Therefore,

$$J_k(x,u+1) = J_k(x+1,u) + c \geq J_k(x,u) \qquad \forall\, u \ \in\ \{0,1,\ldots\},\ \forall\, x \geq S_k.$$

This shows that $u=0$ minimizes $J_k(x,u)$, for all $x \geq S_k$. Now let $x \ \in\ [S_k - n, S_k - n+1),\ n \ \in\ \{1,2,\ldots\}$. Using (2), we have

$$J_k(x,n+m) - J_k(x,n) = J_k(x+n,m) - J_k(x+n,0) \geq 0 \qquad \forall\, m \in\ \{0,1,\ldots\}. \tag{3}$$

However, if $u < n$ then $x+u < S_k$ and

$$J_k(x+u+1,0) - J_k(x+u,0) < J_k(S_k+1,0) - J_k(S_k,0) = -c.$$

Therefore,

$$J_k(x,u+1) = J_k(x+u+1,0) + (u+1)c < J_k(x+u,0) + uc = J_k(x,u) \qquad \forall\, u \ \in\ \{0,1,\ldots\},\ \ n<n. \tag{4}$$

Inequalities (3),(4) show that $u=n$ minimizes $J_k(x,u)$ whenever $x \ \in\ [S_k - n, S_k - n+1)$.

3.18 (Optimal Termination of Sampling) (www)

Let the state x_k be defined as

$$x_k = \begin{cases} T, & \text{if the selection has already terminated} \\ 1, & \text{if the k}^{\text{th}} \text{ object observed has rank 1} \\ 0, & \text{if the k}^{\text{th}} \text{ object observed has rank} < 1 \end{cases}$$

The system evolves according to

$$x_{k+1} = \begin{cases} T, & \text{if } u_k = \text{stop or } x_k = T \\ w_k, & \text{if } u_k = \text{continue} \end{cases}$$

The cost function is given by

$$g_k(x_k,u_k,w_k) = \begin{cases} \frac{k}{N}, & \text{if } x_k = 1 \text{ and } u_k = \text{stop} \\ 0, & \text{otherwise} \end{cases}$$

$$g_N(x_N) = \begin{cases} 1, & \text{if } x_N = 1 \\ 0, & \text{otherwise} \end{cases}$$

Note that if termination is selected at stage k and $x_k \neq 1$ then the probability of success is 0. Thus, if $x_k = 0$ it is always optimal to continue. To complete the model we have to determine $P(w_k \,|\, x_k, u_k) \triangleq P(w_k)$ when the control $u_k = \text{continue}$. At stage k, we have already selected k objects from a sorted set. Since we know nothing else about these objects the new element can, with equal probability, be in any relation with the already observed objects a_j

$$\underbrace{\cdots < a_{i_1} < \cdots < a_{i_2} < \cdots \qquad \cdots < a_{i_k} \cdots}_{k+1 \text{ possible positions for } a_{k+1}}$$

Thus,

$$P(w_k = 1) = \frac{1}{k+1}\,; \qquad P(w_k = 0) = \frac{k}{k+1}$$

Proposition: If $k \in S_N \triangleq \left\{ i \;\middle|\; \left(\frac{1}{N-1} + \cdots + \frac{1}{i}\right) \le 1 \right\}$, then

$$J_k(0) = \frac{k}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{k}\right), \qquad J_k(1) = \frac{k}{N}.$$

Proof: For $k = N-1$,

$$J_{N-1}(0) = \max\Big[\underbrace{0}_{\text{stop}}, \underbrace{E\{w_{N-1}\}}_{\text{continue}}\Big] = \frac{1}{N},$$

and $\mu^*_{N-1}(0) = \text{continue}$, while

$$J_{N-1}(1) = \max\Big[\underbrace{\frac{N-1}{N}}_{\text{stop}}, \underbrace{E\{w_{N-1}\}}_{\text{continue}}\Big] = \frac{N-1}{N}$$

and $\mu^*_{N-1}(1) = \text{stop}$. Note that $N-1 \in S_N$ for all S_N.

Assume the conclusion holds for $J_{k+1}(x_{k+1})$. Then

$$J_k(0) = \max\Big[\underbrace{0}_{\text{stop}}, \underbrace{E\{J_{k+1}(w_k)\}}_{\text{continue}}\Big]$$

$$J_k(1) = \max\Big[\underbrace{\frac{k}{N}}_{\text{stop}}, \underbrace{E\{J_{k+1}(w_k)\}}_{\text{continue}}\Big]$$

Now,

$$\begin{aligned} E\{J_{k+1}(w_k)\} &= \frac{1}{k+1}\frac{k+1}{N} + \frac{k}{k+1}\frac{k+1}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{k+1}\right) \\ &= \frac{k}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{k}\right) \end{aligned}$$

Clearly, then

$$J_k(0) = \frac{k}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{k}\right)$$

and $\mu^*_k(0) = \text{continue}$. If $k \in S_N$,

$$J_k(1) = \frac{k}{N}$$

and $\mu^*_k(1) = \text{stop}$. **Q.E.D.**