Electronic Companion:

On the Futility of Dynamics in Robust Mechanism Design

Santiago Balseiro, Anthony Kim, Daniel Russo

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A Missing Proofs from Section 3

A.1 Proof of Theorem 1

As explained in Section 3.4, we use Lemmas 2 and 3.

Part 1): We use Lemma 2. Taking the infimum over all single-round direct IC/IR mechanisms S on the right-hand side of (4), we obtain for any incentive compatible dynamic mechanism $A \in \mathcal{A}$,

$$\sup_{F \in \mathcal{F}} \mathrm{Regret}(A, F, T) \geq T \cdot \inf_{S \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \mathrm{Regret}(S, \delta_\theta, 1)\,.$$

Then, taking the infimum over all incentive compatible dynamic mechanisms A on the left-hand side of the above, we obtain

$$\mathrm{Regret}(T) = \inf_{A \in \mathcal{A}} \sup_{F \in \mathcal{F}} \mathrm{Regret}(A, F, T) \geq T \cdot \inf_{S \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \mathrm{Regret}(S, \delta_\theta, 1)\,.$$

Now, it remains to show that $\mathrm{Regret}(T) \leq T \cdot \inf_{S \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \mathrm{Regret}(S, \delta_\theta, 1)$. Fix an arbitrary $\epsilon > 0$. By the definition of infimum, there exists a single-round direct IC/IR mechanism S satisfying

$$\sup_{\theta \in \Theta} \mathrm{Regret}(S, \delta_\theta, 1) \leq \inf_{S' \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \mathrm{Regret}(S', \delta_\theta, 1) + \frac{\epsilon}{T}\,.$$

Then,

$$\sup_{F \in \mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) \leq T \cdot \sup_{\theta \in \Theta} \mathrm{Regret}(S, \delta_\theta, 1) \leq T \cdot \inf_{S' \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \mathrm{Regret}(S', \delta_\theta, 1) + \epsilon\,,$$

where the first inequality is by Lemma 3 and the second by the choice of S. Since $S^{\times T}$ is a particular incentive compatible dynamic mechanism for T rounds (see the proof of Lemma 3), it follows that

$$\begin{aligned}\mathrm{Regret}(T) = \inf_{A \in \mathcal{A}} \sup_{F \in \mathcal{F}} \mathrm{Regret}(A, F, T) &\leq \sup_{F \in \mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) \\ &\leq T \cdot \inf_{S' \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \mathrm{Regret}(S', \delta_\theta, 1) + \epsilon\,.\end{aligned}$$

As $\epsilon > 0$ was arbitrary and can be arbitrarily small, it follows that

$$\mathrm{Regret}(T) \leq T \cdot \inf_{S' \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \mathrm{Regret}(S', \delta_\theta, 1)\,.$$

Part 2): For any $\epsilon \geq 0$, assume a single-round direct IC/IR mechanism $S \in \mathcal{S}^{\times 1}$ satisfies

$$\sup_{\theta \in \Theta} \text{Regret}(S, \delta_\theta, 1) \leq \inf_{S' \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \text{Regret}(S', \delta_\theta, 1) + \frac{\epsilon}{T}\,.$$

Then,

$$\begin{aligned}\sup_{F \in \mathcal{F}} \text{Regret}(S^{\times T}, F, T) &\leq T \cdot \sup_{\theta \in \Theta} \text{Regret}(S, \delta_\theta, 1) \\ &\leq T \cdot \inf_{S' \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \text{Regret}(S', \delta_\theta, 1) + \epsilon = \text{Regret}(T) + \epsilon\,,\end{aligned}$$

where the first inequality is by Lemma 3, the second inequality is by the property of S and the last equality is by Part 1.

Part 3): Equivalently, we show that $\arg\min_{A \in \mathcal{A}} \sup_{F \in \mathcal{F}} \text{Regret}(A, F, T)$ is non-empty if and only if $\arg\min_{S \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \text{Regret}(S, \delta_\theta, 1)$ is non-empty. The if direction follows directly from Part 2. If there exists an optimal single-round direct IC/IR mechanism S^* to the single-round problem, the optimal solution S^* satisfies

$$\sup_{\theta \in \Theta} \text{Regret}(S^*, \delta_\theta, 1) \leq \inf_{S \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \text{Regret}(S, \delta_\theta, 1)\,.$$

By Part 2 (with $\epsilon = 0$),

$$\sup_{F \in \mathcal{F}} \text{Regret}((S^*)^{\times T}, F, T) \leq \text{Regret}(T)\,.$$

It follows that the direct static mechanism that repeats S^* T times is an optimal incentive compatible dynamic mechanism in the multi-round problem and, hence, there exists an optimal dynamic mechanism in the multi-round problem. Note the incentive compatibility of the direct static mechanism $(S^*)^{\times T}$ follows from Lemma 2.

For the only-if direction, assume there exists an optimal incentive compatible dynamic mechanism A^* such that $\text{Regret}(T) = \sup_{F \in \mathcal{F}} \text{Regret}(A^*, F, T)$. By Lemma 2, there exists a single-round direct IC/IR mechanism S such that

$$\text{Regret}(T) = \sup_{F \in \mathcal{F}} \text{Regret}(A^*, F, T) \geq T \cdot \sup_{\theta \in \Theta} \text{Regret}(S, \delta_\theta, 1)\,.$$

By Part 1 that $\text{Regret}(T) = T \cdot \inf_{S' \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \text{Regret}(S', \delta_\theta, 1)$, it follows that

$$\inf_{S' \in \mathcal{S}^{\times 1}} \sup_{\theta \in \Theta} \text{Regret}(S', \delta_\theta, 1) \geq \sup_{\theta \in \Theta} \text{Regret}(S, \delta_\theta, 1)\,.$$

The above implies that S is an optimal single-round mechanism because it achieves the single-round minimax regret. Hence, there exists an optimal single-round mechanism in the single-round problem. In particular, the single-round direct IC/IR mechanism constructed from A^*, $S(A^*)$, as in the proof of Lemma 1 is one such single-round mechanism that satisfies the statement of Lemma 2 and, hence, is an optimal solution to the single-round problem.

A.2 Proof of Lemma 1

Note for any single-round direct mechanism S (and the recommended strategy of truthful reporting for the agent) and any point-mass distribution δ_θ, we have $\text{Regret}(S, \delta_\theta, 1) = \text{OPT}(\delta_\theta, 1) - \int_\Omega u(\theta,\omega) dS_\theta(\omega)$. To prove the first and second statements, it suffices to show that a single-round direct mechanism S is incentive compatible (i.e., in the set $\mathcal{S}^{\times 1}$) if and only if its outcome distributions S_θ satisfy the IC and IR constraints as formulated in (3).

For the only-if direction, we proceed as follows. Assume an arbitrary incentive compatible mechanism $S \in \mathcal{S}^{\times 1}$. Recall that S being incentive compatible means $\text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1) \geq \text{AgentUtility}(S, \tilde{\sigma}, F, 1)$ for any distribution F and any feasible agent strategy $\tilde{\sigma}$. For the point-mass distribution δ_θ for $\theta \in \Theta$ and the agent strategy $\tilde{\sigma}$ that deterministically reports $\mathsf{CONTINUE}$ in Round 0 and then shock $\theta' \in \Theta$, the inequality reduces to

$$\int_\Omega v(\theta,\omega) \mathrm{d}S_\theta(\omega) \geq \int_\Omega v(\theta,\omega) \mathrm{d}S_{\theta'}(\omega)$$

which is the IC constraint of (3), in terms of the outcome distributions of S. Now, for the point-mass distribution δ_θ for $\theta \in \Theta$ and the agent strategy $\tilde{\sigma}$ that deterministically reports QUIT in Round 0 and does not participate, the inequality $\text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1) \geq \text{AgentUtility}(S, \tilde{\sigma}, F, 1)$ reduces to

$$\int_\Omega v(\theta,\omega) \mathrm{d}S_\theta(\omega) \geq 0$$

which is the IR constraint of (3). Therefore, the outcome distributions of S satisfy the IC and IR constraints.

We now show the if direction. Assume an arbitrary single-round direct mechanism S (with decision rule π_1) with its outcome distributions satisfying the IC and IR constraints in (3). Let σ^{TR} be the recommended strategy of truthfully reporting for the agent. We consider the following three cases depending on how an arbitrary alternative strategy $\tilde{\sigma}$ reports in Round 0 for each possible distribution $F \in \Delta(\Theta)$. Fix an arbitrary distribution $F \in \Delta(\Theta)$.

Case 1) $\tilde{\sigma}$ deterministically reports $\mathsf{CONTINUE}$ in Round 0

The IC and IR constraints in (3) imply that for each possible shock $\theta_1 \in \Theta$ in Round 1, truthfully reporting the shock θ_1 is weakly better than deterministically reporting some other shock or PASS for the agent. That is,

$$\mathbb{E}_{\pi_1,\sigma^{\text{TR}}}[v(\theta_1, \pi_1(\theta_1, h_1, z_1)) | \theta_1 = \theta] \geq \mathbb{E}_{\pi_1}[v(\theta_1, \pi_1(m_1, h_1, z_1)) | \theta_1 = \theta, m_1 = \hat{m}],$$

for any $\theta \in \Theta$ and $\hat{m} \in \Theta \cup \{\mathsf{PASS}\}$. Note the left-hand side is equal to $\int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega)$ when $\theta_1 = \theta$, and the right-hand side is equal to $\int_\Omega v(\theta,\omega)\mathrm{d}S_{\theta'}(\omega)$ when $\theta_1 = \theta$ and $\hat{m} = \theta'$ and is equal to 0 when $\hat{m} = \mathsf{PASS}$. Hence, truthfully reporting in Round 1 is weakly better than any reporting strategy that is potentially randomized:

$$\mathbb{E}_{\pi_1,\sigma^{\text{TR}}}[v(\theta_1, \pi_1(\theta_1, h_1, z_1)) | \theta_1 = \theta] \geq \mathbb{E}_{\pi_1,\tilde{\sigma}}[v(\theta_1, \pi_1(m_1, h_1, z_1)) | \theta_1 = \theta],$$

for any $\theta \in \Theta$, which follows by averaging the above inequality over possible $\hat{m}$ values under $\tilde{\sigma}$.

Since the last inequality holds for each possible value of θ_1, we average it over $\theta_1 \sim F$ and obtain

$$\text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1) = \mathbb{E}_{\pi_1, \sigma^{\text{TR}}}[v(\theta_1, \omega_1)] \geq \mathbb{E}_{\pi_1, \tilde{\sigma}}[v(\theta_1, \omega_1)] = \text{AgentUtility}(S, \tilde{\sigma}, F, 1)\,.$$

Case 2) $\tilde{\sigma}$ deterministically reports QUIT in Round 0

The IR constraint in (3) implies that for each possible shock $\theta_1 \in \Theta$ in Round 1, truthfully reporting is weakly better for the agent than reporting PASS which yields the utility of 0. Then, for all $\theta \in \Theta$,

$$\mathbb{E}_{\pi_1, \sigma^{\text{TR}}}[v(\theta_1, \pi_1(\theta_1, h_1, z_1)) | \theta_1 = \theta] \geq 0\,,$$

where the left-hand side is equal to $\int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega)$. Averaging the above over $\theta_1 \sim F$, we obtain $\text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1) = \mathbb{E}_{\pi_1, \sigma^{\text{TR}}}[v(\theta_1, \omega_1)] \geq 0$.

Since $\tilde{\sigma}$ reports QUIT in Round 0, the agent does not participate at all and $\text{AgentUtility}(S, \tilde{\sigma}, F, 1) = 0$. Clearly, $\text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1) \geq \text{AgentUtility}(S, \tilde{\sigma}, F, 1)$.

Case 3) $\tilde{\sigma}$ probabilistically reports $\mathsf{CONTINUE}$ or QUIT in Round 0

Let $\tilde{\sigma} = \{\tilde{\sigma}_t\}_{0:1}$ where $m_0 = \tilde{\sigma}_0(h_0^+, y_0)$ can be $\mathsf{CONTINUE}$ or QUIT. From the above cases, we have

$$\begin{aligned}
&\text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1) \geq \mathbb{E}_{\pi_1, \tilde{\sigma}}[v(\theta_1, \omega_1) | m_0 = \mathsf{CONTINUE}] \text{ and} \\
&\text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1) \geq \mathbb{E}_{\pi_1, \tilde{\sigma}}[v(\theta_1, \omega_1) | m_0 = \mathsf{QUIT}]\,.
\end{aligned}$$

Hence,

$$\begin{aligned}
\text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1) &\geq \mathbb{E}_{\pi_1, \tilde{\sigma}}[v(\theta_1, \omega_1) | m_0 = \mathsf{CONT}] \cdot \mathbb{P}(m_0 = \mathsf{CONT}) \\
&\quad + \mathbb{E}_{\pi_1, \tilde{\sigma}}[v(\theta_1, \omega_1) | m_0 = \mathsf{QUIT}] \cdot \mathbb{P}(m_0 = \mathsf{QUIT}) \\
&= \text{AgentUtility}(S, \tilde{\sigma}, F, 1)\,,
\end{aligned}$$

where CONT stands for $\mathsf{CONTINUE}$.

As F was arbitrary, $\text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1) \geq \text{AgentUtility}(S, \tilde{\sigma}, F, 1)$ in all cases for any distribution F. The above cases cover all possibilities for the alternative strategy $\tilde{\sigma}$ and it follows that S is incentive compatible, i.e., $\text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1) \geq \text{AgentUtility}(S, \tilde{\sigma}, F, 1)$ for any distribution F and any feasible agent strategy $\tilde{\sigma}$.

For the last part of the lemma, we show the original objective of (3) and the alternative objective lead to the same value. Fix an arbitrary single-round direct mechanism $S \in \Delta(\Omega)^\Theta$. We have

$$\begin{aligned}
&\sup_{\theta' \in \Theta} \left\{ \text{OPT}(\delta_{\theta'}, 1) - \int_\Omega u(\theta', \omega) dS_{\theta'}(\omega) \right\} \\
&\qquad = \sup_{\theta' \in \Theta} \left\{ \int_\Theta \text{OPT}(\delta_\theta, 1) d\delta_{\theta'}(\theta) - \int_\Theta \int_\Omega u(\theta, \omega) dS_\theta(\omega) d\delta_{\theta'}(\theta) \right\} \\
&\qquad \leq \sup_{F \in \Delta(\Theta)} \left\{ \int_\Theta \text{OPT}(\delta_\theta, 1) dF(\theta) - \int_\Theta \int_\Omega u(\theta, \omega) dS_\theta(\omega) dF(\theta) \right\}\,,
\end{aligned}$$

where the first step is by rewriting the inner expressions and the second step is because point-mass distributions are a subset of all probability distributions supported on Θ, $\Delta(\Theta)$. For the other direction, we note that

$$
\begin{aligned}
&\sup_{F\in\Delta(\Theta)}\left\{\int_{\Theta}\mathrm{OPT}(\delta_\theta,1)\mathrm{d}F(\theta)-\int_{\Theta}\int_{\Omega}u(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta)\right\}\\
&\qquad=\sup_{F\in\Delta(\Theta)}\int_{\Theta}\left(\mathrm{OPT}(\delta_\theta,1)-\int_{\Omega}u(\theta,\omega)\mathrm{d}S_\theta(\omega)\right)\mathrm{d}F(\theta)\\
&\qquad\le\sup_{F\in\Delta(\Theta)}\sup_{\theta\in\Theta}\left\{\mathrm{OPT}(\delta_\theta,1)-\int_{\Omega}u(\theta,\omega)dS_\theta(\omega)\right\}\\
&\qquad=\sup_{\theta\in\Theta}\left\{\mathrm{OPT}(\delta_\theta,1)-\int_{\Omega}u(\theta,\omega)dS_\theta(\omega)\right\}.
\end{aligned}
$$

As S was arbitrary, it follows that the original objective and alternative objective achieve the same value for $S\in\Delta(\Omega)^{\Theta}$. Therefore, the optimization problem (3) and the version with the alternative objective are equivalent in terms of the optimal value and optimal solutions.

A.3 Additional Materials for Section 3.4

We prove Lemmas 2 and 3 in Appendices A.3.1 and A.3.2, respectively. We use Proposition 1 which is proved in Appendix A.3.3.

A.3.1 Proof of Lemma 2

We need to relate the two regret objectives of the multi-round and single-round problems and reduce the multi-round problem to the single-round problem for direct IC/IR mechanisms. The following lemma is useful. It follows from a revelation-principle-type argument and shows that if the agent's distribution is restricted to point-mass distributions, the principal's dynamic mechanism effectively reduces to a single-round direct mechanism with IC/IR properties and we can assume the agent's recommended strategy is the truthful reporting strategy σ^{TR}. See Appendix A.3.4 for the proof of the lemma.

Lemma 1. *For any incentive compatible dynamic mechanism A with a recommended agent strategy σ, there exists a single-round direct IC/IR mechanism, denoted $S(A)$, such that for any $\theta\in\Theta$,*

$$
\mathrm{PrincipalUtility}(A,\sigma,\delta_\theta,T)=T\cdot\mathrm{PrincipalUtility}(S(A),\sigma^{TR},\delta_\theta,1)\,,
$$

where σ^{TR} is the agent's truthful reporting strategy under which the agent participates (i.e., reports CONTINUE *in Round 0) and truthfully reports his shock.*

Using the above lemma, we prove Lemma 2 as follows:

Proof of Lemma 2. Fix an arbitrary incentive compatible dynamic mechanism $A\in\mathcal{A}$. Note that

$$
\sup_{F\in\mathcal{F}}\mathrm{Regret}(A,F,T)\ge\sup_{\theta\in\Theta}\mathrm{Regret}(A,\delta_\theta,T)\,,
$$

since point-mass distributions are a subset of general distributions $\mathcal{F}$ by Assumption 1. We can equivalently write the last expression as

$$\begin{aligned}\sup_{\theta\in\Theta}\text{Regret}(A,\delta_\theta,T) &= \sup_{\theta\in\Theta}\{\text{OPT}(\delta_\theta,T)-\text{PrincipalUtility}(A,\sigma,\delta_\theta,T)\}\\ &= T\cdot\sup_{\theta\in\Theta}\{\text{OPT}(\delta_\theta,1)-\text{PrincipalUtility}(S(A),\sigma^{\text{TR}},\delta_\theta,1)\}\\ &= T\cdot\sup_{\theta\in\Theta}\text{Regret}(S(A),\delta_\theta,1)\,,\end{aligned}$$

where σ is the recommended agent strategy as part of the mechanism A in the first step, $S(A)$ in the second step is the single-round direct IC/IR mechanism derived from A as described in the proof of Lemma 1, and the second step follows from the same lemma and Proposition 1.

Putting the above together, for the single-round direct IC/IR mechanism $S(A)$, we have

$$\sup_{F\in\mathcal{F}}\text{Regret}(A,F,T)\geq T\cdot\sup_{\theta\in\Theta}\text{Regret}(S(A),\delta_\theta,1)\,. \qquad \square$$

A.3.2 Proof of Lemma 3

To prove Lemma 3, we need the following lemmas. The first one is about direct static mechanisms that are simply T repetitions of a single-round direct IC/IR mechanism. The second one is a technical step that involves a variant of the regret notion with a different benchmark other than $\text{OPT}(F,T)$. We prove these lemmas in Appendix A.3.4.

Lemma 2. *Let $S^{\times T}$ denote the direct static mechanism that repeats single-round direct IC/IR mechanism S for T rounds. For any single-round direct IC/IR mechanism $S\in\mathcal{S}^{\times 1}$, $S^{\times T}$ is incentive compatible and*

$$\text{PrincipalUtility}(S^{\times T},\sigma^{TR},F,T)=T\cdot\text{PrincipalUtility}(S,\sigma^{TR},F,1)$$

for any agent's distribution F, where σ^{TR} is the agent's truthful reporting strategy under which the agent participates (i.e., reports CONTINUE *in Round 0) and truthfully reports his shock(s).*

Lemma 3. *For any single-round direct IC/IR mechanism $S\in\mathcal{S}^{\times 1}$,*

$$\sup_{F\in\mathcal{F}}\{\mathbb{E}_{\theta\sim F}[\text{OPT}(\delta_\theta,1)]-\text{PrincipalUtility}(S,F,1)\}\leq\sup_{\theta\in\Theta}\text{Regret}(S,\delta_\theta,1)\,.$$

We now have:

Proof of Lemma 3. Let $S\in\mathcal{S}^{\times 1}$ be any single-round direct IC/IR mechanism and consider the direct static mechanism $S^{\times T}$ which is T repetitions of S. Note $S^{\times T}$ is incentive compatible by Lemma 2. By the definition of Regret notion,

$$\sup_{F\in\mathcal{F}}\text{Regret}(S^{\times T},F,T)=\sup_{F\in\mathcal{F}}\left\{\text{OPT}(F,T)-\text{PrincipalUtility}(S^{\times T},\sigma^{\text{TR}},F,T)\right\}\,,$$

where σ^{TR} is the agent's truthful reporting strategy (i.e., the agent reports CONTINUE in Round 0 and truthfully reports his shocks in future rounds) that is the recommended strategy for direct mechanisms. Note that for any distribution $F \in \mathcal{F}$,

$$\mathrm{OPT}(F,T) \leq \mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\delta_\theta,T)] = T \cdot \mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\delta_\theta,1)]\,,$$

where the inequality is by Assumption 2 and the equality is by Proposition 1. Then,

$$\begin{aligned}\sup_{F\in\mathcal{F}} \mathrm{Regret}(S^{\times T},F,T) &\leq \sup_{F\in\mathcal{F}} \left\{T\cdot \mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\delta_\theta,1)] - \mathrm{PrincipalUtility}(S^{\times T},\sigma^{\mathrm{TR}},F,T)\right\}\\ &= T\cdot \sup_{F\in\mathcal{F}} \left\{\mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\delta_\theta,1)] - \mathrm{PrincipalUtility}(S,\sigma^{\mathrm{TR}},F,1)\right\}\,,\end{aligned}$$

where the last step is by Lemma 2. By Lemma 3, note the optimization problem in the last expression can be upper bounded as follows:

$$\sup_{F\in\mathcal{F}} \left\{\mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\delta_\theta,1)] - \mathrm{PrincipalUtility}(S,\sigma^{\mathrm{TR}},F,1)\right\} \leq \sup_{\theta\in\Theta} \mathrm{Regret}(S,\sigma^{\mathrm{TR}},\delta_\theta,1)\,.$$

It follows that

$$\sup_{F\in\mathcal{F}} \mathrm{Regret}(S^{\times T},F,T) \leq T\cdot \sup_{\theta\in\Theta} \mathrm{Regret}(S,\delta_\theta,1)\,. \qquad \square$$

A.3.3 Proofs of Propositions 1 and 2

Proof of Proposition 1. The proof follows straightforwardly from Proposition 3 in Appendix E. For any $\theta \in \Theta$,

$$\mathrm{OPT}(\delta_\theta,T) = T\cdot \bar{u}(\delta_\theta) = T\cdot \mathrm{OPT}(\delta_\theta,1)\,,$$

by the second part of Proposition 3. Alternatively, we can prove the proposition directly using the same ideas in the proof of Proposition 3. We keep this presentation to avoid repeating proofs. $\square$

Proof of Proposition 2. In what follows, let $\widehat{\mathrm{Regret}} := \inf_{S\in\mathcal{S}^{\times 1}} \sup_{\theta\in\Theta} \mathrm{Regret}(S,\delta_\theta,1)$. Fix an arbitrary incentive compatible dynamic mechanism A. Following the same line of reasoning in the beginning of the proof of Lemma 2, we have for any $\theta \in \Theta$,

$$\mathrm{Regret}(A,\delta_\theta,T) = T\cdot \mathrm{Regret}(S(A),\delta_\theta,1)\,,$$

where $S(A)$ is the single-round direct IC/IR mechanism derived from A as described in the proof of Lemma 1. Note $\sup_{\theta\in\Theta} \mathrm{Regret}(S(A),\delta_\theta,1) \geq \widehat{\mathrm{Regret}}$. By the definition of supremum, for any $\epsilon > 0$, there exists a point-mass distribution δ_{θ^*} for some $\theta^* \in \Theta$ such that

$$\mathrm{Regret}(S(A),\delta_{\theta^*},1) \geq \widehat{\mathrm{Regret}} - \frac{\epsilon}{T}\,.$$

Combining with the above observation, we then have

$$\mathrm{Regret}(A,\delta_{\theta^*},T) \geq T\cdot \widehat{\mathrm{Regret}} - \epsilon\,.$$

Since Assumptions 1 and 2 hold, we have that $\mathrm{Regret}(T) = T\cdot\widehat{\mathrm{Regret}}$ by Theorem 1 and, hence, that

$$\mathrm{Regret}(A,\delta_{\theta^*},T) \geq \mathrm{Regret}(T) - \epsilon\,. \qquad \square$$

A.3.4 Remaining Proofs from Appendix A.3

Proof of Lemma 1. First, we show a construction of a single-round direct IC/IR mechanism which will be our choice of $S(A)$ and then prove the claimed statements.

Let $\{\omega_{\theta,t}\}_{t=1}^{T}$ be a sequence of outcomes realized when the agent plays the recommended strategy σ against the principal's mechanism A when his distribution is δ_θ. When the agent reports QUIT in Round 0 and does not participate, the sequence is simply the no-interaction outcome in all rounds. Consider the following single-round direct mechanism S which is a collection of distributions S_θ on Ω indexed by $\theta \in \Theta$. For any $\theta \in \Theta$ and measurable set $W \subset \Omega$, we define

$$S_\theta(W) := \frac{1}{T}\sum_{t=1}^{T} \mathbb{P}_{\pi,\sigma}\left(\omega_{\theta,t} \in W | F = \delta_\theta\right) ,$$

where the expectation is taken over the randomness of π and σ (here, the shocks are deterministic and equal to θ). We can interpret S_θ as the time-averaged distribution of outcomes when the agent's distribution is the point-mass distribution δ_θ and the agent plays the recommended strategy σ.

Using the representation of S in terms of outcome distributions (as described in Section 3.3), we can show that S satisfies both IC and IR constraints as formulated in the optimization problem (3). For any $\theta, \theta' \in \Theta$,

$$\begin{aligned}
\mathbb{E}_{\omega\sim S_{\theta'}}[v(\theta,\omega)] &= \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} v(\theta,\omega_{\theta',t})\right] \\
&= \frac{1}{T}\,\mathrm{AgentUtility}(A,\sigma',\delta_\theta,T) \\
&\le \frac{1}{T}\,\mathrm{AgentUtility}(A,\sigma,\delta_\theta,T) \\
&= \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} v(\theta,\omega_{\theta,t})\right] \\
&= \mathbb{E}_{\omega\sim S_\theta}[v(\theta,\omega)] ,
\end{aligned}$$

where σ' is an alternative strategy under which the agent, in particular, reports according the recommended strategy σ as if his distribution is $\delta_{\theta'}$ when his actual distribution is δ_θ and the inequality follows from that A is incentive compatible and σ is a utility-maximizing strategy for the agent when, in particular, his distribution is δ_θ. Similarly, for any $\theta \in \Theta$,

$$\begin{aligned}
\mathbb{E}_{\omega\sim S_\theta}[v(\theta,\omega)] &= \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} v(\theta,\omega_{\theta,t})\right] \\
&= \frac{1}{T}\,\mathrm{AgentUtility}(A,\sigma,\delta_\theta,T) \\
&\ge 0 ,
\end{aligned}$$

where the inequality follows because a utility-maximizing agent can guarantee the total utility of at least 0 by not participating. Since A is incentive compatible, the recommended strategy σ ensures the agent obtains a non-negative utility. Hence, S constructed above is a single-round direct IC/IR mechanism.

By construction, we have for any θ,

$$\begin{aligned}\text{PrincipalUtility}(A, \sigma, \delta_\theta, T) &= T \cdot \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} u(\theta, \omega_{\theta,t})\right] \\ &= T \cdot \mathbb{E}_{\omega\sim S_\theta}[u(\theta, \omega)] \\ &= T \cdot \text{PrincipalUtility}(S, \sigma^{\text{TR}}, \delta_\theta, 1)\,. \qquad \square\end{aligned}$$

Proof of Lemma 2. Fix an arbitrary single-round direct IC/IR mechanism $S \in \mathcal{S}^{\times 1}$ with decision rule $\tilde{\pi}$ and let $S^{\times T}$ be the direct static mechanism that repeats S. Abusing notations, we use σ^{TR} to denote the truthful reporting strategy (that reports CONTINUE in Round 0 and truthfully reports the shocks) to be the associated recommended strategy for the agent for both the single-round and multi-round direct mechanisms. In particular, we have that

$$\text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1) \geq \text{AgentUtility}(S, \tilde{\sigma}, F, 1)\,,$$

for every probability distribution F over Θ and every feasible agent strategy $\tilde{\sigma}$.

First, we argue that $\text{PrincipalUtility}(S^{\times T}, \sigma^{\text{TR}}, F, T) = T \cdot \text{PrincipalUtility}(S, \sigma^{\text{TR}}, F, 1)$ for every distribution F. Since the same decision rule $\tilde{\pi}$ is used under $S^{\times T}$ by the principal and the same truthful reporting strategy is used by the agent in each round, the realized distribution of outcomes associated with each shock under F is identical across rounds and, hence, the principal utility restricted to each round is identical across rounds. The principal utility restricted to each round is exactly the principal utility under S (and the agent truthfully reports). Therefore, the principal utility under $S^{\times T}$ is T times the principal utility under S. Similarly, we have that the agent utility restricted to each round under $S^{\times T}$ is exactly the agent utility under S and, hence, that $\text{AgentUtility}(S^{\times T}, \sigma^{\text{TR}}, F, T) = T \cdot \text{AgentUtility}(S, \sigma^{\text{TR}}, F, 1)$ for every distribution F.

Now, we argue that $S^{\times T}$ is incentive compatible. For the sake of contradiction, assume there exists a distribution F' and an agent strategy $\sigma' = \{\sigma'_t\}_{0:T}$ such that $\text{AgentUtility}(S^{\times T}, \sigma^{\text{TR}}, F', T) < \text{AgentUtility}(S^{\times T}, \sigma', F', T)$. We define per-round expected agent utility V_t and principal utility U_t when the principal implements $S^{\times T}$ and the agent plays σ' as

$$V_t = \mathbb{E}[v(\theta_t, \tilde{\pi}(\sigma'_t(\theta_t, h_t^+)))] \quad \text{and} \quad U_t = \mathbb{E}[u(\theta_t, \tilde{\pi}(\sigma'_t(\theta_t, h_t^+)))]$$

for Rounds $t \in [T]$. Note the principal's mechanism has no dependence on histories while the agent's strategy may depend on the augmented history h_t^+. Since σ' outperforms σ^{TR}, there is a particular round t in which V_t is strictly greater than the agent utility achieved under the single-round mechanism S and σ^{TR}. We use the following claim:

Claim 1. *For any t, there is an agent strategy against S that achieves the expected agent utility and principal utility equal to V_t and U_t, respectively.*

Proof. Fix arbitrary $t \in [T]$. The agent can implement the t-th round strategy σ'_t as a standalone agent strategy against S by implementing σ'_0 in Round 0 and then σ'_t in Round 1. To implement σ'_t, the agent internally chooses randomness z_t and y_t and simulates the history $h_t^+ = (F', \theta_{1:t-1}, m_{0:t-1}, \omega_{1:t-1})$ that is needed for σ'_t. This is possible from the knowledge of S and $\{\sigma'_{t'}\}_{0:t-1}$. By construction, when the agent implements the above strategy against S, the expected agent utility and principal utility equal V_t and U_t, respectively. $\square$

By the above claim, there exists an alternative agent strategy against the single-round mechanism S that yields the expected agent utility equal to V_t which is strictly greater than $\text{AgentUtility}(S, \sigma^{\text{TR}}, F', 1)$. This contradicts that S is incentive compatible with respect to the recommended strategy σ^{TR}. $\square$

Proof of Lemma 3. Let $S \in \mathcal{S}^{\times 1}$ be an arbitrary single-round direct IC/IR mechanism and σ^{TR} be the recommended truthful reporting strategy for the agent. Using the representation of S in terms of outcome distributions as described in Section 3.3, we then have

$$
\begin{aligned}
&\sup_{F \in \mathcal{F}} \left\{ \mathbb{E}_{\theta \sim F}[\text{OPT}(\delta_\theta, 1)] - \text{PrincipalUtility}(S, \sigma^{\text{TR}}, F, 1) \right\} \\
&\qquad = \sup_{F \in \mathcal{F}} \left\{ \int_\Theta \text{OPT}(\delta_\theta, 1) \mathrm{d}F(\theta) - \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \right\} \\
&\qquad = \sup_{F \in \mathcal{F}} \int_\Theta \left(\text{OPT}(\delta_\theta, 1) - \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \right) \mathrm{d}F(\theta) \\
&\qquad = \sup_{F \in \mathcal{F}} \int_\Theta \text{Regret}(S, \sigma^{\text{TR}}, \delta_\theta, 1) \mathrm{d}F(\theta) \\
&\qquad \le \sup_{\theta \in \Theta} \text{Regret}(S, \sigma^{\text{TR}}, \delta_\theta, 1) \,.
\end{aligned}
$$

$\square$

B Additional Materials for Section 4.1

B.1 Proof of Proposition 3

As discussed already, Theorem 1 applies and Proposition 3 follows if we show that the single-round direct IC/IR mechanism S^* is an optimal solution to the single-round problem (2) and achieves the value of $\frac{1}{e}$. By Lemma 1, we can equivalently solve the optimization problem (3) with the alternative objective $\sup_{F \in \Delta(\Theta)} \left\{ \int_\Theta \text{OPT}(\delta_\theta, 1) dF(\theta) - \int_\Theta \int_\Omega u(\theta, \omega) dS_\theta(\omega) dF(\theta) \right\}$. Using the notation $\widehat{\text{Regret}}(S, F) := \int_\Theta \text{OPT}(\delta_\theta, 1) dF(\theta) - \int_\Theta \int_\Omega u(\theta, \omega) dS_\theta(\omega) dF(\theta)$, this optimization problem can be written as

$$
\inf_{\substack{S \in \Delta(\Omega)^\Theta: \\ \text{(IC),(IR)}}} \sup_{F \in \Delta(\Theta)} \widehat{\text{Regret}}(S, F) \,,
$$

where S can be any single-round direct IC/IR mechanism. Let $\widehat{\text{Regret}}$ be the corresponding optimal value.

We now solve the above optimization problem by proving the following saddle-point result, which is closely related to a similar result due to Bergemann and Schlag (2008).

Proposition 1. *Let S^* be the randomized posted pricing strategy given in Proposition 3, which is a single-round direct IC/IR mechanism, and the agent's distribution F^* be given by*

$$
F^*(\theta) = \begin{cases} 0, & \text{if } \theta \in [0, \frac{1}{e}) \\ 1 - \frac{1}{e\theta}, & \text{if } \theta \in [\frac{1}{e}, 1) \\ 1, & \text{if } \theta = 1 \end{cases} .
$$

Then, $\widehat{\mathrm{Regret}} = \widehat{\mathrm{Regret}}(S^*, F^*) = \frac{1}{e}$ *and*

$$\widehat{\mathrm{Regret}}(S^*, F) \le \widehat{\mathrm{Regret}}(S^*, F^*) \le \widehat{\mathrm{Regret}}(S, F^*)\,,$$

for any single-round direct IC/IR mechanism $S \in \mathcal{S}^{\times 1}$ *and distribution* $F \in \Delta(\Theta)$.

Proof. When the principal implements a randomized posted pricing strategy, it is best for the agent to truthfully respond, that is, buy the item if the price is lower than his value, and, therefore, the randomized posted pricing strategy satisfies the IC and IR constraints. Furthermore, a randomized posted pricing mechanism has a direct implementation in that the principal can internally draw a random posted price and determine the allocation and payment given the agent's report θ.

First, we prove $\widehat{\mathrm{Regret}}(S^*, F) \le \widehat{\mathrm{Regret}}(S^*, F^*)$ for any agent's distribution F. Note we can represent any single-round direct IC/IR mechanism S with the corresponding interim rules (x, p) where, by standard arguments in mechanism design, the allocation rule x is monotonically non-decreasing and the payment rule p satisfies

$$p(\theta) = p(0) + x(\theta)\cdot\theta - \int_0^\theta x(t)dt\,, \quad \forall\theta \in [0,1]$$

and $p(0) \le 0$. In particular, let (x^*, p^*) be the interim allocation and payment rules of S^* where $x^*(\theta) = 0$ for $\theta \in [0, \frac{1}{e})$ and $x^*(\theta) = 1 + \ln\theta$ for $\theta \in [\frac{1}{e}, 1]$ and $p^*(\theta) = x^*(\theta)\cdot\theta - \int_0^\theta x^*(t)dt$ for all θ.

It suffices to show that F^* is a solution to the following optimization problem:

$$\max_{F \in \Delta(\Theta)} \left\{ \int_\Theta \theta - p^*(\theta) dF(\theta) \right\},$$

where F can be any distribution over Θ. Given x^*, we can simplify p^* as $p^*(\theta) = 0$ for $\theta \in [0, \frac{1}{e})$ and

$$p^*(\theta) = (1 + \ln\theta)\theta - \int_{\frac{1}{e}}^\theta 1 + \ln t dt = \theta - \frac{1}{e}\,,$$

for $\theta \in [\frac{1}{e}, 1]$. Then, the integrand in the objective function is, equivalently, $\theta\mathbf{1}\{\theta < \frac{1}{e}\} + \frac{1}{e}\mathbf{1}\{\theta \ge \frac{1}{e}\}$ and the optimization problem becomes

$$\max_{F \in \Delta(\Theta)} \left\{ \Pr_{\theta\sim F}\left(\theta < \frac{1}{e}\right)\cdot\mathbb{E}_{\theta\sim F}\left[\theta \mid \theta < \frac{1}{e}\right] + \Pr_{\theta\sim F}\left(\theta \ge \frac{1}{e}\right)\cdot\frac{1}{e} \right\}.$$

It follows that any distribution with its support contained in $[\frac{1}{e}, 1]$ is an optimal solution and F^* is one such distribution. Furthermore, we see that the optimization problem has the value of $\frac{1}{e}$ and, so, $\widehat{\mathrm{Regret}} = \frac{1}{e}$.

Next, we show $\widehat{\mathrm{Regret}}(S^*, F^*) \le \widehat{\mathrm{Regret}}(S, F^*)$ for any single-round direct IC/IR mechanism S. Similar to the above argument, we show that S^* is a solution to:

$$\min_{(x,p)} \left\{ \int_\Theta \theta - p(\theta) dF^*(\theta) \text{ s. t. (IC), (IR)} \right\},$$

where (x,p) are over all possible interim rules satisfying the IC/IR constraints. By standard arguments, the payment rule p satisfies $p(\theta) = p(0) + x(\theta)\cdot\theta - \int_0^\theta x(t)dt$ for $\theta \in [0,1]$ and $p(0) \le 0$ and the allocation rule x is monotonically non-decreasing. Then, the above optimization problem becomes

$$\min_{\text{non-decreasing } x, p(0)\le 0} \left\{ -p(0) + \int_0^1 \left(\theta - x(\theta)\cdot\theta + \int_0^\theta x(t)dt\right) f^*(\theta)d\theta \right\} ,$$

where f^* is the probability density function for F^* with $f^*(\theta) = 0$ for $\theta \in [0, \frac{1}{e})$, $f^*(\theta) = \frac{1}{e\theta^2}$ for $\theta \in [\frac{1}{e}, 1)$ and a point-mass of $\frac{1}{e}$ at $\theta = 1$. By changing the ordering of the integrals, $\int_0^1 \int_0^\theta x(t) f^*(\theta) dt d\theta = \int_0^1 \int_t^1 x(t) f^*(\theta) d\theta dt = \int_0^1 (1 - F^*(t))x(t)dt$, and the optimization problem is equivalently

$$\min_{\text{non-decreasing } x, p(0)\le 0} \left\{ -p(0) + \mathbb{E}_{\theta\sim F^*}[\theta] + \int_0^1 \left(-\theta\cdot f^*(\theta) + (1 - F^*(\theta))\right) x(\theta)d\theta \right\} .$$

In the integral, the expression $\phi(\theta) := -\theta \cdot f^*(\theta) + (1 - F^*(\theta))$ can be further simplified as 1 if $\theta \in [0, \frac{1}{e})$, 0 if $\theta \in [\frac{1}{e}, 1)$ and a point-mass of $-\frac{1}{e}$ if $\theta = 1$. Then, the objective function is equal to

$$-p(0) + \mathbb{E}_{\theta\sim F^*}[\theta] + \int_0^{\frac{1}{e}} 1\cdot x(\theta)d\theta + \int_{\frac{1}{e}}^1 0\cdot x(\theta)d\theta - \frac{1}{e}\cdot x(1) .$$

It follows that an optimal solution has $p(0) = 0$ and a non-decreasing $x(\cdot)$ such that $x(\theta) = 0$ for $\theta \in [0, \frac{1}{e})$, $x(\theta) \ge 0$ for $\theta \in [\frac{1}{e}, 1)$ and $x(\theta) = 1$ for $\theta = 1$ (in the almost everywhere sense for $\theta < 1$). Clearly, S^* satisfies these conditions and is, therefore, an optimal solution. □

C Additional Materials for Section 4.2

C.1 Single-Round Problem

We provide further details on the single-round direct IC/IR mechanisms and a justification for the restriction to those with deterministic contracts for the single-round problem. Instead of (2), we equivalently consider the optimization problem (3) with the alternative objective $\sup_{F\in\Delta(\Theta)}\{\int_\Theta \mathrm{OPT}(\delta_\theta, 1)dF(\theta) - \int_\Theta \int_\Omega u(\theta,\omega)dS_\theta(\omega)dF(\theta)\}$, by Lemma 1. Introducing the notation $\widehat{\mathrm{Regret}}(S,F) := \int_\Theta \mathrm{OPT}(\delta_\theta, 1)dF(\theta) - \int_\Theta \int_\Omega u(\theta,\omega)dS_\theta(\omega)dF(\theta)$, this optimization problem can be written as

$$\inf_{\substack{S\in\Delta(\Omega)^\Theta: \\ \text{(IC),(IR)}}} \sup_{F\in\Delta(\Theta)} \widehat{\mathrm{Regret}}(S,F) ,$$

where S can be any single-round direct IC/IR mechanism. Let $\widehat{\mathrm{Regret}}$ be the corresponding optimal value.

We can think of a single-round direct IC/IR mechanism as a collection of distributions S_θ on $\mathbb{R}^+ \times \mathbb{R}$ indexed by $\theta \in [\underline{\theta}, \bar{\theta}]$ such that when the agent reports θ, the outcome is determined by drawing from S_θ, i.e., $(\hat{q},\hat{p}) \sim S_\theta$ for production level $\hat{q}$ and payment $\hat{p}$. Abusing notations, let $q(\theta) = \mathbb{E}_{(\hat{q},\hat{p})\sim S_\theta}[\hat{q}]$ and $p(\theta) = \mathbb{E}_{(\hat{q},\hat{p})\sim S_\theta}[\hat{p}]$ be the interim allocation and payment rules, respectively. Then, the IC and

IR constraints can be expressed as follows:

$$p(\theta) - \theta \cdot q(\theta) \geq p(\theta') - \theta \cdot q(\theta') , \quad \forall \theta, \theta' \in [\underline{\theta}, \bar{\theta}] \tag{IC}$$
$$p(\theta) - \theta \cdot q(\theta) \geq 0 , \quad \forall \theta \in [\underline{\theta}, \bar{\theta}] \tag{IR}$$

Fix an arbitrary single-round direct IC/IR mechanism. Let $V(\theta) = p(\theta) - \theta \cdot q(\theta)$ for $\theta \in [\underline{\theta}, \bar{\theta}]$. Note $V(\theta)$ is convex and by standard arguments (similar to the auction case, e.g., in Chapter 5 in Krishna (2009)), $q(\theta)$ is non-increasing and V is absolutely continuous and $V'(\theta) = -q(\theta)$ where the derivative exists. As $q(\theta)$ is nonnegative, $V(\theta)$ is non-increasing. Furthermore, we can write

$$p(\theta) = V(\bar{\theta}) + \theta \cdot q(\theta) + \int_{\theta}^{\bar{\theta}} q(x)dx , \text{ for } \theta \in [\underline{\theta}, \bar{\theta}] .$$

Using the notation $\widehat{\text{Regret}}$ and noting $\text{OPT}(\delta_\theta, 1) = \bar{R}(\theta)$, we have

$$\widehat{\text{Regret}}(S, F) = \mathbb{E}_{\theta \sim F}[\bar{R}(\theta)] - \mathbb{E}_{\theta \sim F, (\hat{q}, \hat{p}) \sim S_\theta}[R(\hat{q}) - \hat{p}] ,$$

and

$$\widehat{\text{Regret}} = \inf_{\substack{S \in \Delta(\Omega)^\Theta: \\ \text{(IC),(IR)}}} \sup_{F \in \Delta(\Theta)} \widehat{\text{Regret}}(S, F) .$$

Since the revenue function $R(x)$ is concave, $\mathbb{E}_{(\hat{q}, \hat{p}) \sim S_\theta}[R(\hat{q})] \leq R(\mathbb{E}_{(\hat{q}, \hat{p}) \sim S_\theta}[\hat{q}])$ for any θ. Given any single-round direct IC/IR mechanism S, we can potentially improve (but not hurt) its performance by modifying S_θ to always return a deterministic production level $\hat{q}$ that is the average $q(\theta)$:

$$\mathbb{E}_{(\hat{q}, \hat{p}) \sim S_\theta}[R(\hat{q}) - \hat{p}] \leq R(\mathbb{E}_{(\hat{q}, \hat{p}) \sim S_\theta}[\hat{q}]) - \mathbb{E}_{(\hat{q}, \hat{p}) \sim S_\theta}[\hat{p}] = R(q(\theta)) - p(\theta) .$$

Note the IC/IR constraints are still satisfied. Without loss in the minimax regret objective, we can restrict to those single-round IC/IR mechanisms that can be described in terms of a menu of deterministic contracts $(q(\theta), p(\theta))$ for $\theta \in [\underline{\theta}, \bar{\theta}]$. Using Ω^Θ to denote this restricted set of single-round direct mechanisms, the minimax regret for the single-round problem is equal to:

$$\widehat{\text{Regret}} = \inf_{\substack{(q,p) \in \Omega^\Theta: \\ \text{(IC),(IR)}}} \sup_{F \in \Delta(\Theta)} \int_\Theta \bar{R}(\theta) - (R(q(\theta)) - p(\theta)) \, dF(\theta) ,$$

and for $S = (q, p) \in \Omega^\Theta$ and $F \in \Delta(\Theta)$,

$$\widehat{\text{Regret}}(S, F) = \int_\Theta \bar{R}(\theta) - (R(q(\theta)) - p(\theta)) \, dF(\theta) .$$

C.2 Proof of Proposition 4

By Theorem 1, to prove Proposition 4, it suffices that we prove the stated single-round direct IC/IR mechanism is an optimal solution to the single-round problem. For the single-round problem, we consider the equivalent optimization problem (3) with the alternative objective given in Lemma 1 and restrict attention to those single-round direct mechanisms that can be described in terms of a menu of deterministic contracts, denoted Ω^{Θ}; see details in Appendix C.1. More specifically, we consider the following optimization problem

$$\inf_{\substack{(q,p)\in\Omega^{\Theta}:\\ \text{(IC),(IR)}}} \sup_{F\in\Delta(\Theta)} \widehat{\text{Regret}}(S,F)\,,$$

where Ω^{Θ} denotes the restricted class of single-round direct mechanisms and $\widehat{\text{Regret}}(S,F) = \int_{\Theta} \bar{R}(\theta) - (R(q(\theta)) - p(\theta))\, dF(\theta)$ for $S = (q,p) \in \Omega^{\Theta}$ and $F \in \Delta(\Theta)$. Let $\widehat{\text{Regret}}$ be the corresponding optimal value of the optimization problem.

We show the single-round direct IC/IR mechanism given in Proposition 4 is an optimal solution to the above optimization problem via the following saddle-point result.

Proposition 2. *Let S^* be the single-round direct IC/IR mechanism corresponding to the menu of deterministic contracts $\{(q^*(\theta), p^*(\theta)\}_{\theta\in\Theta}$ given in Proposition 4 and the agent's distribution be given by a point-mass of $F^*(\underline{\theta})$, which can be 0, at $\theta = \underline{\theta}$ and a density $\frac{d}{d\theta}F^*(\theta)$ for $\theta \in (\underline{\theta}, \kappa]$ where*

$$F^*(\theta) = e^{-\int_{\theta}^{\kappa} \frac{1}{R'(q^*(x))-x} dx}\,,$$

for the same κ in the definition of S^*. Then, S^* and F^* are well-defined and the minimax regret is $\widehat{\text{Regret}} = \widehat{\text{Regret}}(S^*, F^*) = \int_{\underline{\theta}}^{\bar{\theta}} q^*(x)dx$ and is strictly positive, and

$$\widehat{\text{Regret}}(S^*, F) \le \widehat{\text{Regret}}(S^*, F^*) \le \widehat{\text{Regret}}(S, F^*)\,,$$

for any single-round direct IC/IR mechanism $S \in \mathcal{S}^{\times 1}$ and distribution $F \in \Delta(\Theta)$.

We first show that single-round direct IC/IR mechanism S^* and distribution F^* in the statement of Proposition 2 are well-defined with the claimed characterizations in the following lemma and then prove Proposition 2.

Lemma 4. *The single-round direct IC/IR mechanism S^* and distribution F^* in Propositions 4 and 2 are well-defined. Furthermore, q^* is continuous over the shock space and, in particular, strictly decreases over $[\underline{\theta}, \kappa]$.*

Proof. We proceed in two steps showing S^* and then F^* are well-defined.

(Single-Round Direct IC/IR Mechanism S^*): It suffices to show that a strictly decreasing solution q^* exists to the ordinary differential equation

$$\begin{aligned} &\frac{dq}{d\theta}(\theta) = \frac{-\bar{q}(\theta)}{R'(q(\theta)) - \theta}\,, \text{ for } \theta \in (\underline{\theta}, \kappa) \\ &\lim_{\theta\to\underline{\theta}^+} q(\theta) = \bar{q}(\underline{\theta})\,, \end{aligned} \tag{C-1}$$

for some κ to be determined and then complete it for $\theta = \underline{\theta}$ and $\theta > \kappa$ accordingly. We follow similar reasoning steps as in the proof of Lemma 2 in Carrasco et al. (2018). We equivalently solve the following differential system, with the roles of q and θ interchanged:

$$\begin{aligned} &\frac{d\theta}{dq}(q) = \frac{R'(q) - \theta(q)}{-\bar{q}(\theta(q))}\,,\ \text{for } q \le \bar{q}(\underline{\theta}) \\ &\theta(\bar{q}(\underline{\theta})) = \underline{\theta}\,. \end{aligned} \tag{C-2}$$

As we show, this system has a solution θ^* that is strictly decreasing over a suitable interval and we can invert the relationship between θ and q to obtain a solution q^* to the original differential system with a well-defined κ.

Recall a solution to an ordinary differential equation (ODE) is a continuously differentiable function defined on some interval satisfying the specified relations. Let $\psi(q,\theta) := \frac{R'(q)-\theta}{-\bar{q}(\theta)}$ defined on the domain $D = (0, \bar{q}(\underline{\theta}) + \epsilon] \times [\underline{\theta} - \epsilon, \bar{\theta} + \epsilon]$ for arbitrarily small $\epsilon > 0$; ϵ is there to make the domain an open set. By the assumptions on R, ψ is continuous on the domain. Furthermore, it is continuously differentiable on any closed set of the domain and, hence, locally Lipschitz with respect to θ. For any initial value point in D, there exists a unique solution to the differential equation $\frac{d\theta}{dq}(q) = \psi(q,\theta)$ in a neighborhood of the initial value point (e.g., Theorem 3.1 in Hale (1969)). In particular, the above system of differential equation (C-2) has a unique solution θ^* in a neighborhood of the point $(\bar{q}(\underline{\theta}), \underline{\theta})$.

Let $(\underline{q}, \bar{q}(\underline{\theta})]$ be the left maximal interval of definition of the ordinary differential equation (C-2). We show θ^* is strictly decreasing in this interval. Note if a solution $\theta(\cdot)$ has $\theta'(q) = 0$, then

$$\frac{d^2\theta}{dq^2}(q) = \psi_1(q,\theta) + \psi_2(q,\theta)\cdot\frac{d\theta}{dq} = -\frac{R''(q)}{\bar{q}(\theta)} > 0,$$

where ψ_i denotes the partial derivative with respect to the i-th parameter. Since $(\theta^*)'(\bar{q}(\underline{\theta})) = 0$, θ^* is strictly convex at $q = \bar{q}(\underline{\theta})$ and decreases over $[\bar{q}(\underline{\theta}) - \epsilon, \bar{q}(\underline{\theta})]$ for sufficiently small $\epsilon > 0$. Fix an arbitrary $q \in (\underline{q}, \bar{q}(\underline{\theta}))$ and assume θ^* achieves the maximum at some $x \in [q, \bar{q}(\underline{\theta})]$. Note x cannot be in the interior because the first-order condition $(\theta^*)'(x) = 0$ is satisfied and it would mean θ^* is strictly convex and is increasing to the left or right of x. By the above observation, x cannot be $\bar{q}(\underline{\theta})$. Hence, the maximum is achieved at the left-end $x = q$. As q was arbitrary, the argument extends and it implies θ^* is strictly decreasing over $(\underline{q}, \bar{q}(\underline{\theta})]$.

Now, we invert θ^* to obtain q^* that is a solution to the original differential system (C-1) that we want to solve. Let $\theta^*(\underline{q}) = \sup_{q\in(\underline{q},\bar{q}(\underline{\theta})]} \theta^*(q)$ which may be ∞. If $\theta^*(\underline{q}) < \bar{\theta}$, then $(\theta^*)'(\underline{q}) = \lim_{q\to\underline{q}^+} \psi(q, \theta^*(q))$ would be equal to $\frac{R'(\underline{q})-\theta^*(\underline{q})}{-\bar{q}(\theta^*(\underline{q}))}$ which is not defined, more specially, $R'(\underline{q})$ in the numerator, if $\underline{q} = 0$ but defined if $\underline{q} > 0$. Since we chose the left maximal interval of definition of the ODE, it must be that $\underline{q} = 0$. Then, we let $\kappa = \theta^*(\underline{q})$ and truncate the solution θ^* so that its range is exactly $[\underline{\theta}, \kappa)$. We let q^* be the inverted curve of the truncated solution on $[\underline{\theta}, \kappa)$ which strictly decreases and converges to 0 over the interval and extend $q^*(\theta) = 0$ for $\theta \in [\kappa, \bar{\theta}]$.

In the other case when $\theta^*(\underline{q}) \ge \bar{\theta}$, we truncate the solution θ^* such that its range is exactly $[\underline{\theta}, \bar{\theta}]$ and consider q^* to be the corresponding inverted solution over the interval $[\underline{\theta}, \bar{\theta}]$. By construction, q^* satisfies the desired differential system and stays positive over the whole interval. We choose $\kappa = \bar{\theta}$.

In both cases, since θ^* is continuous at $q = \bar{q}(\underline{\theta})$ with $\theta^*(\bar{q}(\underline{\theta})) = \underline{\theta}$, we have $q^*(\underline{\theta}) = \bar{q}(\underline{\theta})$ and $\lim_{\theta\to\underline{\theta}^+} q^*(\theta) = \bar{q}(\underline{\theta})$. Also, by our choice of κ, $\lim_{\theta\to\kappa^-} q^*(\theta) = q^*(\kappa)$. That is, q^* is continuous over the whole interval $[\underline{\theta}, \bar{\theta}]$.

(Distribution F^*): Given that we have a solution q^* that is strictly decreasing over $[\underline{\theta}, \kappa]$ and continuously differentiable over $(\underline{\theta}, \kappa)$, the fraction $\frac{1}{R'(q^*(\theta))-\theta}$ is well-defined and positive for $\theta \in (\underline{\theta}, \kappa)$. We argue that the integral $\int_\theta^\kappa \frac{1}{R'(q^*(x))-x}dx$ exists over the same interval. If $\kappa = \bar{\theta}$ and q^* stays positive, the integrand is well-defined and continuous over the compact set. Hence, the integral exists. If $\kappa < \bar{\theta}$, then the integrand goes to 0 as x approaches κ and thus bounded. In this case, again, the integral exists.

As θ approaches $\underline{\theta}$, the integral $\int_\theta^\kappa \frac{1}{R'(q^*(x))-x}dx$ can potentially grow to ∞. But, $F^*(\theta) = e^{-\int_\theta^\kappa \frac{1}{R'(q^*(x))-x}dx}$ is absolutely continuous over $(\underline{\theta}, \kappa]$ and the distribution F^* can be described with a point-mass of $\lim_{\theta' \to \underline{\theta}^+} F^*(\theta')$ at $\theta = \underline{\theta}$, which can be 0, and the absolute continuous part with density $f^*(\theta) = \frac{d}{d\theta}F^*(\theta) = F^*(\theta) \cdot \frac{1}{R'(q^*(\theta))-\theta}$. □

Proof of Proposition 2. We restrict, without loss, to those single-round direct mechanisms that can be described in terms of a menu of deterministic contracts $(q(\theta), p(\theta))$ for $\theta \in [\underline{\theta}, \bar{\theta}]$ in our analysis; we use Ω^Θ to denote this class of mechanisms. By well-definedness, we mean both S^* and F^* exist with the stated characterizations. In particular, it would mean that q^* is a continuous monotone function over $[\underline{\theta}, \bar{\theta}]$ and is integrable. The well-definedness of S^* and F^* has been proved in Lemma 4.

For the first part of the saddle-point result, we show that F^* is an optimal solution to $\max_F \widehat{\text{Regret}}(S^*, F)$ which is equivalent to:

$$\max_{F \in \Delta(\Theta)} \left\{ \int_\Theta \bar{R}(\theta) - (R(q^*(\theta)) - p^*(\theta))\, dF(\theta) \right\} .$$

By the definition of p^*, the optimization problem is equivalent to

$$\max_{F \in \Delta(\Theta)} \left\{ \int_\Theta \left(\bar{R}(\theta) - R(q^*(\theta)) + \theta \cdot q^*(\theta) + \int_\theta^{\bar{\theta}} q^*(x) dx \right) dF(\theta) \right\} .$$

The integrand is continuous and its derivative with respect to θ is

$$-\bar{q}(\theta) - (R'(q^*(\theta)) - \theta) \cdot (q^*)'(\theta) ,$$

where we used $\bar{R}'(\theta) = -\bar{q}(\theta)$. Since $(q^*)'(\theta) = \frac{-\bar{q}(\theta)}{R'(q^*(\theta))-\theta}$ for $\theta \in (\underline{\theta}, \kappa)$, the derivative is equal to 0 over the same interval. For $\theta \in [\kappa, \bar{\theta}]$, the integrand is equal to $\bar{R}(\theta)$ and the derivative is equal to $-\bar{q}(\theta)$, which is negative. Since q^* is continuous, it follows that the integrand stays constant for $\theta \in [\underline{\theta}, \kappa]$ and then decreases for $\theta \in [\kappa, \bar{\theta}]$. Since F^* has support equal to exactly $[\underline{\theta}, \kappa]$, it maximizes the objective and is an optimal solution, as desired.

Similarly, for the second part, we show that S^* is an optimal solution to

$$\min_{\substack{S \in \Omega^\Theta: \\ \text{(IC),(IR)}}} \left\{ \int_\Theta \bar{R}(\theta) - (R(q(\theta)) - p(\theta))\, dF^*(\theta) \right\} .$$

By the standard arguments (see Appendix C.1), it suffices to show that S^* is an optimal solution to the following equivalent problem:

$$\min_{\text{non-increasing } q, V(\bar\theta)\geq 0} \left\{ \int_\Theta \left(\bar R(\theta) - R(q(\theta)) + V(\bar\theta) + \theta\cdot q(\theta) + \int_\theta^{\bar\theta} q(x)dx \right) dF^*(\theta) \right\},$$

where $V(\theta) = p(\theta) - \theta \cdot q(\theta)$ for $\theta \in [\underline{\theta}, \bar\theta]$.

Note F^* has a point-mass of $\lim_{\theta'\to\underline{\theta}^+} F^*(\theta')$ which we, for notational convenience, equate to $F^*(\underline{\theta})$ at $\theta = \underline{\theta}$. But it is otherwise absolutely continuous and has a corresponding density function. We denote the cumulative function without the point-mass at $\theta = \underline{\theta}$ by F^*_- with corresponding density $f^*_-(\theta) = \frac{d}{d\theta}F^*(\theta)$ such that $F^*(\theta) = F^*(\underline{\theta}) + F^*_-(\theta)$ for $\theta \in [\underline{\theta}, \bar\theta]$.

Then, we can rewrite the objective function, denoted OBJ, as follows. Note that

$$\begin{aligned}\text{OBJ} = \mathbb{E}_{\theta\sim F^*}[\bar R(\theta)] + V(\bar\theta) - F^*(\underline{\theta}) \left(R(q(\underline{\theta})) - \underline{\theta}\cdot q(\underline{\theta}) - \int_{\underline{\theta}}^{\bar\theta} q(x)dx \right) \\ - \int_{\underline{\theta}}^{\bar\theta} f^*_-(\theta) \left(R(q(\theta)) - \theta\cdot q(\theta) - \int_\theta^{\bar\theta} q(x)dx \right) d\theta\,.\end{aligned}$$

The last term is equivalently $-\int_{\underline{\theta}}^{\bar\theta} f^*_-(\theta)\left(R(q(\theta)) - \theta\cdot q(\theta)\right)d\theta + \int_{\underline{\theta}}^{\bar\theta} F^*_-(\theta) q(\theta) d\theta$ where we changed the order of integrals. Then,

$$\begin{aligned}\text{OBJ} = \mathbb{E}_{\theta\sim F^*}[\bar R(\theta)] + V(\bar\theta) - F^*(\underline{\theta}) \left(R(q(\underline{\theta})) - \underline{\theta}\cdot q(\underline{\theta}) \right) \\ - \int_{\underline{\theta}}^{\bar\theta} f^*_-(\theta) \left(R(q(\theta)) - \left(\theta + \frac{F^*(\theta)}{f^*_-(\theta)}\right)\cdot q(\theta) \right) d\theta\,.\end{aligned}$$

We now show that S^* minimizes OBJ pointwise. Since $q^*(\underline{\theta}) = \bar q(\theta)$, the third term is minimized. Note $\frac{\partial}{\partial\theta}\ln F^*(\theta) = \frac{f^*_-(\theta)}{F^*(\theta)} = \frac{1}{R'(q^*(\theta))-\theta}$ for $\theta \in [\underline{\theta}, \kappa]$. Then, $R'(q^*(\theta)) = \theta + \frac{F^*(\theta)}{f^*_-(\theta)}$ for $\theta \in [\underline{\theta}, \kappa]$ which is exactly the support of f^*_- and where q^* is nonnegative. It follows that the integrand in the fourth term is minimized pointwise and, thus, the fourth term is minimized. Finally, note that $V(\underline{\theta}) = 0$ for S^* and that the second term is minimized. Overall, the objective is minimized and S^* is an optimal solution. This completes the proof that S^* and F^* form a saddle point.

To compute the minimax regret, we recall from the analysis for the first part of the saddle-point result that the integrand, equivalently, $\widehat{\text{Regret}}(S^*, \theta)$, is constant for $\theta \in [\underline{\theta}, \kappa]$ and decreases for $\theta > \kappa$. It follows that the minimax regret is equal to the integrand evaluated at $\theta = \underline{\theta}$ which is $\int_{\underline{\theta}}^{\bar\theta} q^*(x)dx$. The minimax regret is clearly nonnegative because q^* is nonnegative. We argue it is strictly greater than 0. For the sake of contradiction, assume that $\int_{\underline{\theta}}^{\bar\theta} q^*(x)dx = 0$. As q^* is non-increasing and nonnegative, it would follow that $q^*(\theta) = 0$ for $\theta > \underline{\theta}$. Together with $q^*(\underline{\theta}) = \bar q(\theta) > 0$, it would contradict the continuity of q^* which is implied by the well-definedness of S^*. $\square$

D Additional Materials for Section 5.1

D.1 Proof of Proposition 5

We characterize an optimal mechanism using the relax and verify approach of Kakade et al. (2013). Using that shocks are multiplicatively separable, we can write $\theta_t = \tau\gamma_t$ with γ_t drawn i.i.d. from G. We consider a relaxed environment in which γ_t are public and observable by the principal—the agent's only private information is the parameter τ. By the revelation principle, we can restrict attention to direct mechanisms in which the agent reports the parameter τ in Round 0. We denote by $x_t(\tau, \gamma_{1:t})$ and $p_t(\tau, \gamma_{1:t})$ the allocation and payment, respectively, in Round t when the report is τ and the γ-component of the shocks are $\gamma_{1:t} = (\gamma_1, \ldots, \gamma_t)$. Under the ex-ante participation constraint, the optimal performance achievable when the agent's private distribution $F(\cdot;\tau)$ is known (i.e., the parameter τ is known) is $\mathrm{OPT}(F(\cdot;\tau), T) = T\mathbb{E}_{\theta\sim F(\cdot;\tau)}[\theta] = T\tau\mathbb{E}[\gamma]$ because the principal can simply charge an entry-fee equal to the agent's expected value and then allocate the items for free over the rounds. For the entry fee, the principal can, for example, require the evenly-split fixed constant payment $\tau\mathbb{E}[\gamma]$ over the rounds that the agent has to pay upon participating. Therefore, the minimax regret in the multi-round problem is lower bounded as $\mathrm{Regret}(T) \geq \mathrm{Regret}^{\mathrm{RELAX}}(T)$ where

$$
\begin{aligned}
\mathrm{Regret}^{\mathrm{RELAX}}(T) = \inf_{(x,p)} \sup_{\tau\in[0,1]} & \left\{ T\tau\mathbb{E}[\gamma] - \mathbb{E}_{\gamma_{1:T}}\left[\sum_{t=1}^{T} p_t(\tau,\gamma_{1:t})\right]\right\} \\
\text{s.t. } & V(\tau) = \mathbb{E}_{\gamma_{1:T}}\left[\sum_{t=1}^{T} \tau\gamma_t x_t(\tau,\gamma_{1:t}) - p_t(\tau,\gamma_{1:t})\right] \geq 0\,, \quad \forall \tau\in[0,1]\,, \\
& V(\tau) \geq \mathbb{E}_{\gamma_{1:T}}\left[\sum_{t=1}^{T} \tau\gamma_t x_t(\tau',\gamma_{1:t}) - p_t(\tau',\gamma_{1:t})\right]\,, \quad \forall \tau,\tau'\in[0,1]\,, \\
& 0 \leq x_t(\tau,\gamma_{1:t}) \leq 1\,, p_t(\tau,\gamma_{1:t}) \in \mathbb{R}\,.
\end{aligned}
$$

The first constraint is an individual rationality constraint that guarantees that the ex-ante utility of the agent when his true parameter is τ, which is denoted by $V(\tau)$, is non-negative. The second constraint is an incentive compatibility constraint imposing that the agent is better off reporting his true parameter.

We now show that the relaxed problem can be further lower bounded by a single-round problem, i.e., $\mathrm{Regret}^{\mathrm{RELAX}}(T) \geq T\widehat{\mathrm{Regret}}^{\mathrm{RELAX}}$ where

$$
\begin{aligned}
\widehat{\mathrm{Regret}}^{\mathrm{RELAX}} = \inf_{(\hat{x},\hat{p})} \sup_{\tau\in[0,1]} & \left\{\mathbb{E}[\gamma]\left(\tau - \hat{p}(\tau)\right)\right\} \\
\text{s.t. } & \hat{V}(\tau) = \tau\hat{x}(\tau) - \hat{p}(\tau) \geq 0\,, \quad \forall\tau\in[0,1]\,, \\
& \hat{V}(\tau) \geq \tau\hat{x}(\tau') - \hat{p}(\tau')\,, \quad \forall\tau,\tau'\in[0,1]\,, \\
& 0 \leq \hat{x}(\tau) \leq 1\,, \hat{p}(\tau)\in\mathbb{R}\,.
\end{aligned}
$$

We prove the above claim by showing that every feasible mechanism for the multi-round problem corresponding to $\mathrm{Regret}^{\mathrm{RELAX}}(T)$ induces a feasible mechanism for the single-round problem corresponding to $\widehat{\mathrm{Regret}}^{\mathrm{RELAX}}$ that achieves the same objective value (divided by T). For any multi-round mechanism (x,p), consider the single-round mechanism $(\hat{x},\hat{p})$ given by

$$\hat{x}(\tau) = \frac{1}{T\mathbb{E}[\gamma]}\mathbb{E}_{\gamma_{1:T}}\left[\sum_{t=1}^{T}\gamma_t x_t(\tau,\gamma_{1:t})\right] \quad \text{and} \quad \hat{p}(\tau) = \frac{1}{T\mathbb{E}[\gamma]}\mathbb{E}_{\gamma_{1:T}}\left[\sum_{t=1}^{T}p_t(\tau,\gamma_{1:t})\right]$$

for report $\tau \in [0,1]$, which is obtained by averaging the multi-round mechanism over time and the public γ-components of the shocks. By construction, we have that $V(\tau) = T\mathbb{E}[\gamma](\tau\hat{x}(\tau) - \hat{p}(\tau))$ and the IR constraint holds for the single-round mechanism $(\hat{x},\hat{p})$ because $V(\tau) \geq 0$ for all $\tau \in [0,1]$. The IC constraint similarly holds. Additionally, since $\gamma_t \geq 0$ and $x_t(\cdot,\cdot) \in [0,1]$, we have that $0 \leq \hat{x}(\tau) \leq 1$ for $\tau \in [0,1]$. Similarly, we have that $\hat{p}(\tau) \in \mathbb{R}$ for $\tau \in [0,1]$. For any $\tau \in [0,1]$, the inner regret objective value for the multi-round mechanism can be equivalently written as $T\mathbb{E}[\gamma]\,(\tau - \hat{p}(\tau))$ which equals the corresponding objective value for the constructed single-round mechanism (when divided by T), and the claim follows.

Note the single-round problem corresponding to $\widehat{\text{Regret}}^{\text{RELAX}}$, when divided by $\mathbb{E}[\gamma]$, is exactly the single-round problem (3) in the dynamic selling problem for revenue maximization in Section 4.1. Then, Proposition 1 used in the proof of Proposition 3 immediately implies that the regret of the single-round problem is $\widehat{\text{Regret}}^{\text{RELAX}} = \mathbb{E}[\gamma]/e$ and (x^*,p^*) as stated in Proposition 3 is an optimal single-round mechanism. The single-round mechanism (x^*,p^*) induces the following dynamic mechanism for the original, multi-round problem: screen the agent by the parameter τ and charge an entry-fee $Tp^*(\tau)$ in Round 0, and then allocate each item with probability $x^*(\tau)$ in the subsequent rounds. Equivalently, we can screen the agent by the parameter τ in Round 0, and then allocate each item with probability $x^*(\tau)$ and charge $p^*(\tau)$ in the subsequent rounds, which fits the general formulation given in Section 2. We now conclude by arguing that this multi-round mechanism is incentive compatible in the original, unrelaxed environment in which the γ components of the shocks are private. This is because the mechanism does not ask the agent to report the values γ_t and the agent can only influence the mechanism by misreporting his parameter τ, which is never optimal because the mechanism is incentive compatible with respect to parameter τ.

D.2 Proof of Proposition 6

We first reduce determining the optimal regret $\text{Regret}^{\mathcal{S}}(T)$ for direct static mechanisms to a single-round problem as follows. This reduction works for any arbitrary distribution G supported on $\mathbb{R}_+$. Note that

$$\begin{aligned}
\text{Regret}^{\mathcal{S}}(T) &= \inf_{S\in\mathcal{S}^{\times 1}}\sup_{\tau\in[0,1]} \text{Regret}(S^{\times T}, F(\cdot;\tau), T) \\
&= \inf_{S\in\mathcal{S}^{\times 1}}\sup_{\tau\in[0,1]} \left\{T\mathbb{E}_{\theta\sim F(\cdot;\tau)}[\theta] - \text{PrincipalUtility}(S^{\times T}, \sigma^{\text{TR}}, F(\cdot;\tau), T)\right\} \\
&= T\cdot\inf_{S\in\mathcal{S}^{\times 1}}\sup_{\tau\in[0,1]} \left\{\mathbb{E}_{\theta\sim F(\cdot;\tau)}[\theta] - \text{PrincipalUtility}(S, \sigma^{\text{TR}}, F(\cdot;\tau), 1)\right\}\,,
\end{aligned}$$

where the second step follows because the known-distribution benchmark is $\text{OPT}(F,T) = T\mathbb{E}_{\theta\sim F}[\theta]$ for any distribution F and σ^{TR} denotes the recommended truthful reporting strategy for the agent, and the last step follows from Lemma 2 (in Appendix A.3.2). To see $\text{OPT}(F,T) = T\mathbb{E}_{\theta\sim F}[\theta]$ for any distribution F, note that we argued $\text{OPT}(F,T) \leq T\mathbb{E}_{\theta\sim F}[\theta]$ in Section 4.1, because the principal's revenue is at most the agent's surplus subject to the agent's participation. In fact, the principal can fully extract the agent's surplus with the knowledge of F via, say, what is commonly known as the bundling strategy. The principal can, for example, bundle all items and sell the bundle at the expected value (see, e.g., Bakos and Brynjolfsson 1999).[1]

We then use the outcome distribution representation described in Section 3.3 for a mechanism $S \in \mathcal{S}^{\times 1}$ and equivalently write the last optimization problem as follows via the same reasoning in the proof of Lemma 1:

$$T \cdot \inf_{S \in \Delta(\Omega)^{\Theta}} \left\{ \sup_{\tau \in [0,1]} \mathbb{E}_{\theta \sim F(\cdot;\tau)}[\theta] - \mathbb{E}_{\theta \sim F(\cdot;\tau), \omega \sim S_\theta}[u(\theta,\omega)] \;\text{ s.t. }\; (\text{IC}), (\text{IR}) \right\},$$

where the IC/IR constraints are as formulated in (3). Using the interim rules (x,p) to describe single-round direct mechanisms as in Section 4.1, it follows that

$$\text{Regret}^{\mathcal{S}}(T) = T \cdot \inf_{(x,p)} \left\{ \sup_{\tau \in [0,1]} \mathbb{E}_{\theta \sim F(\cdot;\tau)}\left[\theta - p(\theta)\right] \;\text{ s.t. }\; (\text{IC}), (\text{IR}) \right\}.$$

Now, we let G be the exponential distribution with mean 1 and solve for the resulting single-round problem above with $T = 1$. Let $\exp(\tau)$ denote the exponential distribution with mean τ for $\tau \in [0,1]$. In particular, we show $\text{Regret}^{\mathcal{S}}(1) = 1 - \frac{1}{e}$. To see this, note that since $\tau = 1$ is a feasible parameter, we have:

$$\text{Regret}^{\mathcal{S}}(1) \geq \inf_{(x,p)} \left\{ \mathbb{E}_{\theta \sim \exp(1)}\left[\theta - p(\theta)\right] \;\text{ s.t. }\; (\text{IC}), (\text{IR}) \right\} = 1 - 1/e,$$

because, from Myerson (1981), an optimal Bayesian mechanism that maximizes the revenue when the agent's value distribution is exponential with mean 1 is given by $x(\theta) = \mathbf{1}\{\theta \geq 1\}$ and $p(\theta) = \mathbf{1}\{\theta \geq 1\}$, i.e., a posted pricing mechanism with the price of 1. Furthermore, since this mechanism is feasible for the single-round problem, we obtain that

$$\text{Regret}^{\mathcal{S}}(1) \leq \sup_{\tau \in [0,1]} \mathbb{E}_{\theta \sim \exp(\tau)}\left[\theta - \mathbf{1}\{\theta \geq 1\}\right] = \sup_{\tau \in [0,1]} \left\{ \tau - e^{-\frac{1}{\tau}} \right\} = 1 - 1/e,$$

where the last equality follows because $\tau - e^{-\frac{1}{\tau}}$ is increasing in $\tau \in [0,1]$ and the supremum is achieved at $\tau = 1$. It follows that $\text{Regret}^{\mathcal{S}}(1) = 1 - \frac{1}{e}$. The same argument shows that the deterministic posted pricing mechanism with the price of 1 is an optimal single-round direct mechanism in the single-round problem.

Going back to the multi-round problem with T rounds, it follows that $\text{Regret}^{\mathcal{S}}(T) = (1 - 1/e)T$ and an optimal direct static mechanism that achieves this minimax regret is one that repeats the deterministic posted pricing mechanism with the price of 1.

[1] In the bundling mechanism, the principal lets the agent decide whether to continue or quit in Round 0 and then requires the agent to pay a one-time payment of $T\mathbb{E}_{\theta \sim F}[\theta]$ in Round 1 and allocates all items in Round 1 and future rounds. The agent would be indifferent between participating and not participating and when he decides to participate by continuing in Round 0, he would be bound by the bundling contract. The recommended strategy for the agent is to participate.

E Missing Proofs from Section 5.2

For any distribution F, recall that

$$\begin{aligned} \bar{u}(F) := \sup_{S \in \mathcal{S}^{\times 1}} & \int_{\Theta} \int_{\Omega} u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \\ \text{s.t.} & \int_{\Theta} \int_{\Omega} v(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \geq 0 \,. \end{aligned} \tag{E-3}$$

For $\mathbb{E}_{\theta \sim F}[\bar{u}(\delta_\theta)]$, we can equivalently write

$$\begin{aligned} \mathbb{E}_{\theta \sim F}[\bar{u}(\delta_\theta)] = \sup_{S \in \mathcal{S}^{\times 1}} & \int_{\Theta} \int_{\Omega} u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \\ \text{s.t.} & \int_{\Omega} v(\theta, \omega) \mathrm{d}S_\theta(\omega) \geq 0 \,, \quad \forall \theta \in \Theta \,. \end{aligned} \tag{E-4}$$

We will use the following result in the proofs:

Proposition 3. *We have the following relations:*

1. *For any distribution* $F \in \Delta(\Theta)$, $\mathrm{OPT}(F, T) \leq T \cdot \bar{u}(F)$.
2. *For any* $\theta \in \Theta$, $\mathrm{OPT}(\delta_\theta, T) = T \cdot \bar{u}(\delta_\theta)$.
3. *For any distribution* $F \in \Delta(\Theta)$, $\bar{u}(F) \geq \mathbb{E}_{\theta \sim F}[\bar{u}(\delta_\theta)]$.

This is proved in Appendix E.3. The proofs of Propositions 7 and 8 from Section 5.2 are provided in Appendices E.1 and E.2, respectively.

E.1 Proof of Proposition 7

For any distribution $F \in \mathcal{F}$, we have

$$\mathrm{OPT}(F, T) \leq T \cdot \bar{u}(F) = \mathbb{E}_{\theta \sim F}[T \cdot \bar{u}(\delta_\theta)] = \mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\delta_\theta, T)] \,,$$

where the first step is by the first part of Proposition 3, the second by the linearity assumption on $\bar{u}(F)$, and the third by the second part of Proposition 3.

E.2 Proof of Proposition 8

In what follows, we show $\bar{u}(F) \leq \mathbb{E}_{\theta \sim F}[\bar{u}(\delta_\theta)]$ when the stated conditions hold. By Part 3 of Proposition 3, this suffices. Consequently, it would follow that Assumption 2 holds by Proposition 7.

Part 1): In words, the game is such that the payment is part of the outcome and enters linearly with coefficients of opposing signs into the utility functions of the principal and agent. Separating out the payment, the outcome space can be represented as $\Omega = \Omega^0 \times \mathbb{R}$ where Ω^0 is the space of non-payment component of the outcomes and an outcome $\hat{\omega}$ is a pair $(\hat{\omega}^0, \hat{p})$ where $\hat{\omega}^0$ is the non-payment component and $\hat{p}$ is the payment. We use superscript 0 to denote the non-payment parts of the outcome and outcome space. Since the payment enters linearly into the utility functions of the principal and agent, we can represent $u(\theta, (\hat{\omega}^0, \hat{p})) = u^0(\theta, \hat{\omega}^0) + \alpha \cdot \hat{p}$ for some function $u^0 : \Theta \times \Omega^0 \to \mathbb{R}$ and scalar $\alpha \geq 0$ and, similarly, $v(\theta, (\hat{\omega}^0, \hat{p})) = v^0(\theta, \hat{\omega}^0) - \beta \cdot \hat{p}$ for some function $v^0 : \Theta \times \Omega^0 \to \mathbb{R}$ and scalar $\beta > 0$. Note we interpret a payment as a monetary transfer from the agent to the principal and this fixes the signs in front of α and β.

Fix an arbitrary distribution $F \in \mathcal{F}$. Let S be an arbitrary feasible solution for the optimization problem (E-3) defined for $\bar{u}(F)$. We define the payment offset $q_\theta = \frac{1}{\beta} \int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega)$ for all $\theta \in \Theta$. Now, consider a single-round direct mechanism S' where S'_θ is the outcome distribution S_θ modified with the fixed offset q_θ such that to realize an outcome $\hat{\omega} \sim S'_\theta$, we draw $(\hat{\omega}^0, \hat{p}) \sim S_\theta$ and set $\hat{\omega} = (\hat{\omega}^0, \hat{p} + q_\theta)$.

We show that S' is a feasible solution to the optimization problem (E-4) defining $\mathbb{E}_{\theta \sim F}[\bar{u}(\delta_\theta)]$ and that S' obtains the objective value in (E-4) that is at least that obtained by S in (E-3). As S was arbitrary, it would follow that $\bar{u}(F) \leq \mathbb{E}_{\theta \sim F}[\bar{u}(\delta_\theta)]$ and, as F was arbitrary, the proposition statement would follow.

For any $\theta \in \Theta$,

$$
\begin{aligned}
\int_\Omega v(\theta, \omega) \mathrm{d}S'_\theta(\omega) &= \int_{\Omega^0 \times \mathbb{R}} v(\theta, (\omega^0, p + q_\theta)) \mathrm{d}S_\theta((\omega^0, p)) \\
&= \int_{\Omega^0 \times \mathbb{R}} \left(v^0(\theta, \omega^0) - \beta(p + q_\theta)\right) \mathrm{d}S_\theta((\omega^0, p)) \\
&= \int_{\Omega^0 \times \mathbb{R}} \left(v(\theta, (\omega^0, p)) - \beta q_\theta\right) \mathrm{d}S_\theta((\omega^0, p)) \\
&= \int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) - \beta q_\theta \\
&= 0\,,
\end{aligned}
$$

where the last step follows from how the payment offset is defined. Hence, S' is a feasible solution to (E-4).

Similarly, for any $\theta \in \Theta$,

$$
\int_\Omega u(\theta, \omega) \mathrm{d}S'_\theta(\omega) = \int_{\Omega^0 \times \mathbb{R}} u(\theta, (\omega^0, p + q_\theta)) \mathrm{d}S_\theta((\omega^0, p)) = \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) + \alpha q_\theta\,.
$$

Integrating the first and last expressions over Θ, we obtain

$$
\begin{aligned}
\int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S'_\theta(\omega) \mathrm{d}F(\theta) &= \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) + \alpha \int_\Theta q_\theta \mathrm{d}F(\theta) \\
&= \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) + \frac{\alpha}{\beta} \int_\Theta \int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \\
&\geq \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta)\,,
\end{aligned}
$$

where the second-to-last step follows from the definition of the payment offset q_θ and the last step follows since S is a feasible solution to (E-3) and $\int_\Theta \int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta) \geq 0$. Therefore, S' obtains the objective value in (E-4) that is at least that obtained by S in (E-3). As S was arbitrary, $\bar{u}(F) \leq \mathbb{E}_{\theta\sim F}[\bar{u}(\delta_\theta)]$.

Part 2): By the assumption on the game, we have $v(\theta,\omega) \geq 0$ for all $\theta \in \Theta$ and $\omega \in \Omega$. Then, for any single-round direct mechanism S and shock $\theta \in \Theta$,

$$\int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega) \geq 0\,.$$

Clearly, for any $F \in \mathcal{F}$, any feasible solution to the optimization problem (E-3) is a feasible solution to the optimization problem (E-4) and obtains the same objective. The proposition follows.

E.3 Proof of Proposition 3

Part 1): Fix an arbitrary distribution $F \in \Delta(\Theta)$. Assume the principal commits to an incentive compatible dynamic mechanism A and the agent plays the recommended strategy σ. Let $\{\omega_t\}_{t=1}^T$ be the resulting random sequence of realized outcomes. For each $\theta \in \Theta$, we define measure $\mu_\theta(Q) = \frac{1}{T}\sum_{t=1}^T \Pr(\omega_t \in Q \mid \theta_t = \theta)$ for any $Q \subseteq \Omega$ and let S_θ be the corresponding distribution over Ω such that $\omega \sim S_\theta$ means an outcome ω is realized with probability $\mu_\theta(\omega)$. Consider a single-round direct mechanism $S = \{S_\theta\}_{\theta\in\Theta}$ that given a report θ returns an outcome $\omega \sim S_\theta$. We note that

$$\begin{aligned}
\text{PrincipalUtility}(A,\sigma,F,T) &= \mathbb{E}\left[\sum_{t=1}^T u(\theta_t,\omega_t)\right] \\
&= \sum_{t=1}^T \mathbb{E}_{\theta_t}\left[\mathbb{E}[u(\theta_t,\omega_t)|\theta_t]\right] \\
&= \sum_{t=1}^T \mathbb{E}_{\theta\sim F}\left[\mathbb{E}[u(\theta_t,\omega_t)|\theta_t=\theta]\right] \\
&= T\cdot\mathbb{E}_{\theta\sim F}\left[\frac{1}{T}\sum_{t=1}^T \mathbb{E}_{\omega_t|\theta_t=\theta}[u(\theta,\omega_t)]\right] \\
&= T\cdot\mathbb{E}_{\theta\sim F}\left[\mathbb{E}_{\omega\sim S_\theta}[u(\theta,\omega)]\right]\,,
\end{aligned}$$

where the second equality follows from the linearity of expectations and the tower rule, the third from that the idiosyncratic shocks are drawn independently and identically, and the last from the construction of S. Hence, we have

$$\text{PrincipalUtility}(A,\sigma,F,T) = T\cdot\int_\Theta\int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta)\,.$$

Similarly, we have

$$\text{AgentUtility}(A,\sigma,F,T) = T\cdot\int_\Theta\int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta)\,.$$

Since the recommended strategy σ is a utility-maximizing strategy for the agent and the agent can achieve the aggregate utility of 0 by not participating, it must be that $\text{AgentUtility}(A,\sigma,F,T) \geq 0$. It follows that S is a feasible solution for $\bar{u}(F)$ and achieves the objective of $\frac{1}{T}\cdot\text{PrincipalUtility}(A,\sigma,F,T)$. As the dynamic mechanism $A \in \mathcal{A}$ was arbitrary, the first part follows.

Part 2): Since we have the first part, it suffices to show $\text{OPT}(\delta_\theta, T) \geq \bar{u}(\delta_\theta)$ for any $\theta \in \Theta$. Fix an arbitrary $\theta \in \Theta$ and let the agent's distribution be δ_θ. Note $\bar{u}(\delta_\theta)$ is equivalently

$$\begin{aligned} \bar{u}(\delta_\theta) := \sup_{G\in\Delta(\Omega)} & \int_\Omega u(\theta,\omega)\mathrm{d}G(\omega) \\ \text{s.t.} & \int_\Omega v(\theta,\omega)\mathrm{d}G(\omega) \geq 0\,, \end{aligned}$$

where G is an outcome distribution over Ω. For an arbitrary $\epsilon > 0$, let G_ϵ be an outcome distribution that satisfies the IR constraint in the above optimization problem and

$$\int_\Omega u(\theta,\omega)\mathrm{d}G_\epsilon(\omega) \geq \bar{u}(\delta_\theta) - \epsilon\,.$$

Consider the corresponding dynamic mechanism A_ϵ that repeatedly determines an outcome according to G_ϵ in each round. For the recommended strategy, we let the agent participate when his distribution is δ_θ and give the better of the choices of participating or not when his distribution is something else. When the agent's distribution is δ_θ, since G_ϵ satisfies the IR constraint, participating is a utility-maximizing strategy and the agent accepts the outcomes being drawn independently and identically from G_ϵ. The agent's only other option is to not participate which leads to the aggregate utility of 0. When the agent's distribution is not δ_θ, the recommended strategy is such that the agent still follows the strategy. That is, A_ϵ is incentive compatible.

By construction, the aggregate utility of the principal under A_ϵ is at least $T\cdot\bar{u}(\delta_\theta) - \epsilon\cdot T$. As ϵ was arbitrary, this implies $\text{OPT}(\delta_\theta,T) \geq T\cdot\bar{u}(\delta_\theta)$. Combined with the first part, $\text{OPT}(\delta_\theta,T) = T\cdot\bar{u}(\delta_\theta)$.

Part 3): Note we always have $\bar{u}(F) \geq \mathbb{E}_{\theta\sim F}[\bar{u}(\delta_\theta)]$ for all $F \in \Delta(\Theta)$ unconditionally. This is because a feasible solution in the optimization problem in (E-4) is a feasible solution in the optimization problem (E-3) and obtains the same objective value.

F Additional Materials for Section 5.4

We prove Theorem 2 in the next subsection. We prove Propositions 9 and 10 in Appendices F.2–F.4.

F.1 Proof of Theorem 2

We first derive a lower bound on the multi-round minimax regret Regret(T) via the same reasoning used in the proof of Lemma 2. Note that

$$\begin{aligned}
\text{Regret}(T) &= \inf_{A\in\mathcal{A}} \sup_{F\in\mathcal{F}} \text{Regret}(A, F, T) \\
&\geq \inf_{A\in\mathcal{A}} \sup_{\theta\in\Theta} \text{Regret}(A, \delta_\theta, T) \\
&= \inf_{A\in\mathcal{A}} \sup_{\theta\in\Theta} \{\text{OPT}(\delta_\theta, T) - \text{PrincipalUtility}(A, \sigma, \delta_\theta, T)\} \\
&= T \cdot \inf_{A\in\mathcal{A}} \sup_{\theta\in\Theta} \{\text{OPT}(\delta_\theta, 1) - \text{PrincipalUtility}(S(A), \sigma^{\text{TR}}, \delta_\theta, 1)\} \\
&= T \cdot \inf_{A\in\mathcal{A}} \sup_{\theta\in\Theta} \text{Regret}(S(A), \delta_\theta, 1) \\
&\geq T \cdot \inf_{S'\in\mathcal{S}^{\times 1}} \sup_{\theta\in\Theta} \text{Regret}(S', \delta_\theta, 1)\,,
\end{aligned} \tag{F-5}$$

where the second step follows since point-mass distributions are a subset of $\mathcal{F}$ in the inner maximization expression by Assumption 1; the third step is by the regret definition where σ is the corresponding recommended strategy under A; the fourth step is by Proposition 1 and Lemma 1 where $S(A)$ is the single-round direct IC/IR mechanism derived from A as described in the proof of Lemma 1; the second-to-last step is by the regret definition; and the last step follows because the single-round direct IC/IR mechanisms in $\mathcal{S}^{\times 1}$ are a superset of those mechanisms $S(A)$ derived from incentive compatible dynamic mechanisms, i.e., $\{S(A) \mid A \in \mathcal{A}\}$.

Now, define $\Delta = \sup_{F\in\mathcal{F}} \{\text{OPT}(F, T) - \mathbb{E}_{\theta\sim F}[\text{OPT}(\delta_\theta, T)]\}$. Let $\epsilon \geq 0$ be arbitrary and consider a single-round direct IC/IR mechanism $S \in \mathcal{S}^{\times 1}$ satisfying

$$\sup_{\theta\in\Theta} \text{Regret}(S, \delta_\theta, 1) \leq \inf_{S'\in\mathcal{S}^{\times 1}} \sup_{\theta\in\Theta} \text{Regret}(S', \delta_\theta, 1) + \frac{\epsilon}{T}\,. \tag{F-6}$$

Then,

$$\begin{aligned}
&\sup_{F\in\mathcal{F}} \text{Regret}(S^{\times T}, F, T) \\
&\quad= \sup_{F\in\mathcal{F}} \left\{\text{OPT}(F, T) - \text{PrincipalUtility}(S^{\times T}, \sigma^{\text{TR}}, F, T)\right\} \\
&\quad= \sup_{F\in\mathcal{F}} \{\text{OPT}(F, T) - T \cdot \text{PrincipalUtility}(S, \sigma^{\text{TR}}, F, 1)\} \\
&\quad\leq \sup_{F\in\mathcal{F}} \{\mathbb{E}_{\theta\sim F}[\text{OPT}(\delta_\theta, T)] - T \cdot \text{PrincipalUtility}(S, \sigma^{\text{TR}}, F, 1)\} + \Delta \\
&\quad= T \cdot \sup_{F\in\mathcal{F}} \{\mathbb{E}_{\theta\sim F}[\text{OPT}(\delta_\theta, 1)] - \text{PrincipalUtility}(S, \sigma^{\text{TR}}, F, 1)\} + \Delta \\
&\quad\leq T \cdot \sup_{\theta\in\Theta} \text{Regret}(S, \delta_\theta, 1) + \Delta\,,
\end{aligned} \tag{F-7}$$

where the first step is by the definition of Regret notion and σ^{TR} is the truthful reporting strategy recommended for direct mechanisms; the second step is by Lemma 2; the third step is because the supremum operator is sublinear; the second-to-last step is by Proposition 1; and the last step follows from Lemma 3.

Algorithm 1. *Mechanism $A^*(F,T)$*

1. Round 0: The report space is $\mathcal{M} = \{\mathsf{CONTINUE}, \mathsf{QUIT}\}$. If the report is QUIT, the outcome is $\emptyset$ for all rounds. If the report is CONTINUE, we continue as follows.
2. Rounds 1–T: The report space is $\mathcal{M} = \{\theta^1, \theta^2\}$. If the report is θ^1, the outcome is ω^1 with probability q and $\emptyset$ with probability $1-q$. If the report is θ^2, the outcome is ω^2 with probability 1. Note $q = \begin{cases} 1 & \text{, if } f_1 < \frac{1}{2} \\ \frac{1-f_1}{f_1} & \text{, if } f_1 \geq \frac{1}{2} \end{cases}$.

By the above bounds, (F-5) and (F-7), and the property (F-6) of the single-round direct IC/IR mechanism S, it follows that

$$\begin{aligned}
\sup_{F\in\mathcal{F}} \text{Regret}(S^{\times T}, F, T) &\leq T \cdot \sup_{\theta\in\Theta} \text{Regret}(S, \delta_\theta, 1) + \Delta \\
&\leq T \cdot \inf_{S'\in\mathcal{S}^{\times 1}} \sup_{\theta\in\Theta} \text{Regret}(S', \delta_\theta, 1) + \Delta + \epsilon \\
&\leq \text{Regret}(T) + \Delta + \epsilon\,.
\end{aligned}$$

F.2 Proof of Proposition 9

We first show that the optimal performance achievable is $\text{OPT}(F,T) = T \cdot \bar{u}(F)$ and then characterize the single-round full information benchmark $\bar{u}(F)$. Recall $F = (f_1, f_2)$ is the agent's private distribution over Θ where the shock is θ^i with probability f_i for $i = 1,2$ with $f_1 + f_2 = 1$.

Part 1 (Optimal performance). By Proposition 3, note $\text{OPT}(F,T) \leq T \cdot \bar{u}(F)$ for any game. For the game in Table 1b, we show $\text{OPT}(F,T) \geq T \cdot \bar{u}(F)$ and it would follow that $\text{OPT}(F,T) = T \cdot \bar{u}(F)$.

Consider $A^*(F,T)$ in Algorithm 1. We show reporting CONTINUE in Round 0 and then reporting truthfully in Rounds 1–T is optimal (i.e., utility-maximizing) for the agent. Given the agent participates in a round (i.e., Rounds 1–T), truthful reporting is optimal on the per-round basis. If the shock is θ^1, reporting θ^1 yields $-q$ and reporting θ^2 yields $-\infty$. If the shock is θ^2, reporting θ^1 yields $-\infty$ (or 0 if $q = 0$) and reporting θ^2 yields 1. Hence, truthful reporting is optimal in each round. If the agent reports CONTINUE in Round 0 and participates in all the remaining rounds, the overall utility is $T \cdot (-q \cdot f_1 + f_2)$. The overall utility is $T \cdot (-f_1 + f_2)$ if $f_1 < \frac{1}{2}$ and 0 if $f_1 \geq \frac{1}{2}$, which is at least 0 for any distribution F. If the agent reports QUIT in Round 0 and does not participate in the remaining rounds, the utility is 0. Hence, reporting CONTINUE followed by truthful reporting is optimal over the entire horizon.

We use σ^{TR} to denote the agent's utility-maximizing strategy of reporting CONTINUE in Round 0 and then reporting truthfully in Rounds 1–T. For the mechanism $A^*(F,T)$, we let σ^{TR} be the recommended strategy for the agent. As we argued above, $A^*(F,T)$ with the recommended strategy σ^{TR} is incentive compatible. Given that the agent plays σ^{TR}, the principal's utility in each round (i.e., Rounds 1–T) is $q \cdot f_1 = \bar{u}(F)$ and

$$\text{PrincipalUtility}(A^*(F,T), \sigma^{\text{TR}}, F, T) = T \cdot \bar{u}(F)\,.$$

Then, we have

$$\begin{aligned}\mathrm{OPT}(F,T) &= \sup_{A\in\mathcal{A}} \mathrm{PrincipalUtility}(A,\sigma,F,T)\\ &\geq \mathrm{PrincipalUtility}(A^*(F,T),\sigma^{\mathrm{TR}},F,T)\\ &= T\cdot\bar{u}(F)\,,\end{aligned}$$

where σ is the corresponding recommended strategy for the incentive compatible mechanism A in the first step.

Part 2 (Single-round full information benchmark). Note for distribution F, the single-round full information benchmark is

$$\begin{aligned}\bar{u}(F) = \max_{x\in[0,1]} \quad & f_1\cdot x\\ \text{s.t.} \quad & -f_1\cdot x + f_2\cdot 1 \geq 0\,.\end{aligned}$$

To see this, let S be a single-round direct mechanism in the optimization problem defining $\bar{u}(F)$ (in Section 5.2) such that each S_{θ^i} is a distribution over Ω with probabilities given by $S_{\theta^i}(\emptyset)$, $S_{\theta^i}(\omega^1)$, and $S_{\theta^i}(\omega^2)$, and let $x = S_{\theta^1}(\omega^1)$ ($y = S_{\theta^2}(\omega^1)$) be the probability that outcome ω^1 is selected when the shock is θ^1 (θ^2). On the one hand, the mechanism maximizes $\mathbb{E}_{\theta\sim F,\omega\sim S_\theta}[u(\theta,\omega)] = f_1\cdot x + f_2\cdot y$ if S_{θ^1} and S_{θ^2} place as much probability mass as possible on outcome ω^1. We assume $y=0$ without loss; if $f_2>0$, only $y=0$ is feasible because the agent utility will be $-\infty$ and the ex-ante IR constraint will be violated if $y>0$, and if $f_2=0$, y can be any value and we can set $y=0$ without affecting the principal utility. On the other hand, setting x too large might violate the ex-ante IR constraint because outcome ω^1 gives the utility of -1 to the agent when the shock is θ^1. Setting $S_{\theta^2}(\omega^2)=1$ allows for higher values of x at no cost to the principal.

Now, we determine $\bar{u}(F)$. Since $f_1+f_2=1$, we see that the ex-ante IR constraint only binds when $f_1\geq\frac{1}{2}$. Therefore, the optimal solution is $x=1$ when $f_1<\frac{1}{2}$ and $x=\frac{1-f_1}{f_1}\leq 1$ when $f_1\geq\frac{1}{2}$, or, more succinctly, $x=\min\left\{1,\frac{1-f_1}{f_1}\right\}$. This implies

$$\bar{u}(F) = \min\{f_1, 1-f_1\} = \begin{cases} f_1\,, & \text{if } f_1<\frac{1}{2}\,,\\ 1-f_1\,, & \text{if } f_1\geq\frac{1}{2}\,.\end{cases}$$

F.3 First Part of Proposition 10

We prove the first part in this section and the second part in Appendix F.4.

By Lemma 2, for any arbitrary single-round direct IC/IR mechanism $S\in\mathcal{S}^{\times 1}$ and distribution $F\in\mathcal{F}$,

$$\begin{aligned}\mathrm{Regret}(S^{\times T},F,T) &= \mathrm{OPT}(F,T) - \mathrm{PrincipalUtility}(S^{\times T},F,T)\\ &= \mathrm{OPT}(F,T) - T\cdot\mathrm{PrincipalUtility}(S,F,1)\,.\end{aligned}$$

Since $\mathrm{OPT}(F,T) = T\cdot\bar{u}(F)$ by Proposition 9,

$$\mathrm{Regret}(S^{\times T},F,T) = T\cdot\left(\bar{u}(F) - \mathrm{PrincipalUtility}(S,F,1)\right)\,.$$

Algorithm 2. *Dynamic mechanism A parametrized in terms of T_1, T_2, δ, and q_0.*

1. Round 0: The report space is $\mathcal{M} = \{\mathsf{CONTINUE}, \mathsf{QUIT}\}$. If the report is QUIT, the outcome is $\emptyset$ for all rounds. If the report is $\mathsf{CONTINUE}$, we continue.
2. Phase 1 (T_1 rounds): The report space is $\mathcal{M} = \{\theta^1, \theta^2\}$. If the report is θ^1, the outcome is ω^1 with probability q_0 and $\emptyset$ with probability $1 - q_0$. If the report is θ^2, the outcome is ω^2 with probability 1.
3. Round $T_1 + 1$: The report space is $\mathcal{M} = \{\mathsf{CONTINUE}, \mathsf{QUIT}\}$. If the report is QUIT, the outcome is $\emptyset$ for the current and all remaining rounds. If the report is $\mathsf{CONTINUE}$, the outcome is $\emptyset$ for the current round and we continue.
4. Phase 2 (T_2 rounds):
   - From Phase 1, compute the fraction $\hat{f}_1$ of reports of θ^1. Let $\tilde{f}_1 = \hat{f}_1 + \delta$ and $\tilde{q} = \begin{cases} 1 & \text{, if } \tilde{f}_1 < \frac{1}{2} \\ \frac{1-\tilde{f}_1}{\tilde{f}_1} & \text{, if } \tilde{f}_1 \geq \frac{1}{2} \end{cases}$.
   - The report space is $\mathcal{M} = \{\theta^1, \theta^2\}$. If the report is θ^1, the outcome is ω^1 with probability $\tilde{q}$ and $\emptyset$ with probability $1 - \tilde{q}$. If the report is θ^2, the outcome is ω^2 with probability 1.

It suffices to show that

$$\inf_{S \in \mathcal{S}^{\times 1}} \sup_{F \in \mathcal{F}} \{\bar{u}(F) - \text{PrincipalUtility}(S, F, 1)\} = \frac{1}{2}\,. \tag{F-8}$$

Fix an arbitrary single-round direct IC/IR mechanism $S \in \mathcal{S}^{\times 1}$. Using the outcome distribution representation, let each S_{θ^i} be a distribution over Ω with probabilities $\alpha_0 = S_{\theta^1}(\emptyset)$, $\alpha_1 = S_{\theta^1}(\omega^1)$, and $\alpha_2 = S_{\theta^1}(\omega^2)$ and probabilities $\beta_0 = S_{\theta^2}(\emptyset)$, $\beta_1 = S_{\theta^2}(\omega^1)$, and $\beta_2 = S_{\theta^2}(\omega^2)$. Since S satisfies the (IR) constraint for shock θ^1, we have $0 \cdot \alpha_0 - 1 \cdot \alpha_1 - \infty \cdot \alpha_2 \geq 0$ and it follows that $\alpha_1 = \alpha_2 = 0$. Similarly, the (IR) constraint for shock θ^2 implies we have $0 \cdot \beta_0 - \infty \cdot \beta_1 + 1 \cdot \beta_2 \geq 0$ and it follows that $\beta_1 = 0$. Note $\text{PrincipalUtility}(S, F, 1) = \mathbb{E}_{\theta \sim F, \omega \sim S_\theta}[u(\theta, \omega)] = f_1\alpha_1 + f_2\beta_1$ for any $F = (f_1, f_2)$. From the above observations, $\text{PrincipalUtility}(S, F, 1) = 0$ for all F. Then, we have

$$\sup_{F \in \mathcal{F}} \{\bar{u}(F) - \text{PrincipalUtility}(S, F, 1)\} = \sup_{F \in \mathcal{F}} \bar{u}(F) = \sup_{F \in \mathcal{F}} \min\{f_1, 1 - f_1\} = \frac{1}{2}\,,$$

where the second step follows by Proposition 9.

As $S \in \mathcal{S}^{\times 1}$ was arbitrary and $\mathcal{S}^{\times 1}$ is not empty (for example, we can take $\alpha_0 = 1$ and $\beta_1 = 1$),

$$\inf_{S \in \mathcal{S}^{\times 1}} \sup_{F \in \mathcal{F}} \{\bar{u}(F) - \text{PrincipalUtility}(S, F, 1)\} = \frac{1}{2}\,,$$

which is (F-8).

F.4 Second Part of Proposition 10

Consider A in Algorithm 2 which is parametrized in terms of T_1, T_2, δ and q_0. We choose $T_1 = T^{2/3}$, $T_2 = T - T_1 - 1$, $\delta = \sqrt{\frac{\ln T_1}{4T_1}}$ and $q_0 = \frac{1}{\sqrt{T_1}}$. For ease of presentation, we mostly use T_1, T_2, δ and q_0 as parameters and use their values when necessary. We prove the result in three steps. First, we show that truthful reporting is a utility-maximizing strategy for the agent when his distribution

$F = (f_1, f_2)$ satisfies $f_1 \in [0, \frac{1}{1+q_0}]$. Second, we lower bound the principal's utility when the agent reports truthfully. Finally, we analyze the regret of the dynamic mechanism A.

Step 1. Assume the agent's distribution $F = (f_1, f_2)$ satisfies $f_1 \in [0, \frac{1}{1+q_0}]$. Let σ^{TR} denote the strategy that reports CONTINUE in Round 0, reports truthfully during Phase 1, reports CONTINUE if $-\tilde{q} \cdot f_1 + 1 \cdot f_2 \geq 0$ in Round T_1+1, and then reports truthfully during Phase 2 if the game continues to Phase 2. We note that the agent utility is at least 0 under σ^{TR}. In Phase 1, truthful reporting leads to the utility of $T_1 \cdot (-q_0 \cdot f_1 + 1 \cdot f_2) \geq 0$, since $f_1 \leq \frac{1}{1+q_0}$ implies $-q_0 \cdot f_1 + 1 \cdot (1 - f_1) = 1 - (1+q_0) \cdot f_1 \geq 1 - (1+q_0) \cdot \frac{1}{1+q_0} = 0$. If the game continues to Phase 2, it must be that $-\tilde{q} \cdot f_1 + 1 \cdot f_2 \geq 0$ and σ^{TR} leads to the utility of $T_2 \cdot (-\tilde{q} \cdot f_1 + 1 \cdot f_2) \geq 0$ in Phase 2. If the game does not continue, then it leads to the utility of 0 in Phase 2. In expectation, σ^{TR} leads to the utility of at least 0 in Phase 2. Hence, the overall utility is at least 0.

In fact, σ^{TR} is a utility-maximizing strategy for the agent. To see this, we first note truthful reporting is optimal on the per-round basis in each round in Phase 1 and in Phase 2 (for any value of $\tilde{q}$). That is, given the agent participates in a round, truthful reporting is a utility-maximizing strategy for the agent in that round. In each round in Phase 1, if the shock is θ^1, reporting θ^1 yields $-q_0$ and reporting θ^2 yields $-\infty$. If the shock is θ^2, reporting θ^2 yields 1 and reporting θ^1 yields $-\infty$. In each round in Phase 2, for any value of $\tilde{q}$, truthful reporting is optimal for the agent. If the shock is θ^1, reporting θ^1 yields $-\tilde{q}$ and reporting θ^2 yields $-\infty$. If the shock is θ^2, reporting θ^2 yields 1 and reporting θ^1 yields $-\infty$ (or 0 if $\tilde{q} = 0$).

Then, we note the only way for the agent to influence the principal's mechanism A is through reports in Phase 1 which determine the probability $\tilde{q}$ in Phase 2. Intuitively, the agent may consider some non-truthful reporting strategy in Phase 1 and continue to Phase 2 with $\tilde{q}$ determined favorably to benefit himself. Non-truthful reporting can only lead to lower per-round utilities during Phase 1 and each misreport of the shock costs $-\infty$. The agent needs to continue to Phase 2 in order to gain from such non-truthful reporting, but the cost overwhelms the potential gain from Phase 2. It follows that it is optimal for the agent to truthfully report during Phase 1 and given this observation, reporting CONTINUE in Round T_1+1 if $-\tilde{q} \cdot f_1 + 1 \cdot f_2 \geq 0$ can only benefit the agent because truthfully reporting is optimal on the per-round basis in Phase 2 and the utility from Phase 2 is $T_2 \cdot (-\tilde{q} \cdot f_1 + 1 \cdot f_2)$ given the game continues to Phase 2. Note the agent cannot influence the principal's mechanism during Phase 2. Hence, it is not possible to realize a greater utility overall than that achieved under σ^{TR}.

Step 2. Assume the agent's distribution $F = (f_1, f_2)$ satisfies $f_1 \in [0, \frac{1}{1+q_0}]$. Let U_1^{TR} and U_2^{TR} be the principal utility from Phases 1 and 2, respectively, when the principal's mechanism is A and the agent's utility-maximizing strategy is σ^{TR}, such that

$$\mathrm{PrincipalUtility}(A, \sigma^{\mathrm{TR}}, F, T) = U_1^{\mathrm{TR}} + U_2^{\mathrm{TR}}.$$

Note

$$U_1^{\mathrm{TR}} = T_1 \cdot q_0 \cdot f_1 \geq 0. \tag{F-9}$$

We now consider U_2^{TR}. Let $\mathcal{E}$ be the event that the game continues to Phase 2 under σ^{TR}, or equivalently, $-\tilde{q} \cdot f_1 + 1 \cdot f_2 \geq 0$ and $\mathbf{1}_{\mathcal{E}}$ be the indicator that equals to 1 if the event occurs, and 0 otherwise; so, $\mathbf{1}_{\mathcal{E}} = \mathbf{1}\{-\tilde{q} \cdot f_1 + 1 \cdot f_2 \geq 0\}$. We have $U_2^{\mathrm{TR}} = T_2 \cdot \mathbb{E}[\tilde{q} \cdot f_1 \cdot \mathbf{1}_{\mathcal{E}}]$. Note that $\frac{1}{1+q_0} \geq \frac{1}{2}$ because $q_0 \leq 1$. We consider the following cases depending on whether f_1 is above or below $\frac{1}{2}$.

If $f_1 \le 1/2$, then the event $\mathcal{E}$ always occurs because $f_2 = 1 - f_1 \ge f_1$ and $\tilde{q} \in [0,1]$. Therefore,

$$U_2^{\mathrm{TR}} = f_1 \cdot T_2 \cdot \mathbb{E}\left[\min\left\{1, \frac{1-\tilde{f}_1}{\tilde{f}_1}\right\}\right], \tag{F-10}$$

where $\tilde{f}_1$ is always strictly positive because $\tilde{f}_1 = \hat{f}_1 + \delta \ge \delta > 0$.

If $f_1 > 1/2$, whenever the event $\mathcal{E}$ occurs we have $\tilde{f}_1 \ge 1/2$ and, consequently, $\tilde{q} = \frac{1-\tilde{f}_1}{\tilde{f}_1}$. To see this, note that if $\tilde{f}_1 < 1/2$, then $\tilde{q} = 1$, which implies that $-\tilde{q} \cdot f_1 + 1 \cdot f_2 = -f_1 + f_2 < 0$ and event $\mathcal{E}$ does not occur. Then, $-\tilde{q} \cdot f_1 + 1 \cdot f_2 \ge 0$ is equivalent to $\frac{1-f_1}{f_1} \ge \tilde{q} = \frac{1-\tilde{f}_1}{\tilde{f}_1}$. Since the transformation $x \mapsto \frac{1-x}{x}$ is decreasing, the event $\mathcal{E}$ can be equivalently written as $\mathcal{E} = \{\tilde{f}_1 \ge f_1\}$. This implies that

$$\begin{aligned} U_2^{\mathrm{TR}} &= f_1 \cdot T_2 \cdot \mathbb{E}\left[\mathbf{1}\{\tilde{f}_1 \ge f_1\} \cdot \frac{1-\tilde{f}_1}{\tilde{f}_1}\right] \\ &= T_2 \cdot \left((1-f_1) \cdot \Pr(\tilde{f}_1 \ge f_1) - \mathbb{E}\left[\max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\}\right]\right), \end{aligned} \tag{F-11}$$

where we used that $\frac{f_1(1-\tilde{f}_1)}{\tilde{f}_1} = (1-f_1) - \left(1 - \frac{f_1}{\tilde{f}_1}\right)$ and $\mathbf{1}\{\tilde{f}_1 \ge f_1\} \cdot \left(1 - \frac{f_1}{\tilde{f}_1}\right) = \max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\}$.

The following result will be used in bounding the principal's utility. The proof is provided in Appendix F.5.

Lemma 5. *The following hold:*

1. *If $f_1 \le 1/2$, then $\mathbb{E}\left[\min\left\{1, \frac{1-\tilde{f}_1}{\tilde{f}_1}\right\}\right] \ge 1 - 4\delta - \frac{2}{\sqrt{T_1}}$.*
2. *If $f_1 > 1/2$, then $\mathbb{E}\left[\max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\}\right] \le 2\delta + \frac{1}{\sqrt{T_1}}$.*
3. *$\Pr(\tilde{f}_1 \ge f_1) \ge 1 - e^{-2\delta^2 T_1}$.*

Step 3. We next bound the regret of the dynamic mechanism A. For the recommended strategy σ under A, we let σ be the truthful reporting strategy σ^{TR} defined above for $F = (f_1, f_2)$ with $f_1 \in [0, \frac{1}{1+q_0}]$ and σ be any arbitrary utility-maximizing strategy for the agent corresponding to F for F with $f_1 \in (\frac{1}{1+q_0}, 1]$. Note A with the recommended strategy σ is incentive compatible by the argument in Step 1 and the construction of choosing utility-maximizing strategies. Note

$$\begin{aligned} \mathrm{Regret}(A, F, T) &= \mathrm{OPT}(F, T) - \mathrm{PrincipalUtility}(A, \sigma, F, T) \\ &= T \cdot \bar{u}(F) - \mathrm{PrincipalUtility}(A, \sigma, F, T), \end{aligned}$$

by Proposition 9. We upper bound the regret in three separate cases depending on the agent's distribution $F = (f_1, f_2)$. Note $\frac{1}{1+q_0} \ge \frac{1}{2}$ for $T_1 \ge 1$.

If $f_1 \in [0, \frac{1}{2}]$, then $\bar{u}(F) = f_1$. From (F-9) and (F-10), we have

$$
\begin{aligned}
\text{PrincipalUtility}(A, \sigma, F, T) &= \text{PrincipalUtility}(A, \sigma^{\text{TR}}, F, T) \\
&\geq f_1 \cdot T_2 \cdot \mathbb{E}\left[\min\left\{1, \frac{1-\tilde{f}_1}{\tilde{f}_1}\right\}\right].
\end{aligned}
$$

Then,

$$
\begin{aligned}
\frac{1}{T}\text{Regret}(A, F, T) &\leq f_1\left(1 - \frac{T_2}{T}\cdot\mathbb{E}\left[\min\left(1, \frac{1-\tilde{f}_1}{\tilde{f}_1}\right)\right]\right) \\
&\leq f_1\left(1 - \left(1 - \frac{T_1+1}{T}\right)\cdot\left(1 - 4\delta - \frac{2}{\sqrt{T_1}}\right)\right) \\
&= f_1\left(1 - \left(1 - \frac{T_1+1}{T} - 4\delta - \frac{2}{\sqrt{T_1}} + \frac{T_1+1}{T}\cdot\left(4\delta + \frac{2}{\sqrt{T_1}}\right)\right)\right) \\
&\leq f_1\left(\frac{T_1+1}{T} + 4\delta + \frac{2}{\sqrt{T_1}}\right) \\
&\leq 2\delta + \frac{T_1}{T} + \frac{1}{\sqrt{T_1}}, \qquad \text{(F-12)}
\end{aligned}
$$

where the second inequality follows from $T_2 = T - T_1 - 1$ and Part 1 of Lemma 5; the second-to-last inequality follows from dropping the negative term in the resulting expression in the parentheses; and the last inequality follows from $f_1 \leq \frac{1}{2}$ and $T_1 \geq 1$ which implies $T_1 + 1 \leq 2T_1$.

If $f_1 \in (\frac{1}{2}, \frac{1}{1+q_0}]$, then $\bar{u}(F) = 1 - f_1$. From (F-9) and (F-11), we have

$$
\begin{aligned}
\text{PrincipalUtility}(A, \sigma, F, T) &= \text{PrincipalUtility}(A, \sigma^{\text{TR}}, F, T) \\
&\geq T_2 \cdot \left((1-f_1)\cdot\Pr(\tilde{f}_1 \geq f_1) - \mathbb{E}\left[\max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\}\right]\right).
\end{aligned}
$$

Consequently, we obtain

$$
\begin{aligned}
\frac{1}{T}\text{Regret}(A, F, T) &\leq (1-f_1)\left(1 - \frac{T_2}{T}\cdot\Pr(\tilde{f}_1 \geq f_1)\right) + \frac{T_2}{T}\mathbb{E}\left[\max\left(1 - \frac{f_1}{\tilde{f}_1}, 0\right)\right] \\
&\leq (1-f_1)\left(1 - \left(1 - \frac{T_1+1}{T}\right)\cdot\left(1 - e^{-2\delta^2 T_1}\right)\right) + \frac{T_2}{T}\cdot\left(2\delta + \frac{1}{\sqrt{T_1}}\right) \\
&= (1-f_1)\left(1 - \left(1 - \frac{T_1+1}{T} - e^{-2\delta^2 T_1} + \frac{T_1+1}{T}\cdot e^{-2\delta^2 T_1}\right)\right) + \frac{T_2}{T}\cdot\left(2\delta + \frac{1}{\sqrt{T_1}}\right) \\
&\leq (1-f_1)\left(\frac{T_1+1}{T} + e^{-2\delta^2 T_1}\right) + \frac{T_2}{T}\cdot\left(2\delta + \frac{1}{\sqrt{T_1}}\right) \\
&\leq \frac{1}{2}e^{-2\delta^2 T_1} + 2\delta + \frac{T_1}{T} + \frac{1}{\sqrt{T_1}}, \qquad \text{(F-13)}
\end{aligned}
$$

where the second inequality follows from $T_2 = T - T_1 - 1$ and Parts 2 and 3 of Lemma 5; the second-to-last inequality follows from dropping the product term $\frac{T_1+1}{T}\cdot e^{-2\delta^2 T_1}$; and the last inequality follows because $1 - f_1 \leq \frac{1}{2}$, $T_2 \leq T$, and $T_1 \geq 1$ which implies $T_1 + 1 \leq 2T_1$.

If $f_1 \in (\frac{1}{1+q_0}, 1]$, then $\bar{u}(F) = 1 - f_1$. Note that the principal's utility is always at least 0 regardless of the agent's utility-maximizing strategy, i.e., $\text{PrincipalUtility}(A, \sigma, F, T) \geq 0$. Using the last observation, we obtain

$$\frac{1}{T}\,\text{Regret}(A, F, T) \leq 1 - f_1 \leq 1 - \frac{1}{1+q_0} = \frac{q_0}{1+q_0} \leq q_0\,, \tag{F-14}$$

where the last inequality follows because $q_0 \geq 0$.

Combining the upper bounds on the regret in above three cases, (F-12)–(F-14), and using that $q_0 = \frac{1}{\sqrt{T_1}}$, we obtain

$$\begin{aligned}\frac{1}{T}\sup_{F\in\mathcal{F}} \text{Regret}(A, F, T) &\leq 2\delta + \frac{1}{2}e^{-2\delta^2 T_1} + \frac{T_1}{T} + \frac{1}{\sqrt{T_1}}\\ &\leq \frac{(\ln T)^{1/2}}{T^{1/3}} + \frac{1}{2T^{1/3}} + \frac{1}{T^{1/3}} + \frac{1}{T^{1/3}}\\ &= \frac{(\ln T)^{1/2}}{T^{1/3}} + \frac{5}{2T^{1/3}}\,.\end{aligned}$$

where the second inequality follows from our choices for δ and T_1.

F.5 Missing Proofs from Appendix F.4

Proof of Lemma 5. We prove each part at a time. For Part 1, note that the function $x \mapsto \frac{1}{x}$ is convex and a first-order expansion around $\frac{1}{2}$ yields the lower bound $\frac{1}{\tilde{f}_1} \geq 2 - 4\left(\tilde{f}_1 - \frac{1}{2}\right)$. Therefore,

$$\min\left\{1, \frac{1-\tilde{f}_1}{\tilde{f}_1}\right\} \geq \min\left\{1, 1 - 4\left(\tilde{f}_1 - \frac{1}{2}\right)\right\} = 1 - 4\max\left\{\tilde{f}_1 - \frac{1}{2}, 0\right\}$$

Because $\tilde{f}_1 = \hat{f}_1 + \delta$, $\delta \geq 0$, and $f_1 \leq \frac{1}{2}$, we have that

$$\max\left\{\tilde{f}_1 - \frac{1}{2}, 0\right\} \leq \delta + \max\left\{\hat{f}_1 - f_1, 0\right\} \leq \delta + |\hat{f}_1 - f_1|\,,$$

where the last inequality follows because $\max\{x, 0\} \leq |x|$ for all $x \in \mathbb{R}$. Note that $T_1 \cdot \hat{f}_1$ is binomially distributed with T_1 trials and the success probability of f_1. Jensen's inequality and that $\mathbb{E}[\hat{f}_1] = f_1$ imply that

$$\mathbb{E}\left[|\hat{f}_1 - f_1|\right] \leq \sqrt{\text{Var}(\hat{f}_1)} = \sqrt{\frac{f_1(1-f_1)}{T_1}} \leq \frac{1}{2\sqrt{T_1}}\,, \tag{F-15}$$

where the equality follows from the variance formula for a binomially distributed random variable and the last inequality follows because $f_1(1-f_1) \le \frac{1}{4}$ for $f_1 \in [0,1]$. Putting everything together,

$$\begin{aligned}
\mathbb{E}\left[\min\left\{1, \frac{1-\tilde{f}_1}{\tilde{f}_1}\right\}\right] &\ge 1 - 4\mathbb{E}\left[\max\left\{\tilde{f}_1 - \frac{1}{2}, 0\right\}\right] \\
&\ge 1 - 4\delta - 4\mathbb{E}\left[|\hat{f}_1 - f_1|\right] \\
&\ge 1 - 4\delta - \frac{2}{\sqrt{T_1}}\,.
\end{aligned}$$

For Part 2, we use, again, that the function $x \mapsto \frac{1}{x}$ is convex to obtain that a first-order expansion around f_1 yields the lower bound $\frac{1}{\tilde{f}_1} \ge \frac{1}{f_1} - \frac{1}{f_1^2}(\tilde{f}_1 - f_1)$. Therefore,

$$\begin{aligned}
\max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\} &\le \frac{1}{f_1}\max\left\{\tilde{f}_1 - f_1, 0\right\} \\
&\le 2\delta + 2\max\left\{\hat{f}_1 - f_1, 0\right\} \\
&\le 2\delta + 2|\hat{f}_1 - f_1|\,,
\end{aligned}$$

where the first inequality follows from the above lower bound; the second inequality follows from $f_1 > \frac{1}{2}$, $\tilde{f}_1 = \hat{f}_1 + \delta$, and $\delta \ge 0$; and the last is because $\max\{x, 0\} \le |x|$ for all $x \in \mathbb{R}$. Taking expectations and using (F-15), we obtain

$$\mathbb{E}\left[\max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\}\right] \le 2\delta + 2\mathbb{E}\left[|\hat{f}_1 - f_1|\right] \le 2\delta + \frac{1}{\sqrt{T_1}}\,.$$

For Part 3, we use that $\tilde{f}_1 = \hat{f}_1 + \delta$ to obtain

$$\Pr(\tilde{f}_1 \ge f_1) = \Pr(\hat{f}_1 \ge f_1 - \delta) = 1 - \Pr(\hat{f}_1 < f_1 - \delta) \ge 1 - e^{-2\delta^2 T_1}\,,$$

where the last inequality follows from Hoeffding's inequality because $T_1 \cdot \hat{f}_1$ is binomially distributed with T_1 trials and success probability f_1. □

References

Yannis Bakos and Erik Brynjolfsson. Bundling information goods: Pricing, profits, and efficiency. *Management Science*, 45(12):1613–1630, 1999.

Dirk Bergemann and Karl H. Schlag. Pricing without priors. *Journal of the European Economic Association*, 6(2-3):560–569, 2008.

Vinicius Carrasco, Vitor Farinha Luz, Paulo K. Monteiro, and Humberto Moreira. Robust mechanisms: the curvature case. *Economic Theory*, April 2018.

J.K. Hale. *Ordinary Differential Equations*. Pure and applied mathematics : a series of texts and monographs. Wiley-Interscience, 1969.

S. Kakade, I. Lobel, and H. Nazerzadeh. Optimal dynamic mechanism design and the virtual pivot mechanism. *Operations Research*, 61(3):837–854, 2013.

Vijay Krishna. *Auction Theory*. Elsevier, 2 edition, 2009.

Roger B. Myerson. Optimal auction design. *Math. Oper. Res.*, 6(1):58–73, February 1981. ISSN 0364-765X.