Electronic Companion to
"On the Futility of Dynamics in Robust Mechanism Design"

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EC.1   Introduction

In this electronic companion, we present several results that complement the exposition in the main paper. In Section EC.2, we prove a general minimax theorem and an "asymmetric" minimax theorem using a topological approach for the single-round problem. In Section EC.3, we consider welfare maximization in the single-good case and revenue/welfare maximization in the multiple-goods case of the dynamic selling problem in Section 5 of the main paper. Section EC.4 extends our results in the main paper in several directions: alternative benchmarks, serially correlated shock processes, a multiplicative performance guarantee, and a stronger notion of regret. Finally, in Section EC.5, we show equivalence-type connections between the minimax regret and maximin utility objectives for revenue maximization in the dynamic selling mechanism design problem.

EC.2   Single-Round Saddle-Point Theorems under Sufficient Conditions

We prove a general minimax theorem and an "asymmetric" minimax theorem under sufficient conditions. Under the assumption to follow, we show a general saddle-point result for the single-round problem. Our approach is topological in nature and involves endowing the space of (randomized) single-round direct mechanisms $\mathcal{S}$ with the right topology.

Assumption EC.1. *The game satisfies:*

(i) *The outcome space $\Omega \subset \mathbb{R}^m$ is compact and the shock space Θ is finite.*

(ii) *The principal's utility function $u(\theta, \omega)$ is upper semi-continuous in ω for all θ.*

(iii) *The agent's utility function $v(\theta, \omega)$ is continuous in ω for all θ.*

Assumption EC.1 guarantees that the single-round benchmark is well-defined. Parts of the assumption may be relaxed as long as the benchmark is well-defined in applications. In most cases including those considered in this paper, this assumption is satisfied, at least, for an approximate result because the outcome space can be suitably restricted without loss by imposing large bounds which can be shown to be not binding at an optimal solution and the shock space can be discretized. Parts (ii) and (iii) are typical in the mechanism design literature. Note Assumption EC.1 holds trivially for finite discrete games and may be dropped. We can show the following:

Theorem EC.1. *Suppose the game satisfies Assumption EC.1. The minimax regret of the single-round problem satisfies:*

$$\widehat{\mathrm{Regret}} = \min_{\substack{S \in \mathcal{S}: \\ \mathrm{(IC),(IR)}}} \max_{F \in \mathcal{F}} \widehat{\mathrm{Regret}}(S, F) = \max_{F \in \mathcal{F}} \min_{\substack{S \in \mathcal{S}: \\ \mathrm{(IC),(IR)}}} \widehat{\mathrm{Regret}}(S, F).$$

Moreover, there exists a (randomized) single-round direct IC/IR mechanism S^* and a distribution F^* such that $\widehat{\mathrm{Regret}} = \widehat{\mathrm{Regret}}(S^*, F^*)$ and $\widehat{\mathrm{Regret}}(S^*, F) \leq \widehat{\mathrm{Regret}}(S^*, F^*) \leq \widehat{\mathrm{Regret}}(S, F^*)$ for any $S \in \mathcal{S}$ satisfying the IC/IR constraints and $F \in \mathcal{F}$.

Theorem EC.1 shows that the minimax regret is equivalent to the maximin regret for the single-round problem and the minimax regret is achieved by a single-round mechanism. In the maximin regret formulation, nature first chooses the agent's private distribution and then the principal chooses an optimal mechanism based on nature's choice. The result shows there is a distribution that is uniformly challenging for all possible single-round mechanisms. We prove the result in Appendix EC.2.1 using the well-known von Neumann-Fan minimax theorem (see, e.g., Fan 1953).

The assumption that the shock space is finite appears critical for the existence of a saddle point. In Section 7 of the main paper, we exhibit a simple game with a continuous, even compact, shock space that does not admit a worst-case distribution that is uniformly challenging for all mechanisms. When the shock space is arbitrary and the same conditions in Assumption EC.1 otherwise hold, it is possible to show an "asymmetric" minimax theorem, i.e., the saddle point property holds, the minimax regret problem admits an optimal single-round mechanism, but the maximin regret problem

does not necessarily have an optimal worst-case distribution. We prove such an asymmetric minimax result in Appendix EC.2.2.

EC.2.1   Proof of Theorem EC.1

We prove the result in three steps. First, we endow the space of randomized single-round direct mechanisms with a topology. Second, we show that an optimization problem corresponding to the single-round benchmark $\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)]$ admits an optimal solution and the single-round benchmark is upper semi-continuous in F. The single-round benchmark is equivalently the value of the following optimization problem:

$$\int_\Theta \mathrm{OPT}(\theta, 1) \mathrm{d}F(\theta) = \sup_{S \in \mathcal{S}} \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta)$$
$$\text{s.t.} \int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) \geq 0, \quad \forall \theta \in \Theta.$$

That the above optimization problem admits an optimal solution and the single-round benchmark is the objective value will be useful in showing the upper semi-continuity of the single-round benchmark. Finally, we prove the minimax result using the von Neumann-Fan minimax theorem.

Step 1. We endow the space of distributions over outcomes $\Delta(\Omega)$ with the weak* topology. Namely, for a sequence of probability distributions in $\Delta(\Omega)$, we have $G^k \rightharpoonup G$ if and only if $\int_\Omega \psi(\omega) dG^k(\omega) \to \int_\Omega \psi(\omega) dG(\omega)$ for all ψ in the space of continuous functions $C(\Omega, \mathbb{R})$ from Ω to $\mathbb{R}$ with the sup-norm topology. The space $\Delta(\Omega)$ is weak* compact because Ω is compact (see Aliprantis and Border, 2006, Theorem 15.11 in p. 513). We endow the space of randomized single-round mechanisms $\mathcal{S} = \Delta(\Omega)^\Theta$ with pointwise convergence, i.e., for a sequence of mechanisms in $\mathcal{S}$ we have $S^k \to S$ if and only if $S_\theta^k \rightharpoonup S_\theta$ for all $\theta \in \Theta$. By Tychonoff Product Theorem (see Aliprantis and Border, 2006, Theorem 2.61 in p. 52), the product space $\mathcal{S}$ is compact because each factor is compact. Because the shock space is finite, the space of distributions over shocks $\Delta(\Theta) = \left\{ f \in \mathbb{R}_+^{|\Theta|} : \sum_{\theta \in \Theta} f_\theta = 1 \right\}$ is a compact subset of the Euclidean space. We have $F^k \to F$ if and only if $f_\theta^k \to f_\theta$ for all $\theta \in \Theta$.

Step 2. Let $U : \mathcal{S} \times \Delta(\Omega) \to \mathbb{R}$ be the principal's utility functional which is given by $U(S, F) = \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta)$ for $S \in \mathcal{S}$ and $F \in \Delta(\Theta)$. Because Θ is finite, we have $U(S, F) = \sum_{\theta \in \Theta} \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) f_\theta$. A similar definition holds for the agent's utility functional $V : \mathcal{S} \times \Delta(\Omega) \to$

$\mathbb{R}$, that is, $V(S, F) = \int_\Theta \int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta)$ for $S \in \mathcal{S}$ and $F \in \Delta(\Theta)$. We denote by $\mathcal{C}(F) = \{S \in \mathcal{S} : V(S, \theta) \geq 0, \forall \theta \in \Theta\}$ the set of interim individually rational mechanisms when the distribution is F. The single-round benchmark is given by $\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)] = \sup_{S \in \mathcal{C}(F)} U(F, S)$. The following holds:

- The principal's utility functional $U(S, F)$ is jointly upper semi-continuous: Consider sequences $F^k \to F$ and $S^k \to S$. Fix $\theta \in \Theta$. Let $u_\theta^k = \int_\Omega u(\theta, \omega) dS_\theta^k(\omega)$ and $u_\theta = \int_\Omega u(\theta, \omega) dS_\theta(\omega)$. Because $u(\theta, \omega)$ is upper semi-continuous in ω, we have $\limsup_{k \to \infty} u_\theta^k \leq u_\theta$. Because the shock space is finite we obtain, using the multiplication rule for limits, that $\limsup_{k \to \infty} U(S^k, F^k) = \sum_{\theta \in \Theta} \limsup_{k \to \infty} u_\theta^k f_\theta^k \leq \sum_{\theta \in \Theta} u_\theta f_\theta = U(S, F)$.

- The agent's utility functional $V(S, F)$ is jointly continuous: This follows from repeating the same reasoning steps above for the principal's utility functional.

- The feasible set correspondence $\mathcal{C}(F)$ is compact-valued and non-empty: For any fixed distribution F, the set $\{S \in \mathcal{S} : V(S, F) \geq 0\}$ is closed because upper level sets of upper semi-continuous functions are closed (see Aliprantis and Border, 2006, Corollary 2.60 in p. 52). In particular, $\{S \in \mathcal{S} : V(S, \theta) \geq 0\}$ is closed for each $\theta \in \Theta$. Since the intersection of a finite number of closed sets is closed, $\mathcal{C}(F)$ is closed. Because $\mathcal{S}$ is compact, compactness follows because the intersection of a closed set and a compact set is compact. Non-emptiness follows because the trivial mechanism that always determines the no-interaction outcome satisfies the interim IR constraint and is feasible.

- The feasible set correspondence $\mathcal{C}(F)$ is upper hemi-continuous in F: In fact, the feasible set $\mathcal{C}(F)$ is the same closed set for all distributions $F \in \mathcal{F}$ since it is defined in terms of the point-mass distributions $\theta \in \Theta$. Clearly, the feasible set correspondence is upper hemi-continuous.

Berge's Maximum Theorem implies that the optimization problem corresponding to the single-round benchmark admits an optimal solution and the single-round benchmark $\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)]$ is upper semi-continuous in F (see Aliprantis and Border, 2006, Lemma 17.30 in p. 569).

Step 3. Let $V_{\theta, \theta'} : \mathcal{S} \to \mathbb{R}$ be the agent's utility functional when his shock is θ and his report is θ', which is given by $V_{\theta, \theta'}(S) = \int_\Omega v(\theta, \omega) \mathrm{d}S_{\theta'}(\omega)$ for $S \in \mathcal{S}$. We need to show a minimax result for the

following problem:

$$\inf_{\substack{S \in \mathcal{S}: \\ \text{(IC),(IR)}}} \sup_{F \in \Delta(\Omega)} \{\mathbb{E}_{\theta \sim F}[\text{OPT}(\theta, 1)] - U(S, F)\}$$

where (IR) is given by $V_{\theta,\theta}(S) \geq 0$ for all $\theta \in \Theta$ and (IC) is given by $V_{\theta,\theta}(S) \geq V_{\theta,\theta'}(S)$ for all $\theta, \theta' \in \Theta$.

The following hold:

- The space of feasible mechanisms $\hat{\mathcal{S}} := \{S \in \mathcal{S} : \text{(IC) and (IR)}\}$ is compact: Note that $V_{\theta,\theta}(S) = V(S, \theta)$ where we denote by θ the point-mass distribution that takes the value θ with probability one in the expression $V(S, \cdot)$. Joint continuity of V implies that $V(S, \theta)$ is continuous in S for all θ. The IR constraint is closed because upper level sets of continuous functions are closed (see Aliprantis and Border, 2006, Corollary 2.60 in p. 52). A similar argument follows for the IC constraint by considering the functionals $\tilde{V}_{\theta,\theta'} : \mathcal{S} \to \mathbb{R}$ given by $\tilde{V}_{\theta,\theta'}(S) = V_{\theta,\theta}(S) - V_{\theta,\theta'}(S)$ for $\theta, \theta' \in \Theta$. The result follows because $\mathcal{S}$ is compact and the intersection of a finite number of closed sets with a compact set is compact.

- The objective is convex on $\hat{\mathcal{S}} := \{S \in \mathcal{S} : \text{(IC) and (IR)}\}$, concave on $\mathcal{F}$, and lower semi-continuous on $\hat{\mathcal{S}}$, and upper semi-continuous on $\mathcal{F}$: The convexity in S and concavity in F follows because the objective is bilinear since we allow for randomized mechanisms and the feasible sets are convex. Lower semi-continuity in S follows because $U(S, F)$ is upper semi-continuous. We next prove the upper semi-continuity in F. From the previous step, we know that the single-round benchmark $\mathbb{E}_{\theta \sim F}[\text{OPT}(\theta, 1)]$ is upper semi-continuous in F. Moreover, $U(S, F)$ is continuous in F for any fixed S because all linear functionals are continuous in finite dimensional spaces. The result follows because the sum of a continuous function and an upper semi-continuous function is upper semi-continuous.

- The feasible set is non-empty: This follows because the trivial mechanism that always determines the no-interaction outcome, i.e., the outcome $\emptyset$ such that $v(\theta, \emptyset) = 0$ for all θ, satisfies the IC/IR constraints and is feasible.

The theorem then follows from the von Neumann-Fan minimax theorem (Fan, 1953).

EC.2.2 An Asymmetric Saddle-Point Theorem for Arbitrary Shock Spaces

We discuss how to extend our saddle-point result, Theorem EC.1, to arbitrary shock spaces under the following assumption. We adapt the same notations from Appendix EC.2.1:

Assumption EC.2. *The game satisfies:*

(i) *The outcome space Ω is normed and compact, and the shock space Θ is a topological space.*

(ii) *The principal's utility function $u(\theta, \omega)$ is upper semi-continuous in ω for all θ and uniformly bounded.*

(iii) *The agent's utility function $v(\theta, \omega)$ is continuous in ω for all θ.*

We shall prove that:

$$\widehat{\text{Regret}} = \min_{\substack{S \in \mathcal{S}: \\ (\text{IC}),(\text{IR})}} \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F) = \sup_{F \in \mathcal{F}} \min_{\substack{S \in \mathcal{S}: \\ (\text{IC}),(\text{IR})}} \widehat{\text{Regret}}(S, F).$$

Moreover, there exists a (randomized) single-round direct IC/IR mechanism S^* such that $\widehat{\text{Regret}} = \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S^*, F)$.

As in the proof of Theorem EC.1, we endow the space of distributions over outcomes $\Delta(\Omega)$ with the weak* topology. The space of randomized single-round mechanisms is given by $\mathcal{S} = \{S \in \Delta(\Omega)^\Theta : S$ is Borel measurable in $\Theta\}$. We endow $\mathcal{S}$ with pointwise convergence. By Tychonoff Product Theorem (see Aliprantis and Border, 2006, Theorem 2.61 in p. 52), the product space $\Delta(\Omega)^\Theta$ is compact and Hausdorff. The space of measurable single-round mechanisms is closed with respect to the pointwise-convergence topology because $\Delta(\Omega)$ is metrizable (see Aliprantis and Border, 2006, Lemma 4.29 in p. 142). In turn, the space $\Delta(\Omega)$ is metrizable because Ω is a compact and normed (see Aliprantis and Border, 2006, Theorem 15.11 in p. 513). Because the intersection of a compact set and a closed set is compact, we obtain that $\mathcal{S}$ is compact.

From Theorem 2 in Fan (1953), it suffices to show that (i) the space of feasible mechanisms $\hat{\mathcal{S}} := \{S \in \mathcal{S} : (\text{IC}) \text{ and } (\text{IR})\}$ is compact, and (ii) the objective is convex in S, concave in F, and lower semi-continuous in S. For (i), it suffices to show that $V_{\theta,\theta}(S)$ is continuous in S for all θ. Consider a sequence of mechanisms $S^k \to S$, which is equivalent to $S^k_\theta \rightharpoonup S_\theta$ for all θ. Because $V_{\theta,\theta}(S) = \int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega)$, the result follows from weak* convergence in $\Delta(\Omega)$ since $v(\theta, \omega)$ is continuous in ω

for all θ. For (ii), it suffices to show that $U(S, F)$ is upper semi-continuous in S. We equivalently write $U(S, F) = \int_\Theta U_\theta(S) \mathrm{d}F(\theta)$ with $U_\theta(S) = \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega)$. Consider a sequence of mechanisms $S^k \to S$. From weak* convergence in $\Delta(\Omega)$, we obtain that $\limsup_{k\to\infty} U_\theta(S^k) \leq U_\theta(S)$ for all θ because $u(\theta, \omega)$ is upper semi-continuous in ω. Because $U_\theta(S)$ is uniformly bounded, the result follows from the reverse Fatou's lemma.

EC.3 Additional Materials on Dynamic Selling Mechanisms

We consider other versions of the dynamic selling problem presented in Section 5 of the main paper. In Section EC.3.1, we consider welfare maximization in the single-good case. In Section EC.3.2, we consider both revenue and welfare maximization in the multiple-goods case where the principal sells identical copies of n goods, one per good, in each round.

EC.3.1 Welfare Maximization

For welfare maximization in the single-good case, the principal's utility function is $u(\theta, \omega) = \theta \cdot \hat{x}$ for outcome $\omega = (\hat{x}, \hat{p})$. The single-round benchmark is $\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)] = \mathbb{E}_{\theta \sim F}[\theta]$ because $\mathrm{OPT}(\theta, 1) = \theta$ which is achieved by choosing the outcome $(1, 0)$ or, equivalently, always allocating the item at no cost. By Proposition 8 of the main paper, the second part of Assumption 1 of the main paper holds. Furthermore, combining with Proposition 6 of the main paper, we have $\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)] = \bar{u}(F)$ and, hence, $\sup_{F \in \mathcal{F}} \bar{u}(F) \leq 1$. Then, Assumption 1 holds by Proposition 7 of the main paper and Theorem 1 of the main paper applies. We show that the minimax regret for the single-round problem for direct IC/IR mechanisms is 0 and, thus, that for the multi-round problem is 0 because $\mathrm{Regret}(T) = T \cdot \widehat{\mathrm{Regret}}$.

We formally state the minimax regret result as follows:

Theorem EC.2. *For welfare maximization in the dynamic selling mechanism design problem with one good, the minimax regret is 0 and an optimal solution is allocating items for free, which is T repetitions of the same strategy.*

Clearly, allocating items for free satisfies the IC/IR constraints when considered as a single-round direct mechanism. This is a degenerate result with a trivial solution but we think it is interesting

that our general result captures it. Note the optimal multi-round solution of allocating for free is still optimal if the private shock distribution is known to the principal. In this sense, no dynamic/adaptive strategy with sophisticated learning was necessary to begin with and the distributional information of the agent's private distribution was not needed.

Proof of Theorem EC.2. We solve the single-round minimax regret problem for direct IC/IR mechanisms which is

$$\inf_{\substack{S \in \mathcal{S}: \\ (\text{IC}),(\text{IR})}} \sup_{F \in \mathcal{F}} \left\{ \int_\Theta \theta \mathrm{d}F(\theta) - \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \right\}.$$

The trivial single-round mechanism S^* that always allocates the item for free, i.e., S^* is the deterministic mechanism π^* for which $\pi^*(\theta) = (1, 0)$ for all $\theta \in \Theta$, satisfies the IC/IR constraints and is an optimal solution because it achieves $\mathbb{E}_{\theta \sim F}[\theta]$ for any agent's distribution F. It is easy to see that we can take any distribution F and (S^*, F) is a saddle point for the single-round problem. Hence, $\widehat{\text{Regret}} = 0$. By Theorem 1 of the main paper, we obtain $\text{Regret}(T) = 0$ and repeating S^* over T rounds is an optimal solution with the minimax regret of $\text{Regret}(T) = 0$. $\qquad\square$

EC.3.2 Multiple Goods

Building on Section 5 of the main paper and Section EC.3.1, we consider the multiple-goods version of the dynamic selling mechanism design problem where the principal has n goods and sells independent copies (or units) of these goods, one per good, in each round. In particular, this is an application where the agent's private shock is multidimensional. For welfare maximization, it is straightforward to see that the same strategy from the single-good case of giving away for free is an optimal solution and achieves the minimax regret of 0 in both the single-round and multi-round problems. For revenue maximization, we show below that repeatedly selling each good separately according to the randomized posted pricing mechanism given in Theorem 13 of the main paper for the single-good case is optimal.

In each round, the agent sees n goods (that is, copies of) and realizes his value for each good in the range $[0, 1]$ according to a private shock distribution F. The values of the goods can be arbitrarily correlated but are drawn from the joint distribution independently across rounds. The shock space is $\Theta = [0, 1]^n$ and the agent's private shock distribution F is a distribution over Θ. The outcome space is $\Omega = \{0, 1\}^n \times \mathbb{R}$. Using superscript i to denote the coordinate corresponding to the i-th

good, an outcome $\omega = (\hat{x}, \hat{p}) \in \Omega$ is given by allocations $(\hat{x}^1, \ldots, \hat{x}^n)$ of the goods and payment $\hat{p}$. Given an outcome $\omega = (\hat{x}, \hat{p})$, the agent's utility function is $v(\theta, \omega) = \sum_{i=1}^n \theta^i \cdot \hat{x}^i - \hat{p}$. For revenue maximization, the principal's utility function is $u(\theta, \omega) = \hat{p}$, and for welfare maximization, the principal's utility function is $u(\theta, \omega) = \sum_{i=1}^n \theta^i \cdot \hat{x}^i$.

The main result is as follows.

Theorem EC.3. *For revenue maximization in the dynamic selling mechanism design problem with n goods, the minimax regret is $\frac{n}{e}T$ and an optimal solution is T repetitions of selling each good separately via the randomized posted pricing mechanism S^* given in Theorem 13 of the main paper. For welfare maximization, the minimax regret is 0 and an optimal solution is allocating goods for free, which is T repetitions of the same strategy.*

The proof for the welfare maximization part is almost identical to the single-good case in Section EC.3.1 and thus omitted. Henceforth, we discuss the revenue maximization part. In each round, we implement the single-good solution (x^*, p^*) for each good to determine $\hat{x}^i = x^*(\theta^i)$ and $\hat{p}^i = p^*(\theta^i)$. Then, the overall allocation and payment are $(\hat{x}^1, \ldots, \hat{x}^n)$ and $\hat{p} = \sum_i \hat{p}^i$. Note $\mathrm{OPT}(\theta, 1) = \sum_{i=1}^n \theta^i$ because the principal can extract the full surplus of the agent by allocating all the goods and charging $\sum_{i=1}^n \theta^i$. Hence, the single-round benchmark is $\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)] = \mathbb{E}_{\theta \sim F}[\sum_{i=1}^n \theta^i]$. Since this is a game where the payment enters linearly into the utility functions of the principal and agent (see Proposition 8 in Section 3.3 of the main paper), the linearity condition in $\bar{u}(F)$ holds. By the linearity condition and Proposition 6 of the main paper, $\bar{u}(F)$ is equal to the single-round benchmark and $\sup_{F \in \mathcal{F}} \bar{u}(F) = \sup_{F \in \mathcal{F}} \mathbb{E}_{\theta \sim F}[\sum_{i=1}^n \theta^i] \leq n$. By Proposition 7 of the main paper, Assumption 1 of the main paper holds. The optimal performance achievable is $\mathrm{OPT}(F, T) = T \cdot \bar{u}(F)$ and it is achieved by a bundling-type mechanism that knows the agent's private distribution a priori as in the single-good case.

As before, the general result (Theorem 1 of the main paper) applies. To show the claimed dynamic mechanism is an optimal solution to the multi-round problem, we show that the single-round solution of selling each good separately is an optimal solution to the single-round minimax regret problem, via a saddle-point formulation. Note Kocyigit et al. (2018) have recently shown the same single-round result. We prove it for completeness below.

Theorem EC.4. *Let $S^{*,n}$ denote the single-round direct IC/IR mechanism that separately implements $S^* = (x^*, p^*)$ for each good and $F^{*,n}$ denote the agent's distribution with $F^{*,n}(\theta) = F^*(\min_i \theta^i)$ for*

any $\theta \in \Theta$ (i.e., perfectly correlated values) where S^* and F^* are as given in Theorems 13 and 21 of the main paper. Then, $\widehat{\text{Regret}} = \widehat{\text{Regret}}(S^{*,n}, F^{*,n}) = \frac{n}{e}$ and

$$\widehat{\text{Regret}}(S^{*,n}, F) \leq \widehat{\text{Regret}}(S^{*,n}, F^{*,n}) \leq \widehat{\text{Regret}}(S, F^{*,n}),$$

for any $S \in \mathcal{S}$ satisfying the IC/IR constraints and $F \in \mathcal{F}$.

Proof. First, we show that $F^{*,n}$ is a solution to

$$\max_{F \in \mathcal{F}} \widehat{\text{Regret}}(S^{*,n}, F).$$

The above expression is equivalent to

$$\max_{F \in \mathcal{F}} \mathbb{E}_{\theta \sim F}\left[\sum_{i=1}^{n}(\theta^i - p^*(\theta^i))\right] = \max_{F \in \mathcal{F}} \sum_{i=1}^{n} \mathbb{E}_{\theta^i \sim F^i}\left[\theta^i - p^*(\theta^i)\right],$$

where F^i is the marginal distribution of F for the i-th good. Interchanging the maximum and summation, the last expression is upper bounded by

$$\sum_{i=1}^{n} \max_{F^i \sim \Delta(\Theta^i)} \mathbb{E}_{\theta^i \in F^i}\left[\theta^i - p^*(\theta^i)\right],$$

where we can independently choose distributions F_i over $\Theta^i := [0,1]$ as one-dimensional distributions. By Theorem 21 of the main paper, an optimal one-dimensional distribution F^i in each summand is F^* which yields the value of $\frac{1}{e}$ and the upper bound evaluates to $\frac{n}{e}$. Since $F^{*,n}$ is a distribution with marginal distributions equal to F^* for each good, we have $\mathbb{E}_{\theta \sim F}[\sum_i(\theta^i - p^*(\theta^i))] = \frac{n}{e}$ and, hence, the distribution is a solution to the original maximization we started out with. Furthermore, it follows that $\widehat{\text{Regret}}(S^{*,n}, F^{*,n}) = \frac{n}{e}$.

Second, we show that $S^{*,n}$ is a solution to

$$\min_{\substack{S \in \mathcal{S}: \\ (\text{IC}),(\text{IR})}} \widehat{\text{Regret}}(S, F^{*,n}).$$

Substituting in $F^{*,n}$, the above optimization problem reduces to

$$\min_{\substack{S=(x,p)\in\mathcal{S}: \\ (\text{IC}),(\text{IR})}} \mathbb{E}_{\phi \sim F^*}[n\phi - p(\phi, \ldots, \phi)]$$

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where (x, p) are the interim allocation and payment rules. We relax the IC/IR constraints by restricting to the one-dimensional space where the agent's per-good shocks are the same. Recall:

$$\theta^\mathsf{T} \cdot x(\theta) - p(\theta) \geq \theta^\mathsf{T} \cdot x(\theta') - p(\theta'), \quad \forall \theta, \theta' \in [0, 1]^n. \tag{IC}$$

$$\theta^\mathsf{T} \cdot x(\theta) - p(\theta) \geq 0, \quad \forall \theta \in [0, 1]^n. \tag{IR}$$

The relaxed IC/IR constraints are:

$$\phi \cdot \sum_{i=1}^n x^i(\phi \cdot \vec{\mathbf{1}}) - p(\phi \cdot \vec{\mathbf{1}}) \geq \phi \cdot \sum_{i=1}^n x^i(\phi' \cdot \vec{\mathbf{1}}) - p(\phi' \cdot \vec{\mathbf{1}}), \quad \forall \phi, \phi' \in [0, 1]. \tag{IC'}$$

$$\phi \cdot \sum_{i=1}^n x^i(\phi \cdot \vec{\mathbf{1}}) - p(\phi \cdot \vec{\mathbf{1}}) \geq 0, \quad \forall \phi \in [0, 1]. \tag{IR'}$$

Note $\vec{\mathbf{1}}$ is the all-ones vector $(1, \ldots, 1)$. Then, the value of the optimization problem is lower bounded by

$$\min_{\substack{S=(x,p)\in\mathcal{S}: \\ \text{(IC'),(IR')}}} \mathbb{E}_{\phi \sim F^*}[n\phi - p(\phi, \ldots, \phi)]. \tag{EC-1}$$

We can transform a single-round mechanism $S = (x, p)$ for multiple goods to $\tilde{S} = (\tilde{x}, \tilde{p})$ for a single good by taking $\tilde{x}(\phi) = \frac{1}{n} \sum_{i=1}^n x^i(\phi \cdot \vec{\mathbf{1}})$ and $\tilde{p}(\phi) = \frac{1}{n} p(\phi \cdot \vec{\mathbf{1}})$ for report $\phi \in [0, 1]$. For any multi-good single-round mechanism S satisfying the relaxed IC/IR constraints, the transformation yields a single-good single-round mechanism satisfying the original IC/IR constraints for the single-good case. Furthermore, note that the multi-good mechanism implementing a single-good single-round direct IC/IR mechanism separately for each good satisfies the relaxed IC/IR constraints and yields the same single-good mechanism via the transformation. Hence, (EC-1) is equal to the following in terms of the objective value:

$$\min_{\substack{\tilde{S}=(\tilde{x},\tilde{p})\in\tilde{\mathcal{S}}: \\ \text{(IC''),(IR'')}}} n \cdot \mathbb{E}_{\phi \sim F^*}[\phi - \tilde{p}(\phi)],$$

where $\tilde{\mathcal{S}}$ denotes the single-round direct mechanisms for the single-good case and the IC/IR constraints are for the single-good case. This is equal to $\frac{n}{e}$ from the single-round minimax regret determined for the single-good case in Section 5 of the main paper.

To complete, we note $S^{*,n}$ is an optimal solution to (EC-1) and, hence, the original optimization problem because it is a feasible solution and achieves the lower bound $\frac{n}{e}$. More specifically,

$\widehat{\mathrm{Regret}}(S^{*,n}, F^{*,n})$ is equal to

$$n \cdot \mathbb{E}_{\phi \sim F^*}[\phi - p^*(\phi)] = n \cdot \widetilde{\mathrm{Regret}} = \frac{n}{e},$$

where $\widetilde{\mathrm{Regret}}$ is the single-good single-round minimax regret (so, it is equal to $\widehat{\mathrm{Regret}} = \frac{1}{e}$ in Section 5 of the main paper). $\qquad\square$

EC.4 Extensions

In this section, we extend our results in the main paper in several directions. First, we show our results still hold for other alternative benchmarks that are considered in the learning literature. Second, we consider serially correlated shock processes and show our results still apply. Third, we consider multiplicative performance guarantees and prove analogous results connecting the multi-round and single-round problems. Fourth, we explore a stronger notion of regret in which the agent plays a utility-maximizing strategy that is the least favorable for the principal.

EC.4.1 Two Alternative Benchmarks

Instead of the optimal performance achievable $\mathrm{OPT}(F, T)$, we consider two different alternative benchmarks and show our results in the main paper still hold for the minimax regret defined with respect to these benchmarks. The first one is $T \cdot \bar{u}(F)$ which can be thought of as a stronger benchmark than $\mathrm{OPT}(F, T)$ since $T \cdot \bar{u}(F) \geq \mathrm{OPT}(F, T)$ by Proposition 6 of the main paper. It is equivalently the first-best performance that the principal can achieve in the full-information version of the multi-round problem. The second one is $T \cdot \tilde{u}(F)$ where $\tilde{u}(F)$ is the performance of an optimal single-round incentive compatible and individually rational mechanism in the single-round problem. This benchmark has been studied previously in the literature and can be thought of as a weaker benchmark as we will show $T \cdot \tilde{u}(F) \leq \mathrm{OPT}(F, T)$. Interestingly, the same general results from Section 3 of the main paper hold for these benchmarks under a stronger assumption than Assumption 1, and the main observation is that the alternative benchmarks and $\mathrm{OPT}(F, T)$ coincide for point-mass distributions which form the worst cases in so far as determining the minimax regret.

For the agent's distribution F, we define

$$\tilde{u}(F) := \sup_{\substack{S \in \mathcal{S}: \\ \text{(IC),(IR)}}} \int_\Theta \int_\Omega u(\theta, \omega) dS_\theta(\omega) dF(\theta) \,,$$

which can be thought of as the "second-best" benchmark in the single-round problem when F is known to the principal. We have $\tilde{u}(F) \leq \bar{u}(F)$, because we can relax the IC constraint in the definition of $\tilde{u}(F)$ and solve for the best outcome distribution for each shock separately and this leads to an upper bound of $\mathbb{E}_{\theta \sim F}[\bar{u}(\theta)]$ which is at most $\bar{u}(F)$. The multi-round benchmark $T \cdot \tilde{u}(F)$ is a weaker benchmark in the sense that $T \cdot \tilde{u}(F) \leq \text{OPT}(F, T)$, because the principal can repeat the single-round solution (or approximately optimal) to the optimization problem defining $\tilde{u}(F)$ and realize the performance of $T \cdot \tilde{u}(F)$ since a utility-maximizing strategy for the agent is to participate and then truthfully report in each round. In the case of revenue maximization in the dynamic selling mechanism design problem with one good, $\bar{u}(F)$ evaluates to the average shock, equivalently, the full surplus of the agent, and $\tilde{u}(F)$ evaluates to the optimal revenue achievable, say, by posting the Myerson's price, i.e., $\max_{x \geq 0} x \cdot \Pr_{\theta \sim F}(\theta \geq x)$ (Myerson, 1981).

To distinguish the regret notions with respect to different benchmarks, we use $\text{Regret}^{\text{OPT}}$ to denote

$$\text{Regret}^{\text{OPT}}(A^T, F, T) := \text{OPT}(F, T) - \text{PrincipalUtility}(A^T, B^*(A^T, F), F, T)$$

which is the original regret notion as defined in Section 2 of the main paper and $\text{Regret}^{\text{FB}}$ (FB for "first-best") and $\text{Regret}^{\text{SB}}$ (SB for "second-best") to denote, respectively,

$$\text{Regret}^{\text{FB}}(A^T, F, T) := T \cdot \bar{u}(F) - \text{PrincipalUtility}(A^T, B^*(A^T, F), F, T) \,, \text{ and}$$
$$\text{Regret}^{\text{SB}}(A^T, F, T) := T \cdot \tilde{u}(F) - \text{PrincipalUtility}(A^T, B^*(A^T, F), F, T) \,.$$

We distinguish $\text{Regret}^{\text{OPT}}(T)$, $\text{Regret}^{\text{FB}}(T)$ and $\text{Regret}^{\text{SB}}(T)$ similarly.

Of the three variants, the $\text{Regret}^{\text{SB}}(T)$ notion is closely related to the standard regret notion in the learning literature that considers the best fixed "action" in hindsight, which naturally corresponds to the best fixed single-round mechanism in our setting, which is repeated across the time horizon. Since a dynamic mechanism can potentially do better, this regret can be negative sometimes. In particular, Amin et al. (2013) and subsequent works studied the $\text{Regret}^{\text{SB}}(T)$ notion (what they call "strategic regret") for the restricted class of dynamic posted pricing strategies for the problem of repeatedly

selling a single good.

As the following theorem shows, we obtain identical results with respect to above alternative benchmarks with the same minimax regret and structural characterization of an optimal dynamic mechanism. The single-round minimax regret problem is as defined in Section 2 of the main paper and the single-round benchmark is $\mathbb{E}_{\theta \sim F}[\text{OPT}(\theta, 1)]$. Instead of Assumption 1 of the main paper, we assume a stronger assumption in terms of $\bar{u}(F)$ for the stronger benchmark $T \cdot \bar{u}(F)$; this is stronger by Proposition 7 of the main paper. See Appendix A.1 for its proof and further details.

Theorem EC.5. *Suppose* $\sup_{F \in \mathcal{F}} \bar{u}(F) < \infty$ *and* $\bar{u}(F) = \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)]$ *for all* $F \in \mathcal{F}$. *Using* $\text{Regret}^{(\cdot)}$ *to denote both* $\text{Regret}^{\mathsf{FB}}$ *and* $\text{Regret}^{\mathsf{SB}}$ *notions, the following statements hold with respect to both notions:* $\text{Regret}^{(\cdot)}(T) = T \cdot \widehat{\text{Regret}}$. *For any* $\epsilon \geq 0$, *if a single-round direct IC/IR mechanism* S *satisfies*

$$\sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F) \leq \widehat{\text{Regret}} + \frac{\epsilon}{T},$$

then,

$$\sup_{F \in \mathcal{F}} \text{Regret}^{(\cdot)}(S^{\times T}, F, T) \leq \text{Regret}^{(\cdot)}(T) + \epsilon.$$

There exists an optimal dynamic mechanism in the multi-round minimax regret problem with respect to the $\text{Regret}^{(\cdot)}$ *notion if and only if there exists an optimal single-round direct IC/IR mechanism in the single-round minimax regret problem.*

EC.4.2 Arbitrary Shock Processes

We consider more general arbitrary shock processes under which the agent's private shocks may be correlated across rounds. For example, in the dynamic selling mechanism design problem, the agent's private value may be given by permanent and transitory components. His permanent component v_0 is drawn from a privately known distribution and transitory components ϵ_t are drawn from separate privately known distributions over the rounds such that his private shock in Round t is $\theta_t = v_0 + \epsilon_t$. For comparison, Carrasco et al. (2015) considered arbitrary shock processes for the maximin utility objective; we further discuss Carrasco et al. (2015) in Section EC.5. The repeated i.i.d. setting described in Section 2 of the main paper and considered throughout the paper is a special case where the shocks are independently and identically drawn from a fixed underlying distribution. In this section, we show that the same general results from Section 3 of the main paper still hold in the general shock process setting.

To distinguish the shock processes, we use superscript T. We use $\mathcal{F}^T$ to denote the set of possible T-round shock processes with a support contained in Θ, i.e., $\mathcal{F}^T := \Delta(\Theta^T)$, and F^T to denote a particular T-round shock process. The minimax regret value of the multi-round problem is, using the same Regret notation,

$$\text{Regret}(T) = \inf_{A^T \in \mathcal{A}^T} \sup_{F^T \in \mathcal{F}^T} \text{Regret}(A^T, F^T, T), \tag{EC-2}$$

where function Regret explicitly takes a T-round shock process. Generalizing $\text{OPT}(F, T)$, the multi-round benchmark is $\text{OPT}(F^T, T)$ which is the optimal performance achievable when the principal knows the agent's private shock process F^T:

$$\text{OPT}(F^T, T) = \sup_{A^T \in \mathcal{A}^T} \text{PrincipalUtility}(A^T, B^*(A^T, F^T), F^T, T),$$

where, as in the repeated i.i.d. setting, $B^*(A^T, F^T)$ is the agent's utility-maximizing strategy chosen in the principal's favor. Hence, the regret for a dynamic mechanism A^T when the agent's T-round shock process is F^T is the difference between the optimal performance achievable and the actual performance achieved:

$$\text{Regret}(A^T, F^T, T) = \text{OPT}(F^T, T) - \text{PrincipalUtility}(A^T, B^*(A^T, F^T), F^T, T).$$

The corresponding single-round problem in the general shock process setting is still the same problem as in the repeated i.i.d. setting (in Section 2 of the main paper); the agent's shock is drawn from a single-round distribution known only to the agent.

We use the same notations from the main paper to refer to the repeated i.i.d. setting such that F without superscript T denotes a single-round distribution and $\text{Regret}(A^T, F, T)$ is the regret of a dynamic mechanism A^T when the agent's private distribution is F in the repeated i.i.d. setting. We use both $F^{\times T}$ and $(F)^{\times T}$ to denote the shock process in which the per-round shocks are drawn i.i.d. from F in the general shock process setting. Whether the Regret notation refers to the repeated i.i.d. setting or the general shock process setting will be clear from the context and the parameters; in particular, $\text{Regret}(A^T, F, T) = \text{Regret}(A^T, F^{\times T}, T)$ for $A^T \in \mathcal{A}^T$ and $F \in \mathcal{F}$.

We have the following result:

Theorem EC.6. *Assume* $\sup_{F^T \in \mathcal{F}^T} \text{OPT}(F^T, T) < \infty$ *and* $\text{OPT}(F^T, T) \leq \mathbb{E}_{t \sim [T], \theta \sim (F^T)_t}[\text{OPT}(\theta^{\times T}, T)]$ *for all* $F^T \in \mathcal{F}^T$, *where* $t \sim [T]$ *means we draw a round uniformly at random and* $\theta \sim (F^T)_t$ *means we draw a shock from the marginal shock distribution of* F^T *in Round* t. *Then, the following statements*

hold in the general shock process setting: $\mathrm{Regret}(T) = T \cdot \widehat{\mathrm{Regret}}$. *For any $\epsilon \geq 0$, if a single-round direct IC/IR mechanism S satisfies*

$$\sup_{F \in \mathcal{F}} \widehat{\mathrm{Regret}}(S, F) \leq \widehat{\mathrm{Regret}} + \frac{\epsilon}{T},$$

then,

$$\sup_{F^T \in \mathcal{F}^T} \mathrm{Regret}(S^{\times T}, F^T, T) \leq \mathrm{Regret}(T) + \epsilon.$$

There exists an optimal dynamic mechanism in the multi-round minimax regret problem if and only if there exists an optimal single-round direct IC/IR mechanism in the single-round minimax regret problem.

Instead of Assumption 1 of the main paper, we have analogous assumptions in terms of $\mathrm{OPT}(F^T, T)$. The right-hand side of the second assumption can be equivalently written as $\sum_{t=1}^{T} \mathbb{E}_{\theta \sim (F^T)_t}[\mathrm{OPT}(\theta, 1)]$, by Proposition 2 of the main paper. We defer the proof of Theorem EC.6 and further details to Appendix A.2. In the same appendix, we also prove analogous results to Propositions 6, 7 and 8 of the main paper in terms of a generalization of $\bar{u}(F)$ in the general shock process setting. Therefore, Theorem EC.6 holds for all games with payments that enter linearly into the utility functions of the principal and agent or with a nonnegative utility function for the agent; in particular, it holds for all applications considered in Sections 5–7 of the main paper. Similar to the repeated i.i.d. setting, the class of constant shock processes, which are equivalently point-mass distributions in the repeated i.i.d. setting, form the worst cases in the general shock process setting; we refer to Proposition 5 of the main paper and note the same proof still works.

EC.4.3 Multiplicative Guarantees

We show that our results and analyses of the main paper extend to analogous results for a multiplicative performance metric. In particular, the best multiplicative guarantee for the multi-round problem is equal to the best multiplicative guarantee for the corresponding single-round problem and the principal can achieve the best multiplicative guarantee arbitrarily closely by repeating a single-round mechanism and exactly by repeating, if it exists, an optimal single-round mechanism to the single-round problem.

The minimax regret and regret are standard performance metrics in the sequential learning literature.

If a learning algorithm has the minimax regret of $o(T)$, the worst-case regret when averaged over the rounds diminishes towards 0 as the time horizon T increases and the algorithm achieves the optimal performance (or some suitable benchmark) asymptotically. Alternatively, we can consider multiplicative guarantees in terms of a ratio, similar to the approximation and competitive ratios that are common in the theoretical computer science literature. This is a reasonable performance metric that is scale-free and may be more interpretable than a difference in absolute terms; at least 50% of the benchmark versus at most \$50 less than the benchmark.

Suppose there exist some constants $0 < L < U < \infty$ such that $\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)] \in [L, U]$ and $\mathrm{OPT}(F, T) \in [LT, UT]$ for all distributions $F \in \mathcal{F}$, i.e., bounded from above and away from 0. We define the multi-round multiplicative guarantee $\mathrm{Ratio}(T)$ and the corresponding multi-round problem as

$$\mathrm{Ratio}(T) := \sup_{A \in \mathcal{A}} \inf_{F \in \mathcal{F}} \frac{\mathrm{PrincipalUtility}(A, B^*(A, F), F, T)}{\mathrm{OPT}(F, T)} \,. \tag{EC-3}$$

Note $\mathrm{Ratio}(T) \leq 1$ since the realized performance of a dynamic mechanism is upper bounded by the optimal performance achievable with the knowledge of the agent's distribution. It is also at least 0 because the principal can guarantee the total utility of 0 via the trivial mechanism that always forces the no-interaction outcome. Similarly, we define the single-round multiplicative guarantee $\widehat{\mathrm{Ratio}}$ and the corresponding single-round problem for direct IC/IR mechanisms where the agent reports truthfully as

$$\widehat{\mathrm{Ratio}} := \sup_{\substack{S \in \mathcal{S}: \\ (IC),(IR)}} \inf_{F \in \mathcal{F}} \frac{\int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta)}{\int_\Theta \mathrm{OPT}(\theta, 1) \mathrm{d}F(\theta)} \,, \tag{EC-4}$$

which is in the interval $[0, 1]$. To see $\widehat{\mathrm{Ratio}} \leq 1$, we note $\int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \leq \int_\Theta \mathrm{OPT}(\theta, 1) \mathrm{d}F(\theta)$ for any single-round direct IC/IR mechanism S and distribution F because any single-round direct IC/IR mechanism is a feasible solution in the following optimization program that equivalently defines $\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)]$ and achieves the objective value equal to the integral expression:

$$\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)] = \sup_{S \in \mathcal{S}} \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta)$$

$$\text{s.t.} \int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) \geq 0, \quad \forall \theta \in \Theta \,.$$

The lower bound is by the same argument as above.

Similar to the general minimax regret result, Theorem 1 of the main paper, we can show the following. We defer its proof to Appendix A.3.

Theorem EC.7. *Suppose Assumption 1 of the main paper holds and there exist some constants* $0 < L < U < \infty$ *such that* $\mathbb{E}_{\theta \sim F}[\text{OPT}(\theta, 1)] \in [L, U]$ *and* $\text{OPT}(F, T) \in [LT, UT]$ *for all* $F \in \mathcal{F}$. *Then,* $\text{Ratio}(T) = \widehat{\text{Ratio}}$. *For any* $\epsilon \geq 0$, *if a single-round direct IC/IR mechanism S satisfies*

$$\inf_{F \in \mathcal{F}} \frac{\int_\Theta \int_\Omega u(\theta, \omega)\, dS_\theta(\omega)\, dF(\theta)}{\int_\Theta \text{OPT}(\theta, 1)\, dF(\theta)} \geq \widehat{\text{Ratio}} - \frac{\epsilon L}{U},$$

then,

$$\inf_{F \in \mathcal{F}} \frac{\text{PrincipalUtility}(A, B^*(A, F), F, T)}{\text{OPT}(F, T)} \geq \text{Ratio}(T) - \epsilon.$$

There exists an optimal dynamic mechanism in the multi-round problem (EC-3) *if and only if there exists an optimal single-round direct IC/IR mechanism in the single-round problem* (EC-4).

We remark that for the same game, the minimax regret and multiplicative guarantee are, in general, different. For example, in the case of revenue maximization in the dynamic selling problem with one good (Section 5 of the main paper), the minimax regret is $\frac{T}{e}$ while the multiplicative guarantee is unbounded whenever zero is the lowest value for the shock.

EC.4.4 Principal Pessimism

We consider a stronger notion of minimax regret under which the agent can choose to play any utility-maximizing strategy. We show most of our results of the main paper continue to hold with no modification with respect to the stronger notion of minimax regret and analyze in terms of what we call the principal pessimism constraint. Throughout the main paper, we assumed the agent plays a best-response strategy chosen in the principal's favor and the uncertainty that the principal faces is essentially in terms of the agent's distribution only. Equivalently, if there are multiple utility-maximizing strategies for the agent, the agent plays one that also maximizes the principal utility. Alternatively, we can allow for a different kind of "tie-breaking" possibility in which the agent plays any utility-maximizing strategy and, in particular, one that minimizes the principal utility among such utility-maximizing strategies. This leads to a stronger and more robust notion of minimax regret under which guarantees hold for any shock distribution and any best-response strategy for the agent.

The minimax regret corresponding to the agent playing a utility-maximizing strategy that also max-

imizes the principal utility can be equivalently written as

$$\text{Regret}(T) = \inf_{A^T \in \mathcal{A}^T} \sup_{F \in \mathcal{F}} \inf_{B \in \mathcal{B}(A^T, F)} \text{Regret}(A^T, B, F, T),$$

where $\text{Regret}(A^T, B, F, T) = \text{OPT}(F, T) - \text{PrincipalUtility}(A^T, B, F, T)$. For the stronger minimax regret notion, the innermost infimum becomes a supremum as in

$$\inf_{A^T \in \mathcal{A}^T} \sup_{F \in \mathcal{F}} \sup_{B \in \mathcal{B}(A^T, F)} \text{Regret}(A^T, B, F, T),$$

and the uncertainty that the principal faces is both in terms of the agent's distribution and best-response strategy in the worst-case sense. Going back to the main paper where an optimal dynamic mechanism repeats a single-round direct IC/IR mechanism, truthful reporting is one utility-maximizing strategy and the regret guarantees hold for this strategy and, also, for any utility-maximizing strategy chosen in the principal's favor. But it is possible that the regret can be greater for other utility-maximizing strategies.[1] In what follows, for any principal's mechanism A^T and agent's distribution F, let $B^*(A^T, F)$ be a utility-maximizing strategy that, if multiple ones exist, minimizes the principal utility among such utility-maximizing strategies. We refer to such strategy as a *principal-pessimistic utility-maximizing strategy*. We use the same notations as introduced in Section 2 of the main paper for the stronger notions of regret and minimax regret. Note we keep the same optimal performance achievable $\text{OPT}(F, T)$ from the main paper (even though, it is defined in terms of $B^*(A^T, F)$) and it is what the principal can achieve when he knows the agent's distribution F and the agent plays a utility-maximizing strategy in the principal's favor; this is the best-case scenario from the principal's perspective. Except for $\text{OPT}(F, T)$, $B^*(A^T, F)$ will be a principal-pessimistic utility-maximizing strategy, which affects, in particular, the regret and minimax regret.

Our main general result (Theorem 1 of the main paper) still holds with suitable changes (in choosing a utility-maximizing strategy) in the proof to show that the multi-round minimax regret is T times the single-round minimax regret and we can achieve this minimax regret arbitrarily closely by repeating a single-round mechanism. By the same revelation-principle-type argument applied for principal-

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[1]Different selections of the agent's utility-maximizing strategies lead to different principal utilities and, hence, regrets. Consider a principal selling a single good to an agent where the agent's value for the good is $\theta = \frac{1}{2}$. Assume the principal's mechanism is given the agent's report $\hat{\theta}$: 1) charge $\frac{1}{2}$ and allocate with probability 1 if $\hat{\theta} > \frac{1}{2}$; 2) charge $\frac{1}{4}$ and allocate with probability $\frac{1}{2}$ if $\hat{\theta} = \frac{1}{2}$; and 3) do not allocate if $\hat{\theta} < \frac{1}{2}$. This mechanism satisfies the IC/IR constraints. For the agent, truthful reporting is a utility-maximizing strategy and leads to the agent utility of 0 and the principal utility of $\frac{1}{4}$, but other reporting strategies are also utility-maximizing and can lead to a different principal utility. Reporting some $\hat{\theta} < \frac{1}{2}$ leads to a smaller principal utility of 0 and reporting some $\hat{\theta} > \frac{1}{2}$ leads to a greater principal utility of $\frac{1}{2}$, while both lead to the same agent utility of 0 as truthful reporting.

pessimistic utility-maximizing strategies, we can relate the performance of a dynamic mechanism to that of a single-round direct IC/IR mechanism S that satisfies the *principal pessimism constraint*:

$$\int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega) \le \int_\Omega u(\theta,\omega)\mathrm{d}S_{\theta'}(\omega)\,, \quad \forall \theta \in \Theta, \theta' \in \mathcal{B}^*(S,\theta)$$

$$\int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega) \le 0\,, \quad \forall \theta \in \Theta_0$$

(PP)

where $\mathcal{B}^*(S,\theta) = \{\theta' \in \Theta \mid \int_\Omega v(\theta,\omega)\mathrm{d}S_{\theta'}(\omega) = \int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega)\}$ is the set of utility-maximizing re-ports for the agent when his shock is θ given the IC constraint holds and $\Theta_0 = \{\theta \in \Theta \mid \int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega) = 0\}$ is the set of shocks for which truthful reporting leads to the agent utility of 0. We say a direct IC/IR mechanism is principal-pessimistic (PP) if it satisfies the PP constraint above. The first part of PP constraint, given the IC constraint holds, stipulates that truthful reporting leads to the lowest principal utility among all best responses for the agent, and the second part stipulates that the prin-cipal utility is at most 0 if truthful reporting leads to the agent utility of 0. Note if S satisfies the IC/IR constraints, $\theta \in \mathcal{B}^*(S,\theta)$ for all $\theta \in \Theta$. The IC/IR/PP constraints together imply that truthful reporting is an interim principal-pessimistic utility-maximizing strategy. We obtain the following single-round minimax regret problem for single-round direct IC/IR/PP mechanisms:

$$\widehat{\mathrm{Regret}} := \inf_{\substack{S \in \mathcal{S}: \\ (\mathrm{IC}),(\mathrm{IR}),(\mathrm{PP})}} \sup_{F \in \mathcal{F}} \widehat{\mathrm{Regret}}(S,F)\,. \tag{EC-5}$$

Repeating a single-round direct IC/IR/PP mechanism that achieves a regret arbitrarily close to the single-round minimax regret $\widehat{\mathrm{Regret}}$ obtains a regret correspondingly close to the multi-round minimax regret in the multi-round problem. If there exists an optimal solution to (EC-5), repeating the optimal single-round solution is optimal and achieves the multi-round minimax regret exactly. We provide the formal statement below and refer to Appendix A.4.1 for proof details.

Theorem EC.8. *Suppose Assumption 1 of the main paper holds. The following statements hold with respect to the stronger notion of regret and minimax regret where the agent plays a principal-pessimistic utility-maximizing strategy:* $\mathrm{Regret}(T) = T \cdot \widehat{\mathrm{Regret}}$. *For any $\epsilon \ge 0$, if a single-round direct IC/IR/PP mechanism S satisfies*

$$\sup_{F \in \mathcal{F}} \widehat{\mathrm{Regret}}(S,F) \le \widehat{\mathrm{Regret}} + \frac{\epsilon}{T}\,,$$

then,

$$\sup_{F \in \mathcal{F}} \mathrm{Regret}(S^{\times T},F,T) \le \mathrm{Regret}(T) + \epsilon\,.$$

There exists an optimal dynamic mechanism in the multi-round minimax regret problem if and only if there exists an optimal single-round direct IC/IR/PP mechanism in the single-round minimax regret problem (EC-5).

One way to find an optimal solution to the single-round problem is to solve the relaxed version without the PP constraint and show the solution to the relaxed version satisfies the PP constraint and, thus, is also a solution to the original version. Note we already have optimal solutions to the relaxed versions in Sections 5–7 of the main paper. We can characterize the PP constraint in these applications and show the optimal single-round direct IC/IR mechanisms satisfy the PP constraint. For instance, for the dynamic selling mechanism design problem with a single good, the if-and-only-if condition for the PP constraint is roughly that any flat part of the interim allocation rule, if exists, is closed on the right side. The optimal single-round direct IC/IR mechanism in Theorem 13 of the main paper has an interim allocation rule that is continuous and indeed satisfies the PP constraint. Consequently, Theorems 13, 14, and 15 (when we let $x^*(0) = 1 - c$) of the main paper and Theorems EC.2 and EC.3 of this electronic companion still hold the same with respect to the stronger notion of minimax regret. We defer further details to Appendix A.4.2.

In the most general setting, it is not clear how to mathematically formalize the PP constraint in a tractable way and we do not have an explicit characterization, but we believe other results still hold with respect to the stronger notions of regret and minimax regret.

EC.5   Connections to Maximin Utility Objective

We show equivalence-type connections between the minimax regret and maximin utility objectives for revenue maximization in the single-good case of the dynamic selling mechanism design problem (considered in Section 5 of the main paper). Similar connections hold more generally for other robust mechanism design problems as long as a saddle-point result exists for the minimax regret objective and a corresponding saddle-point result exists for the maximin utility objective (with the known mean). Carrasco et al. (2015) recently showed a similar false-dynamics result for the multi-round dynamic selling problem with respect to the maximin utility objective where the principal maximizes the minimum utility achieved in the worst-case sense and only the mean of the agent's shock distribution is known a priori. More specifically, we show the minimax regret criterion and maximin utility criterion with a known mean share the same single-round saddle-point problem involving direct

IC/IR mechanisms. For the principal-agent model with hidden costs, we are not aware of a multi-round false-dynamics result but our analysis of the single-round minimax regret problem is, again, similar to Carrasco et al. (2018) that considered a single-round utility-maximization problem subject to a nonlinear quality cost function with respect to the other robust objective.

In contrast to the minimax regret objective, the maximin utility objective without any additional distributional information (other than the shock space Θ) leads to a trivial answer in the case of revenue maximization in the dynamic selling mechanism design problem. The worst-case is when the agent has the lowest possible value and the principal will accordingly price the good at this value always. If the lowest possible value is 0, then the maximin utility would be 0. The maximin utility objective leads to more meaningful solutions with additional fixed-moment-type distributional information. In particular, Carrasco et al. (2015) considered a dynamic setting where the agent's values within $[0,1]$ follow an unknown arbitrary value process over T rounds with the property that each per-round marginal distribution has the same mean known a priori to the principal. They showed a saddle-point result where the optimal dynamic mechanism is T repetitions of a single-round mechanism and the worst-case value process is one where a value is drawn once from a specified distribution and is fixed across all rounds. Interestingly, the worst-case value process is such that T repetitions of a single-round mechanism is still optimal even when the value process is known to the principal.

In Section 5 of the main paper, we considered the same multi-round problem with respect to the minimax regret objective without any additional distributional information (other than the shock space Θ) and value processes were such that per-round value distributions are identical and independent. Our version of the multi-round problem does not allow the above worst-case value process in which values are not drawn independently from the same marginal shock distribution. Furthermore, there is no saddle-point result for the multi-round problem in our setting.

Despite these differences, our results and Carrasco et al. (2015) have similar analyses and solution structures. This is because both papers rely on essentially the same single-round saddle-point problem involving direct IC/IR mechanisms. Strictly speaking, the single-round problem with the maximin utility objective and a known mean is "finer-grained" in the sense that we can use the solutions to this problem to find a solution to the single-round problem with the minimax regret objective. This is assuming we have saddle-point results with respect to both objectives which were indeed proved, independently, in Carrasco et al. (2015) and in Section 5 of the main paper; they allow us to change

the order of the infimum and supremum in the single-round problems.

In the remainder, we describe the connection via single-round problems. We start with the saddle-point result for the single-round problem for direct IC/IR mechanisms with respect to the minimax regret objective. Recall the single-round benchmark $\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)]$ is equal to $\mathbb{E}_{\theta \sim F}[\theta]$ from Section 5 of the main paper. Denoting by $\hat{\mathcal{S}} = \{S \in \mathcal{S} : (\mathrm{IC}), (\mathrm{IR})\}$ the set of single-round direct IC/IR mechanisms, we have

$$\widehat{\mathrm{Regret}} = \inf_{S \in \hat{\mathcal{S}}} \sup_{F \in \mathcal{F}} \widehat{\mathrm{Regret}}(S, F) = \sup_{F \in \mathcal{F}} \inf_{S \in \hat{\mathcal{S}}} \widehat{\mathrm{Regret}}(S, F)$$

$$= \sup_{F \in \mathcal{F}} \inf_{S \in \hat{\mathcal{S}}} \left\{ \mathbb{E}_{\theta \sim F}[\theta] - \mathrm{PrincipalUtility}(S, \sigma^{\mathrm{TR}}, F, 1) \right\},$$

where σ^{TR} denotes the truthful reporting strategy. Let $\mathcal{F}_\mu = \{F \in \mathcal{F} \mid \mathbb{E}_{\theta \sim F}[\theta] = \mu\}$, i.e., distributions with the mean equal to μ. The optimization problem can be equivalently written as

$$\widehat{\mathrm{Regret}} = \sup_{\mu \in [0,1]} \sup_{F \in \mathcal{F}_\mu} \inf_{S \in \hat{\mathcal{S}}} \left\{ \mu - \mathrm{PrincipalUtility}(S, \sigma^{\mathrm{TR}}, F, 1) \right\}$$

$$= \sup_{\mu \in [0,1]} \left\{ \mu - \inf_{F \in \mathcal{F}_\mu} \sup_{S \in \hat{\mathcal{S}}} \mathrm{PrincipalUtility}(S, \sigma^{\mathrm{TR}}, F, 1) \right\}$$

$$= \sup_{\mu \in [0,1]} \left\{ \mu - \sup_{S \in \hat{\mathcal{S}}} \inf_{F \in \mathcal{F}_\mu} \mathrm{PrincipalUtility}(S, \sigma^{\mathrm{TR}}, F, 1) \right\},$$

where the second equation follows by extracting the mean μ from the objective and accounting for the negative sign in front of the principal utility expression, and the last by the saddle-point result for the maximin utility objective with a known mean from Carrasco et al. (2015). Now, assume we have saddle-point solutions (S_μ^*, F_μ^*) for this problem for all possible $\mu \in [0, 1]$. We arrive at a saddle-point solution for the single-round problem with respect to the minimax regret objective by choosing (S_μ^*, F_μ^*) for μ that maximizes the quantity

$$\mu - \mathrm{PrincipalUtility}(S_\mu^*, \sigma^{\mathrm{TR}}, F_\mu^*, 1).$$

Note the connection holds when the saddle-point results exist with respect to both objectives, which we indeed have.

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A   Additional Materials for Section EC.4

A.1   Two Alternative Benchmarks

Before proving Theorem EC.5, we prove the following proposition relating $\bar{u}(F)$ and $\tilde{u}(F)$ which will be used in proving Theorem EC.5.

Proposition EC.9. *We have the following relations:*

1. *For any distribution $F \in \mathcal{F}$, $\tilde{u}(F) \leq \bar{u}(F)$.*

2. *For any point-mass distribution $\theta \in \Theta$, $\tilde{u}(\theta) = \bar{u}(\theta)$.*

Proof. We prove the first part. Let F be any arbitrary distribution. We equivalently write $\tilde{u}(F)$ as

$$\tilde{u}(F) = \sup_{S \in \mathcal{S}} \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta)$$

$$\text{s.t.} \int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) \geq \int_\Omega v(\theta, \omega) \mathrm{d}S_{\theta'}(\omega), \quad \forall \theta, \theta' \in \Theta \tag{EC-6}$$

$$\int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) \geq 0, \quad \forall \theta \in \Theta.$$

Relaxing the IC constraint, we obtain the optimization problem (6) of the main paper that defines $\mathbb{E}_{\theta \sim F}[\bar{u}(\theta)]$. Hence, $\tilde{u}(F) \leq \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)]$. By Part 3 of Proposition 6 of the main paper, it follows that $\tilde{u}(F) \leq \bar{u}(F)$.

We now show that $\tilde{u}(\theta)$ is equal to $\bar{u}(\theta)$ for any θ. Fix an arbitrary θ. Let G_θ be an arbitrary outcome distribution over Ω that is feasible for the following equivalent form of the optimization problem defining $\bar{u}(\theta)$:

$$\bar{u}(\theta) = \sup_{G \in \Delta(\Omega)} \int_\Omega u(\theta, \omega) \mathrm{d}G(\omega)$$

$$\text{s.t.} \int_\Omega v(\theta, \omega) \mathrm{d}G(\omega) \geq 0 \,.$$

Consider the single-round direct IC/IR mechanism S that given report θ' returns a random outcome drawn from G_θ if $\mathbb{E}_{\omega \sim G_\theta}[v(\theta', \omega)] \geq 0$, and the no-interaction outcome otherwise. Mechanism S provides two possibilities and chooses the better possible outcome (or a distribution of outcomes) for each θ' if the agent's private shock was θ'. Hence, by construction, S satisfies the IC/IR constraints and is a feasible solution in the optimization problem defining $\tilde{u}(\theta)$. Furthermore, $\mathbb{E}_{\omega \sim S_\theta}[u(\theta, \omega)] = \mathbb{E}_{\omega \sim G_\theta}[u(\theta, \omega)]$ and G_θ and S obtain the same objectives in respective optimization problems. As G_θ was arbitrary, it follows that $\tilde{u}(\theta) \geq \bar{u}(\theta)$. Combined with the observation that $\tilde{u}(F) \leq \bar{u}(F)$ for any distribution F, it follows that $\tilde{u}(\theta) = \bar{u}(\theta)$. $\qquad\square$

We now prove Theorem EC.5 below:

Proof of Theorem EC.5. We show that Lemmas 3 and 4 of the main paper hold for the Regret$^{\mathsf{FB}}$ and Regret$^{\mathsf{SB}}$ notions. Then, Theorem 1 of the main paper would follow for these notions by the same reasoning steps used in the proof for the Regret$^{\mathsf{OPT}}$ notion in Appendix A.1 of the main paper. The assumptions of the theorem are similar to Assumption 1 in the main paper. The first part ensures the benchmarks and minimax regrets are well-defined and the second part, the linearity assumption, is more stringent and needed for our false-dynamics results.

We start by showing Lemma 3 of the main paper holds with respect to the alternative benchmarks. We use the same reasoning used for the Regret$^{\mathsf{OPT}}$ notion. Fix an arbitrary dynamic mechanism $A^T \in \mathcal{A}^T$. Then,

$$\begin{aligned}
\sup_{F \in \mathcal{F}} \mathrm{Regret}^{\mathsf{FB}}(A^T, F, T) &\geq \sup_{\theta \in \Theta} \mathrm{Regret}^{\mathsf{FB}}(A^T, \theta, T) \\
&= \sup_{\theta \in \Theta} \left\{ T \cdot \bar{u}(\theta) - \mathrm{PrincipalUtility}(A^T, B^*(A^T, \theta), \theta, T) \right\} \\
&= T \cdot \sup_{\theta \in \Theta} \left\{ \bar{u}(\theta) - \mathrm{PrincipalUtility}(S(A^T), \sigma^{\mathrm{TR}}, \theta, 1) \right\} \\
&= T \cdot \sup_{\theta \in \Theta} \left\{ \mathrm{OPT}(\theta, 1) - \mathrm{PrincipalUtility}(S(A^T), \sigma^{\mathrm{TR}}, \theta, 1) \right\} \\
&= T \cdot \sup_{\theta \in \Theta} \widehat{\mathrm{Regret}}(S(A^T), \theta) \\
&= T \cdot \sup_{F \in \mathcal{F}} \widehat{\mathrm{Regret}}(S(A^T), F) \,,
\end{aligned}$$

where the first step is because point-mass distributions are a subset of general distributions $\mathcal{F}$; the

second step is by the definition of the Regret$^{\mathsf{FB}}$ notion; the third step is by Lemma 16 of the main paper and $S(A^T)$ is the single-round direct IC/IR mechanism derived from A^T as described in the proof of the lemma; the fourth step is because $\mathrm{OPT}(\theta, 1) = \bar{u}(\theta)$ which is by Proposition 6 of the main paper; the second-to-last step is by the definition of the $\widehat{\mathrm{Regret}}$ notion; and the last step is by Lemma 17 of the main paper. Hence, Lemma 3 holds for the Regret$^{\mathsf{FB}}$ notion.

For the Regret$^{\mathsf{SB}}$ notion, we follow the same reasoning with Regret$^{\mathsf{SB}}$ and $\tilde{u}(F)$ in places of Regret$^{\mathsf{FB}}$ and $\bar{u}(F)$, respectively. The fourth step in the above sequence still follows from Proposition 6 of the main paper and Proposition EC.9 which imply that $\mathrm{OPT}(\theta, 1) = \bar{u}(\theta) = \tilde{u}(\theta)$ for $\theta \in \Theta$.

We now show Lemma 4 of the main paper holds with respect to the alternative benchmarks. Let S be an arbitrary single-round direct IC/IR mechanism and consider the dynamic mechanism $S^{\times T}$ which is T repetitions of S. Then,

$$
\begin{aligned}
\sup_{F \in \mathcal{F}} \mathrm{Regret}^{\mathsf{FB}}(S^{\times T}, F, T) &= \sup_{F \in \mathcal{F}} \left\{ T \cdot \bar{u}(F) - \mathrm{PrincipalUtility}(S^{\times T}, B^*(S^{\times T}, F), F, T) \right\} \\
&= \sup_{F \in \mathcal{F}} \left\{ T \cdot \mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)] - \mathrm{PrincipalUtility}(S^{\times T}, B^*(S^{\times T}, F), F, T) \right\} \\
&= T \cdot \sup_{F \in \mathcal{F}} \left\{ \mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)] - \mathrm{PrincipalUtility}(S, B^*(S, F), F, 1) \right\} \\
&\leq T \cdot \sup_{F \in \mathcal{F}} \widehat{\mathrm{Regret}}(S, F),
\end{aligned}
$$

where the first step is by the definition of the Regret$^{\mathsf{FB}}$ notion; the second step is because for any distribution $F \in \mathcal{F}$,
$$
T \cdot \bar{u}(F) = T \cdot \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)] = T \cdot \mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)],
$$
by the linearity assumption on $\bar{u}(F)$ and Proposition 6 of the main paper; the second-to-last step is by Lemma 18 of the main paper; and the last step is because truthful reporting is a utility-maximizing strategy for the agent but may not be one that also maximizes the principal's utility among utility-maximizing strategies. Hence, Lemma 4 holds for the Regret$^{\mathsf{FB}}$ notion.

Similarly, we follow the same reasoning for the Regret$^{\mathsf{SB}}$ notion in terms of $\tilde{u}(F)$. The second step in the above sequence will be an inequality and it follows because
$$
T \cdot \tilde{u}(F) \leq T \cdot \bar{u}(F) = T \cdot \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)] = T \cdot \mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)],
$$
where we used Proposition EC.9, the linearity assumption on $\bar{u}(F)$ and Proposition 6 of the main paper, in that order. $\qquad \square$

A.2 Arbitrary Shock Processes

We first prove Theorem EC.6 and then introduce a generalization of $\bar{u}(F)$ and prove analogous results to Propositions 6, 7 and 8 of the main paper in the general shock process setting.

Proof of Theorem EC.6. Note the boundedness assumption on $\mathrm{OPT}(F^T, T)$ corresponds to the first part of Assumption 1 in the main paper and guarantees the benchmarks and minimax regrets are well-defined in the same manner. We prove the analogues of Lemmas 3 and 4 of the main paper for the general shock process setting. Then, the theorem statements would follow directly from the

analogues via the same reasoning steps in the proof of Theorem 1 of the main paper.

For an analogue of Lemma 3 of the main paper, we proceed as follows. Note that for any dynamic mechanism A^T,

$$\sup_{F^T \in \mathcal{F}^T} \text{Regret}(A^T, F^T, T) \geq \sup_{F \in \mathcal{F}} \text{Regret}(A^T, (F)^{\times T}, T)$$

because the set of shock processes where per-round shocks are drawn independently and identically is a subset of the set of general T-round shock processes. Note the right-hand side is the regret of the mechanism A^T in the repeated i.i.d. setting where the agent's distribution can be any distribution $F \in \mathcal{F}$. It follows from Lemma 3 of the main paper that there exists some single-round direct IC/IR mechanism S such that

$$\sup_{F^T \in \mathcal{F}^T} \text{Regret}(A^T, F^T, T) \geq T \cdot \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F) \,.$$

This is an analogous lower bound to Lemma 3 of the main paper for the general shock process setting.

We now show an analogous upper bound to Lemma 4 of the main paper. Let S be an arbitrary single-round direct IC/IR mechanism. For any $F^T \in \mathcal{F}^T$, we have

$$\text{Regret}(S^{\times T}, F^T, F) = \text{OPT}(F^T, T) - \text{PrincipalUtility}(S^{\times T}, B^*(S^{\times T}, F^T), F^T, T)$$
$$\leq \text{OPT}(F^T, T) - \text{PrincipalUtility}(S^{\times T}, \sigma^{\text{TR}}, F^T, T) \,,$$

where the first step is by the definition of the Regret notion and the second step is because truthful reporting σ^{TR} (so, the agent participates) is a utility-maximizing strategy for the agent that may or may not maximize the principal utility. We can upper bound $\text{OPT}(F^T, T)$ as follows:

$$\text{OPT}(F^T, T) \leq \mathbb{E}_{t \sim [T], \theta \sim (F^T)_t}[\text{OPT}(\theta^{\times T}, T)] = \frac{1}{T} \sum_{t=1}^{T} \mathbb{E}_{\theta \sim (F^T)_t}[\text{OPT}(\theta, T)] = \sum_{t=1}^{T} \mathbb{E}_{\theta \sim (F^T)_t}[\text{OPT}(\theta, 1)] \,,$$

where the first step is by the second assumption of the theorem and the last step is by Proposition 2 of the main paper. Then,

$$\text{Regret}(S^{\times T}, F^T, F) \leq \sum_{t=1}^{T} \mathbb{E}_{\theta \sim (F^T)_t}[\text{OPT}(\theta, 1)] - \text{PrincipalUtility}(S^{\times T}, \sigma^{\text{TR}}, F^T, T)$$
$$= \sum_{t=1}^{T} \left(\mathbb{E}_{\theta \sim (F^T)_t}[\text{OPT}(\theta, 1)] - \text{PrincipalUtility}(S, \sigma^{\text{TR}}, (F^T)_t, 1) \right)$$
$$= \sum_{t=1}^{T} \widehat{\text{Regret}}(S, (F^T)_t)$$
$$\leq T \cdot \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F) \,,$$

where the second step follows from that since the principal repeats S and the agent repeats the same strategy of truthful reporting, the rounds become independent and the principal utility in each round can be written in terms of the marginal shock distributions; the third step is by the definition of the $\widehat{\text{Regret}}$ notion; and the last step is by upper bounding each summand. As F^T was arbitrary, it follows

that
$$\sup_{F^T \in \mathcal{F}^T} \text{Regret}(S^{\times T}, F^T, T) \leq T \cdot \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F),$$

which is the upper bound analogue of Lemma 4 of the main paper. □

We now define a generalization of $\bar{u}(F)$ for T-round shock processes and prove analogous results to Propositions 6, 7 and 8 of the main paper. For T-round shock processes $F^T \in \mathcal{F}^T$, we define $\bar{u}(F^T)$ as follows:

$$\bar{u}(F^T) := \sup_{S_1,\ldots,S_T} \mathbb{E}_{\theta \sim F^T, \omega \sim \prod_{t=1}^T S_{t,\theta_t}} \left[\sum_{t=1}^T u(\theta_t, \omega_t) \right]$$

$$\text{s.t. } \mathbb{E}_{\theta \sim F^T, \omega \sim \prod_{t=1}^T S_{t,\theta_t}} \left[\sum_{t=1}^T v(\theta_t, \omega_t) \right] \geq 0,$$
(EC-7)

where the supremum is over sequences of single-round direct mechanisms $\{S_t\}_{t \in [T]}$ where each mechanism S_t is a collection of outcome distributions $\{S_{t,\theta}\}_{\theta \in \Theta}$ and the expectation is with respect to the shocks $\theta = (\theta_1, \ldots, \theta_T)$ determined according to the T-round shock process F^T and the outcomes $\omega = (\omega_1, \ldots, \omega_T)$ where the t-th round outcome ω_t is determined according to the t-th single-round mechanism S_t and the t-th shock θ_t, i.e., $\omega_t \sim S_{t,\theta_t}$, in the product notation. Similar to $\bar{u}(F)$ in the repeated i.i.d. setting, we can think of $\bar{u}(F^T)$ as the first-best performance that the principal can achieve subject to the ex-ante IR constraint when the shock process and per-round shocks are known. Correspondingly, the linearity condition for $\bar{u}(F^T)$ can be stated $\bar{u}(F^T) = \sum_{t=1}^T \mathbb{E}_{\theta \sim (F^T)_t}[\bar{u}(\theta)]$ for $F^T \in \mathcal{F}^T$. This is indeed a generalization of the linearity condition for $\bar{u}(F)$ considered in the main paper as the following proposition shows:

Proposition EC.10. *If $\bar{u}(F^T) = \sum_{t=1}^T \mathbb{E}_{\theta \sim (F^T)_t}[\bar{u}(\theta)]$ for all $F^T \in \mathcal{F}^T$, then $\bar{u}(F) = \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)]$ for all $F \in \mathcal{F}$.*

Proof. Fix any $F \in \mathcal{F}$. Note $\bar{u}((F)^{\times T}) \geq T \cdot \bar{u}(F)$ because the right-hand side is the supremum value of (EC-7) for $(F)^{\times T}$ when the IR constraint is imposed in each round instead of once over all rounds. We also have $\bar{u}((F)^{\times T}) \leq T \cdot \bar{u}(F)$ because any feasible solution $\{S_t\}$ to (EC-7) for $(F)^{\times T}$ can be aggregated by uniformly randomizing over $\{S_t\}$ into a feasible solution in the optimization problem defining $\bar{u}(F)$ that achieves $\frac{1}{T}$ times the objective value achieved by $\{S_t\}$ in (EC-7). Then, we have $\bar{u}((F)^{\times T}) = T \cdot \bar{u}(F)$. By the linearity condition in the general shock process setting,

$$\bar{u}((F)^{\times T}) = \sum_{t=1}^T \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)] = T \cdot \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)].$$

Combining the above two observations, it follows that $\bar{u}(F) = \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)]$. □

The following proposition shows analogous results similar to those in Propositions 6 and 7 of the main paper. It relates $\bar{u}(F^T)$ and the optimal performance achievable $\text{OPT}(F^T, T)$.

Proposition EC.11. *The following hold:*

1. *For any $F^T \in \mathcal{F}^T$, $\text{OPT}(F^T, T) \leq \bar{u}(F^T)$.*

2. *For any $F^T \in \mathcal{F}^T$, $\bar{u}(F^T) \geq \sum_{t=1}^T \mathbb{E}_{\theta \sim (F^T)_t}[\bar{u}(\theta)]$.*

3. If $\sup_{F^T \in \mathcal{F}^T} \bar{u}(F^T) < \infty$, then $\sup_{F^T \in \mathcal{F}^T} \mathrm{OPT}(F^T, T) < \infty$.

4. If $\bar{u}(F^T) = \sum_{t=1}^{T} \mathbb{E}_{\theta \sim (F^T)_t}[\bar{u}(\theta)]$ for all $F^T \in \mathcal{F}^T$, then $\mathrm{OPT}(F^T, T) \leq \mathbb{E}_{t \sim [T], \theta \sim (F^T)_t}[\mathrm{OPT}(\theta^{\times T}, T)]$ for all $F^T \in \mathcal{F}^T$.

Proof. Part 1): Fix an arbitrary T-round shock process $F^T \in \mathcal{F}^T$. Assume the principal commits to any arbitrary dynamic mechanism A^T and the agent plays the strategy $B^*(A^T, F^T)$. Let $\{\omega_t\}_{t=1}^T$ be the resulting random sequence of realized outcomes. If the agent reports QUIT in Round 0 and does not participate, the sequence is the sequence of the no-interaction outcome. For each $t \in [T]$ and $\theta \in \Theta$, we define measure $\mu_{t,\theta}(Q) = \mathrm{Pr}(\omega_t \in Q \mid \theta_t = \theta)$ for any $Q \subseteq \Omega$ and let $S_{t,\theta}$ be the corresponding distribution over Ω such that $\omega \sim S_{t,\theta}$ means an outcome ω is realized with probability $\mu_{t,\theta}(\omega)$.

For each $t \in [T]$, we define a single-round direct mechanism $S_t = \{S_{t,\theta}\}_{\theta \in \Theta}$ that given a report θ returns an outcome $\omega \sim S_{t,\theta}$. Consider the sequence of single-round direct mechanisms $\{S_t\}_{t \in [T]}$ thus constructed. We show that $\{S_t\}$ is a feasible solution to (EC-7) and achieves the objective value that is equal to PrincipalUtility$(A^T, B^*(A^T, F^T), F^T, T)$. As the dynamic mechanism A^T was arbitrarily chosen, the proposition statement would follow.

First, we note that

$$
\begin{aligned}
\mathrm{AgentUtility}(A^T, B^*(A^T, F^T), F^T, T) &= \mathbb{E}\left[\sum_{t=1}^{T} v(\theta_t, \omega_t)\right] \\
&= \sum_{t=1}^{T} \mathbb{E}_{\theta_t \sim (F^T)_t}\left[\mathbb{E}[v(\theta_t, \omega_t)|\theta_t]\right] \\
&= \sum_{t=1}^{T} \mathbb{E}_{\theta_t \sim (F^T)_t}\left[\mathbb{E}_{\omega_t \sim S_{t,\theta_t}}[v(\theta_t, \omega_t)]\right] \\
&= \mathbb{E}_{\theta \sim F^T, \omega \sim \prod_{t=1}^T S_{t,\theta_t}}\left[\sum_{t=1}^{T} v(\theta_t, \omega_t)\right],
\end{aligned}
$$

where the second equality follows from the linearity of expectations and the tower rule, the third from that the t-th round idiosyncratic shock can be thought to be drawn independently from the marginal distribution $(F^T)_t$, and the last from the construction of $\{S_t\}_{t \in [T]}$. Since the agent's strategy $B^*(A^T, F^T)$ is a utility-maximizing strategy and the agent can achieve the aggregate utility of 0 by not participating, it must be that AgentUtility$(A^T, B^*(A^T, F^T), F^T, T) \geq 0$. Hence, $\{S_t\}$ is a feasible solution to (EC-7).

Similarly, we have

$$
\mathrm{PrincipalUtility}(A^T, B^*(A^T, F^T), F^T, T) = \mathbb{E}_{\theta \sim F^T, \omega \sim \prod_{t=1}^T S_{t,\theta_t}}\left[\sum_{t=1}^{T} u(\theta_t, \omega_t)\right].
$$

It follows that $\{S_t\}$ is a feasible solution to (EC-7) and achieves the objective value of PrincipalUtility$(A^T, B^*(A^T, F^T), F^T, T)$. This completes the proof.

Part 2): This is because the right-hand side is the supremum value of (EC-7) with the IR constraint for each shock in each round which is a more constrained version of (EC-7).

Part 3): This part follows from the first part. Taking the supremum of both sides of the first part, we obtain

$$\sup_{F \in \mathcal{F}} \text{OPT}(F^T, T) \le \sup_{F \in \mathcal{F}} \bar{u}(F^T) < \infty \,.$$

Part 4): Let $F^T \in \mathcal{F}^T$ be an arbitrary T-round shock process. Then,

$$\text{OPT}(F^T, T) \le \bar{u}(F^T) = \sum_{t=1}^{T} \mathbb{E}_{\theta \sim (F^T)_t}[\bar{u}(\theta)] = \sum_{t=1}^{T} \mathbb{E}_{\theta \sim (F^T)_t}[\text{OPT}(\theta, 1)] \,,$$

where the first step is by the first part, the second step is by the linearity assumption on $\bar{u}(F^T)$ and the last step is by Proposition 6 of the main paper. The last expression is equivalently

$$\sum_{t=1}^{T} \mathbb{E}_{\theta \sim (F^T)_t}[\text{OPT}(\theta, 1)] = T \cdot \mathbb{E}_{t \sim [T], \theta \sim (F^T)_t}[\text{OPT}(\theta, 1)] = \mathbb{E}_{t \sim [T], \theta \sim (F^T)_t}[\text{OPT}(\theta, T)] \,,$$

where we used Proposition 2 of the main paper. $\qquad\square$

The following is the analogue of Proposition 8 of the main paper in the general shock process setting. It holds for games with payments that enter linearly into the utility functions of the principal and agent or with a nonnegative utility function for the agent:

Proposition EC.12. *Assume the game satisfies either conditions of Proposition 8 of the main paper. Then, $\bar{u}(F^T) = \sum_{t=1}^{T} \mathbb{E}_{\theta \sim (F^T)_t}[\bar{u}(\theta)]$ for all $F^T \in \mathcal{F}^T$.*

Proof. By the second part of Proposition EC.11, it remains to show $\bar{u}(F^T) \le \sum_{t=1}^{T} \mathbb{E}_{\theta \sim (F^T)_t}[\bar{u}(\theta)]$. It suffices to show that $\bar{u}(F^T) \le \sum_{t=1}^{T} \bar{u}((F^T)_t)$. It then would follow that

$$\bar{u}(F^T) \le \sum_{t=1}^{T} \bar{u}((F^T)_t) = \sum_{t=1}^{T} \mathbb{E}_{\theta \sim (F^T)_t}[\bar{u}(\theta)] \,,$$

where the equality is by the linearity condition in the repeated i.i.d. setting which holds by Proposition 8 of the main paper.

Recall that the optimization problem (EC-7) is

$$\sup_{S_1, \dots, S_T} \mathbb{E}_{\theta \sim F^T, \omega \sim \prod_{t=1}^{T} S_{t, \theta_t}} \left[\sum_{t=1}^{T} u(\theta_t, \omega_t) \right]$$

$$\text{s.t. } \mathbb{E}_{\theta \sim F^T, \omega \sim \prod_{t=1}^{T} S_{t, \theta_t}} \left[\sum_{t=1}^{T} v(\theta_t, \omega_t) \right] \ge 0 \,.$$

Note $\sum_{t=1}^{T} \bar{u}((F^T)_t)$ in the value of the following optimization problem which is a modified version

of (EC-7) with the nonnegativity constraint for each round:

$$\sup_{S_1,\ldots,S_T} \mathbb{E}_{\theta\sim F^T,\omega\sim\prod_{t=1}^T S_{t,\theta_t}} \left[\sum_{t=1}^T u(\theta_t,\omega_t) \right] \tag{EC-8}$$

$$\text{s.t. } \mathbb{E}_{\hat{\theta}\sim(F^T)_t,\hat{\omega}\sim S_{t,\hat{\theta}}} \left[v(\hat{\theta},\hat{\omega}) \right] \geq 0, \quad \forall t \in [T].$$

Part 1: We use the same representation used in the statement of Proposition 8 of the main paper for separating out the payment from the outcome, outcome space and utility functions of the principal and agent. We refer to the statement of Proposition 8 of the main paper for further details of the representation.

Fix an arbitrary T-round shock process $F^T \in \mathcal{F}^T$. Let $\{S_t\}$ be an arbitrary sequence of single-round direct mechanisms that is a feasible solution to (EC-7). For $t \in [T]$ and $\hat{\theta} \in \Theta$, let the payment offset be defined as

$$q_{t,\hat{\theta}} = \frac{1}{\beta} \left(\mathbb{E}_{\hat{\omega}\sim S_{t,\hat{\theta}}}[v(\hat{\theta},\hat{\omega})] - \frac{1}{T} \cdot \mathbb{E}_{\theta\sim F^T,\omega\sim\prod_{t=1}^T S_{t,\theta_t}} \left[\sum_{t=1}^T v(\theta_t,\omega_t) \right] \right).$$

Note the second term in the payment offset is a constant that does not depend on t or $\hat{\theta}$. By construction, $\mathbb{E}_{\theta\sim F^T}[\sum_{t=1}^T q_{t,\theta_t}] = 0$.

Consider the sequence of single-round direct mechanisms $\{S_t'\}$ constructed as follows. For each $t \in [T]$, S_t' is the single-round direct mechanism where for each $\hat{\theta} \in \Theta$, $S_{t,\hat{\theta}}'$ is the outcome distribution $S_{t,\hat{\theta}}$ modified with the fixed offset $q_{t,\hat{\theta}}$ such that to realize an outcome $\hat{\omega} \sim S_{t,\hat{\theta}}'$, we draw $(\hat{\omega}^0,\hat{p}) \sim S_{t,\hat{\theta}}$ and set $\hat{\omega} = (\hat{\omega}^0,\hat{p}+q_{t,\hat{\theta}})$. Note $q_{t,\hat{\theta}}$ is the same constant adjustment for every realized outcome.

Note, for each $t \in [T]$,

$$
\begin{aligned}
\mathbb{E}_{\substack{\hat{\theta}\sim(F^T)_t\\\hat{\omega}\sim S_{t,\hat{\theta}}'}} \left[v(\hat{\theta},\hat{\omega}) \right] &= \mathbb{E}_{\substack{\hat{\theta}\sim(F^T)_t\\(\hat{\omega}^0,\hat{p})\sim S_{t,\hat{\theta}}}} \left[v(\hat{\theta},(\hat{\omega}^0,\hat{p}+q_{t,\hat{\theta}})) \right] \\
&= \mathbb{E}_{\substack{\hat{\theta}\sim(F^T)_t\\\hat{\omega}\sim S_{t,\hat{\theta}}}} \left[v(\hat{\theta},\hat{\omega}) \right] - \beta \cdot \mathbb{E}_{\hat{\theta}\sim(F^T)_t}[q_{t,\hat{\theta}}] \\
&= \mathbb{E}_{\substack{\hat{\theta}\sim(F^T)_t\\\hat{\omega}\sim S_{t,\hat{\theta}}}} \left[v(\hat{\theta},\hat{\omega}) \right] - \left(\mathbb{E}_{\substack{\hat{\theta}\sim(F^T)_t\\\hat{\omega}\sim S_{t,\hat{\theta}}}} \left[v(\hat{\theta},\hat{\omega}) \right] - \frac{1}{T} \cdot \mathbb{E}_{\substack{\theta\sim F^T\\\omega\sim\prod_{t=1}^T S_{t,\theta_t}}} \left[\sum_{t=1}^T v(\theta_t,\omega_t) \right] \right) \\
&= \frac{1}{T} \cdot \mathbb{E}_{\substack{\theta\sim F^T\\\omega\sim\prod_{t=1}^T S_{t,\theta_t}}} \left[\sum_{t=1}^T v(\theta_t,\omega_t) \right] \\
&\geq 0,
\end{aligned}
$$

where the second step follows because

$$v(\hat{\theta},(\hat{\omega}^0,\hat{p}+q_{t,\hat{\theta}})) = v^0(\hat{\theta},\hat{\omega}^0) - \beta(\hat{p}+q_{t,\hat{\theta}}) = v(\hat{\theta},(\hat{\omega}^0,\hat{p})) - \beta q_{t,\hat{\theta}}$$

and the payment offset can be separated out, the third step follows by substituting in the payment offsets, and the last step is because $\{S_t\}$ is a feasible solution to (EC-7). Hence, $\{S_t'\}$ is a feasible

solution to (EC-8).

Furthermore, note that

$$
\begin{aligned}
\mathbb{E}_{\substack{\theta \sim F^T \\ \omega \sim \prod_{t=1}^T S'_{t,\theta_t}}} \left[\sum_{t=1}^T u(\theta_t, \omega_t) \right] &= \sum_{t=1}^T \mathbb{E}_{\substack{\hat{\theta} \sim (F^T)_t \\ \hat{\omega} \sim S'_{t,\hat{\theta}}}} \left[u(\hat{\theta}, \hat{\omega}) \right] \\
&= \sum_{t=1}^T \mathbb{E}_{\substack{\hat{\theta} \sim (F^T)_t \\ (\hat{\omega}^0, \hat{p}) \sim S_{t,\hat{\theta}}}} \left[u(\hat{\theta}, (\hat{\omega}^0, \hat{p} + q_{t,\hat{\theta}})) \right] \\
&= \sum_{t=1}^T \mathbb{E}_{\substack{\hat{\theta} \sim (F^T)_t \\ (\hat{\omega}^0, \hat{p}) \sim S_{t,\hat{\theta}}}} \left[u(\hat{\theta}, (\hat{\omega}^0, \hat{p})) \right] + \alpha \cdot \sum_{t=1}^T \mathbb{E}_{\substack{\hat{\theta} \sim (F^T)_t \\ (\hat{\omega}^0, \hat{p}) \sim S_{t,\hat{\theta}}}} \left[q_{t,\hat{\theta}} \right] \\
&= \sum_{t=1}^T \mathbb{E}_{\substack{\hat{\theta} \sim (F^T)_t \\ \hat{\omega} \sim S_{t,\hat{\theta}}}} \left[u(\hat{\theta}, \hat{\omega}) \right] + \alpha \cdot \sum_{t=1}^T \mathbb{E}_{\hat{\theta} \sim (F^T)_t} \left[q_{t,\hat{\theta}} \right] \\
&= \mathbb{E}_{\substack{\theta \sim F^T \\ \omega \sim \prod_{t=1}^T S_{t,\theta_t}}} \left[\sum_{t=1}^T u(\theta_t, \omega_t) \right] + \alpha \cdot \mathbb{E}_{\theta \sim F^T} \left[\sum_t q_{t,\theta_t} \right] \\
&= \mathbb{E}_{\substack{\theta \sim F^T \\ \omega \sim \prod_{t=1}^T S_{t,\theta_t}}} \left[\sum_{t=1}^T u(\theta_t, \omega_t) \right] ,
\end{aligned}
$$

where the third step follows by separating out the payment offset and the last step follows because $\mathbb{E}_{\theta \sim F^T} \left[\sum_t q_{t,\theta_t} \right] = 0$. It follows that $\{S'_t\}$ is a feasible solution to (EC-8) that achieves the same objective value as $\{S_t\}$ in (EC-7).

As $\{S_t\}$ was arbitrarily chosen, it follows that (EC-7) $\leq \sum_{t=1}^T \bar{u}((F^T)_t)$. Since F^T was arbitrary, the proposition follows.

Part 2: Assume the utility function of the agent is always nonnegative. Let $F^T \in \mathcal{F}^T$ be an arbitrary T-round shock process. Any feasible solution to (EC-7) satisfies the nonnegativity constraint in each round, because the agent's utility function is nonnegative, and, hence, is a feasible solution to (EC-8). In addition, note that the objective functions of (EC-7) and (EC-8) are identical. Therefore, (EC-7) $\leq$ (EC-8) for any $F^T \in \mathcal{F}^T$ and the proposition follows. $\qquad \square$

A.3 Multiplicative Guarantees

Suppose there exist some constants $0 < L < U < \infty$ such that $\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)] \in [L, U]$ and $\mathrm{OPT}(F, T) \in [LT, UT]$ for all distributions $F \in \mathcal{F}$. Recall that $\mathrm{Ratio}(T) \in [0, 1]$ and $\widehat{\mathrm{Ratio}} \in [0, 1]$. We introduce a parametrized notion of regret suitable for our analysis of the multiplicative guarantee. For any $\lambda \in [0, 1]$, we define

$$
\mathrm{Regret}(T, \lambda) := \inf_{A \in \mathcal{A}} \sup_{F \in \mathcal{F}} \mathrm{Regret}(A, F, T, \lambda),
$$

where

$$
\mathrm{Regret}(A, F, T, \lambda) := \lambda \cdot \mathrm{OPT}(F, T) - \mathrm{PrincipalUtility}(A, B^*(A, F), F, T).
$$

Note $\mathrm{Regret}(T,\lambda)$ is monotonically increasing in λ. For the single-round minimax regret problem, we similarly define

$$\widehat{\mathrm{Regret}}(\lambda) := \inf_{\substack{S\in\mathcal{S}:\\ (\mathrm{IC}),(\mathrm{IR})}} \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S,F,\lambda)\,,$$

where

$$\widehat{\mathrm{Regret}}(S,F,\lambda) := \lambda\cdot\int_\Theta \mathrm{OPT}(\theta,1)\mathrm{d}F(\theta) - \int_\Theta\int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta)\,.$$

We first prove the following proposition relating the multiplicative guarantees and the parametrized regret notions.

Proposition EC.13. *We have the following relations:*

1. $\mathrm{Ratio}(T) = \sup\{\lambda\in[0,1]\mid \mathrm{Regret}(T,\lambda)\le 0\}$

2. $\widehat{\mathrm{Ratio}} = \sup\{\lambda\in[0,1]\mid \widehat{\mathrm{Regret}}(\lambda)\le 0\}$

Proof. We prove the first relation. The second relation follows similarly and we omit the proof.

Note that by the definition of infimum, for any $\epsilon > 0$, there exists a dynamic mechanism A_ϵ such that $\inf_{F\in\mathcal{F}}\frac{\mathrm{PrincipalUtility}(A_\epsilon,B^*(A_\epsilon,F),F,T)}{\mathrm{OPT}(F,T)} \ge \mathrm{Ratio(T)} -\epsilon$, or

$$\frac{\mathrm{PrincipalUtility}(A_\epsilon,B^*(A_\epsilon,F),F,T)}{\mathrm{OPT}(F,T)} \ge \mathrm{Ratio(T)} -\epsilon\,,$$

for all distributions $F\in\mathcal{F}$. The above expression can be rearranged:

$$\epsilon\cdot\mathrm{OPT}(F,T) \ge \mathrm{Ratio}(T)\cdot\mathrm{OPT}(F,T) - \mathrm{PrincipalUtility}(A_\epsilon,B^*(A_\epsilon,F),F,T)\,.$$

Since $\mathrm{OPT}(F,T)\le UT$,

$$\begin{aligned}
\mathrm{Regret}(T,\mathrm{Ratio}(T)) &\le \sup_{F\in\mathcal{F}}\left\{\mathrm{Ratio}(T)\cdot\mathrm{OPT}(F,T) - \mathrm{PrincipalUtility}(A_\epsilon,B^*(A_\epsilon,F),F,T)\right\}\\
&\le \epsilon UT\,.
\end{aligned}$$

As $\epsilon > 0$ was arbitrary, it follows that $\mathrm{Regret}(T,\mathrm{Ratio}(T))\le 0$ and $\mathrm{Ratio}(T)$ is in the set $\{\lambda\in[0,1]\mid \mathrm{Regret}(T,\lambda)\le 0\}$.

We now show there cannot be a $\lambda > \mathrm{Ratio}(T)$ in the set because reversing the above reasoning leads to a contradiction. Assume there exists $\lambda > \mathrm{Ratio}(T)$ for which

$$\mathrm{Regret}(T,\lambda) = \inf_A \sup_F \left\{\lambda\cdot\mathrm{OPT}(F,T) - \mathrm{PrincipalUtility}(A,B^*(A,F),F,T)\right\}\le 0\,.$$

Then, for any $\epsilon > 0$, there exists a dynamic mechanism A_ϵ for which the inner supremum is at most ϵLT, or

$$\epsilon LT \ge \lambda\cdot\mathrm{OPT}(F,T) - \mathrm{PrincipalUtility}(A_\epsilon,B^*(A_\epsilon,F),F,T)\,,$$

for any distribution $F\in\mathcal{F}$. After rearranging and using $\mathrm{OPT}(F,T)\ge LT$, we have

$$\frac{\mathrm{PrincipalUtility}(A_\epsilon,B^*(A_\epsilon,F),F,T)}{\mathrm{OPT}(F,T)} \ge \lambda - \frac{\epsilon LT}{\mathrm{OPT}(F,T)} \ge \lambda - \epsilon\,,$$

for any distribution $F \in \mathcal{F}$. Taking the infimum over $F \in \mathcal{F}$ on the leftmost expression, we obtain

$$\inf_{F \in \mathcal{F}} \frac{\text{PrincipalUtility}(A_\epsilon, B^*(A_\epsilon, F), F, T)}{\text{OPT}(F, T)} \geq \lambda - \epsilon \,,$$

and as ϵ was arbitrary, this would mean $\text{Ratio}(T) \geq \lambda$, a contradiction. Hence, the first relation follows. $\qquad\square$

We now prove Theorem EC.7:

Proof of Theorem EC.7. Similar to the proof of Theorem 1 of the main paper, we prove using analogues of Lemmas 3 and 4 of the main paper in terms of the parametrized regret notions. Since λ is a scalar multiplier in front of the benchmarks, all the propositions and lemmas used to prove these analogues still apply with the same proofs and the analogues of Lemmas 3 and 4 of the main paper follow with little changes in their proofs but with the multiplier λ. Omitting the proofs, we state and use the following analogues of Lemmas 3 and 4 of the main paper. For the lower bound, for any $\lambda \in [0, 1]$ and dynamic mechanism A^T, there exists a single-round direct IC/IR mechanism S such that

$$\sup_{F \in \mathcal{F}} \text{Regret}(A^T, F, T, \lambda) \geq T \cdot \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F, \lambda) \,. \tag{EC-9}$$

For the upper bound, for any $\lambda \in [0, 1]$ and single-round direct IC/IR mechanism S,

$$\sup_{F \in \mathcal{F}} \text{Regret}(S^{\times T}, F, T, \lambda) \leq T \cdot \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F, \lambda) \,. \tag{EC-10}$$

(First Part): Fix an arbitrary $\lambda \in [0, 1]$. Taking the infimum over all single-round direct IC/IR mechanisms S on the right-hand side of the lower bound (EC-9), we have for any dynamic mechanism A^T,

$$\sup_{F \in \mathcal{F}} \text{Regret}(A^T, F, T, \lambda) \geq T \cdot \inf_{\substack{S \in \mathcal{S}: \\ (\text{IC}),(\text{IR})}} \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F, \lambda) = T \cdot \widehat{\text{Regret}}(\lambda) \,.$$

Taking the infimum over all dynamic mechanisms A^T on the left-hand side of the above, we obtain

$$\text{Regret}(T, \lambda) = \inf_{A^T \in \mathcal{A}^T} \sup_{F \in \mathcal{F}} \text{Regret}(A^T, F, T, \lambda) \geq T \cdot \widehat{\text{Regret}}(\lambda) \,.$$

Then, for any $\lambda \in [0, 1]$,

$$\text{Regret}(T, \lambda) \geq T \cdot \widehat{\text{Regret}}(\lambda) \,.$$

This implies whenever $\text{Regret}(T, \lambda) \leq 0$, $\widehat{\text{Regret}}(\lambda) \leq 0$. By Proposition EC.13, it follows that $\text{Ratio}(T) \leq \widehat{\text{Ratio}}$.

Again, fix an arbitrary $\lambda \in [0, 1]$. Let $\epsilon > 0$ be arbitrary and S be a single-round direct IC/IR mechanism satisfying

$$\sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F, \lambda) \leq \widehat{\text{Regret}}(\lambda) + \frac{\epsilon}{T} \,.$$

Note that such mechanism S exists by the definition of infimum. By the upper bound (EC-10) and

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the property of S,

$$\sup_{F \in \mathcal{F}} \text{Regret}(S^{\times T}, F, T, \lambda) \leq T \cdot \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F, \lambda) \leq T \cdot \widehat{\text{Regret}}(\lambda) + \epsilon \,.$$

It follows that

$$\text{Regret}(T, \lambda) = \inf_{A^T \in \mathcal{A}^T} \sup_{F \in \mathcal{F}} \text{Regret}(A^T, F, T, \lambda) \leq \sup_{F \in \mathcal{F}} \text{Regret}(S^{\times T}, F, T, \lambda) \leq T \cdot \widehat{\text{Regret}}(\lambda) + \epsilon \,.$$

As $\epsilon > 0$ was arbitrary and can be made arbitrarily small, it follows that

$$\text{Regret}(T, \lambda) \leq T \cdot \widehat{\text{Regret}}(\lambda) \,.$$

As $\lambda \in [0,1]$ was arbitrary, the above holds for any $\lambda \in [0,1]$. In particular, $\widehat{\text{Regret}}(\lambda) \leq 0$ implies $\text{Regret}(T, \lambda) \leq 0$. By Proposition EC.13, we have $\text{Ratio}(T) \geq \widehat{\text{Ratio}}$. Combining with the earlier observation that $\text{Ratio}(T) \leq \widehat{\text{Ratio}}$, we have the first part.

(Second Part): For any $\epsilon \geq 0$, let S be a single-round direct IC/IR mechanism satisfying

$$\inf_{F \in \mathcal{F}} \frac{\int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta)}{\int_\Theta \text{OPT}(\theta, 1) \mathrm{d}F(\theta)} \geq \widehat{\text{Ratio}} - \frac{\epsilon L}{U} \,.$$

Then, for any distribution $F \in \mathcal{F}$, $\frac{\int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta)}{\int_\Theta \text{OPT}(\theta, 1) \mathrm{d}F(\theta)} \geq \widehat{\text{Ratio}} - \frac{\epsilon L}{U}$. After rearranging terms and upper bounding $\int_\Theta \text{OPT}(\theta, 1) \mathrm{d}F(\theta)$ by U, we obtain that for any $F \in \mathcal{F}$,

$$\widehat{\text{Ratio}} \cdot \int_\Theta \text{OPT}(\theta, 1) \mathrm{d}F(\theta) - \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \leq \frac{\epsilon L}{U} \cdot \int_\Theta \text{OPT}(\theta, 1) \mathrm{d}F(\theta) \leq \epsilon L \,.$$

Then, $\sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F, \widehat{\text{Ratio}}) \leq \epsilon L$ and by the upper bound (EC-10) for $\lambda = \widehat{\text{Ratio}}$,

$$\sup_{F \in \mathcal{F}} \text{Regret}(S^{\times T}, F, T, \widehat{\text{Ratio}}) \leq T \cdot \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F, \widehat{\text{Ratio}}) \leq \epsilon L T \,.$$

Substituting $\text{Ratio}(T) = \widehat{\text{Ratio}}$ which is by Part 1 into the leftmost expression, we obtain for any $F \in \mathcal{F}$,

$$\text{Ratio}(T) \cdot \text{OPT}(F, T) - \text{PrincipalUtility}(S^{\times T}, B^*(S^{\times T}, F), F, T) \leq \epsilon L T \,.$$

After rearranging terms and lower bounding $\text{OPT}(F, T)$ by LT, we have that for any $F \in \mathcal{F}$,

$$\frac{\text{PrincipalUtility}(S^{\times T}, B^*(S^{\times T}, F), F, T)}{\text{OPT}(F, T)} \geq \text{Ratio}(T) - \frac{\epsilon L T}{\text{OPT}(F, T)} \geq \text{Ratio}(T) - \epsilon \,.$$

Taking the infimum of the leftmost expression over $F \in \mathcal{F}$,

$$\inf_{F \in \mathcal{F}} \frac{\text{PrincipalUtility}(S^{\times T}, B^*(S^{\times T}, F), F, T)}{\text{OPT}(F, T)} \geq \text{Ratio}(T) - \epsilon \,.$$

(Third Part): For the if direction, assume there exists an optimal single-round direct IC/IR mechanism

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S^* in the single-round problem. The optimal mechanism S^* satisfies

$$\inf_{F \in \mathcal{F}} \frac{\int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S^*_\theta(\omega) \mathrm{d}F(\theta)}{\int_\Theta \mathrm{OPT}(\theta, 1) \mathrm{d}F(\theta)} \geq \widehat{\mathrm{Ratio}} \,.$$

Then, by Part 2 with $\epsilon = 0$, it follows that

$$\inf_{F \in \mathcal{F}} \frac{\mathrm{PrincipalUtility}((S^*)^{\times T}, B^*((S^*)^{\times T}, F), F, T)}{\mathrm{OPT}(F, T)} \geq \mathrm{Ratio}(T) \,.$$

This implies that the static mechanism $(S^*)^{\times T}$ that repeats S^* is optimal in the multi-round problem. Hence, an optimal dynamic mechanism exists in the multi-round problem.

For the only-if direction, assume there exists an optimal dynamic mechanism A^* in the multi-round problem. In particular, $\inf_{F \in \mathcal{F}} \frac{\mathrm{PrincipalUtility}(A^*, B^*(A^*, F), F, T)}{\mathrm{OPT}(F,T)} \geq \mathrm{Ratio}(T)$. Then, for any distribution $F \in \mathcal{F}$, we have $\frac{\mathrm{PrincipalUtility}(A^*, B^*(A^*, F), F, T)}{\mathrm{OPT}(F,T)} \geq \mathrm{Ratio}(T)$, or

$$\mathrm{Ratio}(T) \cdot \mathrm{OPT}(F, T) - \mathrm{PrincipalUtility}(A^*, B^*(A^*, F), F, T) \leq 0 \,.$$

Taking the supremum over $F \in \mathcal{F}$ on the left-hand side,

$$\sup_{F \in \mathcal{F}} \mathrm{Regret}(A^*, F, T, \mathrm{Ratio}(T)) \leq 0 \,.$$

By the lower bound (EC-9) for $\lambda = \mathrm{Ratio}(T)$, there exists a single-round direct IC/IR mechanism S such that

$$\sup_{F \in \mathcal{F}} \widehat{\mathrm{Regret}}(S, F, \mathrm{Ratio}(T)) \leq \sup_{F \in \mathcal{F}} \mathrm{Regret}(A^*, F, T, \mathrm{Ratio}(T)) \leq 0 \,.$$

Since $\mathrm{Ratio}(T) = \widehat{\mathrm{Ratio}}$ by Part 1, $\sup_{F \in \mathcal{F}} \widehat{\mathrm{Regret}}(S, F, \widehat{\mathrm{Ratio}}) \leq 0$. This implies that for any $F \in \mathcal{F}$,

$$\widehat{\mathrm{Ratio}} \cdot \int_\Theta \mathrm{OPT}(\theta, 1) \mathrm{d}F(\theta) - \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \leq 0 \,.$$

Rearranging terms, we obtain

$$\widehat{\mathrm{Ratio}} \leq \frac{\int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta)}{\int_\Theta \mathrm{OPT}(\theta, 1) \mathrm{d}F(\theta)} \,.$$

Taking the infimum over $F \in \mathcal{F}$ on the right-hand side,

$$\widehat{\mathrm{Ratio}} \leq \inf_{F \in \mathcal{F}} \frac{\int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta)}{\int_\Theta \mathrm{OPT}(\theta, 1) \mathrm{d}F(\theta)} \,.$$

Then, S is an optimal single-round direct IC/IR mechanism in the single-round problem and the result follow. For S, we can take the single-round direct IC/IR mechanism constructed from A^* via the method described in the proof of Lemma 16 of the main paper. The same single-round mechanism satisfies the lower bound (EC-9). $\qquad\square$

A.4 Principal Pessimism

A.4.1 Proof of Theorem EC.8

We can prove Theorem EC.8 by the same reasoning as in the proof of Theorem 1 of the main paper using Lemmas 3 and 4 of the main paper with respect to the stronger notions of regret and minimax regret and single-round direct IC/IR/PP mechanisms. In particular, Lemma 3 of the main paper will show the existence of a single-round direct IC/IR/PP mechanism for any dynamic mechanism and Lemma 4 of the main paper will hold for any single-round direct IC/IR/PP mechanism. These lemmas hold with almost identical proofs with modifications only in how we break ties, i.e., the agent plays a utility-maximizing strategy that minimizes, instead of maximizing, the principal utility among utility-maximizing strategies. Below, we sketch the proofs of Lemmas 3 and 4 of the main paper with respect to the stronger notions and single-round direct IC/IR/PP mechanisms.

In the proof of Lemma 3 of the main paper with respect to the original notions of regret and minimax regret, we use Lemmas 16 and 17 of the main paper. Instead of single-round direct IC/IR mechanisms, we can prove these lemmas for single-round direct IC/IR/PP mechanisms. Then, Lemma 3 of the main paper would follow in terms of single-round direct IC/IR/PP mechanisms with respect to the stronger notions of regret and minimax regret by the same proof with the modified versions of these lemmas.

Lemma 17 of the main paper still holds for any single-round direct IC/IR/PP mechanisms without any modification in its proof. For Lemma 16 of the main paper, the same construction provided in its proof yields a single-round direct IC/IR/PP mechanism S satisfying the lemma statement for any given dynamic mechanism A^T. That is, we construct $S = \{S_\theta\}_{\theta \in \Theta}$ from the random sequences of outcomes when the principal commits to a dynamic mechanism A^T and the agent plays $B^*(A^T, \theta)$ for point-mass distributions $\theta \in \Theta$. As before, the IC/IR constraints follow from that $B^*(A^T, \theta)$ is a utility-maximizing strategy for the agent when his shock is θ and guarantees the agent utility of at least 0. For the PP constraint, we prove by contradiction. Suppose there exist θ and θ' such that $\mathbb{E}_{\omega \sim S_\theta}[v(\theta, \omega)] = \mathbb{E}_{\omega \sim S_{\theta'}}[v(\theta, \omega)]$ and $\mathbb{E}_{\omega \sim S_\theta}[u(\theta, \omega)] > \mathbb{E}_{\omega \sim S_{\theta'}}[u(\theta, \omega)]$. Then, the agent can implement the strategy $B^*(A^T, \theta')$, instead of $B^*(A^T, \theta)$, when his distribution is $\{\theta\}$ in the multi-round problem and obtain the utility of

$$
\begin{aligned}
\text{AgentUtility}(A^T, B^*(A^T, \theta'), \theta, T) &= T \cdot \mathbb{E}_{\omega \sim S_{\theta'}}[v(\theta, \omega)] \\
&= T \cdot \mathbb{E}_{\omega \sim S_\theta}[v(\theta, \omega)] \\
&= \text{AgentUtility}(A^T, B^*(A^T, \theta), \theta, T),
\end{aligned}
$$

which is the same utility as under $B^*(A^T, \theta)$. At the same time, the principal utility will be strictly lower:

$$
\begin{aligned}
\text{PrincipalUtility}(A^T, B^*(A^T, \theta'), \theta, T) &= T \cdot \mathbb{E}_{\omega \sim S_{\theta'}}[u(\theta, \omega)] \\
&< T \cdot \mathbb{E}_{\omega \sim S_\theta}[u(\theta, \omega)] \\
&= \text{PrincipalUtility}(A^T, B^*(A^T, \theta), \theta, T).
\end{aligned}
$$

This contradicts that $B^*(A^T, \theta)$ is a principal-pessimistic utility-maximizing strategy when the agent's distribution is $\{\theta\}$. For the second part of the PP constraint, suppose there exists θ such that $\mathbb{E}_{\omega \sim S_\theta}[v(\theta, \omega)] = 0$ and $\mathbb{E}_{\omega \sim S_\theta}[u(\theta, \omega)] > 0$. The agent can choose to not participate and obtain the

same agent utility of 0 as under $B^*(A^T, \theta)$ and the principal utility will be 0 which is strictly worse than that achieved under $B^*(A^T, \theta)$. Again, this contradicts the choice of $B^*(A^T, \theta)$. Hence, the singe-round direct mechanism constructed is IC/IR/PP and the rest of the proof of Lemma 16 of the main paper follows.

For Lemma 4 of the main paper with respect to the original notions of regret and minimax regret, we use Lemma 18 of the main paper in its proof. We can show Lemma 18 of the main paper still holds with respect to the stronger notions of regret and minimax regret. We adapt the same proof but tie break accordingly in that a best-response strategy for the agent is a utility-maximizing strategy that also minimizes the principal utility among utility-maximizing strategies.

Given Lemma 18 of the main paper, with the same lemma statement, we follow the same reasoning steps to prove Lemma 4 of the main paper in terms of single-round direct IC/IR/PP mechanisms and with respect to the stronger notions of regret and minimax regret. Let S be any single-round direct IC/IR/PP mechanism and consider the dynamic mechanism $S^{\times T}$ which is T repetitions of the mechanism S. After skipping the steps that use the same reasoning outlined in the proof of Lemma 4 of the main paper, we have

$$\sup_{F \in \mathcal{F}} \text{Regret}(S^{\times T}, F, T) \le T \cdot \sup_{F \in \mathcal{F}} \left\{ \mathbb{E}_{\theta \sim F}[\text{OPT}(\theta, 1)] - \text{PrincipalUtility}(S, B^*(S, F), F, 1) \right\}.$$

We claim that the right-hand side is equal to $\sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F)$ for any single-round direct IC/IR/PP mechanism. This is because for any single-round direct IC/IR/PP mechanism S and distribution F, the principal-pessimistic utility-maximizing strategy $B^*(S, F)$ may be assumed to be to participate and truthfully report. Given that the agent participates, the IC/IR constraints and the first part of the PP constraint imply that truthful reporting is a utility-maximizing strategy that also minimizes the principal utility among utility-maximizing strategies that participate and yields the agent utility of at least 0. The IR constraint and the second part of the PP constraint imply that if truthful reporting leads to the agent utility of 0, the principal utility is at most 0. Since not participating leads to the agent and principal utilities of 0, it is a principal-pessimistic utility-maximizing strategy for the agent to participate and then report truthfully. Then, it follows that

$$\sup_{F \in \mathcal{F}} \text{Regret}(S^{\times T}, F, T) \le T \cdot \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F).$$

A.4.2   Characterizations in Specific Settings

For the applications considered in Sections 5–7 of the main paper, we show the following characterizations of the PP constraint and that the optimal single-round direct IC/IR mechanisms found for these applications satisfy the PP constraint:

Proposition EC.14. *For revenue maximization in the dynamic selling mechanism design problem with one good, a single-round direct IC/IR mechanism (x, p) satisfies the PP constraint if and only if 1) there exists no $\theta' < \theta \in [0, 1]$ such that $x(\theta') < x(\theta)$ and $x(\hat{\theta}) = x(\theta')$ for all $\hat{\theta} \in [\theta', \theta)$, and 2) $p(\theta) \le 0$ for all θ such that $\theta \cdot x(\theta) - p(\theta) = 0$.*

Proposition EC.15. *For the principal-agent model with hidden costs, a (deterministic) single-round direct IC/IR mechanism (q, z) satisfies the PP constraint if and only if 1) there exists no interval (θ', θ) where $q(\hat{\theta}) = q_0$ for $\hat{\theta} \in (\theta', \theta)$ for some q_0 and at least one of following sets of conditions holds:*

1. $q(\theta) < q_0$ and $R(q_0) - \theta \cdot q_0 < R(q(\theta)) - \theta \cdot q(\theta)$

2. $q(\theta') > q_0$ and $R(q_0) - \theta \cdot q_0 < R(q(\theta')) - \theta \cdot q(\theta')$,

and 2) $R(q(\theta)) - z(\theta) \leq 0$ for all θ such that $z(\theta) - \theta \cdot q(\theta) = 0$.

Proposition EC.16. *For the dynamic resource allocation problem without monetary transfers, a single-round direct IC/IR mechanism x satisfies the PP constraint if and only if the probabilistic allocation rule x is constant.*

Except for the last one, the if-and-only-if conditions for PP consist of two parts for the two parts of the PP constraint. The second part of the conditions follows directly from the second part of the PP constraint and is written in terms of the corresponding interim rules in respective problems. The first part of the conditions is more complicated and, intuitively, it says that any flat part of the interim allocation rule $x(\cdot)$, if exists, is closed on the right side in the dynamic selling problem or satisfies some similar condition on both sides in the principal-agent model with hidden costs. In particular, if the interim allocation rule $x(\cdot)$ is continuous, the first part of the conditions will be satisfied in these problems.

For revenue maximization in the dynamic selling problem with a single good, the optimal single-round direct IC/IR mechanism in Theorem 13 of the main paper has a continuous interim allocation rule and the interim rule is such that the principal utility is 0 whenever the agent utility is 0. By Proposition EC.14, the optimal single-round solution is PP. The optimal single-round solution $S^{*,n}$ in Theorem EC.3 in the multiple-goods case satisfies the PP constraint because it uses S^* in Theorem 13 of the main paper on each good and inherits the same properties from the single-good case. For welfare maximization in both the single-good and multiple-goods cases, the principal's utility function $u(\cdot, \cdot)$ coincides with the agent's utility function $v(\cdot, \cdot)$ when there is no payment and the realized principal utility is the same for all utility-maximizing strategies for the agent. That is, tie-breaking among utility-maximizing strategies is meaningless and all utility-maximizing strategies are also principal-pessimistic utility-maximizing strategies by default. Hence, the optimal mechanism of allocating items for free in Theorem EC.2 and the same mechanism in the multiple-goods case in Theorem EC.3 satisfy the PP constraint.

Similarly, in the principal-agent model, the optimal single-round direct IC/IR mechanism in Theorem 14 of the main paper is deterministic and has a continuous interim allocation rule. It also satisfies the second part of the if-and-only-if condition, and by Proposition EC.15, it satisfies the PP constraint. For the dynamic resource allocation problem, there are multiple optimal single-round solutions in Theorem 15 of the main paper and, in particular, the constant probabilistic allocation rule $x^*(\theta) = 1 - c$ for all $\theta \in [0, 1]$ is optimal. By Proposition EC.16, it satisfies the PP constraint.

We prove the above propositions below.

Revenue Maximization in the Dynamic Selling Problem with One Good   We primarily work with the interim allocation rule $x(\cdot)$ and payment rule $p(\cdot)$. The IC/IR/PP constraints can be

equivalently written as

$$\theta \cdot x(\theta) - p(\theta) \geq \theta \cdot x(\theta') - p(\theta'), \quad \forall \theta, \theta' \in [0,1] \tag{IC}$$

$$\theta \cdot x(\theta) - p(\theta) \geq 0, \quad \forall \theta \in [0,1] \tag{IR}$$

$$p(\theta) \leq p(\theta'), \quad \forall \theta \in [0,1], \theta' \in \mathcal{B}^*(\theta)$$
$$p(\theta) \leq 0, \quad \forall \theta \in \Theta_0 \tag{PP}$$

where $\mathcal{B}^*(\theta) = \{\theta' \in \Theta \mid \theta \cdot x(\theta') - p(\theta') = \theta \cdot x(\theta) - p(\theta)\}$ and $\Theta_0 = \{\theta \in \Theta \mid \theta \cdot x(\theta) - p(\theta) = 0\}$. Note if a single-round direct mechanism satisfies the IC/IR constraints, x is non-decreasing and p is given by the payment equivalence formula and, similarly, non-decreasing. Let $V(\theta) = \theta \cdot x(\theta) - p(\theta)$. Note $V(\theta) = \max_{\theta'}\{\theta \cdot x(\theta') - p(\theta')\}$. Equivalently, the utility curve $V(\cdot)$ is supported by lines with slope-intercept pairs $(x(\theta'), -p(\theta'))$ for $\theta' \in [0,1]$ and, in particular, the line with slope-intercept pair $(x(\theta), -p(\theta))$ goes through the point $(\theta, V(\theta))$.

Proof of Proposition EC.14. It is straightforward to see that the second part of the if-and-only-if condition is equivalent to the second part of the PP constraint in terms of the interim rules. It remains to prove that the first part corresponds to the first part of the PP constraint.

(If part): We prove the contrapositive. Assume the first part of the PP constraint is not satisfied at θ. Note $\theta \neq 0$. Given that IC/IR constraints are satisfied, the first part of the PP constraint not holding at θ means that truthful reporting is not a principal-pessimistic utility-maximizing strategy when the realized value is θ. There exists another utility-maximizing report θ' leading to outcome $(x(\theta'), p(\theta'))$ with a strictly worse principal utility, i.e., $p(\theta') < p(\theta)$. Since the payment rule is non-decreasing, it must be that $\theta' < \theta$. Since

$$V(\theta) = \theta \cdot x(\theta) - p(\theta) = \theta \cdot x(\theta') - p(\theta'),$$

and $\theta \neq 0$, it follows that $x(\theta') < x(\theta)$. Note the line with slope-intercept pair $(x(\theta'), -p(\theta'))$ goes through $(\theta, V(\theta))$. For this to happen, x has to be constant over the interval $[\theta', \theta]$. If there exists some $\hat{\theta} \in [\theta', \theta]$ at which $x(\hat{\theta}) > x(\theta')$ (this is the only possibility since x is non-decreasing), the line with slope-intercept pair $(x(\hat{\theta}), -p(\hat{\theta}))$ leads to a strictly higher utility for the agent at the realized value θ than the line with slope-intercept pair $(x(\theta'), -p(\theta'))$. This is because the point $(\hat{\theta}, V(\hat{\theta}))$ is above or on the line with slope-intercept pair $(x(\theta'), -p(\theta'))$ and the line with slope-intercept pair $(x(\hat{\theta}), -p(\hat{\theta}))$ goes through the point and has a greater slope. This would contradict that reporting θ' achieves $V(\theta)$ when the realized value is θ. For θ and θ', we have $x(\theta') < x(\theta)$ and x constant over $[\theta', \theta)$.

(Only if part): We similarly prove the contrapositive. Assume there exists a pair $\theta' < \theta$ such that $x(\theta') < x(\theta)$ and $x(\hat{\theta}) = x(\theta')$ for $\hat{\theta} \in [\theta', \theta)$. Since the utility curve $V(\cdot)$ is convex and absolutely continuous, the line with slope-intercept pair $(x(\theta'), -p(\theta'))$ which is tangent to V on $[\theta', \theta)$ contains the point $(\theta, V(\theta))$. This implies

$$V(\theta) = \theta \cdot x(\theta) - p(\theta) = \theta \cdot x(\theta') - p(\theta'),$$

and reporting θ' and truthfully reporting θ lead to the same agent utility when the realized value is θ. By the payment equivalence formula, $p(\hat{\theta}) = p(\theta')$ for $\hat{\theta} \in [\theta', \theta)$ and, in particular, $p(\theta') < p(\theta)$. Then, reporting θ' leads to a strictly worse principal utility. The first part of the PP constraint is not satisfied at θ. $\square$

Principal-Agent Model with Hidden Costs This is after the without-loss restriction of the single-round direct IC/IR mechanisms to those that can be described as a menu of deterministic contracts $\{(q(\theta), z(\theta))\}_{\theta \in [\underline{\theta}, \bar{\theta}]}$. By standard arguments, $q(\cdot)$ is non-increasing and $z(\cdot)$ is given by the payment equivalence formula. See Appendix D.1 of the main paper for details. In this setting, the IC/IR/PP constraints in consideration are

$$z(\theta) - \theta \cdot q(\theta) \geq z(\theta') - \theta \cdot q(\theta'), \quad \forall \theta, \theta' \in [\underline{\theta}, \bar{\theta}] \tag{IC}$$

$$z(\theta) - \theta \cdot q(\theta) \geq 0, \quad \forall \theta \in [\underline{\theta}, \bar{\theta}] \tag{IR}$$

$$R(q(\theta)) - z(\theta) \leq R(q(\theta')) - z(\theta'), \quad \forall \theta \in [\underline{\theta}, \bar{\theta}], \theta' \in \mathcal{B}^*(\theta)$$
$$R(q(\theta)) - z(\theta) \leq 0, \quad \forall \theta \in \Theta_0 \tag{PP}$$

where $\mathcal{B}^*(\theta) = \{\theta' \in \Theta \mid z(\theta') - \theta \cdot q(\theta') = z(\theta) - \theta \cdot q(\theta)\}$ and $\Theta_0 = \{\theta \in \Theta \mid z(\theta) - \theta \cdot q(\theta) = 0\}$. In what follows, let $V(\theta) = z(\theta) - \theta \cdot q(\theta)$ for $\theta \in [\underline{\theta}, \bar{\theta}]$. As in the dynamic selling problem, we can describe the utility curve $V(\cdot)$ as the upper envelope of lines with slope-intercept pairs $(-q(\theta), z(\theta))$ for $\theta \in [\underline{\theta}, \bar{\theta}]$.

Proof of Proposition EC.15. The second part of the if-and-only-if condition is exactly the second part of the PP constraint written in terms of the interim rules. We prove the first part is equivalent to the first part of the PP constraint.

(If part): We prove the contrapositive. Assume the first part of the PP constraint is not satisfied and reporting some θ' yields the same agent utility as truthful reporting but less principal utility when the realized cost is θ, that is:

$$z(\theta') - \theta \cdot q(\theta') = z(\theta) - \theta \cdot q(\theta) \text{ and}$$
$$R(q(\theta')) - z(\theta') < R(q(\theta)) - z(\theta).$$

Combining the two relations, we obtain

$$R(q(\theta')) - \theta \cdot q(\theta') < R(q(\theta)) - \theta \cdot q(\theta). \tag{EC-11}$$

In particular, this implies $q(\theta) \neq \bar{q}(\theta) = \arg\max_{q \geq 0}\{R(q) - \theta \cdot q\}$. There are two cases depending on how $q(\theta)$ and $\bar{q}(\theta)$ compare.

If $q(\theta) < \bar{q}(\theta)$, it must be that $q(\theta') > q(\theta)$ for (EC-11) to hold. As q is non-increasing, it follows that $\theta' < \theta$. We claim q is constant over $[\theta', \theta)$. If otherwise, $q(\hat{\theta}) < q(\theta')$ for some $\hat{\theta} \in [\theta', \theta)$ and this would mean the line with slope-intercept pair $(-q(\hat{\theta}), z(\hat{\theta}))$ leads to a higher utility at realized cost θ than the line with slope-intercept pair $(-q(\theta'), z(\theta'))$. Note the line with slope-intercept pair $(-q(\hat{\theta}), z(\hat{\theta}))$ goes through $(\hat{\theta}, V(\hat{\theta}))$ which is above or on the line with slope-intercept pair $(q(\theta'), z(\theta'))$ and has a greater slope. This would contradict the choice of θ'. For the interval (θ', θ), we have q constant over the interval and the first set of conditions hold.

If $q(\theta) > \bar{q}(\theta)$, then $q(\theta') < q(\theta)$ and $\theta' > \theta$. By a similar argument as above, it follows that q is constant over $(\theta, \theta']$. With the roles of θ and θ' reversed, we have that the second set of conditions hold.

(Only if part): We prove the contrapositive. Assume there exists an interval (θ', θ) within which q is constant, say, equal to q_0. Assume the first set of conditions hold. The argument is similar when

the second set of conditions hold instead. We have $q(\theta) < q_0$ and $R(q_0) - \theta \cdot q_0 < R(q(\theta)) - \theta \cdot q(\theta)$. We show that truthful reporting is not a principal-pessimistic utility-maximizing strategy when the realized cost is θ because reporting some $\hat{\theta} \in (\theta', \theta)$ is a utility-maximizing strategy that gives a worse principal utility. Fix an arbitrary $\hat{\theta} \in (\theta', \theta)$. In the interval $[\hat{\theta}, \theta)$, q is equal to q_0 and z is also constant, say z_0. Then, the utility curve is on the line with slope-intercept pair $(-q_0, z_0)$ on interval $[\hat{\theta}, \theta)$. Since it is absolutely continuous, $(\theta, V(\theta))$ is also on the same line. Hence,

$$V(\theta) = z(\theta) - \theta \cdot q(\theta) = z(\hat{\theta}) - \theta \cdot q(\hat{\theta}),$$

and reporting $\hat{\theta}$ is a utility-maximizing strategy when the realized cost is θ. Combining the above with the assumption that $R(q_0) - \theta \cdot q_0 < R(q(\theta)) - \theta \cdot q(\theta)$, we obtain

$$R(q(\hat{\theta})) - z(\hat{\theta}) < R(q(\theta)) - z(\theta).$$

This shows reporting $\hat{\theta}$ leads to a strictly worse principal utility. It follows that the first part of the PP constraint does not hold when realized cost is θ. $\qquad\square$

Resource Allocation Problem without Monetary Transfers We represent a single-round direct mechanism with its interim allocation rule $x : [0, 1] \to [0, 1]$. The IC/IR/PP constraints are

$$\theta \cdot x(\theta) \geq \theta \cdot x(\theta'), \quad \forall \theta, \theta' \in [0, 1] \tag{IC}$$

$$\theta \cdot x(\theta) \geq 0, \quad \forall \theta \in [0, 1] \tag{IR}$$

$$(\theta - c) \cdot x(\theta) \leq (\theta - c) \cdot x(\theta'), \quad \forall \theta \in [0, 1], \theta' \in \mathcal{B}^*(\theta)$$
$$(\theta - c) \cdot x(\theta) \leq 0, \quad \forall \theta \in \Theta_0 \tag{PP}$$

where $\mathcal{B}^*(\theta) = \{\theta' \in [0, 1] \mid \theta \cdot x(\theta') = \theta \cdot x(\theta)\}$ and $\Theta_0 = \{\theta \in [0, 1] \mid \theta \cdot x(\theta) = 0\}$. Recall that $c \in (0, 1)$.

Proof of Proposition EC.16. We exhaustively go through the following cases and show that the proposition statement holds. As discussed in Appendix E of the main paper, we parametrize single-round direct IC/IR mechanisms in terms of $0 \leq x_0 \leq x_1 \leq 1$ such that $x(0) = x_0$ and $x(\theta) = x_1$ for $\theta \in (0, 1]$.

Case 1) $x_0 = x_1$

In this case, the probabilistic allocation rule x is constant. That is, any report θ leads to the same probability of allocation. Then, any report leads to the same principal and agent utilities and the first part of the PP constraint holds trivially. For the second part, we divide into two subcases depending on whether $x_0 = 0$ or $x_0 > 0$. If $x_0 = 0$, the corresponding single-round mechanism does not allocate at all and the principal and agent utilities will be 0. Hence, the second part of the PP constraint holds. If $x_0 > 0$, then Θ_0 consists of exactly $\theta = 0$ for which the principal utility evaluates to $-c \cdot x_0 < 0$. Again, the second part of the PP constraint holds. It follows that when the allocation rule is constant, the single-round direct IC/IR mechanism satisfies the PP constraint.

Case 2) $x_0 < x_1$

For $\theta \in (0, 1]$, any reports $\theta' \in [0, 1]$ lead to either the same agent utility or a strictly smaller agent utility compared to truthful reporting. Reporting $\theta' \in (0, 1]$ leads to the same allocation

probability and, hence, the same principal and agent utilities. Reporting $\theta' = 0$ leads to the agent utility of $\theta \cdot x_0$ which is less than that of $\theta \cdot x_1$ under truthful reporting. When $\theta = 0$, all reports $\theta' \in [0, 1]$ lead to the agent utility of 0, but can lead to different principal utilities. Reporting $\theta' = 0$ leads to the principal utility of $-c \cdot x_0$ whereas reporting any $\theta' \in (0, 1]$ leads to the principal utility of $-c \cdot x_1$. Since $c > 0$ and $x_0 < x_1$, reporting some $\theta' \in (0, 1]$ leads to the same agent utility but a strictly smaller principal utility compared to truthful reporting. Hence, the PP constraint does not hold.

It follows that only the single-round direct IC/IR mechanisms with $x_0 = x_1$, i.e., with a constant probability allocation rule, satisfy the PP constraint. $\quad\square$