Now,

$$
\begin{aligned}
E\{J_{k+1}(w_k)\} &= \frac{1}{k+1}\frac{k+1}{N} + \frac{k}{k+1}\frac{k+1}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{k+1}\right) \\
&= \frac{k}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{k}\right)
\end{aligned}
$$

Clearly, then:

$$J_k(0) = \frac{k}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{k}\right)$$

and $\mu_k^*(0) = \text{continue}$. If $k \in S_N$,

$$J_k(1) = \frac{k}{N}$$

and $\mu_k^*(1) = \text{stop}$. **Q.E.D.**

Proposition If $k \notin S_N$:

$$J_k(0) = J_k(1) = \frac{\delta - 1}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{\delta - 1}\right)$$

where δ is the minimum element of S_N.

Proof For $k = \delta - 1$:

$$
\begin{aligned}
J_k(0) &= \frac{1}{\delta}\frac{\delta}{N} + \frac{\delta - 1}{\delta}\frac{\delta}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{\delta}\right) \\
&= \frac{\delta - 1}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{\delta - 1}\right)
\end{aligned}
$$

$$
\begin{aligned}
J_k(1) &= \max\left[\frac{\delta - 1}{N}, \frac{\delta - 1}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{\delta - 1}\right)\right] \\
&= \frac{\delta - 1}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{\delta - 1}\right)
\end{aligned}
$$

and $\mu_{\delta-1}^*(0) = \mu_{\delta-1}^*(1) = \text{continue}$.

Assume the proposition is true for $J_k(x_k)$. Then:

$$J_{k-1}(0) = \frac{1}{k}J_k(1) + \frac{k-1}{k}J_k(0) = J_k(0)$$

and $\mu_{k-1}^*(0) = \text{continue}$.

$$
\begin{aligned}
J_{k-1}(1) &= \max\left[\frac{1}{k}J_k(1) + \frac{k-1}{k}J_k(0), \frac{k-1}{N}\right] \\
&= \max\left[\frac{\delta - 1}{N}\left(\frac{1}{N-1} + \cdots + \frac{1}{\delta - 1}\right), \frac{k-1}{N}\right] \\
&= J_k(0)
\end{aligned}
$$

and $\mu^*_{k-1}(1) =$ continue. **Q.E.D.**

Thus the optimum policy is to continue until the δ^{th} object, where δ is the minimum integer such that $\left(\frac{1}{N-1} + \cdots + \frac{1}{\delta}\right) \leq 1$, and then stop at the first time an element is observed with largest rank.

Exercise 4.19

a) Let the state $x_k \in \{T, \bar{T}\}$ where T represents the driver having parked before reaching the kth spot. Let the control at each parking spot $u_k \in \{P, \bar{P}\}$ where P represents the choice to park in the kth spot. Let the disturbance w_k equal 1 if the kth spot is free; otherwise it equals 0. Clearly, we have the control constraint that $u_k = \bar{P}$, if $x_k = T$ or $w_k = 0$. The cost associated with parking in the kth spot is:

$$g_k(\bar{T}, P, 1) = k$$

If the driver has not parked upon reaching his destination, he incurs a cost $g_N(\bar{T}) = C$. All other costs are zero. The system evolves according to:

$$x_{k+1} = \begin{cases} T, & \text{if } x_k = T \text{ or } u_k = P \\ \bar{T}, & \text{otherwise} \end{cases}$$

Once the driver has parked, his remaining cost is zero. Thus, we can define F_k to be the expected remaining cost, given that the driver has not parked before the kth spot. (Note that this is simply $J_k(\bar{T})$). The DP algorithm is given by:

$$\begin{aligned}
F_0 &= C \\
F_k &= \min_{u_k \in \{P, \bar{P}\}} \mathop{E}_{w_k} \left\{ g_k(\bar{T}, u_k, w_k) + J_{k-1}(x_{k-1}) \right\} \\
F_k &= \min \Big[\underbrace{p\big[k + J_{k-1}(T)\big]}_{park,\ free} + \underbrace{qJ_{k-1}(\bar{T})}_{park,\ not\ free} \ , \ \underbrace{J_{k-1}(\bar{T})}_{don't\ park} \Big]
\end{aligned}$$

But since $J_i(T) = 0 \quad \forall i$:

$$\begin{aligned}
F_k &= \min\big[pk + qF_{k-1}, F_{k-1}\big] \qquad (1) \\
&= p\min\big[k, F_{k-1}\big] + qF_{k-1}
\end{aligned}$$