Credit: Jimmy Qin for 3.3 and Q1

HW2 solutions

3.3 Problem Formulation:

As given in section 1.4 of the book; let the forecasting process be represented by:

$$y_{k+1} = q_k.$$

$$q_k = \begin{cases} 0 \text{ with prob. } p_{small} \\ 1 \text{ with prob. } p_{large} \end{cases} \text{ : Forecast for demand at next time step.}$$

$$w_{k+1} \sim Q_{q_k}$$: q_k determines whether the next step disturbance is sampled from Q_{large} or Q_{small}. (the demand distributions).

- Dynamics: $\begin{pmatrix} x_{k+1} = f(x_k, u_k, w_k) = x_k + u_k - w_k \\ y_{k+1} = q_k \end{pmatrix}$

Augmented state: $\tilde{x}_k = (x_k, y_k)$.

- Cost functions:

$g_k(x_k, u_k, w_k) = cu_k + r(x_k + u_k - w_k)$ where $r(x)$
$= p\max\{0, -x\} + h\max\{0, x\}$

- DP Algorithm: $J_N(x_N, y_N) = g(x_N) = 0$

$$J_k(x_k, y_k) = \min_{u_k \in U_k(x_k)} E_{w_k}\left[g_k(x_k, u_k, w_k) + \sum_{i=0}^{1} p_i J_{k+1}(f(x_k, u_k, w_k), i) \right] y_k$$

$$J_k(x_k, y_k) = \min_{u_k \in U_k(x_k)} E_{w_k}\Big[g_k(x_k, u_k, w_k) + p_{small}\, J_{k+1}\big(f_k(x_k,u_k,w_k), 0\big) + p_{large}\, J_{k+1}\big(f_k(x_k,u_k,w_k), 1\big) \,\Big|\, y_k \Big].$$

Note the conditioning on y_k as $w_k \sim Q_{y_k}$

- Define $z_k = x_k + u_k$. As in class, one can show that $z_k = \max\{S_k(y_k), x_k\}$ is optimal.

For $z \geq x$, let

$$Q_k^*(x, z, y) = E\Big[c(z-x) + r(z-w_k) + \sum_{i=0}^{1} p_i\, J_{k+1}^*(z-w_k, i) \,\Big|\, y \Big]$$

$$= E\Big[cz + r(z-w_k) + \sum_{i=0}^{1} p_i\, J_{k+1}^*(z-w_k, i) \,\Big|\, y \Big] - cx$$

$$= G_k(z, y) - cx$$

As in class; define $S_k(y) = \arg\min_{z \in \mathbb{R}} G_k(z, y)$. The proof for coercivity of $G_k(\cdot, y)$ follow from the same argument ~~in~~ class.

The proof of convexity follows by the induction argument by showing that $J_k^*(x, y_k) = \min_{z \geq x} G_k(z, y_k) - cx$ is convex. Same arguments as in class: except linear combination of J_{k+1}^* which ~~Q_k is convex as $r(\cdot)$ is convex for all w_k~~ preserves convexity.

- Note; for $k=N-1$, $G_{N-1}(z, y_{N-1}) = E\left[cz + r(\cancel{z} - w_{N-1}) \,\middle|\, y_{N-1} \right]$
as $J^*_N(x_N, y_N) = 0$.

- $$\mu^*_{N-1}(x, y_{\cancel{N}}) = \begin{cases} s_{N-1}(y) - x & : \text{if } x < s_{N-1}(y) \\ 0 & : \text{otherwise} \end{cases}$$

- Similarly,
$$\mu^*_k(x, y) = \begin{cases} s_k(y) - x & : \text{if } x < s_k(y) \\ 0 & : \text{otherwise} \end{cases}$$

- The threshholds now depend on ~~[illegible]~~ y which indicate the distribution of the next time step demand.

Q1: Let start backwards: For $t = N$,

$$J_N^*(x) = (x - y_N)^T Q (x - y_N) = x^T Q x - 2 y_N^T Q x + y_N^T Q y_N.$$

Pattern matching with the form of cost-to-go functions,

$$K_N = Q; \quad q_N = -Q^T y_N; \quad r_N = y_N^T Q y_N.$$

which is quadratic in x

For $t = N-1$,

$$J_{N-1}^*(x) = \min_u \left[g_{N-1}(x,u) + E_w \left[J_N^* \left(f(x,u,w) \right) \right] \right].$$

- $g_{N-1}(x,u) = (x - y_{N-1})^T Q (x - y_{N-1}) + u^T R u$, which is convex in both (x,u).
- $(x,u) \rightarrow f(x,u,w)$ is affine in (x,u), J_N^* is convex in its argument and taking expectation preserves convexity $\Rightarrow E_w \left[J_N^* (f(x,u,w)) \right]$ is convex in u.
- $h_{N-1}(x,u) = g_{N-1}(x,u) + E_w \left[J_N^* (f(x,u,w)) \right]$ is therefore quadratic in u. (in fact it is easy to argue that $h(\cdot)$ is ~~convex~~, quadratic in x also)

$$u_{N-1}^*(x) = \arg\min_u h_{N-1}(x,u).$$

$$\cancel{h_{N-1}(u) = u^T R u + E_w \left[(A x_{N-1} + B u + w_{N-1})^T K_N (A x_{N-1} + B u + w_{N-1}) + 2 q_N^T (A x_{N-1} \right.}$$

$$h_{N-1}(x,u) = u^TRu + E_w\Big[(Ax+Bu+w_{N-1})^T K_N (Ax+Bu+w_{N-1}) + 2q_N^T(Ax+Bu+w_{N-1}) + r_N\Big]$$

$$= u^TRu + u^T(B^TK_NB)u + (2x^TA^TK_NB + 2q_N^TB)u$$

(ignoring all terms not involving u).

- set $\nabla_u h_{N-1}(x,u) = 0$

$$\Rightarrow \quad u = -(R+B^TK_NB)^{-1}(B^TK_N^TAx + B^Tq_N)$$

$$\Rightarrow \quad u^*_{N-1}(x) = -(R+B^TK_NB)^{-1}(B^TK_N^TAx + B^Tq_N)$$

$$= L_{N-1}x + m_{N-1}$$

- where we can easily read off the formulae for L_{N-1}, m_{N-1}.
- Noteworthy point is that it is an affine function of x.

__Induction:__ We want to show that $J^*_{N-1}(x)$ is convex in x and is quadratic of the form $x^TK_{N-1}x + 2q_{N-1}^Tx + r_{N-1}$. Note that

$$J^*_{N-1}(x) = h(x, u^*(x)) = h(x, L^*_{N-1}x + m_{N-1})$$

- As $h(x,u)$ is quadratic in u, x and $L_{N-1}x + m_{N-1}$ is affine in x, it is easy to argue that $J^*_{N-1}(x)$ is quadriatic in x.
- The values of $K_{N-1}, q_{N-1}, r_{N-1}$ can be seen by simply ~~plugs~~ writing out $h(x, L_{N-1}x + m_{N-1})$.