Solutions to DP HW
Daniel Russo

Volume I, Problem 1.23

For the problem of minimizing

$$\mathbb{E}_{\{w_k\}}\left[\alpha^N g_N(x_N) + \sum_{k=0}^{N-1} \alpha^k g_k(x_k, u_k, w_k)\right]$$

The DP algorithm takes the form:

$$J_N(x) = \alpha^N g_N(x) \qquad \forall x$$

$$J_k(x) = \min_{u \in U(x)} \mathbb{E}\left[g_k(x_k, u, w_k) + J_{k+1}\left(f_k(x_k, u, w_k)\right)\right]$$

Define $V_k(x) = \frac{1}{\alpha^k} J_k(x)$.

Then,

$$V_N(x) = g_N(x)$$

$$V_k(x) = \min_{u \in U(x)} \left[g_k(x_k, u_k, w_k) + \alpha V_{k+1}\left(f_k(x_k, u, w_k)\right)\right]$$

Volume I, Problem 1.24

(a) Take an arbitrary policy $\pi = (\mu_0, \ldots, \mu_{N-1})$

Let $\pi^k = (\mu_k, \ldots, \mu_{N-1})$ denote the "tail policy"

Define $J_k^{\pi_k}(x) = \mathbb{E}^{\pi_k}\left[\exp\left\{\sum_{i=k}^{N-1} g_i(x_i, u_i, w_i) + g_N(x_N)\right\} \Big| x_k = x\right]$

Then,

$$J_k^{\pi_k}(x) = \mathbb{E}^{\pi_k}\left[\exp\{g_k(x_k, u_k, w_k)\}\exp\left\{\sum_{i=k+1}^{N-1} g_i(x_i, u_i, w_i) + g_N(x_N)\right\} \Big| x_k = x\right]$$

$$= \mathbb{E}^{\pi_k}\left[\exp\{g_k(x_k, u_k, w_k)\} J_{k+1}^{\pi_{k+1}}(x_{k+1}) \Big| x_k = x\right]$$

$$= \mathbb{E}^{\pi_k}\left[\exp\{g_k(x_k, u_k, w_k)\} J_{k+1}^{\pi_{k+1}}(f(x_k, u_k, w_k)) \Big| x_k = x\right]$$

uses the tower property of conditional expectation

where we write $J_N^{\pi_N}(x) = g_N(x)$ for notational simplification.

set

$$J_N^*(x) = g_N(x)$$

$$J_k^*(x) = \min_{u \in U(x)} \mathbb{E}\left[\exp\{g_k(x_k, u_k, w_k)\} J_{k+1}(f_k(x_k, u_k, w_k)) \Big| x_k = x\right]$$

Then, $J_k^{\pi_k^*}(x) = J_k^*(x)$ for the policy $\pi^* = (\mu_0^*, \ldots, \mu_{N-1}^*)$ that attains the minimum above $\forall x$.

For any other policy π, it is easy to show

$$J_{N-1}^{\pi^{N-1}}(x) \geq \min_{\pi^{N-1}} J_{N-1}^{\pi^{N-1}}(x) = J_{N-1}^*(x) \quad \forall x$$

Proceeding by induction, if $J_{k+1}^{\pi^{k+1}}(x) \geq J_{k+1}^*(x) \quad \forall x$

Then

$$J_k^{\pi^k}(x) \geq \min_{u \in U(x)} \mathbb{E}\left[\exp\{g_k(x,u,w)\}\, J_{k+1}^{\pi^{k+1}}\left(f_k(x,u,w_k)\right)\right]$$

$$\geq \min_{u \in U(x)} \mathbb{E}\left[\exp\{g_k(x,u,w)\}\, J_{k+1}^{*}\left(f_k(x,u,w_k)\right)\right]$$

$$= J_{k+1}^{*}(x)$$

(b)

Assume $g_k(x,u,w) = \tilde{g}_k(x,u) \quad \forall w.$

Then,

$$\log J_k^*(x) = \log\left(\min_{u \in U(x)} \tilde{g}(x,u)\, \mathbb{E}\left[J_{k+1}^*\left(f(x,u,w)\right)\right]\right)$$

$$= \min_{u \in U(x)} \tilde{g}(x,u) + \log \mathbb{E}\left[\exp\{\log J_{k+1}^*(f(x,u,w)\}\right]$$

Which gives a recursion in terms of $\log J_k^*$.