_{\mu_\gamma} \preceq \cdots \preceq T_{\mu_\gamma} J_\mu \preceq J_\mu$. Hence,

$$\frac{J_{\mu_\gamma} - J_\mu}{\gamma} \preceq \frac{T_{\mu_\gamma} J_\mu - J_\mu}{\gamma} = -\,(TJ_\mu - J_\mu)$$

Left multiplying by ν, and using $TJ_\mu \preceq J_\mu$

$$\frac{\ell(\mu_\gamma) - \ell(\mu)}{\gamma} \leq -\|TJ_\mu - J_\mu\|_{1,\nu}$$

Taking $\gamma \to 0$ gives the result.

Table of Contents

Weighted policy iteration

Policy iteration is

$$\mu_{k+1} \in \operatorname*{argmin}_{\mu \in M} T_\mu J_{\mu_k},$$

where the minimization is performed elementwize. As long as $w(s) > 0$, policy iteration can be written minimizing a scalarized objective:

$$\mu_{k+1} \in \operatorname*{argmin}_{\mu \in M} w\left(T_\mu J_{\mu_k}\right)$$

Define the weighted policy iteration cost function

$$\mathcal{B}(\bar{\mu}|w, J_\mu) = wT_{\bar{\mu}} J_\mu = \sum_s w(s)(T_{\bar{\mu}} J_\mu)(s) = \sum_s w(s) Q_\mu(s, \bar{\mu}(s))$$

We will connect policy gradient to the constrained policy iteration scheme

$$\mu_{\theta_{k+1}} = \operatorname*{argmin}_{\mu_\theta \in M_\Theta} \mathcal{B}(\mu_\theta | w, J_{\mu_{\theta_k}})$$

that restricts to the parameterized policy class when minimizing.

A policy gradient formula

*Overload notation: $\mathcal{B}(\theta|\nu, J) \equiv \mathcal{B}(\mu_\theta|\nu, J)$.,
and $\nu_\mu := \sum_{k=0}^{\infty} \alpha^k \nu P_\mu^k$.

Under appropriate smoothness conditions

$$\nabla \ell(\theta) = \nabla_{\bar{\theta}} \mathcal{B}(\bar{\theta}|\nu_{\mu_\theta}, J_{\mu_\theta})\Big|_{\bar{\theta}=\theta} = \mathbb{E}_{s\sim\nu_{\mu_\theta}} \left[\nabla_{\bar{\theta}} Q_{\mu_\theta}(s, \mu_{\bar{\theta}}(s))\Big|_{\bar{\theta}=\theta} \right]$$

Policy gradient is the iteration

$$\theta_{i+1} = \theta_i - \gamma_i \nabla_{\bar{\theta}} \mathcal{B}(\bar{\theta}|\nu_{\mu_{\theta_i}}, J_{\mu_{\theta_i}})\Big|_{\bar{\theta}=\theta_i}$$

This is similar to weighted PI restricted to M_Θ, except:

1. PG takes incremental gradient steps rather than solving $\min_{\bar{\theta}} \mathcal{B}(\bar{\theta}|w, J_{\mu_{\theta_i}})$ to optimality.
2. PG uses state-relevance weights $\nu_{\mu_{\theta_i}}$ induced by applying μ_{θ_i}.

Deriving the policy gradient formula

A first-order approximation to J_{μ_θ}

Under appropriate smoothness conditions

$$J_{\mu_{\bar\theta}} = J_{\mu_\theta} + (I - \alpha P_{\mu_\theta})^{-1} \left(T_{\mu_{\bar\theta}} J_{\mu_\theta} - J_{\mu_\theta} \right) + O(\|\bar\theta - \theta\|_2^2),$$

Proof Set $T_\theta \equiv T_{\mu_\theta}$, $P_\theta \equiv P_{\mu_\theta}$, $g_\theta \equiv g_{\mu_\theta}$.
By the variational form of Bellman's equation,

$$\begin{aligned} J_{\bar\theta} &= J_\theta + (I - \alpha P_{\bar\theta})^{-1} (T_{\bar\theta} J_\theta - J_\theta) \\ &= J_\theta + (I - \alpha P_\theta)^{-1} (T_{\bar\theta} J_\theta - J_\theta) + e_\theta \end{aligned}$$

where

$$\begin{aligned} e_\theta &= \left([I - \alpha P_{\bar\theta}]^{-1} - [I - \alpha P_\theta]^{-1} \right) (T_{\bar\theta} J_\theta - J_\theta) \\ &= \left([I - \alpha P_{\bar\theta}]^{-1} - [I - \alpha P_\theta]^{-1} \right) (g_{\bar\theta} - g_\theta + \alpha [P_{\bar\theta} - P_\theta] J_\theta) \end{aligned}$$

is $O(\|\bar\theta - \theta\|_2^2)$ assuming $\theta \mapsto P_\theta$ and $\theta \mapsto g_\theta$ are differentiable.

Deriving the policy gradient formula (2)

Under appropriate smoothness conditions

$$J_{\mu_{\bar{\theta}}} = J_{\mu_\theta} + (I - \alpha P_{\mu_\theta})^{-1} \left(T_{\mu_{\bar{\theta}}} J_{\mu_\theta} - J_{\mu_\theta} \right) + O(\|\bar{\theta} - \theta\|_2^2),$$

Left multiplying each side by $(1-\alpha)\nu$ gives the following:

Under appropriate smoothness conditions

$$\ell(\bar{\theta}) = \ell(\theta) + \nu_{\mu_\theta} \left(T_{\mu_{\bar{\theta}}} J_{\mu_\theta} - J_{\mu_\theta} \right) + O(\|\bar{\theta} - \theta\|_2^2),$$

Differentiating with respect to $\bar{\theta}$ gives,

$$\nabla \ell(\theta) = \nabla_{\bar{\theta}}\, \nu_{\mu_\theta} \left(T_{\mu_{\bar{\theta}}} J_{\mu_\theta} \right) \Big|_{\bar{\theta}=\theta} = \nabla_{\bar{\theta}}\, \mathcal{B}(\bar{\theta} | \nu_{\mu_\theta}, J_{\mu_\theta}) \Big|_{\bar{\theta}=\theta}$$

Table of Contents

Main insight of Bhandari and Russo (2020)

Two conditions ensure $\ell(\cdot)$ has no suboptimal stationary points

1. The policy class is closed under policy improvement
   - A PI update to $\mu \in M_\Theta$ can be solved within the policy class:

$$\min_{\mu_\theta \in M_\Theta} \mathcal{B}(\mu_\theta | \nu, J_\mu) = \min_{\bar{\mu} \in M} \mathcal{B}(\bar{\mu} | \nu, J_\mu)$$

   - A sense in which the policy class is sufficient for control.
   - Much weaker than requiring it represents (nearly) all policies
   - Necessarily stronger than requiring $\mu^* \in M_\Theta$.

2. First order methods can solve the PI problem
   - $\theta \mapsto \mathcal{B}(\mu_\theta \mid \nu, J_\mu)$ has no suboptimal stationary points.
   - E.g. this single period objective is often convex...

Example: linear quadratic control (deterministic for this talk)

States $s_t \in \mathbb{R}^n$ and actions $a_t \in \mathbb{R}^k$

$$\text{Minimize} \quad \sum_{t=0}^{\infty} \alpha^t \left(s_t^\top R s_t + a_t^\top U a_t \right)$$

$$\text{Subject to} \quad s_t = A s_{t-1} + B a_{t-1}$$

Linear policies: $\mu_\theta(s) = \theta s$.

Total cost $\ell(\mu_\theta) = \mathbb{E}_{s_0 \sim \nu}[J_{\mu_\theta}(s_0)]$ is messy and nonconvex:

$$\begin{aligned} J_{\mu_\theta}(s_0) &= \sum_{t=0}^{\infty} \alpha^t \left(s_t^\top R s_t + s_t^\top \theta^\top U \theta s_t \right) \\ &= s_0^\top \underbrace{\left(\sum_{t=0}^{\infty} \alpha^t \left[(A + B\theta)^t \right]^\top \left(R + \theta^\top U \theta \right) \left[(A + B\theta)^t \right] \right)}_{K_\theta} s_0 \end{aligned}$$

since $s_t = A s_{t-1} + B\theta s_{t-1} = \cdots = (A + B\theta)^t s_0$

Policy iteration for LQ control –Kleinman (1968)

The policy improvement objective is quadratic in a:

$$Q_{\mu_\theta}(s,a) = a^\top U a + \alpha (As + Ba)^\top K_\theta (As + Ba)$$

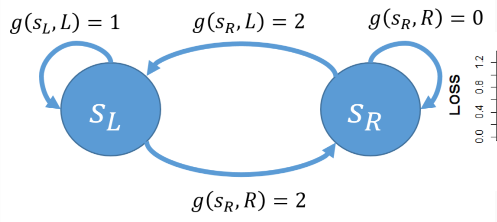
Condition 1: The policy class M_Θ is closed under PI updates?

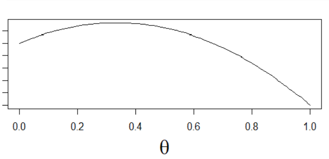
- Yes. $\bar{\theta}s = \operatorname{argmin}_a Q(s,a)$ for all s
  (Updated parameter is $\bar{\theta} = \alpha \left(U + \alpha B^\top K_\theta B \right)^{-1} B^\top K_\theta A$)

Condition 2 The PI obj. has no suboptimal stationary points?

- Holds since $\bar{\theta} \mapsto \mathbb{E}_{s\sim\nu}\left[Q_{\mu_\theta}(s, \bar{\theta}s) \right]$ is quadratic

Back to the bad example for policy gradient





- $\mu_\theta(s_L) = \mu_\theta(S_R) = \theta =$ "Probability of Right."

The parameterized policy class is not closed

For $\theta = .1$,

$$L = \operatorname*{argmin}_{a \in \{L,R\}} Q_{\mu_\theta}(S_L, a)$$

whereas

$$R = \operatorname*{argmin}_{a \in \{L,R\}} Q_{\mu_\theta}(S_R, a)$$

Approximation with rich policy classes?

Punchline: if the policy iteration problem can be nearly solved within the policy class, then stationary points of the PG objective are nearly optimal.

Approximation with rich policy classes

- **Condition 1b**: Policy class is closed under approximate PI

$$\text{For any } \mu \in M_\Theta, \quad \min_{\mu_\theta \in M_\Theta} \mathcal{B}(\mu_\theta | \nu, J_\mu) \leq \min_{\bar{\mu} \in M} \mathcal{B}(\bar{\mu} | \nu, J_\mu) + \epsilon$$

- **Condition 2a** The PI objective has no suboptimal stationary points
  - $\theta \mapsto \mathbb{E}_{s \sim \nu}[Q_\mu(s, \mu_\theta(s))]$ has no suboptimal stationary points for any distribution ν over $\mathcal{S}$.

Theorem (Informal)

Under conditions 1b and 2a (and mild regularity conditions), if θ is a stationary point of $\ell(\cdot)$ then

$$\ell(\mu_\theta) \leq \min_{\mu \in M} \ell(\mu) + \frac{\kappa_\nu}{(1-\alpha)^2} \cdot \epsilon$$

where

$$k_\nu = \sup_{J_\mu \,:\, \mu \in M_\Theta} \frac{\|J_\mu - TJ_\mu\|_{1,\nu_{\mu^*}}}{\|J_\mu - TJ_\mu\|_{1,\nu}}$$

Table of Contents

Actor critic form of the policy gradient theorem

Actor critic form the policy gradient theorem

When U is finite, $\mu_\theta(s,u)$ is the differentiable probability of playing control u in state s, $S \sim \nu_{\mu_\theta}$, and $U|S \sim \mu_\theta(S,\cdot)$,

$$\begin{aligned}\nabla \ell(\theta) &= \nabla_{\bar\theta}\, \mathbb{E}\left[\sum_{u\in U} Q_{\mu_\theta}(S,u)\mu_{\bar\theta}(u|S)\right]\Bigg|_{\bar\theta=\theta} \\ &= \mathbb{E}\left[Q_{\mu_\theta}(S,U)\nabla_\theta \log \mu_\theta(S,U)\right]\end{aligned}$$

Proof The first equality was already established. The second equality uses that

$$\begin{aligned}\left[\sum_{u\in U} Q_{\mu_\theta}(S,u)\nabla_\theta \mu_\theta(S,u)\right] &= \left[\sum_{u\in U} Q_{\mu_\theta}(S,u)\left(\nabla_\theta \log \mu_\theta(S,u)\right)\mu_\theta(S,u)\right] \\ &= \left[\sum_{u\in U} Q_{\mu_\theta}(S,U)\nabla_\theta \log \mu_\theta(S,U)\right].\end{aligned}$$

Actor critic algorithms

The policy gradient theorem suggests we can approximate gradients by approximating Q_{μ_θ} with by some parameterized cost-to-go function $\hat{Q}^{\eta}$.

Actor critic algorithms

1. **(Critic)** Track estimates $\hat{Q}^{\eta_n}$ of the current policy using approximate policy evaluation (either Monte-Carlo or TD)
2. (**Actor)** Run policy gradient with approximate gradients

$$\nabla \ell(\theta_n) \approx \mathbb{E}\left[\hat{Q}^{\eta_n}(S, U) \nabla_\theta \log \mu_\theta(S, U)\right]$$

Usual suggestion is to run the actor at a slower timescale (e..g with smaller stepsizes) than the critic, so gradient estimates can track the changing policy.

Choosing a critic with compatible function approximation

Konda and Tsitsiklis [2000] and Sutton et al. [2000].
"*The key observation in this paper is that in actor-critic methods, the actor parameterization and the critic parameterization need not, and should not be chosen independently. Rather, an appropriate approximation architecture for the critic is directly prescribed by the parameterization used in actor.*"

Choosing a critic with compatible function approximation

- Set $\psi_\theta(s,u) = \nabla_\theta \log \mu_\theta(s,u) \in \mathbb{R}^d$ for $\theta \in \mathbb{R}^d$
- $\psi_\theta^i \in \mathbb{R}^{S\times U}$ the i-th component.
- The policy gradient theorem says, for $S \sim \nu_{\mu_\theta}$ and $U|S \sim \mu_\theta(S,\cdot)$,

  $$\frac{\partial}{\partial\theta_i}\ell(\theta) = \mathbb{E}\left[Q_{\mu_\theta}(S,U)\psi_\theta^i(S,U)\right] := \langle Q_{\mu_\theta}\,,\,\psi_\theta^i\rangle_\theta$$

  The gradient is a weighted inner product between the features $(\psi_\theta^i : i = 1,\cdots,d)$ and the true Q function.
- Define the span of the features $\Psi_\theta = \{\sum_{i=1}^d a_i\psi_\theta^i : a_1,\cdots,a_d \in \mathbb{R}\}$.
- Then

  $$\frac{\partial}{\partial\theta_i}\ell(\theta) = \langle Q_{\mu_\theta}\,,\,\psi_\theta^i\rangle_\theta = \langle \Pi_\theta Q_{\mu_\theta}\,,\,\psi_\theta^i\rangle_\theta$$

  where Π_θ is the projection onto Ψ_θ w.r.t $\|Q\|_\theta = \sqrt{\langle x\,,\,x\rangle_\theta}$.
- Exact gradient don't require learning Q_{μ_θ}, just the low rank approximation $\Pi_\theta Q_{\mu_\theta}$ onto the span of the compatible features.

Example: softmax features

Suppose

$$\mu_\theta(s,u) \propto \exp\left\{\phi(s,u)^\top \theta\right\}.$$

Then, the compatible features of the critic

$$\psi_\theta(s,u) = \nabla \log \mu_\theta(s,u) = \phi(s,u) - \sum_{u' \in U} \mu_\theta(s,u)\phi(s,u')$$

are the features vector used by the policy after "centering."

Reducing variance by subtracting a baseline

Intuitively, Q_{μ_θ} is not a natural way of "scoring" descent directions because it is not centered. It could be large and positive for all values of (s, u). Subtracting a baseline reduces variance.

Lemma(Baseline invariance) For any $b : \mathcal{S} \to \mathbb{R}$,

$$\nabla \ell(\theta) = \mathbb{E}_\theta \left[\left(Q_{\mu_\theta}(S, U) - b(S) \right) \nabla_\theta \log \mu_\theta(S, U) \right]$$

Proof. We show the baseline term has mean zero by showing its conditional expectation equals zero:

$$\begin{aligned}
\mathbb{E}\left[b(S) \nabla \log \mu_\theta(S, U) \mid S = s \right] &= b(s) \mathbb{E}\left[\nabla \log \mu_\theta(S, U) \mid S = s \right] \\
&= b(s) \sum_{u \in U} \mu_\theta(s, u) \nabla \log \mu_\theta(s, u) \\
&= b(s) \sum_{u \in U} \frac{\mu_\theta(s, u)}{\mu_\theta(s, u)} \nabla \mu_\theta(s, u) \\
&= b(s) \nabla_\theta \sum_{u \in U} \mu_\theta(s, u) = \nabla_\theta [1] = 0.
\end{aligned}$$

Idealized baseline and advantage functions

Roughly speaking, an optimal choice of baseline would be to center Q_{μ_θ}. Define the advantage function

$$A_{\mu_\theta}(s,u) = Q_{\mu_\theta}(s,u) - J_{\mu_\theta}(s)$$

with the implicit baseline $J_{\mu_\theta}(s) = \sum_{u \in U} Q_{\mu_\theta}(s,u)\mu_\theta(s,u)$.

Advantage form the policy gradient theorem:

$$\nabla \ell(\theta) = \mathbb{E}_\theta \left[A_{\mu_\theta}(S,U) \nabla_\theta \log \mu_\theta(S,U) \right].$$

Score (s,u) relative to an average control decision under $\mu_\theta(s,\cdot)$.

Table of Contents

Unnatural policy parameterization

Policy gradient

$$\begin{aligned}\theta_{n+1} &= \theta_n - \gamma \nabla \ell(\theta_n) \\ &= \underset{\theta \in \mathbb{R}^d}{\operatorname{argmin}} \langle \nabla \ell(\theta_n)\,,\, \theta - \theta_n \rangle + \frac{1}{2\gamma} \|\theta - \theta_n\|_2^2\end{aligned}$$

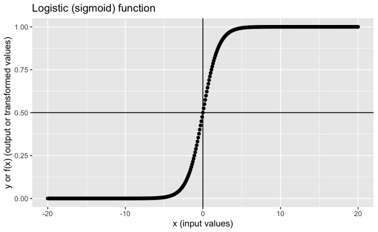
But euclidean distance with respect to the parameter may be a natural distance with respect to policies.

Natural gradient methods address this issue. They were adapted to reinforcement learning by Kakade [2002]. The popular TRPO method Schulman et al. [2015] is a variant of the idea.

Example: softmax polices

Consider the softmax policy $\mu_\theta(s,u) \propto \exp\{\phi(s,u)^\top \theta\}$, in a special case with one state, two actions, $\theta \in \mathbb{R}^d$:

$$\mu_\theta(2) = 1 - \mu_\theta(1) = \frac{e^\theta}{1+e^\theta} = \frac{1}{1+e^{-\theta}}.$$



In solving,

$$\min_\theta Q(1-\mu_\theta(2)) + Q(\mu_\theta(2))$$

gradient descent suffers from vanishing gradients.

Example continued

Set $c(\mu_\theta) = Q(1 - \mu_\theta(2)) + Q(\mu_\theta(2))$.
Natural gradient descent computes

$$\theta_{n+1} = \theta_n - \gamma \left(\frac{\partial \mu_\theta}{\partial \theta} \right)^{-1} \frac{\partial c(\mu_\theta)}{\partial \theta}$$

Can be thought of as a two step process:

1. Compute gradient in natural parameterization $g = \frac{\partial}{\partial \mu} c(\mu)|_{\mu = \mu_\theta}$.
2. Compute local change $d\theta$ in θ needed to mimic this gradient: $\frac{\partial \mu_\theta}{\partial \theta} d\theta = g$.

Regularization in different metrics

Return to

$$\theta_{n+1} = \theta_n - \gamma \nabla \ell(\theta_n) = \underset{\theta \in \mathbb{R}^d}{\operatorname{argmin}} \langle \nabla \ell(\theta_n)\,,\, \theta - \theta_n \rangle + \frac{1}{2\gamma} \|\theta - \theta_n\|_2^2 .$$

We decided the regularizer $\|\theta - \theta_n\|_2^2$ is problematic.

A simple idea is to use a metric $d : M \times M \to \mathbb{R}$ on the space of policies rather than parameters, and then run the iteration

$$\theta_{n+1} = \underset{\theta \in \mathbb{R}^d}{\operatorname{argmin}} \langle \nabla \ell(\theta_n)\,,\, \theta - \theta_n \rangle + \frac{1}{\gamma} d(\mu_{\theta_n}, \mu_\theta)$$

Such an algorithm is invariant to the choice of parameterization (by definition), and only depends on the policy class $M_\Theta = \{\mu_\theta : \theta \in \Theta\}$.

Reflects that we

Background: KL Divergence and Fisher Information

Define the Kullback-Leibler divergence

$$D_{\mathrm{KL}}(p||q) = \sum_i p_i \log \left(\frac{p_i}{q_i} \right).$$

- This is the expected value (under p) of the log likelihood ratio.
- Often a better measure than $\|p - q\|_2$ of the distance between probability vectors.
- It is not a metric, since it is symmetric and does not obey the triangle inequality. But is non-negative, convex, and obeys an information-projection property that substitutes for the triangle inequality.

Background: KL Divergence and Fisher Information

For sufficiently smooth $\theta \mapsto p_\theta(x)$, define the Fisher information

$$I_\theta = \mathbb{E}\left[\nabla \log p_\theta(X)(\nabla \log p_\theta(X))^\top\right] = \mathbb{E}\left[\nabla^2 \log p_\theta(X)\right]$$

where the second equality relies on special properties of score functions.

Connection between Fisher information and KL divergence

Under appropriate smoothness conditions,

$$\begin{aligned} D_{\mathrm{KL}}(p_\theta, p_{\theta'}) &= (\theta - \theta')^\top I_\theta(\theta - \theta') + O(\|\theta - \theta'\|^2) \\ &= \|\theta - \theta'\|_{2,I_\theta}^2 + O(\|\theta - \theta'\|^2) \end{aligned}$$

Natural gradient descent in general

Consider the algorithm

$$\theta_{n+1} = \underset{\theta \in \mathbb{R}^d}{\text{argmin}} \langle \nabla \ell(\theta_n) \, , \, \theta - \theta_n \rangle + \frac{1}{2\gamma} d(\mu_{\theta_n}, \mu_\theta)$$

where we choose the KL divergence of the state-action distribution

$$d(\mu, \mu') = \sum_s \nu_\mu(s) D_{\mathrm{kl}}(\mu(s, \cdot) \,||\, ||\mu(s, \cdot))$$

As $\gamma \to 0$, this update is *steepest descent in the Fisher information metric*

$$\theta_{n+1} = \theta_n - \gamma I_\theta^{-1} \nabla \ell(\theta_n).$$

for

$I_\theta = \mathbb{E}\left[\nabla^2 \log \mu_\theta(S, U)\right] = \sum_s \nu_{\mu_\theta}(s) \mathbb{E}\left[\nabla^2 \log \mu_\theta(s, U) \mid S = s\right].$

Sham M Kakade. A natural policy gradient. In *Advances in neural information processing systems*, pages 1531–1538, 2002.

Vijay R Konda and John N Tsitsiklis. Actor-critic algorithms. In *Advances in neural information processing systems*, pages 1008–1014, 2000.

John Schulman, Sergey Levine, Pieter Abbeel, Michael Jordan, and Philipp Moritz. Trust region policy optimization. In *International conference on machine learning*, pages 1889–1897, 2015.

Richard S Sutton, David A McAllester, Satinder P Singh, and Yishay Mansour. Policy gradient methods for reinforcement learning with function approximation. In *Advances in neural information processing systems*, pages 1057–1063, 2000.