On the Futility of Dynamics in Robust Mechanism Design

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Abstract

We consider a principal who repeatedly interacts with a strategic agent holding private information over T rounds. In each round, the agent observes an idiosyncratic shock drawn independently and identically from a distribution known to the agent but not to the principal. When the principal commits to a dynamic mechanism, the agent best-responds to maximize his aggregate utility over the whole time horizon. The principal's goal is to design a dynamic mechanism to minimize the worst-case regret, that is, the largest difference possible between the aggregate utility he could obtain if he knew the agent's distribution and the actual aggregate utility he obtains. We identify a broad class of games in which the principal's optimal mechanism is static without any meaningful dynamics. More specifically, we show the minimax regret is T times the minimax regret of a single-round problem and an optimal mechanism, if it exists, repeats T times an optimal single-round mechanism in the single-round problem. The class of games includes repeatedly selling identical units of a single good or multiple goods, repeatedly contracting, and repeatedly allocating a resource without money. Outside this class of games, we construct an example in which a dynamic mechanism provably outperforms any static mechanisms. Finally, we show our techniques extend and similar results hold in other related settings.

1 Introduction

Individuals increasingly have repeated interactions with the same online platform. Commuters check the same ride-hailing app each morning, freelancers frequently hunt for short-term work on the same online marketplace, and advertisers bid daily on the same ad exchange. These interactions generate data the platform could use to personalize future offerings. Sometimes, the objectives of the platform and users are aligned – such as when an online music service recommends songs tailored to a user's tastes and the user's experience improves as accurate data is gathered. But often, the incentives of the platform and users are misaligned and repeated interactions become more complicated when a user strategically responds to the platform's strategy. Consider a platform that targets discount coupons at users who appear price-sensitive. This incentivizes loyal price-insensitive customers to mimic those who are not, complicating any inference from past data. Similar concerns arise in online ad exchanges – where, due to ad targeting, a meaningful fraction of auctions contain only a single bidder with a significantly high bid and appropriately setting reserve prices is a key driver of revenue – or online freelancing platforms – where a freelancer might reject an otherwise profitable contract to avoid signaling they are open to working for a low wage in the future.

In such environments, the platform could employ a myriad of dynamic strategies under which the offers available to an individual depend on all the past interactions. How much additional benefit can be derived from such dynamic strategies when an individual is strategic? We use the language of robust mechanism design to formalize a sharp impossibility result. We identify a broad class of problems in which an optimal dynamic mechanism is static and simply repeats a single-round mechanism over and over. In this sense, the platform cannot benefit by using a more complex mechanism with meaningful dynamics, including any schemes that attempt to infer the private information of an individual and exploit this information using, for example, dynamic schemes (Baron and Besanko, 1984; Bakos and Brynjolfsson, 1999; Jackson and Sonnenschein, 2007) that link together outcomes across periods. Intuitively, dynamic mechanisms that adapt based on previous actions can be manipulated by a strategic individual to induce future outcomes that are beneficial for him at the expense of the platform. Therefore, the platform finds it optimal to commit to implementing a static mechanism that does not exploit the individual's private information beyond what is known at the beginning of their repeated interactions.

For these problems, our results could be interpreted as a negative result showing the impossibility of learning and exploiting the private information of a strategic individual. Viewed more positively, these results lead to massive simplification in that static mechanisms are not only robust to strategic manipulations but also optimal, allowing the platform to search over the more tractable space of single-round mechanisms. In addition, these results justify the use of simple mechanisms with substantial practical advantages: static mechanisms are simple to implement and alleviate the need for individuals to engage in complex strategic behavior. Interestingly, for some other problems, it is still possible for the platform to implement a dynamic mechanism and perform strictly better than implementing static mechanisms.

1.1 Contributions

We study a model where a principal and a strategic agent repeatedly play a game over a discrete-time finite horizon of length T. In each round, the agent privately observes an idiosyncratic shock drawn independently and identically from a distribution known to the agent but not to the principal, and the principal and agent interact through the game to realize an outcome and respective utilities. Both parties derive aggregate utility equal to the sum of their utilities across individual rounds. When the principal commits to a dynamic mechanism, the agent is strategic in the sense that he plays a best-response strategy to maximize his aggregate utility. Drawing inspiration from the enormous literature on dynamic learning in non-strategic environments (see, e.g., Kleinberg and Leighton 2003; Besbes and Zeevi 2009), we measure the performance of a dynamic mechanism through its worst-case regret, that is, the largest difference possible between the aggregate utility he could obtain if he knew the agent's distribution and the actual aggregate utility he obtains. The principal's objective is minimax regret and the principal designs a dynamic mechanism to minimize the worst-case regret.

Under a general assumption, we provide false-dynamics results for a broad class of games, showing the principal's optimal mechanism is static without any meaningful dynamics. More specifically, we show the minimax regret is T times the minimax regret of a single-round problem and repeating T times a (near) optimal single-round mechanism from the single-round problem is correspondingly (near) optimal in the multi-round problem. Moreover, an optimal dynamic mechanism exists in the multi-round problem if and only if an optimal single-round mechanism exists in the single-round problem. Our formal analysis relies on leveraging point-mass distributions as worst-case distributions. When restricted to point-mass distributions, we obtain a static information structure where the agent's shock is constant and an optimal dynamic mechanism for the principal is static and repeats a single-round mechanism. Under the general assumption, the optimality of static mechanisms extends to all possible distributions. We explain the general assumption in terms of two opposing effects of shock uncertainty – information asymmetry and trade across shocks – and provide sufficient conditions for the assumption to hold. To the best of our knowledge, the second effect of trade across shocks is novel and may be of independent interests. When the assumption does not hold, we show that static mechanisms are still near-optimal to the extent that it nearly holds. Furthermore, we provide saddle-point theorems in connection to the existence of an optimal dynamic mechanism and relate our robust mechanism design problem to a Bayesian mechanism design problem via saddle-point properties and saddle points. The saddle-point theorems allow interpreting the multi-round and single-round problems as simultaneous-move zero-sum games between the principal who chooses a mechanism and nature who adversarially determines the agent's shock distribution to maximize the principal's regret.

When the general assumption does not hold, it is possible that static mechanisms are not optimal and we show a specific game in which a dynamic mechanism provably outperforms any static mechanisms. For specific applications of our general false-dynamics results, we consider 1) the dynamic selling mechanism design problem where a seller sells independent units of a single or multiple goods sequentially over time to a buyer and maximizes either revenue or welfare; 2) the principal-agent model with hidden costs where a principal has a nonlinear revenue function and repeatedly contracts with an agent to produce at particular output levels, and 3) the repeated resource allocation problem without monetary transfers where a social planner allocates a costly resource in settings where monetary transfers are not allowed. In all these applications, the general assumption holds and an optimal mechanism for the multi-period problem simply repeats an optimal mechanism for the single-round problem. Finally, we extend our results in several directions and discuss connections to other related settings: alternative benchmarks, serially correlated shock processes, a multiplicative performance guarantee, a stronger notion of regret, and the maximin utility objective.

1.2 Related Work

We discuss connections between our work and several streams of literature including Bayesian dynamic mechanism design, robust mechanism design and strategic learning.

Bayesian mechanism design. This stream of literature studies Bayesian mechanism design problems where the principal and agent share a common known prior over the distribution of shocks. Our results are closely related to a recurring phenomenon in Bayesian dynamic mechanism design, known as false dynamics, where optimal mechanisms do not display meaningful dynamics in sequential problems with static information (Laffont and Tirole, 1993; Börgers et al., 2015). This phenomenon was first observed by Baron and Besanko (1984), who considered a continuing relationship between a firm who reports cost information and a regulator who grants a license to operate and specifies quantity of production for the firm. Similar results hold in many other dynamic allocation models either with static or dynamic information, which we overview below. In comparison, our paper is the first to study false-dynamics in a general class of problems with respect to a minimax regret objective. Interestingly, this robust formulation appears to yield qualitatively different results than the classical Bayesian analysis and require substantially different mathematical techniques.

Static information models assume the agent's shock is constant over time, and they can be further refined into static screening and sequential screening models. In static screening models, the agent knows the realization of all shocks before signing the contract and the principal has a Bayesian prior over the distribution of shocks. Baron and Besanko (1984) shows the optimal contract does not exhibit any meaningful dynamics as any effort by the principal to learn the realization of the agent's shock can be exploited by the agent to his advantage. Our work is more aligned with sequential screening models in which the agent does not know the realization of shocks at the point of contracting and the agent's initial private information is his distribution of shocks. The principal does not know the agent's distribution of shocks and instead has a Bayesian prior over this distribution. The principal screens the agent first based on his distribution and then based on the realization of his private shock. Courty and Li (2000) shows that the optimal mechanism is dynamic in the sense that the principal screens the agent over time. The optimality of dynamic contracts holds even under more stringent participation constraints (Krähmer and Strausz, 2015; Bergemann et al., 2017).

In dynamic information models, the agent's shocks change over time. When the principal knows the distribution of shocks and shocks are independent over time, Bakos and Brynjolfsson (1999) shows that the principal can profit by bundling all items. By the law of large number, the distribution of shocks for the bundle tends to concentrate around its mean and the principal can price the bundle to extract all gains from trade. Along similar lines, Kakade et al. (2013) and Pavan et al. (2014) study dynamic information models when the agent has some private payoff-relevant information at the point of contracting. These models display a kind of false dynamics, as optimal mechanisms may determine the outcomes of all future periods in the first round. But, unlike our problem, optimal mechanisms generally depend on the problem's time horizon and can vastly outperform the optimal mechanism for a single period problem.

Several papers study these Bayesian problems with additional constraints on the mechanism under which false-dynamics disappears and there are important benefits to adaptive mechanisms. Krishna et al. (2013), Ashlagi et al. (2016) and Balseiro et al. (2018) study the design of dynamic mechanisms under more stringent liquidity or participation constraints such as restricting an agent to derive a non-negative utility from every period. The mechanisms proposed in these papers are adaptive in nature and outperform the optimal static mechanism. By contrast, our results reveal a much stronger collapse of dynamics for a class of problems with the robust minimax regret objective: an optimal mechanism for a multi-period problem simply repeats the optimal mechanism for a static problem and this remains true even when these participation constraints are imposed.

Robust mechanism design. Although Bayesian models have appealing philosophical foundations, the resulting mechanisms sometimes place impractical requirements on the prior information of the designer. Wilson (1987) argues that mechanisms should not excessively rely on probabilistic assessments on the agents' types. Our work contributes to the robust mechanism design literature that was pioneered by Bergemann and Schlag (2008, 2011). In Bergemann and Schlag (2008), the authors consider the single-round problem where the principal (i.e., seller) sells a good to an agent (i.e., buyer) to minimize the worst-case regret without the knowledge of the agent's distribution. In Bergemann and Schlag (2011), they consider the same problem but with the imperfect information that the agent's true distribution is in some neighborhood of a known model distribution. The latter work considers both the minimax regret objective and a maximin utility objective, where the principal wants to maximize the minimum utility attained among a restricted class of problem instances. When the agent's shock distribution is fixed and known, minimizing expected regret is equivalent to maximizing expected utility. Regret formulations influence nature's objective when it selects worst-case shock distributions, typically leading to more meaningful results than maximin utility formulations.

A recent paper by Carrasco et al. (2015) is perhaps the most closely related work to ours. They show a similar false-dynamics result in an auction setting with respect to the maximin utility objective where the principal knows the mean of the unknown distribution and maximizes the worst-case utility over distributions that are potentially correlated over time. The techniques used in Carrasco et al. (2015) are different and rely on distributions being correlated across time. They bound the worst-case utility of any dynamic mechanisms by considering a worst-case distribution which is perfectly correlated across time and then invoking standard false-dynamics results from the Bayesian literature. Their approach does not readily apply to our minimax regret setting in which distributions are restricted to be independent across time. Our approach is different and relies on considering point masses as worst-case distributions, which are not feasible in the setup of Carrasco et al. (2015) because of their moment constraints. In Section 3.5, we show that their approach can be extended to our setting but leads to results with more restrictive assumptions on the primitives. Furthermore, our results apply to a broader class of games. Finally, in this paper, we uncover a connection between the maximin utility objective with constraints on the mean of the unknown distribution and the minimax regret objective by showing that optimal mechanisms in both problems have similar structures. To the best of our knowledge, there are no other false-dynamics results in dynamics settings with robust objectives.

The maximin utility objective has been further studied considerably in the single-round selling problem without the knowledge of the buyer's distribution but with some moment information. Carrasco et al. (2015) and Kos and Messner (2015) consider a similar setting where the principal (i.e., seller) knows the mean and/or variance of the agent's (i.e., buyer) distribution only. Other known moment-information assumptions include an arbitrary moment in terms of a continuous differentiable function (Carrasco et al., 2015) and first n moments (Carrasco et al., 2018a). Similarly, Pınar and Kızılkale (2017) study a discrete environment where the principal has information about the mean and the upper bound on the agent's valuations and the agent's value can be one of finitely many possible values. For selling multiple goods to one agent, Carrasco et al. (2015) consider a principal who knows the means of the marginal distributions of the agent's valuations for the goods, and Carroll (2017) considers a principal who knows the marginal distributions but not the agent's joint valuation distribution. More recently, Carrasco et al. (2018b) consider a more general problem with the maximin utility objective and nonlinearity in preferences, and Kocyigit et al. (2018) consider a single-round selling problem with the minimax regret objective where the principal sells multiple goods to one agent.

Strategic learning. At least for a special case of our current problem, it is known from prior work that regret must grow linearly with the problem's time horizon (Amin et al., 2013). Such results show that the principal bears a cost of asymmetric information that does not vanish regardless of the length of the time horizon. This formalizes the common folklore that learning about a strategic agent is fundamentally more difficult than learning about a myopic one. However, such results do not speak to the potential benefits of dynamic mechanisms over static ones, which is our main contribution. In addition, it is worth mentioning that the lower bound in Amin et al. (2013) only applies to posted-price mechanisms for dynamic pricing problems. In comparison, this paper provides a general reduction for general dynamic mechanisms and a broader class of games, and, moreover, identifies a sufficient condition on the primitives of the game under which complex dynamic mechanisms offer no benefit over static ones. Nevertheless, a clever lower bound technique in their proofs appears to be useful more broadly and is one key component of our analysis.

Numerous papers in the learning theory literature consider assumptions that enable efficient dynamic learning on the part of the principal. Most notably, positive results are available if the agent is myopic, meaning he always tries to optimize the next interaction without internalizing the consequences his actions have on future interactions (e.g., Kleinberg and Leighton 2003). The same models and techniques apply to learning some feature of a population of agents who each interact with the principal only once and, therefore, are optimally myopic.

Relaxing the assumption that the agent is entirely myopic, Amin et al. (2013, 2014), Mohri and Munoz (2014, 2015) and Golrezaei et al. (2019) consider a setting were the agent is forward-looking but far less patient than the principal, usually modeled through unequal discount factors. Intuitively, this mismatch in time-preferences allows the principal to offer improved payouts in early periods in exchange for information that could be used to exploit the agent in later interactions. Not surprisingly, the performance guarantees obtained by these papers degrade as the difference in the discount factors becomes small. A classic game-theoretic literature on bargaining offers a similar insight about the advantages reaped by more patient players (e.g., Segal 2003). However, these papers assume that the discount factor of the agent is known by the principal. Estimating discount factors in dynamic discrete choice models is challenging and, in general, discount factors cannot be identified from choice data without further restrictions (see, e.g., Rust 1994; Magnac and Thesmar 2002). Many online platforms, moreover, are characterized by short planning horizons and a high frequency of transactions. While both parties might discount future payoffs, the difference in the discount factors can be expected to be small as planning horizons span only for weeks or months.

Another line of work treats auction models where there are multiple agents (i.e., buyers) and agents' values are drawn i.i.d. from a value distributions unknown to the principal (i.e., seller) (e.g., Kanoria and Nazerzadeh (2019)). A key assumption of this line of work is that agents have the same distribution of values. Using a so-called second-price auction with lazy personalized reserves, the principal can decouple the learning problem across agents and use the bids of competitors to optimize the reserve price of each agent independently and in an incentive compatible way. A distinguishing feature of our model is that the principal and agent are placed on a more equal footing. They are both forward-looking, are equally patient (i.e., the same discount factor), and information revealed by the agent could be used to exploit him in future periods (rather than other similar agents).

2 Model

We consider a discrete-time finite horizon setting where a principal and an agent repeatedly play a game over T rounds. The game is a tuple (Ω, Θ, u, v) where Ω is a set of outcomes, Θ is a set of idiosyncratic shocks of the agent, $u : \Theta \times \Omega \to \mathbb{R}$ is the utility function of the principal and $v : \Theta \times \Omega \to \mathbb{R}$ is the utility function of the agent. We assume Ω contains a designated no-interaction outcome denoted by $\emptyset$ for which both the principal and agent obtain utility of 0. The agent has a private shock distribution F over Θ which is known only to him and realizes his private shock $\theta_t \sim F$ which is drawn independently and identically in each round. For simplicity, we assume no discounting but our results would still hold if both the principal and agent have the same discount factor. For example, in the dynamic selling problem where the principal sells independent items to the agent, the shock of the agent is his private willingness-to-pay for an item and the outcome is the allocation of the item and the agent's payment.

We assume the principal commits to a dynamic mechanism which is a tuple $A = (\{\mathcal{M}_t\}_{0:T}, \{\pi_t\}_{1:T})$ where $\mathcal{M}_t$ is the report space and $\pi_t : \mathcal{M}_t \times \mathcal{H}_t \times \mathbb{W} \to \Omega$ is the decision rule for Round t. The decision rule determines the current outcome $\omega_t \in \Omega$ depending on the current report $m_t \in \mathcal{M}_t$ and history $h_t \in \mathcal{H}_t$ of all observable previous reports and realized outcomes and may be randomized in terms of a private random variable w defined over a probability space $(\mathbb{W}, \mathcal{W}, \mathbb{P}_w)$, as in $\pi_t(m_t, h_t, w)$.[1] Let the history available in Round t, denoted h_t, be all reports and outcomes up to Round t. That is, $h_t = (m_{0:t-1}, \omega_{1:t-1})$ where $m_{t'}$ and $\omega_{t'}$ denote the agent's report and realized outcome in Round t', respectively. The set of all possible histories that can be observed in Round t is $\mathcal{H}_t = \prod_{t'=0}^{t-1} \mathcal{M}_{t'} \times \Omega^{t-1}$ for $t \geq 1$. In Round 0, there is no outcome to be determined and there is no decision rule, but report $m_0 \in \mathcal{M}_0$ can affect future decision rules. To allow the agent to not participate in the mechanism, we assume that $\mathcal{M}_0$ contains a special report denoted by QUIT. If the agent reports QUIT in Round 0, he does not participate and the outcome is understood to be the no-interaction outcome over the whole time horizon.

[1] The random variable w and outcome ω have similar-looking notations but we primarily use the outcome ω throughout the paper and seldom use the random variable w explicitly such that there is no confusion.

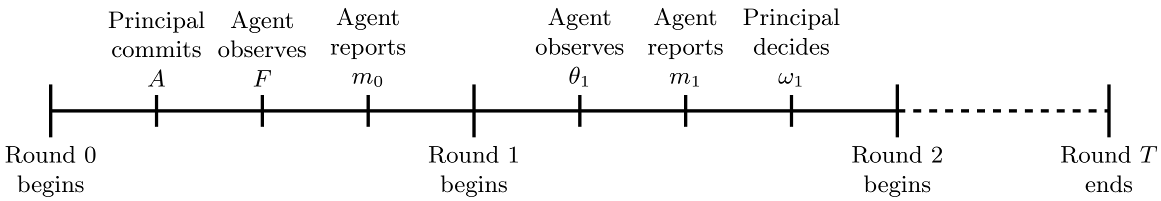

Figure 1: The order of events over the time horizon

For full generality, we consider dynamic mechanisms that specify a report space $\mathcal{M}_t$ in each round.[2] The principal is flexible and can implement any arbitrary dynamic mechanism with per-round decision rules that may be linked across rounds and depend on the history of past interactions. For example, in the dynamic selling problem, the principal may post a reserve price that he adjusts dynamically, bundle the current item and future items together, and/or provide some discount scheme that offers a future item at a low price if the current item is bought at a high price, etc.

The agent's strategy is a sequence $B = \{\sigma_t\}_{0:T}$ where $\sigma_0 : \mathcal{F} \times \mathbb{Y} \to \mathcal{M}_0$ is the agent's strategy for Round 0 that gives a report $m_0 \in \mathcal{M}_0$ given his distribution $F \in \mathcal{F}$ and $\sigma_t : \Theta \times \mathcal{H}_t^+ \times \mathbb{Y} \to \mathcal{M}_t$ is the agent's strategy for Round $t \geq 1$ that gives a report $m_t \in \mathcal{M}_t$ given his realized shock $\theta_t \in \Theta$ and his history $h_t^+ \in \mathcal{H}_t^+$. The agent's strategy can be randomized in terms of a private random variable y over a probability space $(\mathbb{Y}, \mathcal{Y}, \mathbb{P}_y)$. The agent's history in Round t, denoted h_t^+, is the history with the additional information of the distribution of shocks and all previous realized shocks, that is, $h_t^+ = (F, \theta_{1:t-1}, h_t) = (F, \theta_{1:t-1}, m_{0:t-1}, \omega_{1:t-1})$. Correspondingly, let $\mathcal{H}_t^+$ be the set of all possible agent's histories in Round t. Our model assumes an ex-ante participation constraint because the agent can determine whether to participate or not in Round 0 while knowing his distribution but not the realization of future shocks. Figure 1 summarizes the order of events over the time horizon.

Given the principal's dynamic mechanism A, the agent's strategy B and distribution F and time horizon T, the principal's total expected utility is defined as:

$$\text{PrincipalUtility}(A, B, F, T) := \mathbb{E}\left[\sum_{t=1}^{T} u\left(\theta_t, \pi_t(\sigma_t(\theta_t, h_t^+), h_t)\right)\right].$$

[2]Note the report space $\mathcal{M}_t$ can depend on the history and may be different depending on the previous reports and outcomes. For notational simplicity, we assume that $\mathcal{M}_t$ is the same for any history by replacing it with the union of all possible $\mathcal{M}_t$'s attributed with the corresponding histories. Then, in each round, only the subset of the report space that corresponds to the actual history will be considered by both the principal and agent.

Similarly, we can define the agent's total expected utility:

$$\text{AgentUtility}(A, B, F, T) := \mathbb{E}\left[\sum_{t=1}^{T} v\left(\theta_t, \pi_t(\sigma_t(\theta_t, h_t^+), h_t)\right)\right].$$

The above expectations are with respect to the internal randomness of the principal's mechanism A (in terms of w), the agent's strategy B (in terms of y) and the agent's per-round shocks which are drawn from F. The random variables w and y are omitted for notational simplicity and will be omitted when the context is clear.[3] Throughout the paper, the principal and agent utilities will refer to the above expected quantities.[4]

The agent is strategic and will adapt to the principal's choice of A. We denote by $\mathcal{B}(A, F) = \arg\max_{B \in \mathcal{B}} \text{AgentUtility}(A, B, F, T)$ the set of all utility-maximizing strategies for the agent for the dynamic mechanism A and distribution F, where $\mathcal{B}$ is the set of all possible agent strategies. Note that, in principle, the agent could have multiple utility-maximizing strategies and different utility-maximizing strategies can lead to different total expected utilities for the principal. Following standard approaches in the mechanism design literature, we assume the agent is indifferent between utility-maximizing strategies and plays one chosen in the principal's favor. That is, the agent plays a utility-maximizing strategy $B^*(A, F)$ that maximizes the principal's utility among his utility-maximizing strategies in $\mathcal{B}(A, F)$.[5] In other words, the principal would provide a menu (since he does not know the agent's distribution F) of recommended utility-maximizing strategies $B^*(A, \cdot)$ such that $B^*(A, F) \in \mathcal{B}(A, F)$ for all distribution F and the agent will play the recommended strategy corresponding to his actual distribution. In particular, we will show an optimal dynamic mechanism is a direct incentive compatible mechanism and our guarantees will hold if the agent truthfully reports.[6]

[3] Without loss of generality, we assume the random variables w and y are determined before the first round. The random variables can still be arbitrary; for example, the random variables can be a collection of independent random variables drawn separately for individual rounds or a sequence of correlated random variables drawn according to some process. In particular, the random variable w can be a tuple of independent uniform random variables over $[0, 1]$ and the principal's mechanism can use these more generally via the inverse transform sampling method where the transform functions can arbitrarily depend on the history.

[4] The principal utility can be interpreted as some quantitative objective that the principal wants to maximize such as the revenue or welfare in the dynamic selling problem. The principal may not directly observe his utility if it depends on the agent's private shocks, for example, in the welfare maximization.

[5] Note the strategy $B^*(A, F)$ may not exist. For ease of presentation, we assume that the utility-maximizing strategy $B^*(A, F)$ for the agent exists for any dynamic mechanism A and private distribution F. Without the assumption, we can instead reason with sequences of "approximately" utility-maximizing strategies and define AgentUtility and PrincipalUtility using limit superior and inferior.

[6] This is similar to well-established results obtained by use of the revelation principle and phrased in terms of incentive compatibility where the existence of a truthtelling equilibrium is shown for a direct mechanism and the performance guarantees are proved at the truthtelling equilibrium. It is possible there are other equilibria in a given direct mechanism. See Bergemann and Morris (2013) for a discussion on truthful versus full implementations in a different robust mechanism design setting.

Stronger guarantees without the assumption of the principal choosing the agent's utility-maximizing strategy are possible and can be obtained for our results; we defer further discussion to Section A.4 of the electronic companion.

For the agent's distribution F, the principal achieves the following regret for dynamic mechanism A:

$$\mathrm{Regret}(A, F, T) := \mathrm{OPT}(F, T) - \mathrm{PrincipalUtility}(A, B^*(A, F), F, T)\,,$$

where $\mathrm{OPT}(F, T)$ is the optimal performance achievable when the principal knows the agent's private distribution F. The principal's objective is to design a dynamic mechanism A that minimizes the worst-case regret objective. We define the optimal worst-case regret value as the minimax regret value of the multi-round problem:

$$\mathrm{Regret}(T) := \inf_{A \in \mathcal{A}} \sup_{F \in \mathcal{F}} \mathrm{Regret}(A, F, T)\,, \tag{1}$$

where $\mathcal{A}$ is the set of dynamic mechanisms and $\mathcal{F} := \Delta(\Theta)$ is the set of all distributions over Θ. Our robust mechanism design problem can be interpreted as a sequential zero-sum game between the principal who first chooses a dynamic mechanism and nature who then adversarially determines the agent's shock distribution to maximize the principal's regret.

The principal benchmarks his performance against the optimal performance achievable when the agent's distribution is known which we formally define as:

$$\mathrm{OPT}(F, T) := \sup_{A \in \mathcal{A}} \mathrm{PrincipalUtility}(A, B^*(A, F), F, T)\,,$$

where the dynamic mechanism A ranges over all possible dynamic mechanisms $\mathcal{A}$. This is a Bayesian dynamic mechanism design problem in which the principal is informed about the agent's distribution and commits to a dynamic mechanism A and the agent plays a best-response strategy $B^*(A, F)$ chosen in the principal's favor. The optimal mechanism when the distribution is known can be characterized recursively using the promised utility framework pioneered by Green (1987); Spear and Srivastava (1987); Thomas and Worrall (1990). For example, when shocks are independently distributed, $\mathrm{OPT}(F, T)$ can be computed with a dynamic program in which the continuation utility of the agent is used as the state variable. Alternatively, when the number of rounds is large, asymptotically optimal mechanisms can be provided in certain settings (see, e.g., Fudenberg et al. 1994; Jackson and Sonnenschein 2007). Note the regret is defined with respect to the optimal performance achievable in foresight via a dynamic mechanism, but we can also define the regret differently. We consider two alternative regret notions and show our results still hold nevertheless in Section EC.4.1 of the electronic companion.

Single-Round Problem For the special case when $T = 1$, we have the single-round problem where the principal and agent plays the game (Ω, Θ, u, v) for exactly one round. For our analysis, we are interested in the simpler space of single-round direct mechanisms. We denote a randomized single-round direct mechanism by $S : \Theta \to \Delta(\Omega)$, which maps a report $\theta \in \Theta$ to a probability distribution S_θ over outcomes Ω, i.e., $S_\theta \in \Delta(\Omega)$. We denote by $\mathcal{S}$ the set of all randomized direct mechanisms. A randomized direct mechanism $S \in \mathcal{S}$ is incentive compatible (IC) if the agent is better off reporting his shock truthfully, i.e., truthful reporting is a utility-maximizing strategy for the agent:

$$\int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) \geq \int_\Omega v(\theta, \omega) \mathrm{d}S_{\theta'}(\omega) \,, \quad \forall \theta, \theta' \in \Theta \,. \tag{IC}$$

Additionally, the mechanism is individually rational (IR) if the agent's expected utility is non-negative under truthful reporting:

$$\int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) \geq 0 \,, \quad \forall \theta \in \Theta \,. \tag{IR}$$

We introduce a regret notion for the single-round problem in which the agent is assumed to participate and report truthfully under a direct IC/IR mechanism S:

$$\widehat{\mathrm{Regret}}(S, F) := \int_\Theta \mathrm{OPT}(\theta, 1) \mathrm{d}F(\theta) - \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \,,$$

where $\mathrm{OPT}(\theta, 1)$ is equivalently

$$\mathrm{OPT}(\theta, 1) = \sup_{G \in \Delta(\Omega)} \int_\Omega u(\theta, \omega) \mathrm{d}G(\omega) \quad \text{s.t.} \quad \int_\Omega v(\theta, \omega) \mathrm{d}G(\omega) \geq 0 \,.$$

Therefore, the single-round benchmark $\mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)]$ can be thought of as a "first-best" benchmark without IC constraints in which the principal chooses, for each shock, the best possible distribution over outcomes subject to an IR constraint. We define the minimax regret value for the single-round problem when restricted to single-round direct IC/IR mechanisms as follows:

$$\widehat{\text{Regret}} := \inf_{\substack{S\in\mathcal{S}: \\ \text{(IC),(IR)}}} \sup_{F\in\mathcal{F}} \widehat{\text{Regret}}(S,F)\,. \tag{2}$$

Note the above notion and Regret notion are different because truthful reporting may not be a utility-maximizing strategy that also maximizes the principal utility among utility-maximizing strategies for the agent.[7] Furthermore, the IR constraint imposed on single-round direct mechanisms is interim in the sense that the agent knows both his distribution and the realization of the shock before he decides to participate and, therefore, there is one round of reporting (as opposed to the two rounds of reporting and the ex-ante IR constraint in the single-round version with $T = 1$ with respect to the Regret notion). The single-round benchmark $\int_\Theta \text{OPT}(\theta, 1)\mathrm{d}F(\theta)$ is not necessarily equal to the optimal performance achievable in one round, i.e., $\text{OPT}(F, 1)$, for all distributions F, but it is easier to characterize in most cases since $\text{OPT}(\theta, 1)$ is so and will play a critical role in our analysis. Overall, the $\widehat{\text{Regret}}$ notion leads to a more tractable problem than the Regret notion.

Notations To indicate different time horizons when both the multi-round and single-round problems are considered, we use superscripts T and 1, respectively. $\mathcal{A}^T$ is the set of dynamic mechanisms for T rounds and, with some abuse of notations, $\mathcal{A}^1$ is the set of single-round general mechanisms with one round of reporting (i.e., there is no $\mathcal{M}_0$) that may not be direct and have an arbitrary report space. When we refer to a single-round mechanism in this paper, it is in $\mathcal{A}^1$ and has one round of reporting. Similarly, we use A^T to denote a dynamic mechanism for T rounds and A^1 to denote a single-round mechanism. For the set of (randomized) single-round direct mechanisms and a particular such mechanism, we use $\mathcal{S}$ and S, respectively, without a superscript. We also consider dynamic mechanisms that are T repetitions of a single-round mechanism and use $(\cdot)^{\times T}$, and, also, without parentheses, to indicate T repetitions; for example, both $(S)^{\times T}$ and $S^{\times T}$ will be T repetitions of a single-round direct mechanism S. For a dynamic mechanism that repeats a single-round mechanism and a single-round mechanism considered in the multi-round problem (the latter for the $T = 1$ version), the report space $\mathcal{M}_0$ in Round 0 is understood to be $\{\mathsf{CONTINUE}, \mathsf{QUIT}\}$ by default. If the agent reports $\mathsf{CONTINUE}$ in Round 0, the agent participates in all T rounds where the per-round

[7] For example, consider a principal selling a single good to an agent where the agent's value for the good is $\theta = \frac{1}{2}$. Assume the principal's mechanism is to, given the agent's report $\hat{\theta}$, allocate with probability 1 and charge $\frac{1}{2}$ if $\hat{\theta} > \frac{1}{2}$ and not allocate if $\hat{\theta} \le \frac{1}{2}$. Truthful reporting is a utility-maximizing strategy, but reporting some $\hat{\theta} > \frac{1}{2}$ is also a utility-maximizing strategy for the agent that leads to a greater principal utility of $\frac{1}{2}$.

mechanism is the single-round mechanism. If the agent reports QUIT, the agent does not participate and the outcome is the no-interaction outcome over the whole time horizon.

For the set of agent strategies $\mathcal{B}$ and a particular strategy B, we use the superscripts T and 1 and the T-repetition notations in the same manner, with the reporting strategy σ_0 in Round 0 to be specified appropriately in the text. The time horizon of the agent's utility-maximizing strategy will be determined by the first argument; for example, $B^*(A^T, F)$ will be for T rounds and $B^*(A^1, F)$ will be for one round. When the agent's private shock distribution is a point-mass distribution $\{\theta\}$, i.e., the shock is θ with probability 1, we also use θ instead, as in $B^*(S, \theta)$ and $\mathrm{Regret}(A^T, \theta, T)$, for simplicity. For $a < b$, we use $a : b$ to denote the sequence $a, a+1, \ldots, b$ such that $\{\mathcal{M}_t\}_{0:T}$ in the mechanism $A = (\{\mathcal{M}_t\}_{0:T}, \{\pi_t\}_{1:T})$ is the sequence $\mathcal{M}_0, \ldots, \mathcal{M}_T$ and $m_{0:t-1}$ in the history $(m_{0:t-1}, \omega_{1:t-1})$ is the sequence $m_0, \ldots, m_{t-1}$.

3 General Results

In this section, we provide our main results. For a general class of games satisfying a sufficient condition, we determine the minimax regret of the multi-round problem to be T times that of the single-round minimax regret problem and show an optimal dynamic mechanism is static, i.e., it simply repeats a single-round mechanism, without any meaningful dynamics. More specifically, we use a revelation-principle-type argument to lower bound the minimax regret of the multi-round problem in terms of the single-round problem and then show that the lower bound can be achieved arbitrarily closely by repeating a single-round direct IC/IR mechanism. The lower bound can be achieved exactly by repeating, if it exists, an optimal single-round mechanism to the single-round problem. Furthermore, we discuss the sufficient condition in economic terms, how an optimal solution to the single-round problem and, hence, an optimal dynamic mechanism can be determined via a saddle-point result, and connections to a Bayesian mechanism design problem via a multi-round saddle-point result. For space considerations, we defer all proofs to Appendix A.

3.1 Optimality of Static Mechanisms

We prove our results for the general class of games satisfying the following assumption:

Assumption 1. $\sup_{F \in \mathcal{F}} \mathrm{OPT}(F, T) < \infty$, *and* $\mathrm{OPT}(F, T) \leq \mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, T)]$ *for all* $F \in \mathcal{F}$.

The first part of Assumption 1 on the boundedness of $\mathrm{OPT}(F, T)$ is technical and guarantees the benchmarks and minimax regrets are well-defined in both the multi-round and single-round problems.[8] It holds in rather general settings even when the outcome space Ω is not bounded. The second part is more stringent but, as we will show later, holds for all games with either 1) monetary transfers that enter linearly into the utility functions of the principal and agent, or 2) a non-negative utility function for the agent. Assumption 1 holds, in particular, for all applications to be considered in Sections 5–7. The second part is crucial for our false-dynamics results. In Section 4, we construct a simple example in which it does not hold and prove formally that a particular dynamic mechanism outperforms any static mechanisms. In Section 3.3, we further discuss an economic interpretation of Assumption 1 and, also, the near-optimality of static mechanisms to the extent that it holds.

The following theorem shows a complete reduction from the multi-round problem to the single-round problem in terms of their objective values, their (nearly) optimal solutions, and the existence of optimal solutions.

Theorem 1. *Suppose Assumption 1 holds. Then,* $\mathrm{Regret}(T) = T \cdot \widehat{\mathrm{Regret}}$. *For any* $\epsilon \geq 0$, *if a single-round direct IC/IR mechanism* S *satisfies*

$$\sup_{F \in \mathcal{F}} \widehat{\mathrm{Regret}}(S, F) \leq \widehat{\mathrm{Regret}} + \frac{\epsilon}{T}\,,$$

then,

$$\sup_{F \in \mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) \leq \mathrm{Regret}(T) + \epsilon\,.$$

There exists an optimal dynamic mechanism in the multi-round minimax regret problem (1) *if and only if there exists an optimal single-round direct IC/IR mechanism in the single-round minimax regret problem* (2)*.*

The minimax regret in the multi-round problem is T times the minimax regret in the single-round problem. If we solve for a single-round direct IC/IR mechanism that is (nearly) optimal in the

[8] In the multi-round problem, the first part would imply that for any F, $\mathrm{OPT}(F, T)$ is bounded from above. Since the principal can guarantee his utility to be at least 0 by implementing the trivial mechanism that always determines the no-interaction outcome, $\mathrm{OPT}(F, T)$ is also at least 0 and, thus, well-defined. Correspondingly, the minimax regret is bounded from above and below and, thus, well-defined. For an upper bound, consider the trivial mechanism that always determines the no-interaction outcome. Then, the worst-case regret for the principal is $\sup_{F \in \mathcal{F}} \mathrm{OPT}(F, T)$ which is bounded and, hence, $\mathrm{Regret}(T)$ is at most the same quantity. For a lower bound, $\mathrm{Regret}(T) \geq 0$ by the definition of $\mathrm{OPT}(F, T)$. For the single-round problem, Proposition 2, to be presented in Section 3.2, implies $\sup_{F \in \mathcal{F}} \mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, 1)] < \infty$ and a similar reasoning applies. Note, in particular, that the single-round mechanism that always determines the no-interaction outcome satisfies the IC/IR constraints and is a feasible solution in the single-round problem.

single-round problem, then simply repeating this mechanism T times is (nearly) optimal in the multi-round problem. An optimal dynamic mechanism may not exist, that is, the infimum is not achieved, but it always holds unconditionally, by the definition of infimum, that for any $\epsilon > 0$, there exists a single-round direct IC/IR mechanism satisfying the stated condition and, hence, repeating it T times achieves the regret within ϵ of the minimax regret, $\text{Regret}(T)$, in the multi-round problem. We also note the existential equivalence statement further implies that whenever there exists an optimal dynamic mechanism, there also exists a static mechanism that is optimal. We assume no discounting in above results, but when the principal and agent discount future payoffs using the same discount factor $\gamma \in (0, 1)$, the same results would still hold with minimal changes and the minimax regret would be linear in the effective time horizon $T_\gamma := 1 + \gamma + \cdots + \gamma^{T-1}$.

Intuitively speaking, similar to the Bayesian false-dynamics results (see, e.g., Börgers et al. (2015)), a static information structure where the agent's shock is constant arises and leads to false-dynamics results. At the very least, the principal should consider point-mass distributions and can restrict attention to this subset of $\mathcal{F}$. In the Bayesian false-dynamics results, the agent's shock is drawn once from a distribution (known to the principal) and is fixed over the whole time horizon. In our setting, there is no prior distribution known to the principal but we still have a similar static information structure, after restricting to point-mass distributions, in that the possibility of the agent's distribution being any point-mass distribution leads to the consideration that the agent's shock can be any value in the shock space and is fixed over the time horizon. It then suffices for the principal to implement a static mechanism that repeats a single-round mechanism, because the agent does not care about the sequence but only the aggregate distribution of outcomes since his shock is constant and he best-responds only to this extent. The principal can design such aggregate distributions for each possible point-mass distribution via a single-round direct mechanism. Interestingly, under Assumption 1, implementing a static mechanism derived via this reasoning is sufficient for all distributions $\mathcal{F}$. We provide a more complete sketch of the proof in Section 3.2 and further discuss Assumption 1 in Section 3.3. In Section 3.4, we discuss saddle-point theorems in connection to the existence of an optimal dynamic mechanism and, hence, a static one, and in Section 3.5, we explicitly relate our robust mechanism design problem to a Bayesian mechanism design problem via saddle-point properties and saddle points.

Depending on the value of the minimax regret for the single-round problem, different interpretations are possible. If $\widehat{\text{Regret}} = 0$, repeating a single-round mechanism obtains essentially the optimal performance achievable with the knowledge of the agent's private distribution. Then, the distributional information is not necessary and learning/adaptive schemes are not beneficial to begin with and the multi-round problem is easy. On the other hand, if $\widehat{\text{Regret}} > 0$, the minimax regret for the multi-round problem is linear in the time horizon and repeating a single-round mechanism is still a (near) optimal dynamic mechanism. Even when there is potential value of learning from the agent's behavior (that is, we can do strictly better with the knowledge of the agent's distribution), leveraging a learning/adaptive scheme to strictly increase the principal's aggregate utility is impossible. The welfare maximization in the dynamic selling mechanism design problem (Section 5) is of the former kind. The revenue maximization in the same problem, the principal-agent contract model (Section 6) and the dynamic resource allocation problem without monetary transfers (Section 7) are of the latter. We defer further details to respective sections.

Our model as stated assumes an ex-ante participation constraint in the sense that the agent determines whether to participate or not in Round 0 while knowing his distribution but not the realization of future shocks. This participation constraint is standard in the dynamic screening literature (see, e.g., Courty and Li 2000). Our results above readily extend to other, more stringent participation constraints that have been considered in the literature such as the dynamic individual rationality constraint (Kakade et al., 2013; Pavan et al., 2014), ex-post individual rationality constraint (Ashlagi et al., 2016), and periodic individual rationality constraint (Krishna et al., 2013; Balseiro et al., 2018). This is because the minimax regret can be achieved by repeating a single-round direct IC and IR mechanism, which naturally satisfies the latter participation constraints.[9]

3.2 Proof Sketch for Theorem 1

To prove our main theorem, we show a lower bound and an upper bound on the minimax regret of the multi-round problem in terms of the single-round problem. Essentially, the multi-round problem reduces to the single-round problem when the principal restricts to point-mass distributions and this restricting is without loss for the principal under Assumption 1. Both the lower bound and upper bound arguments rely crucially on how the single-round benchmark $\mathbb{E}_{\theta\sim F}[\text{OPT}(\theta, 1)]$ relates to the optimal performance achievable $\text{OPT}(F, T)$ in T rounds. As already mentioned, point-mass distributions will be important and we have the following result on the benchmark. When the distribution is a point-mass and is known to the principal, the agent holds no private information and

[9] Strictly speaking, a single-round direct IC/IR mechanism can be randomized and the ex-post individual rationality constraint might not hold over all random choices of the mechanism (over T rounds since repeated). It would hold if the single-round mechanism is deterministic or satisfies the IR constraint in the ex-post sense. This is the case for all applications considered in Sections 5–7.

the principal can (nearly) attain $\mathrm{OPT}(\theta, T)$ by simply repeating a (near) optimal mechanism that (nearly) attains $\mathrm{OPT}(\theta, 1)$ over T rounds.

Proposition 2. *For any $\theta \in \Theta$, $\mathrm{OPT}(\theta, T) = T \cdot \mathrm{OPT}(\theta, 1)$.*

The following result lower bounds the regret of any dynamic mechanism in terms of the $\widehat{\mathrm{Regret}}$ notion and it directly implies the lower bound on the minimax regret, $\mathrm{Regret}(T) \geq T \cdot \widehat{\mathrm{Regret}}$.

Lemma 3 (Lower Bound)**.** *For any dynamic mechanism A^T, there exists a single-round direct IC/IR mechanism S such that*

$$\sup_{F \in \mathcal{F}} \mathrm{Regret}(A^T, F, T) \geq T \cdot \sup_{F \in \mathcal{F}} \widehat{\mathrm{Regret}}(S, F)\,. \tag{3}$$

To see the lower bound $\mathrm{Regret}(T) \geq T \cdot \widehat{\mathrm{Regret}}$, we first take the infimum over all single-round direct IC/IR mechanisms S on the right-hand side and then take the infimum over all dynamic mechanisms A^T on the left-hand side. To prove the lemma, we use a revelation-principle-type argument to reduce the multi-round problem to the single-round problem. The main idea is that we focus on point-mass distributions and impose structural constraints (the IC/IR constraints) as we effectively shrink the time horizon; this idea also appears in Amin et al. (2013). More specifically, we can construct a single-round direct mechanism S from any dynamic mechanism A^T by letting S_θ for $\theta \in \Theta$ to be the aggregate distribution of expected outcomes when the agent's distribution is the point-mass distribution θ and the agent best-responds. By construction, truthfully reporting a shock θ under S gives the same utilities to both parties, when scaled by T, as implementing the best-response strategy $B^*(A^T, \theta)$ under A^T. Then, the resulting single-round mechanism S is incentive compatible and individually rational because the best-response strategy $B^*(A^T, \theta)$ is optimal for the point-mass distribution θ compared to the best-response strategies corresponding to other point-mass distributions and it guarantees the agent utility of at least 0 in the multi-round problem.

The next result upper bounds the regret of static mechanisms in the multi-round problem under Assumption 1. For any single-round direct IC/IR mechanism, the regret incurred by repeating it T times is no greater than T times its regret in the single-round problem with respect to the $\widehat{\mathrm{Regret}}$ notion.

Lemma 4 (Upper Bound)**.** *Suppose Assumption 1 holds. For every single-round direct IC/IR mech-*

anism S,

$$\sup_{F\in\mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) \le T \cdot \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F)\,. \tag{4}$$

The lemma implies the upper bound on the minimax regret, $\mathrm{Regret}(T) \le T \cdot \widehat{\mathrm{Regret}}$. By the definition of infimum, for any $\epsilon > 0$, there exists a single-round direct IC/IR mechanism S satisfying $\sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F) \le \widehat{\mathrm{Regret}} + \frac{\epsilon}{T}$. Then, by the above lemma, $\sup_{F\in\mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) \le T \cdot \widehat{\mathrm{Regret}} + \epsilon$ and as $\epsilon > 0$ was arbitrary, it follows that $\mathrm{Regret}(T) \le T \cdot \widehat{\mathrm{Regret}}$ because repetitions of single-round mechanisms are a subset of all dynamic mechanisms $\mathcal{A}$. If an optimal single-round solution exists, then the same argument with $\epsilon = 0$ shows that repeating it T times achieves the minimax regret exactly.

For a sketch of the proof of the lemma, we note that when the principal implements $(S)^{\times T}$ for a single-round direct IC/IR mechanism S, the agent always participates and reporting truthfully is a utility-maximizing strategy regardless of his distribution of shocks. More generally, when the principal repeats a single-round mechanism, the agent's best-response strategy is also repeated in the same manner and the individual rounds correspondingly decouple. Then, for any distribution $F \in \mathcal{F}$, each shock independently and identically drawn from it can be thought of as an independent worst-case shock value for each round separately, and the same performance guarantee in terms of the worst-case regret against point-mass distributions extend to that against any distributions under Assumption 1.

Combining Lemmas 3 and 4, we can prove Theorem 1. From the above discussion, we already have $\mathrm{Regret}(T) = T \cdot \widehat{\mathrm{Regret}}$ and the second part. Similarly, we can prove the third part about the existence of optimal mechanisms from these lemmas. For a complete proof, we refer to Appendix A.1. We refer to Appendix A.2 for proofs of Lemmas 3 and 4 and to Appendix A.2.3 for that of Proposition 2.

To conclude, we note restricting to point-mass distributions is reasonable in hindsight because point-mass distributions happen to be the right class of "worst-case" distributions in that for any dynamic mechanism, there exists a point-mass distribution for the agent against which the dynamic mechanism is forced to obtain a regret at least the minimax regret. See the following proposition; its proof is provided in Appendix A.2.3.

Proposition 5. *Suppose Assumption 1 holds. For any dynamic mechanism A^T and $\epsilon > 0$, there exists a point-mass distribution θ such that* $\mathrm{Regret}(A^T, \theta, T) \ge \mathrm{Regret}(T) - \epsilon$.

3.3 On Assumption 1

In this subsection, we provide an economic interpretation of the second part of Assumption 1, provide sufficient conditions for this part to hold, and show that static mechanisms are still near-optimal to the extent that the gap $\mathrm{OPT}(F,T) - \mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta,T)]$ is small for all $F \in \mathcal{F}$ when it does not hold. Due to space considerations, all proofs will be deferred to Appendix A.4.

The second part of Assumption 1 imposes conditions on the optimal performance achievable $\mathrm{OPT}(F,T)$ which is the objective value of the Bayesian dynamic mechanism problem where the principal knows the agent's distribution F but not the per-round shocks. Intuitively, the inequality states that, for every distribution of shocks, the principal prefers to face a random agent with known, constant shocks drawn from the distribution than a single agent with uncertain shocks drawn i.i.d. from the same distribution. To connect back to our robust mechanism design problem, the principal can use $\mathrm{OPT}(F,T)$ to obtain insights on how to design a dynamic mechanism and which distributions are challenging. It then makes sense for the principal to account for the point-mass distributions to minimize his regret since the principal preferring to face a random agent with constant shocks implies $\mathrm{OPT}(F,T)$ will be maximal for point-mass distributions. Furthermore, his performance against point-mass distributions will likely carry over to all possible distributions because each shock drawn i.i.d. can be treated as a separate worst-case shock value in each round. Whether the inequality holds or not depends on the effects of shock uncertainty on $\mathrm{OPT}(F,T)$.

Shock uncertainty impacts the second part of Assumption 1 through two opposing effects: information asymmetry and trade across shocks. The first effect is related to the incentive compatibility constraints and is a negative effect, i.e., the higher the shock uncertainty, the worse for the principal. Uncertainty in the shocks leads to larger information asymmetry between the principal and agent. To elicit the agent's private information, the principal needs to concede information rents to the agent. Informationally, the principal thus prefers to face random agents with known, constant shocks. The second effect is related to the individual rationality constraints and is a positive effect, i.e., the higher the shock uncertainty, the better for the principal. The principal can exploit the randomness of the shocks to implement a broader set of outcomes. This is achieved by offsetting a shock-outcome pair that is only favorable to the principal with another pair that is favorable to the agent but not too unfavorable to the principal. Therefore, by trading across rounds in which the realizations of shocks are different, the principal can achieve higher utilities.

Because these effects are opposing, whether the inequality holds or not depends on which effect dominates. When the first effect dominates, it is reasonable that the principal prefers to face a random agent with known, constant shocks drawn from the distribution. In particular, the second effect disappears and the first effect dominates by default in games with money because the principal can use payments to balance utilities on the per-round-per-shock basis and, naturally, in games in which the individual rationality constraint does not bind; see Proposition 8 below. Note we can interpret the two effects similarly if we write the right-hand side of the inequality as $T \cdot \mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta, 1)]$ instead. Since the game is repeated, $\mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta, T)] = T \cdot \mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta, 1)]$ for all distribution $F \in \mathcal{F}$. The modified right-hand side is what the principal achieves in the full information setting with restriction on trade across shocks since the IR constraint is for each shock in each round and he has to decide on the outcomes on the per-round-per-shock basis. But, writing the right-hand side as $\mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta, T)]$ shows more directly why it is sufficient for the principal to restrict attention to point-mass distributions in our setting.[10]

We next present simple sufficient conditions for the second part of Assumption 1 by focusing on a related single-round benchmark with an ex-ante IR constraint, which allows to isolate the trade-across-shocks effect. We define

$$\begin{aligned}\bar{u}(F) := \sup_{S\in\mathcal{S}} &\int_\Theta \int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta) \\ \text{s.t.} &\int_\Theta \int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta) \geq 0\,,\end{aligned}$$

which can be thought of as the "first-best" benchmark with an ex-ante IR constraint and without the IC constraint. It is related to the optimal performance achievable $\mathrm{OPT}(F, T)$ as follows:

Proposition 6. *We have the following relations:*

1. *For any distribution $F \in \mathcal{F}$, $\mathrm{OPT}(F, T) \leq T \cdot \bar{u}(F)$.*
2. *For any point-mass distribution $\theta \in \Theta$, $\mathrm{OPT}(\theta, T) = T \cdot \bar{u}(\theta)$.*
3. *For any distribution $F \in \mathcal{F}$, $\bar{u}(F) \geq \mathbb{E}_{\theta\sim F}[\bar{u}(\theta)]$.*

[10] The difference between the right- and left-hand sides seems similar to the decision-theoretic notion of expected value of perfect information (EVPI) which measures the price a decision maker is willing to pay to gain access to perfect information and is always non-negative. However, the shocks are drawn independently in every round in the definition of EVPI instead of being drawn once at the beginning and remaining constant throughout the rounds in the second part of Assumption 1. If we write the right-hand side as $T \cdot \mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta, 1)]$, it is still different because while the shocks are drawn independently in every round, the IR constraint is now imposed for each shock and each round.

In particular, $\mathrm{OPT}(\theta, 1) = \bar{u}(\theta)$ for any $\theta \in \Theta$ and the benchmark in the single-round problem can be equivalently written as $\mathbb{E}_{\theta\sim F}[\bar{u}(\theta)]$. Furthermore, $\bar{u}(F)$ being bounded and linear in F implies Assumption 1:

Proposition 7. *If $\sup_{F\in\mathcal{F}} \bar{u}(F) < \infty$, then the first part of Assumption 1 holds. If $\bar{u}(F) = \mathbb{E}_{\theta\sim F}[\bar{u}(\theta)]$ for all $F \in \mathcal{F}$, then the second part of Assumption 1 holds.*

Note that the inequality $\bar{u}(F) \geq \mathbb{E}_{\theta\sim F}[\bar{u}(\theta)]$ always holds and the linearity condition on $\bar{u}(F)$ is satisfied when equality holds. Put concisely, the linearity condition is a statement of the optimality of mechanisms without trade across shocks in the full information version of the multi-round problem. Note $T \cdot \bar{u}(F)$ is the optimal performance achievable under the full information where the principal knows the agent's distribution F and also observes the agent's per-round shocks. When the linearity condition holds, we have $T \cdot \bar{u}(F) = T \cdot \mathbb{E}_{\theta\sim F}[\bar{u}(\theta)]$ for any distribution $F \in \mathcal{F}$ and the right-hand side can be interpreted as the performance achievable when the principal designs an outcome distribution for each shock in each round separately subject to the IR constraint restricted to that shock in that round. Equivalently, the principal can design an optimal mechanism by focusing on each shock in each round independently without linking decisions across rounds and across shocks within a round.[11]

When the linearity condition holds, the trade-across-shocks effect is not in effect in the full-information setting and, by Proposition 7, in the Bayesian setting. The next proposition shows the linearity condition holds for games where payments enter linearly into the utility functions of the principal and agent and for games where the utility function of the agent is always nonnegative. In particular, it covers all applications to be considered in Sections 5–7.

Proposition 8. *Assume the game is such that either 1) the outcome space factorizes as $\Omega = \Omega^0 \times \mathbb{R}$ and the utility functions can be written $u(\theta, (\omega^0, p)) = u^0(\theta, \omega^0) + \alpha p$ and $v(\theta, (\omega^0, p)) = v^0(\theta, \omega^0) - \beta p$ for all outcomes $(\omega^0, p) \in \Omega^0 \times \mathbb{R}$ for some functions $u^0 : \Theta \times \Omega^0 \to \mathbb{R}$ and $v^0 : \Theta \times \Omega^0 \to \mathbb{R}$ and scalars $\alpha \geq 0$ and $\beta > 0$, or 2) $v(\theta, \omega) \geq 0$ for all $\theta \in \Theta$ and $\omega \in \Omega$. Then, $\bar{u}(F) = \mathbb{E}_{\theta\sim F}[\bar{u}(\theta)]$ for all $F \in \mathcal{F}$ and, therefore, the second part of Assumption 1 holds.*

When Assumption 1 does not hold, static mechanisms may no longer be optimal. In Section 4, we provide a specific example and show a dynamic mechanism outperforms static mechanisms. But, we can

[11] Interestingly, this multi-round consideration can be equivalently formulated in the single-round setting and the linearity condition can be stated with $T = 1$. This is because our setting is a repeated i.i.d. setting where shocks are drawn independently and identically from a distribution and the decisions linked across rounds in the multi-round setting can be equivalently realized as decisions linked across shocks in the single-round setting.

still show static mechanisms are near-optimal to the extent that the gap $\mathrm{OPT}(F,T)-\mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta,T)]$ is small for all $F\in\mathcal{F}$. Intuitively speaking, restricting to point-mass distributions might still be a good idea for the principal when the gap is small. The following theorem effectively generalizes Theorem 1.

Theorem 9. *Assume* $\mathrm{OPT}(F,T)<\infty$ *for all* $F\in\mathcal{F}$*. For any* $\epsilon\geq 0$*, if a single-round direct IC/IR mechanism* S *satisfies*

$$\sup_{F\in\mathcal{F}}\widehat{\mathrm{Regret}}(S,F)\leq \inf_{\substack{S'\in\mathcal{S}:\\ \text{(IC),(IR)}}}\sup_{F\in\mathcal{F}}\widehat{\mathrm{Regret}}(S',F)+\frac{\epsilon}{T}\,,$$

then,

$$\sup_{F\in\mathcal{F}}\mathrm{Regret}(S^{\times T},F,T)-\mathrm{Regret}(T)\leq \sup_{F\in\mathcal{F}}\left\{\mathrm{OPT}(F,T)-\mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta,T)]\right\}+\epsilon\,.$$

3.4 On the Existence of an Optimal Dynamic Mechanism and Saddle-Point Theorems

Theorem 1 shows that the principal can obtain a regret arbitrarily close to the minimax regret by repeating a single-round mechanism and exactly the minimax regret by repeating an optimal solution to the single-round minimax regret problem if it exists. Furthermore, whenever there exists an optimal dynamic mechanism in the multi-round problem, an optimal solution to the single-round problem exists and, in turn, an optimal dynamic mechanism that is static (i.e., repeats a single-round mechanism) exists in the multi-round problem. Hence, in light of Theorem 1, it suffices that we directly solve for an optimal single-round mechanism for the single-round problem and this is our approach via saddle-point results in the specific applications considered in Sections 5–7.

Often, we can cast the single-round problem as a simultaneous-move zero-sum game between the principal and nature and show it admits an optimal solution via a saddle-point result that says 1) the saddle-point property (or, the duality gap of 0) holds, i.e.,

$$\inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}}\sup_{F\in\mathcal{F}}\widehat{\mathrm{Regret}}(S,F)=\sup_{F\in\mathcal{F}}\inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}}\widehat{\mathrm{Regret}}(S,F)\,,$$

and 2) there exists a saddle point (S^*,F^*), which is a single-round direct IC/IR mechanism S^* and a distribution F^* such that

$$\widehat{\text{Regret}}(S^*, F) \leq \widehat{\text{Regret}}(S^*, F^*) \leq \widehat{\text{Regret}}(S, F^*)$$

for any $S \in \mathcal{S}$ satisfying the IC/IR constraints and $F \in \mathcal{F}$. Note that 2) implies 1) but does not necessarily hold when 1) does. If a saddle point or, equivalently, a Nash equilibrium exists, the minimax regret of the single-round problem is exactly the value of the zero-sum game, i.e., $\widehat{\text{Regret}} = \widehat{\text{Regret}}(S^*, F^*)$, and S^* is an optimal single-round solution.

A saddle-point theorem can be shown to hold either from first principles or topologically. The former approach is constructive and it involves exploiting the structure of the particular application at hand to explicitly derive a saddle point. In the dynamic selling problem (Section 5) and the principal-agent model with hidden costs (Section 6), we derive a saddle point in this way and this yields an explicit characterization of an optimal dynamic mechanism that simply repeats a single-round mechanism. In the resource allocation problem without monetary transfers (Section 7), saddle points do not exist. Instead, we show an "asymmetric" saddle-point result where only the saddle-point property holds and one of the minimax and maximin regret formulations admits an optimal solution. It happens we can find an optimal solution to the single-round minimax regret problem and, thus, an optimal dynamic mechanism.

The latter approach is non-constructive and it involves imposing some topological structure on the game and then leveraging existing minimax theorems from the literature. However, establishing a saddle-point result for the single-round problem in full generality is challenging, and the resource allocation problem without monetary transfers hints at the difficulty, because saddle points do not exist even for this simple game. We can still establish a general saddle-point result under general conditions and the assumption of a finite shock space, and an asymmetric saddle-point result when the game has a continuous shock space. Due to space considerations, we state and prove these results in Section EC.2 of the electronic companion.

3.5 A Multi-Round Saddle-Point Theorem and Connections to Bayesian Mechanism Design

In the current formulation, the multi-round problem does not admit a saddle-point result in that there is no worst-case distribution for the agent that is uniformly challenging for all possible dynamic mechanisms. This is because for each possible distribution F, there are mechanisms tailored to the distribution that achieve $\mathrm{OPT}(F,T)$ arbitrarily closely and, hence, incur regret arbitrarily close to 0. Interestingly, a saddle-point result can be recovered if nature is allowed to use mixed strategies and randomize over distributions in $\mathcal{F}$, and this leads to a Bayesian mechanism design interpretation of the multi-round problem in which the principal has a Bayesian prior over the space of possible distributions.

Thus motivated, we introduce the space $\Delta(\mathcal{F})$ of all possible distributions over distributions, i.e., Bayesian priors, and the following equivalent formulation of the multi-round problem:

$$\inf_{A\in\mathcal{A}} \sup_{G\in\Delta(\mathcal{F})} \mathbb{E}_{F\sim G}\left[\mathrm{Regret}(A,F,T)\right] .$$

This is equivalent to the original formulation, $\inf_{A\in\mathcal{A}}\sup_{F\in\mathcal{F}}\mathrm{Regret}(A,F,T)$, because nature cannot benefit from randomizing over distributions. Since the space $\mathcal{F}$ is convex, an additional layer of randomization can be compounded by the principal into a single, payoff-equivalent distribution over shocks. The corresponding dual problem is

$$\sup_{G\in\Delta(\mathcal{F})} \inf_{A\in\mathcal{A}} \mathbb{E}_{F\sim G}\left[\mathrm{Regret}(A,F,T)\right] ,$$

where the inner infimum of the maximin regret can be equivalently written as the difference

$$\mathbb{E}_{F\sim G}[\mathrm{OPT}(F,T)] - \sup_{A\in\mathcal{A}} \mathbb{E}_{F\sim G}[\mathrm{PrincipalUtility}(A,B^*(A,F),F,T)] ,$$

where the first quantity is the optimal performance achievable when F is known averaged according to G and the second quantity is the principal utility under an optimal Bayesian mechanism where the principal knows F is drawn from the known prior G. In other words, in the maximin regret formulation, nature adversarially chooses a Bayesian prior over $\mathcal{F}$ and then the principal responds by choosing a Bayesian optimal mechanism based on the prior.

The next result shows connections between the modified formulation of the multi-round problem and the single-round problem in terms of the saddle-point properties and saddle points.

Theorem 10. *Suppose Assumption 1 holds. The saddle-point property holds for the single-round minimax regret problem if and only if the saddle-point property holds for the multi-round minimax regret problem. Moreover, if a saddle point (S^*,F^*) exists in the single-round problem, then the repeated mechanism $(S^*)^{\times T}$ and the distribution over distributions G^* that assigns probability $F^*(\theta)$*

$u(\cdot,\cdot)$, $v(\cdot,\cdot)$	$\emptyset$	ω^1	ω^2
θ^1	0,0	1, -1	0,$-\infty$
θ^2	0,0	1,$-\infty$	0, 1

Table 1: A game (Ω, Θ, u, v) with outcome space $\Omega = \{\emptyset, \omega^1, \omega^2\}$ (with the no-interaction outcome $\emptyset$), shock space $\Theta = \{\theta^1, \theta^2\}$, and utility functions of the principal u and agent v in matrix representation.

to point-mass distribution θ for all $\theta \in \Theta$ form a saddle point $((S^*)^{\times T}, G^*)$ in the multi-round problem. Conversely, if a saddle point exists in the multi-round problem, then a saddle point exists in the single-round problem (see the proof for a construction).

This means that we can similarly cast our multi-round problem as a simultaneous-move zero-sum game between the principal and nature and solve for a Nash equilibrium, assuming a saddle-point result holds for the single-round problem. More importantly, the above theorem shows that the multi-round minimax regret problem reduces to the Bayesian mechanism design problem where the objective is the expected regret and the principal implements a Bayesian optimal mechanism against a worst-case prior, that is, a distribution over distributions. To prove the result, we use our general results from Theorem 1, a classical result in the analysis of online algorithms known as Yao's principle (Yao, 1977), and an extension of the classical false-dynamics results of Baron and Besanko (1984) in a Bayesian setting. We prove it in Appendix A.5.

4 A Game with Effective Dynamics

We show a game for which Assumption 1 does not hold and our general result, Theorem 1, does not apply. For this game, we prove that repeating a single-round mechanism leads to a linear minimax regret but that implementing a dynamic mechanism leads to a sublinear minimax regret. The performance gap shows that dynamic or adaptive schemes can effectively take advantage of the history of past outcomes and reports by the agent when Assumption 1 does not hold.

Consider the game in Table 1. For a real-life example, the game can model repeated interactions between a farm owner (the principal) and a volunteer (the agent). On each day, the three possible outcomes are no assignment ($\emptyset$) which means the volunteer is not called in and stays in his home; assignment to a hard task (ω^1) which means the volunteer works on certain chores that need a lot of effort at the farm; and assignment to an easy task (ω^2) which means the volunteer works on other kinds of chores that are less strenuous. A hard task can be cleaning stalls and transporting manure that the farm owner actually cares about and hopes that the volunteer would do. An easy task can be feeding and playing with farm animals that the volunteer is more excited about. The shocks θ^1 and θ^2 may correspond to the agent's mood or sense of responsibility on each day that makes him prefer one type of task over the other and his distribution F may correspond to his inherent inclination with regards to these tasks. A contractual agreement would be formed between the two parties when the farm owner commits to a particular work schedule and the volunteer participates.

For this game, the optimal performance achievable is $\mathrm{OPT}(F,T) = T \cdot \bar{u}(F)$. Let $F = (f_1, f_2)$ be the agent's private distribution over Θ where the shock is θ^i with probability f_i for $i = 1, 2$. Note for distribution F, the single-round ex-ante benchmark is

$$\begin{aligned}\bar{u}(F) = \max_{x\in[0,1]} \quad & f_1 \cdot x \\ \text{s.t.} \quad & -f_1 \cdot x + f_2 \cdot 1 \geq 0\,.\end{aligned}$$

To see this, let S be a single-round direct mechanism in the optimization problem defining $\bar{u}(F)$ such that each S_{θ^i} is a distribution over Θ with probabilities given by $S_{\theta^i}(\emptyset), S_{\theta^i}(\omega^1), S_{\theta^i}(\omega^2)$ and $x = S_{\theta^1}(\omega^1)$ ($y = S_{\theta^2}(\omega^1)$) be the probability that outcome ω^1 is selected when the shock is θ^1 (θ^2). On the one hand, the mechanism maximizes $\mathbb{E}_{\theta\sim F,\omega\sim S_\theta}[u(\theta,\omega)] = f_1 \cdot x + f_2 \cdot y$ if S_{θ^1} and S_{θ^2} place as much probability mass as possible on outcome ω^1. We assume $y = 0$ without loss because if $f_2 > 0$, only $y = 0$ is feasible because the agent utility will be $-\infty$ and the ex-ante IR constraint will be violated if $y > 0$, and if $f_2 = 0$, y can be any value and we can set $y = 0$ without affecting the principal utility. On the other hand, setting x too large might violate the ex-ante IR constraint because outcome ω^1 gives the utility of -1 to the agent when the shock is θ^1. Setting $S_{\theta^2}(\omega^2) = 1$ allows for higher values of x at no cost to the principal. Since $f_1 + f_2 = 1$, we see that the ex-ante IR constraint only binds when $f_1 \geq \frac{1}{2}$. Therefore, the optimal solution is $x = 1$ when $f_1 < \frac{1}{2}$ and $x = \frac{1-f_1}{f_1} \leq 1$ when $f_1 \geq \frac{1}{2}$, or, more succinctly, $x = \min\left\{1, \frac{1-f_1}{f_1}\right\}$. This implies

$$\bar{u}(F) = \min\{f_1, 1-f_1\} = \begin{cases} f_1\,, & \text{if } f_1 < \frac{1}{2}\,, \\ 1-f_1\,, & \text{if } f_1 \geq \frac{1}{2}\,. \end{cases}$$

For point-mass distributions, we have $\bar{u}(\theta^1) = \bar{u}(\theta^2) = 0$. While the first part of Assumption 1 holds, the second part does not hold. For example, when $F = (\frac{1}{2}, \frac{1}{2})$, we have $\bar{u}(F) = \frac{1}{2}$ but $\mathbb{E}_{\theta\sim F}[\bar{u}(\theta)] = f_1 \cdot \bar{u}(\theta^1) + f_2 \cdot \bar{u}(\theta^2) = 0$; that is, $\mathrm{OPT}(F,T) > \mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta,T)]$.

Intuitively, because of the trade-across-shocks effect, the distributional information is important for the principal to maximize his utility in this game. The only way for the principal to generate a strictly positive utility is deciding on the outcome ω^1 which causes a negative utility to the agent. To guarantee the agent's overall utility to be nonnegative and ensure the agent is, at least, indifferent between participating and not participating, the principal also needs to decide on the outcome ω^2 when the agent's private shock is θ^2. The principal would need to balance between these two decisions and this is where knowing the distribution F can be helpful. In particular, the trade-across-shocks effect disappears when the distribution is a point mass.

The following proposition shows a gap between what is achievable by repeating a single-round mechanism versus by implementing a dynamic mechanism with some meaningful dynamics.

Proposition 11. *For the game in Table 1, a separation exists:*

1. *For any T, the minimax regret of static mechanisms that repeat single-round mechanisms is $(1-\sqrt{2}/2)\cdot T$, i.e.,*

$$\inf_{A^1\in\mathcal{A}^1}\sup_{F\in\mathcal{F}}\text{Regret}((A^1)^{\times T},F,T)=\left(1-\frac{\sqrt{2}}{2}\right)\cdot T\approx 0.292893\cdot T\,.$$

2. *For any T, there exists a dynamic mechanism A^T such that*

$$\sup_{F\in\mathcal{F}}\text{Regret}(A^T,F,T)=O\left(\ln(T)^{1/2}T^{2/3}\right)\,.$$

The first part of the proposition shows that any repetition of a single-round mechanism necessarily incurs a linear regret. We prove it by characterizing an optimal static mechanism that repeats a single-round mechanism. First, we show that outcome probabilities under an equilibrium (obtained by composing the mechanism with the agent's best-response strategy) are the same for all shock distributions for which the agent decides to participate. This follows because the mechanism is static (i.e., it does not explicitly screen the agent for his distribution) and the shock distribution is not payoff relevant. We can characterize that the agent participates when his shock probability f_1 is lower than a fixed threshold $\hat{f}_1$ and optimize over the threshold to minimize the regret.

For the second part of the proposition, we design a dynamic mechanism A^T with two phases. In the first phase, the principal implements some default mechanism that places positive probabilities on outcomes ω^1 and ω^2 when the reported shocks are θ^1 and θ^2, respectively, and induces the agent to truthfully report his per-round shocks, as the present disutility from misreporting overwhelms any potential future gains. Then in the second phase, the principal infers the agent's distribution F from the reports in the first phase and implements a better-tuned mechanism which is a version of the optimal ex-ante mechanism that is perturbed to account for the statistical errors introduced by inferring F with limited samples. Using standard concentration inequalities, we can balance the loss of offering suboptimal mechanisms in the first and second phases. By choosing the number of rounds in the first phase to grow sublinearly relative to the time horizon, we can show that the dynamic mechanism incurs a sublinear regret. We provide the proof of Proposition 11 and further details in Appendix B.

While we do not provide details due to the space consideration, 1) we can still show a separation between static mechanisms and dynamics mechanisms when the utility functions of the principal and agent are bounded, i.e., the entries in the matrices in Table 1 are bounded; and 2) we can show the class of sequential screening mechanisms, which ask the agent to report his distribution and then implement a static mechanism based on the screened information, performs better than the class of (naive) static mechanisms considered above but still obtains the minimax regret that is linear in the time horizon (approximately, $0.217812 \cdot T$). Interestingly, our general results in Section 3 imply that sequential screening mechanisms offer no benefits over static mechanisms when Assumption 1 holds.

For comparison, we consider the game in Table 1 coupled with payments for which Propositions 7 and 8 then apply to show Assumption 1 holds. Let the modified outcome space be $\Omega' = \Omega \times \mathbb{R}$ and an outcome is a pair $(\hat{\omega}, \hat{p}) \in \Omega'$. The principal's utility function is $u'(\theta, (\hat{\omega}, \hat{p})) = u(\theta, \hat{\omega}) + \hat{p}$ and the agent's utility function is $v'(\theta, (\hat{\omega}, \hat{p})) = v(\omega, \hat{\omega}) - \hat{p}$. For the modified game with payments, note that $\bar{u}(\theta^1) = 0$ which is achieved when the outcome is $(\omega^1, -1)$ with probability 1 and $\bar{u}(\theta^2) = 1$ which is achieved when the outcome is $(\omega^2, 1)$ with probability 1. Our false-dynamics results apply and the gap between static mechanisms and dynamic mechanisms disappears.

Proposition 12. *For the modified game $(\Omega', \Theta, u', v')$ with payments, the minimax regret is 0 and an optimal mechanism repeats a direct IC/IR mechanism that determines the outcome $(\omega^1, -1)$ with probability 1 for the report θ^1 and the outcome $(\omega^2, 1)$ with probability 1 for the report θ^2.*

With monetary transfers available, the principal can be more flexible in designing a dynamic mechanism but the optimal performance achievable $\mathrm{OPT}(F, T)$ under the knowledge of the agent's distribution, which is his benchmark, can also improve. Nevertheless, there is no longer a gap between static mechanisms and dynamic mechanisms and a static mechanism is optimal. Going back to the real-life example of a farm owner and a volunteer, it is interesting to note that the way the farm owner generates his utility changes when payments are allowed. In the original game, the farm owner has to convince the volunteer to accept ω^1 when his shock is θ^1 with promises that he will be given ω^2 when his shock is θ^2. The farm owner has to learn the volunteer's distribution F over time and adjust so that he obtains some utility while the volunteer is still incentivized to participate. In contrast, in the modified game, the farm owner generates his utility by extracting surplus from the volunteer when his shock is θ^2. The farm owner treats the volunteer like a worker and compensates him accordingly when his shock is θ^1 and like a visitor and charges for feeding and playing with farm animals when his shock is θ^2.[12] See Appendix B.4 for the proof of Proposition 12.

5 Dynamic Selling Mechanism

We apply the general results from Section 3 in the dynamic selling mechanism design problem for revenue and welfare maximization in both the single-good case (selling one good in each round) and the multiple-goods case (selling n goods in each round), where the agent's shock is multidimensional in the latter. Since monetary transfers are allowed and utilities are quasilinear, Proposition 8 implies that the second part of Assumption 1 holds and Theorem 1 applies as long as $\sup_{F\in\mathcal{F}} \mathrm{OPT}(F,T) < \infty$. In each version of the dynamic selling problem in terms of the objective (revenue vs. welfare) and number of goods (single vs. multiple), we explicitly determine an optimal single-round solution and, hence, the optimal dynamic mechanism that repeats it over T rounds. The objectives of revenue maximization and welfare maximization differ only in the principal's utility function, but we find the minimax regrets of the respective single-round problems to be greater than 0 in the former and exactly 0 in the latter and, consequently, have different interpretations. Due to space considerations, we consider revenue maximization in the single-good case in this section and defer other variants to Appendices EC.3.1 and EC.3.2 of the electronic companion.

Consider a repeated setting where the principal (i.e., seller) sells independent and identical items to a strategic agent (i.e., buyer) over T rounds and seeks to maximize the revenue; this is the single-good case. The items are being sold one by one sequentially and in each round, the agent realizes his value

[12] The same farm example works if the farm owner pays the volunteer a fixed daily wage and the agent's utility in the game in Table 1 is after accounting for the fixed wage. In the modified game, the farm owner is more flexible and can now adjust the wage on the daily basis. The comparison between these games is more about the flexibility via monetary transfers than about the presence of monetary transfers. In the storyline presented, the fixed wage can be thought to have been normalized to 0.

(equivalently, his willingness to pay) for the current item which is drawn from an underlying private distribution known only to him. The principal does not know the agent's private value distribution except that the agent's value is in the range $[0, 1]$.

In the language of the general model, the agent's shock is his private value for the item and the shock space is $\Theta = [0, 1]$. The outcome space is $\Omega = \{0, 1\} \times \mathbb{R}$ and an outcome is $\omega = (\hat{x}, \hat{p}) \in \Omega$ where $\hat{x}$ is the allocation and $\hat{p}$ is the payment, i.e., whether or not the item is allocated to the agent and the payment the agent makes to the principal. Given an outcome $\omega = (\hat{x}, \hat{p})$, the agent's utility function is $v(\theta, \omega) = \theta \cdot \hat{x} - \hat{p}$ and the principal's utility function is $u(\theta, \omega) = \hat{p}$. Abusing notations, we break down a decision rule π_t in the principal's mechanism $A = (\{\mathcal{M}_t\}_{0:T}, \{\pi_t\}_{1:T})$ into corresponding allocation rule $x_t : \mathcal{M}_t \times \mathcal{H}_t \times \mathbb{W} \to \{0, 1\}$ and payment rule $p_t : \mathcal{M}_t \times \mathcal{H}_t \times \mathbb{W} \to \mathbb{R}$. We do similarly for single-round direct IC/IR mechanisms. Furthermore, we use $x(\cdot)$ and $p(\cdot)$ to denote the interim rules and the pair (x, p) to represent a single-round mechanism when convenient.

Note that $\mathrm{OPT}(\theta, 1) = \theta$ for all $\theta \in \Theta$ which is because the principal can extract the full surplus of the agent and satisfy the IR constraint by charging the agent's value. Then, the single-round benchmark is $\mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta, 1)] = \mathbb{E}_{\theta\sim F}[\theta]$. Moreover, $\bar{u}(F) = \mathbb{E}_{\theta\sim F}[\theta]$ by Propositions 6 and 8. The optimal performance achievable $\mathrm{OPT}(F, T)$ in T rounds is exactly $T \cdot \bar{u}(F) = T \cdot \mathbb{E}_{\theta\sim F}[\theta]$. From Proposition 6, we already know $\mathrm{OPT}(F, T) \leq T \cdot \bar{u}(F)$ for all $F \in \mathcal{F}$. For achievability, we can argue that for any agent's distribution F, there exists a dynamic mechanism that, equipped with the knowledge of and tailored for F, extracts all of the agent's surplus. Via what is commonly known as the bundling strategy, the principal can, for example, bundle all items and sell the bundle at the expected value (see, e.g., Bakos and Brynjolfsson 1999).[13] The optimal performance can be achieved asymptotically even under more stringent participation constraints, i.e., $\liminf_{T\to\infty} \mathrm{OPT}(F, T)/T = \bar{u}(F)$; see, e.g., Balseiro et al. 2018.

Assumption 1 holds by Proposition 7 and Theorem 1 applies; the first part of the assumption follows from $\sup_{F\in\mathcal{F}} \bar{u}(F) = \sup_{F\in\mathcal{F}} \mathbb{E}_{\theta\sim F}[\theta] = 1$ and the second part from Proposition 8. We show the minimax regret is $\frac{1}{e}T$ and an optimal dynamic mechanism is T repetitions of a randomized posted pricing mechanism to be specified below. This implies the impossibility of leveraging any learning or adaptive scheme against a strategic agent:

[13] In the bundling mechanism, the principal lets the agent decide whether to continue or quit in Round 0 and then requires the agent to pay a one-time payment of $T \cdot \bar{u}(F)$ in Round 1 and allocates all items in Round 1 and future rounds. The agent would be indifferent between participating and not participating and when he decides to participate by continuing in Round 0, he would be bound by the bundling contract.

Theorem 13. *For revenue maximization in the dynamic selling mechanism design problem with one good, the minimax regret is $\frac{1}{e}T$ and an optimal solution is T repetitions of the randomized posted pricing mechanism S^* with price distribution Φ^* given by*

$$\Phi^*(p) = \begin{cases} 0, & \text{if } p \in [0, \frac{1}{e}) \\ 1 + \ln p, & \text{if } p \in [\frac{1}{e}, 1] \end{cases},$$

such that the interim allocation and payment rules are $x^*(\theta) = \Phi^*(\theta)$ and $p^*(\theta) = \left[\theta - \frac{1}{e}\right]_+$ for $\theta \in [0,1]$.

For comparison, Amin et al. (2013) showed a lower bound of $\frac{1}{12}T$ for the restricted class of dynamic posted pricing mechanisms; they considered a slightly different benchmark but their results still hold in our setting (see Section EC.4.1 of the electronic companion). We consider more general dynamic mechanisms and determine an optimal dynamic mechanism in this larger class with the performance that matches the improved lower bound of $\frac{1}{e}T$. To prove Theorem 13, it suffices we show that the randomized posted pricing mechanism S^* is a solution to the single-round minimax regret problem and $\widehat{\text{Regret}} = \frac{1}{e}$ via a saddle-point result, by Theorem 1. Its proof is deferred to Appendix C. An analogous result restricted to randomized posted pricing strategies exists due to Bergemann and Schlag (2008). Our single-round saddle-point result is for the slightly more general class of single-round direct IC/IR mechanisms and is still obtained with the same optimality structure and minimax regret value.[14]

6 Principal-Agent Model with Hidden Costs

We consider a repeated principal-agent problem that captures various applications such as retail franchising, labor contracts and procurement contracts. Similar to revenue maximization in the dynamic selling problem, we show that the minimax regret is linear in T and an optimal mechanism is static and repeats a single-round mechanism. Even though there is an opportunity to adapt to the agent's behavior, the principal does not realize any additional gain from implementing a dynamic mechanism. Due to nonlinearity in the problem, our analysis is more involved.

[14]To see the connection, note that for any single-round direct IC/IR mechanism S, there exists a randomized posted pricing strategy with nearly matching interim allocation and payment rules, that is, over $[0,1]$ except a set of measure 0. For example, we can interpret a suitable extension and modification of the interim allocation rule of S as the cumulative distribution function from which posted prices are randomly drawn.

More formally, the principal repeatedly contracts with the agent to produce output on his behalf and obtains revenue $R(\hat{q})$ when the agent produces $\hat{q}$ units of output which is publicly observable. A contract specifies a payment $\hat{z}$ from the principal to the agent as a function of the number of output units $\hat{q}$. The agent has a private marginal production cost $\theta \in [\underline{\theta}, \bar{\theta}]$ where $0 < \underline{\theta} < \bar{\theta} < \infty$ which is assumed to be independently and identically distributed according to distribution F across the rounds. The agent observes his private cost and then decides on the production level $\hat{q}$ in each round. When he produces $\hat{q}$ units of output and receives a payment $\hat{z}$, his utility is $\hat{z} - \theta \cdot \hat{q}$ where θ is his marginal cost for that round. The principal does not know the agent's private distribution F but only that realized costs are in the range $[\underline{\theta}, \bar{\theta}]$. We assume $R(x)$ is a strictly increasing, strictly concave function that is twice continuously differentiable on $(0, \infty)$ with $R(0) = 0$ and $\lim_{x \to 0} R'(x) = \infty$; for example, $R(x) = \sqrt{x}$.

In terms of our general model, the agent's shock is his marginal cost of production and $\Theta = [\underline{\theta}, \bar{\theta}]$. The outcome space is $\Omega = \mathbb{R}_+ \times \mathbb{R}$ and an outcome $\omega = (\hat{q}, \hat{z}) \in \Omega$ is a pair of the production level $\hat{q}$ and the payment $\hat{z}$. When the outcome is $\omega = (\hat{q}, \hat{z})$ in a round, the agent's utility function is $v(\theta, \omega) = \hat{z} - \theta \cdot \hat{q}$ and the principal utility function is $u(\theta, \omega) = R(\hat{q}) - \hat{z}$. For notational convenience, we represent a decision rule π in terms of the allocation rule q and payment rule z, as in $S = (q, z)$ where $q : \Theta \times \mathbb{W} \to \mathbb{R}_+$ and $z : \Theta \times \mathbb{W} \to \mathbb{R}$ for a single-round direct IC/IR mechanism S.

Since monetary transfers are allowed, the principal would set payments so that the IR constraint of the agent binds. Denote by $\bar{q}(\theta) = \arg\max_{x \geq 0}\{R(x) - \theta \cdot x\}$ the optimal production level when the agent's shock is known. The first-best mechanism involves requesting the agent to produce $\hat{q} = \bar{q}(\theta)$ units and paying the agent the minimum amount $\hat{z} = \theta \cdot \bar{q}(\theta)$ that makes him indifferent between participating or not in the contract (see, e.g., Laffont and Martimort 2001). Let $\bar{R}(\theta) = \max_{x \geq 0}\{R(x) - \theta \cdot x\}$ be the first-best utility of the principal when the shock is known to be θ; this is equivalently $\mathrm{OPT}(\theta, 1)$ when the agent's distribution is the point-mass distribution $\{\theta\}$. Because of the assumptions on $R(\cdot)$, $\bar{q}(\theta)$ is uniquely defined such that $R'(\bar{q}(\theta)) = \theta$ and $\bar{R}(\theta)$ is a strictly decreasing convex function. Since this is a game covered by Proposition 8, the linearity condition for $\bar{u}(F)$ holds and the single-round benchmark is equal to $\bar{u}(F) = \mathbb{E}_{\theta \sim F}\left[\bar{R}(\theta)\right]$ by Proposition 6. Note $\mathrm{OPT}(F, T)$ is equal to $T \cdot \bar{u}(F)$ via a bundling-type mechanism similar to the dynamic selling mechanism case.[15] Under more stringent participation constraints, $T \cdot \bar{u}(F)$ is still attainable asymptotically as in $\liminf_{T \to \infty} \mathrm{OPT}(F, T)/T =$

[15]For example, the principal lets the agent decide whether to continue or quit in Round 0. Then, he requires a one-time payment of $T \cdot \bar{u}(F)$ in Round 1 and, in return, lets the agent decide on the output levels and keep the revenue from the outputs in Round 1 and all the remaining rounds. This would yield the principal utility of $T \cdot \bar{u}(F)$ and the agent utility of 0, that is, the agent will be indifferent between participating and not participating.

$\bar{u}(F)$; see, e.g., Appendix D.2 from Balseiro et al. 2018.

Note Assumption 1 holds from Proposition 7 since $\sup_{F\in\mathcal{F}} \bar{u}(F) = \sup_{F\in\mathcal{F}} \mathbb{E}_{\theta\sim F}[\bar{R}(\theta)] \leq \bar{R}(\underline{\theta})$ and $\bar{u}(F)$ satisfies the linearity condition. Then, Theorem 1 applies to show we can essentially achieve the minimax regret by repeating a single-round mechanism. We formally state the main result of the section as follows:

Theorem 14. *For the principal-agent model with hidden costs, the minimax regret of the multi-round problem is cT for some constant $c > 0$ and an optimal solution is T repetitions of offering the menu of deterministic contracts $\{(q^*(\theta), z^*(\theta)\}_{\theta\in\Theta}$, which is a single-round direct IC/IR mechanism. The allocation rule is continuous and satisfies the differential equation characterization*

$$(q^*)'(\theta) = -\frac{\bar{q}(\theta)}{R'(q^*(\theta)) - \theta}\,, \text{ for } \theta \in (\underline{\theta}, \kappa)\,,$$

with boundary conditions $q^*(\underline{\theta}) = \bar{q}(\underline{\theta})$ and $q^*(\theta) = 0$ for $\theta \in [\kappa, \bar{\theta}]$, where κ is the smallest cost for which q^* equals to 0 and is assumed to be $\bar{\theta}$ if q^* is positive over $[\underline{\theta}, \bar{\theta}]$, and the payment rule is given by

$$z^*(\theta) = \theta \cdot q^*(\theta) + \int_{\theta}^{\bar{\theta}} q^*(x)dx\,.$$

For Theorem 14, we prove the stated single-round direct IC/IR mechanism is an optimal solution to the single-round minimax regret problem via a saddle-point result. Since the revenue function R is concave, no randomization is needed and we can restrict our search of an optimal solution to those single-round direct IC/IR mechanisms that can be described in terms of a menu of deterministic contracts $(q(\theta), z(\theta))$ for $\theta \in [\underline{\theta}, \bar{\theta}]$ where the contract terms are all deterministic without randomization. We refer to Appendix D.1 for further details on the single-round problem including the IC/IR constraints and to Appendix D.2 for the proof of Theorem 14.

7 Resource Allocation without Monetary Transfers

In this section, we consider a dynamic resource allocation problem without monetary transfers. In particular, a social planner is repeatedly allocating a costly resource in settings where monetary transfers may not be practical for legal, ethical and various other reasons. Real-life applications include an organization or government allocating an internal resource and a nurse attending a patient (see, e.g., Guo and Hörner (2015); Balseiro et al. (2017)). We show that the minimax regret of the multi-round problem is linear in T and an optimal solution simply repeats a single-round mechanism.

Our model is closely related to Guo and Hörner (2015). The principal repeatedly allocates an independent and identical unit of a resource in each round over the time horizon. In each round, the agent privately observes his current value for the resource which is drawn independently and identically from an underlying distribution known to him. The principal does not know the agent's distribution nor per-round values but knows that the per-round values are in the range $[0, 1]$. The outcome is a singleton $\omega = \hat{x}$ where $\hat{x}$ is the allocation, i.e., whether or not the resource is allocated to the agent. The principal incurs an opportunity cost $c \in (0, 1)$ when allocating the resource.[16] Equivalently, the principal wants to allocate the resource only when the value exceeds the cost, but the agent wants to be allocated always.

In the formal language of Section 2, the agent's shock is his value for the resource and $\Theta = [0, 1]$. The outcome space is $\Omega = \{0, 1\}$ and an outcome $\omega = \hat{x} \in \Omega$ is the allocation $\hat{x}$. When the outcome is $\omega = \hat{x}$ in a round, the agent's utility function is $v(\theta, \hat{x}) = \theta \cdot \hat{x}$ and the principal's utility function is $u(\theta, \hat{x}) = (\theta - c) \cdot \hat{x}$ where c is the fixed opportunity cost. For notational convenience, we represent a decision rule π in terms of the allocation rule $x : \Theta \times \mathbb{W} \to \{0, 1\}$ for a single-round direct IC/IR mechanism. Abusing notations, we represent the interim allocation rule with $x : \Theta \to [0, 1]$ with the understanding that when the agent reports θ, the probability of allocation is $x(\theta)$.

Note the agent's utility function is always nonnegative and, by Proposition 8, the linearity condition holds for $\bar{u}(F)$, that is, $\bar{u}(F) = \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)]$ for all $F \in \mathcal{F}$. Since the principal would want to allocate only if $\theta \geq c$ when the agent's shock is θ, $\mathrm{OPT}(\theta, 1) = \max\{\theta - c, 0\}$ for any $\theta \in \Theta$. By Proposition 6 and the linearity condition, the single-round benchmark is equivalently $\bar{u}(F) = \mathbb{E}_{\theta \sim F}[\max\{\theta - c, 0\}]$ and it is bounded because $\sup_{F \in \mathcal{F}} \bar{u}(F) \leq 1 - c$. Then, Assumption 1 holds from Propositions 7. Our general results from Section 3 apply and we obtain the following:

Theorem 15. *For the dynamic resource allocation problem without monetary transfers, the minimax regret of the multi-round problem is $c(1 - c)T$ and an optimal solution is T repetitions of the probabilistic allocation rule x^* where $x^*(\theta) = 1 - c$ for $\theta \in (0, 1]$ and $x^*(0)$ can be any probability in the range $[0, 1 - c]$.*

We can show that the probability allocation rule x^* is an optimal solution to the single-round problem,

[16] Instead of a fixed opportunity cost c, we can alternatively think the cost per allocation to be a random Bernoulli variable with the average of c and the principal only observes the cost after an allocation decision and reasons in terms of the average cost c. Furthermore, we focus on the cases where $c \in (0, 1)$ because there exists a trivial optimal solution when $c = 0$ or $c = 1$. If $c = 0$, always allocating is optimal. If $c = 1$, not allocating is optimal.

but not by finding a saddle point. Interestingly, saddle points do not exist for the corresponding single-round minimax regret problem in contrast to preceding sections and we instead show an asymmetric saddle-point result. See the proof and details in Appendix E.

8 Extensions and Discussion

Our results can be extended in several directions. Due to space considerations, we describe these briefly below and provide details in Sections EC.4 and EC.5 of the electronic companion. First, we show our results still hold for other alternative multi-round benchmarks that are considered in the learning literature. Instead of the optimal performance achievable $\mathrm{OPT}(F, T)$, we consider $T \cdot \bar{u}(F)$ which is a stronger benchmark by Proposition 6 and a weaker benchmark which naturally corresponds to the performance achievable by repeating the best fixed single-round IC/IR mechanism (i.e., the best fixed "action" in hindsight). The latter has been considered by Amin et al. (2013) and subsequent works. Second, our results also apply in the general shock process setting where the agent's shocks can be serially correlated according to a stochastic process that is known to the agent but not to the principal. This is a natural generalization of the repeated i.i.d. setting considered thus far where the per-round shocks are drawn independently and identically from an underlying distribution. As the set of shock processes is more general, the multi-round minimax regret problem is more challenging for the principal. Not surprisingly, the constant shock processes where the agent's shock is fixed over the whole time horizon are the corresponding counterparts of point-mass distributions which are worst-cases in the repeated i.i.d. setting.

Third, we prove analogous results in terms of multiplicative performance guarantees instead of regrets. Similar to the approximation and competitive ratios in the theoretical computer science literature, we consider the multiplicative performance guarantee that is the ratio of the principal utility and the optimal performance achievable, as in $\mathrm{PrincipalUtility}(A, B^*(A, F), F, T)/\, \mathrm{OPT}(F, T)$, and the principal's goal is to maximize the worst-case ratio. Under Assumption 1, we can show that the multi-round multiplicative guarantee is equal to an appropriately defined single-round multiplicative guarantee and an optimal mechanism, if it exists, is static in that it repeats a single-round mechanism. Fourth, we consider a stronger notion of regret in which the agent plays a utility-maximizing strategy that is least favorable for the principal. Under this alternative tie-breaking possibility, the worst-case uncertainty that the principal faces is in both the agent's distribution and his utility-maximizing strategy. We can analyze in terms of what we call the principal pessimism constraint and show our main general result (Theorem 1) and those in Sections 5–7 still hold with respect to this stronger and more robust notion of minimax regret. Finally, we discuss some connections to the maximin utility objective for revenue maximization in the dynamic selling problem with a single good (Section 5). Despite differences in the settings and objectives, our results with respect to the minimax regret objective and those in Carrasco et al. (2015) with respect to the maximin utility objective have similar analyses and solution structures. We show this is because both papers rely on essentially the same single-round saddle-point problem involving direct IC/IR mechanisms and show equivalence-type connections using saddle-point results.

9 Conclusion

In this paper, we proved false-dynamics results for a finite horizon setting where the principal and agent repeatedly play a game for a general class of games. Our results hold for games that satisfy a general assumption (Assumption 1) that include all games with linear dependence on monetary transfers or in which the agent utility function is always nonnegative. In particular, this includes the dynamic selling problem, the principal-agent model with hidden costs and resource allocation without monetary transfers, and we determined the minimax regret and characterized an optimal dynamic mechanism that simply repeats a single-round mechanism in these applications. When the general assumption does not hold, it is possible that a dynamic mechanism can outperform static mechanisms, i.e., those that repeat a single-round mechanism, and we showed a separation in terms of performance between dynamic and static mechanisms for a specific game. Furthermore, we showed our techniques extend and similar false-dynamics results hold in other settings in the electronic companion.

For future research, it would be interesting to better understand Assumption 1 and find a more general class of games where false-dynamics-type results hold. On the other hand, it would be also interesting to further explore where false-dynamics-type results do not hold and identify the class of games in which there is a separation between dynamic and static mechanisms, that is, dynamics strictly helps.

Other possible research directions include restricting the space of distributions and considering multiple agents. Point-mass distributions happen to be the right class of worst-case distributions in our analysis, and it is possible that false-dynamics-type results do not hold when we rule out point-mass distributions or restrict to, say, a finite family of distributions. We considered one strategic agent who is forward-looking and responds to the principal's mechanism. When there are multiple forward-looking agents, equilibrium considerations become important as the outcome may differ depending on whether the agents know each other's distribution or not, which, in turn, may affect the principal's optimal dynamic mechanism.

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A Missing Proofs from Section 3

A.1 Proof of Theorem 1

As explained in Section 3.2, we use Lemmas 3 and 4.

Part 1): We use Lemma 3. Taking the infimum over all single-round direct IC/IR mechanisms S on the right-hand side of (3), we obtain for any dynamic mechanism A^T,

$$\sup_{F\in\mathcal{F}} \mathrm{Regret}(A^T, F, T) \geq T \cdot \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F) = T \cdot \widehat{\mathrm{Regret}}\,.$$

Then, taking the infimum over all dynamic mechanisms A^T on the left-hand side of the above, we obtain

$$\mathrm{Regret}(T) = \inf_{A^T\in\mathcal{A}^T} \sup_{F\in\mathcal{F}} \mathrm{Regret}(A^T, F, T) \geq T \cdot \widehat{\mathrm{Regret}}\,.$$

Now, it remains to show that $\mathrm{Regret}(T) \leq T \cdot \widehat{\mathrm{Regret}}$. Fix an arbitrary $\epsilon > 0$. By the definition of infimum, there exists a single-round direct IC/IR mechanism S satisfying

$$\sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F) \leq \widehat{\mathrm{Regret}} + \frac{\epsilon}{T}\,.$$

Then,

$$\sup_{F\in\mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) \leq T \cdot \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F) \leq T \cdot \widehat{\mathrm{Regret}} + \epsilon\,,$$

where the first inequality is by Lemma 4 and the second by the choice of S. Since $S^{\times T}$ is a particular dynamic mechanism for T rounds, it follows that

$$\mathrm{Regret}(T) = \inf_{A^T\in\mathcal{A}^T} \sup_{F\in\mathcal{F}} \mathrm{Regret}(A^T, F, T) \leq \sup_{F\in\mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) \leq T \cdot \widehat{\mathrm{Regret}} + \epsilon\,.$$

As $\epsilon > 0$ was arbitrary and can be arbitrarily small, it follows that

$$\mathrm{Regret}(T) \leq T \cdot \widehat{\mathrm{Regret}}\,.$$

Part 2): For any $\epsilon \geq 0$, assume a single-round direct IC/IR mechanism S satisfies

$$\sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F) \leq \widehat{\mathrm{Regret}} + \frac{\epsilon}{T}\,.$$

Then,

$$\sup_{F\in\mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) \leq T \cdot \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F) \leq T \cdot \widehat{\mathrm{Regret}} + \epsilon = \mathrm{Regret}(T) + \epsilon\,,$$

where the first inequality is by Lemma 4, the second inequality is by the property of S and the last equality is by Part 1.

Part 3): The if direction follows directly from Part 2. If there exists an optimal single-round direct

IC/IR mechanism S^* to the single-round problem, the optimal solution S^* satisfies

$$\sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S^*, F) \le \widehat{\mathrm{Regret}}\,.$$

By Part 2 (with $\epsilon = 0$),

$$\sup_{F\in\mathcal{F}} \mathrm{Regret}((S^*)^{\times T}, F, T) \le \mathrm{Regret}(T)\,.$$

It follows that the static mechanism that repeats S^* T times is an optimal dynamic mechanism in the multi-round problem and, hence, there exists an optimal dynamic mechanism in the multi-round problem.

For the only-if direction, assume there exists an optimal dynamic mechanism A^* such that $\mathrm{Regret}(T) = \sup_{F\in\mathcal{F}} \mathrm{Regret}(A^*, F, T)$. By Lemma 3, there exists a single-round direct IC/IR mechanism S such that

$$\mathrm{Regret}(T) = \sup_{F\in\mathcal{F}} \mathrm{Regret}(A^*, F, T) \ge T \cdot \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F)\,.$$

By Part 1 that $\mathrm{Regret}(T) = T \cdot \widehat{\mathrm{Regret}}$, it follows that

$$\widehat{\mathrm{Regret}} \ge \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F)\,.$$

The above implies that S is an optimal single-round mechanism because it achieves the single-round minimax regret $\widehat{\mathrm{Regret}}$. Hence, there exists an optimal single-round mechanism in the single-round problem. In particular, the single-round direct IC/IR mechanism constructed from A^*, $S(A^*)$, as in the proof of Lemma 16 is one such single-round mechanism that satisfies the statement of Lemma 3 and, hence, is an optimal solution to the single-round problem.

A.2 Additional Materials for Section 3.2

We prove Lemmas 3 and 4 in Appendices A.2.1 and A.2.2, respectively. We use Proposition 2 which is proved in Appendix A.2.3.

A.2.1 Proof of Lemma 3

We need to relate the two regret objectives, Regret and $\widehat{\mathrm{Regret}}$, and reduce the multi-round problem to the single-round problem for direct IC/IR mechanisms. The following lemmas are useful. The first lemma follows from a revelation-principle-type argument and shows that if the agent's distribution is restricted to point-mass distributions, the principal's dynamic mechanism effectively reduces to a single-round direct mechanism with IC/IR properties and we can assume the agent's utility-maximizing strategy chosen in the principal's favor is truthful reporting σ^{TR}. The second one is a simple observation on $\widehat{\mathrm{Regret}}$. See Appendix A.3 for the proofs of the lemmas.

Lemma 16. *For any dynamic mechanism A^T, there exists a single-round direct IC/IR mechanism, denoted $S(A^T)$, such that for any $\theta \in \Theta$,*

$$\mathrm{PrincipalUtility}(A^T, B^*(A^T, \theta), \theta, T) = T \cdot \mathrm{PrincipalUtility}(S(A^T), \sigma^{TR}, \theta, 1)\,,$$

where σ^{TR} is the agent's strategy under which the agent participates (i.e., reports CONTINUE *in Round 0) and truthfully reports his shock.*

Lemma 17. *For any single-round direct IC/IR mechanism S,*

$$\sup_{\theta\in\Theta} \widehat{\text{Regret}}(S,\theta) = \sup_{F\in\mathcal{F}} \widehat{\text{Regret}}(S,F)\,.$$

Using the above lemmas, we prove Lemma 3:

Proof of Lemma 3. Fix an arbitrary dynamic mechanism $A^T \in \mathcal{A}^T$. Note that

$$\sup_{F\in\mathcal{F}} \text{Regret}(A^T,F,T) \geq \sup_{\theta\in\Theta} \text{Regret}(A^T,\theta,T)\,,$$

since point-mass distributions are a subset of general distributions $\mathcal{F}$. We can equivalently write the last expression as

$$\begin{aligned}
\sup_{\theta\in\Theta} \text{Regret}(A^T,\theta,T) &= \sup_{\theta\in\Theta} \big\{\text{OPT}(\theta,T) - \text{PrincipalUtility}(A^T,B^*(A^T,\theta),\theta,T)\big\}\\
&= T\cdot \sup_{\theta\in\Theta} \big\{\text{OPT}(\theta,1) - \text{PrincipalUtility}(S(A^T),\sigma^{\text{TR}},\theta,1)\big\}\\
&= T\cdot \sup_{\theta\in\Theta} \widehat{\text{Regret}}(S(A^T),\theta)\,,
\end{aligned}$$

where $S(A^T)$ is the single-round direct IC/IR mechanism derived from A^T as described in the proof of Lemma 16 and the second step follows from the same lemma and Proposition 2. By Lemma 17,

$$T\cdot \sup_{\theta\in\Theta} \widehat{\text{Regret}}(S(A^T),\theta) = T\cdot \sup_{F\in\mathcal{F}} \widehat{\text{Regret}}(S(A^T),F)\,.$$

Putting the above together, for the single-round direct IC/IR mechanism $S(A^T)$, we have

$$\sup_{F\in\mathcal{F}} \text{Regret}(A^T,F,T) \geq T\cdot \sup_{F\in\mathcal{F}} \widehat{\text{Regret}}(S(A^T),F)\,. \qquad \square$$

A.2.2 Proof of Lemma 4

To prove Lemma 4, we need the following lemma. It is about dynamic mechanisms that are simply T repetitions of a single-round mechanism. More specifically, we show that the agent's utility-maximizing strategy for such mechanism can be assumed to be, similarly, T repetitions of a utility-maximizing strategy for the single-round mechanism (interpreting no participation as T repetitions of the same strategy). We prove the lemma in Appendix A.3.

Lemma 18. *Let $(A^1)^{\times T}$ denote the dynamic mechanism that repeats single-round mechanism A^1 for T rounds. For any single-round mechanism A^1 and agent's distribution F,*

$$\text{PrincipalUtility}((A^1)^{\times T},B^*((A^1)^{\times T},F),F,T) = T\cdot \text{PrincipalUtility}(A^1,B^*(A^1,F),F,1)\,.$$

We now have:

Proof of Lemma 4. Let S be any single-round direct IC/IR mechanism and consider the dynamic mechanism $S^{\times T}$ which is T repetitions of S. By the definition of Regret notion,

$$\sup_{F\in\mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) = \sup_{F\in\mathcal{F}} \left\{\mathrm{OPT}(F, T) - \mathrm{PrincipalUtility}(S^{\times T}, B^*(S^{\times T}, F), F, T)\right\} .$$

Note that for any distribution $F \in \mathcal{F}$,

$$\mathrm{OPT}(F, T) \le \mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta, T)] = T \cdot \mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta, 1)] \,,$$

where the inequality is by Assumption 1 and the equality is by Proposition 2. Then,

$$\begin{aligned}
\sup_{F\in\mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) &\le \sup_{F\in\mathcal{F}} \left\{T \cdot \mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta, 1)] - \mathrm{PrincipalUtility}(S^{\times T}, B^*(S^{\times T}, F), F, T)\right\} \\
&= T \cdot \sup_{F\in\mathcal{F}} \{\mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta, 1)] - \mathrm{PrincipalUtility}(S, B^*(S, F), F, 1)\} \\
&\le T \cdot \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F) \,,
\end{aligned}$$

where the second-to-last step is by Lemma 18 and the last step is because truthful reporting is a utility-maximizing strategy for the agent but may not be one that also maximizes the principal's utility, equivalently, minimizes the principal's regret, among utility-maximizing strategies. It follows that

$$\sup_{F\in\mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) \le T \cdot \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F) \,. \qquad \square$$

A.2.3 Proofs of Propositions 2 and 5

Proof of Proposition 2. The proof follows straightforwardly from Proposition 6 in Section 3.3. For any $\theta \in \Theta$,

$$\mathrm{OPT}(\theta, T) = T \cdot \bar{u}(\theta) = T \cdot \mathrm{OPT}(\theta, 1) \,,$$

by the second part of Proposition 6. Alternatively, we can prove the proposition directly using the same ideas in the proof of Proposition 6. We keep the current presentation to avoid repeating proofs. $\square$

Proof of Proposition 5. Fix an arbitrary dynamic mechanism A^T. Following the same line of reasoning in the beginning of the proof of Lemma 3, we have for any $\theta \in \Theta$,

$$\mathrm{Regret}(A^T, \theta, T) = T \cdot \widehat{\mathrm{Regret}}(S(A^T), \theta) \,,$$

where $S(A^T)$ is the single-round direct IC/IR mechanism derived from A^T as described in the proof of Lemma 16. Note $\sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S(A^T), F) \ge \widehat{\mathrm{Regret}}$. For any $\epsilon > 0$, there exists a distribution F_ϵ such that

$$\widehat{\mathrm{Regret}}(S(A^T), F_\epsilon) \ge \widehat{\mathrm{Regret}} - \epsilon \,.$$

Then,

$$\begin{aligned}\widehat{\text{Regret}}(S(A^T), F_\epsilon) &= \int_\Theta \text{OPT}(\theta, 1)\text{d}F_\epsilon(\theta) - \int_\Theta \int_\Omega u(\theta, \omega)\text{d}S(A^T)_\theta(\omega)\text{d}F_\epsilon(\theta) \\ &= \int_\Theta \left(\text{OPT}(\theta, 1) - \int_\Omega u(\theta, \omega)\text{d}S(A^T)_\theta(\omega)\right) \text{d}F_\epsilon(\theta) \\ &= \int_\Theta \widehat{\text{Regret}}(S(A^T), \theta) dF_\epsilon(\theta)\,.\end{aligned}$$

Hence,

$$\int_\Theta \widehat{\text{Regret}}(S(A^T), \theta) dF_\epsilon(\theta) \geq \widehat{\text{Regret}} - \epsilon\,,$$

and there must exist some θ^* in the support of F_ϵ for which $\widehat{\text{Regret}}(S(A^T), \theta^*) \geq \widehat{\text{Regret}} - \epsilon$. For the point-mass distribution θ^*, we have

$$\text{Regret}(A^T, \theta^*, T) = T \cdot \widehat{\text{Regret}}(S(A^T), \theta^*) \geq T \cdot \widehat{\text{Regret}} - \epsilon T\,.$$

As ϵ was arbitrary, we can choose $\epsilon' = \frac{\epsilon}{T}$ and the above argument for $F_{\epsilon'}$ shows a point-mass distribution θ^* such $\text{Regret}(A^T, \theta^*, T) \geq T \cdot \widehat{\text{Regret}} - \epsilon$. Since Assumption 1 holds and $\text{Regret}(T) = T \cdot \widehat{\text{Regret}}$ by Theorem 1,

$$\text{Regret}(A^T, \theta^*, T) \geq \text{Regret}(T) - \epsilon\,.$$

□

A.3 Remaining Proofs from Appendix A.2

Proof of Lemma 16. First, we show a construction of a single-round direct IC/IR mechanism which will be our choice of $S(A^T)$ and then prove the claimed statements. Let $\{\omega_{\theta,t}\}_{t=1}^T$ be a sequence of outcomes realized when the agent plays $B^*(A^T, \theta)$ against the principal's mechanism A^T. For each $\theta \in \Theta$ and $\omega \in \Omega$, let $\mu_\theta(\omega) = \frac{1}{T}\sum_{t=1}^T \mathbf{1}\{\omega_{\theta,t} = \omega\}$. As $\{\omega_{\theta,t}\}_{t=1}^T$ is random, μ_θ is a random measure with a well-defined distribution. When the agent reports QUIT in Round 0 and does not participate, the sequence is simply the no-interaction outcome in all rounds.

Consider the following single-round direct mechanism S which is a collection of distributions S_θ on Ω indexed by $\theta \in \Theta$. Each S_θ is induced by drawing a μ_θ and then drawing $\omega \sim \mu_\theta$ such that when the agent reports θ, the principal determines the outcome by drawing from S_θ, that is, $\omega \sim S_\theta$. For any $\theta \in \Theta$ and measurable function f, note that

$$\mathbb{E}_{\omega \sim S_\theta}[f(\omega)] = \mathbb{E}_{\mu_\theta \sim S_\theta, \omega \sim \mu_\theta}\left[f(\omega)\right] = \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^T f(\omega_{\theta,t})\right]\,.$$

We can show that S satisfies both IC and IR constraints. For any $\theta, \theta' \in \Theta$,

$$\begin{aligned}
\mathbb{E}_{\omega\sim S_{\theta'}}[v(\theta,\omega)] &= \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} v(\theta,\omega_{\theta',t})\right] \\
&= \frac{1}{T}\,\mathrm{AgentUtility}(A^T, B^*(A^T,\theta'),\theta,T) \\
&\le \frac{1}{T}\,\mathrm{AgentUtility}(A^T, B^*(A^T,\theta),\theta,T) \\
&= \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} v(\theta,\omega_{\theta,t})\right] \\
&= \mathbb{E}_{\omega\sim S_{\theta}}[v(\theta,\omega)]\,,
\end{aligned}$$

where the inequality follows from that $B^*(A^T,\theta)$ is a utility-maximizing strategy for the agent when his shock is θ. Similarly, for any $\theta \in \Theta$,

$$\begin{aligned}
\mathbb{E}_{\omega\sim S_{\theta}}[v(\theta,\omega)] &= \frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} v(\theta,\omega_{\theta,t})\right] \\
&= \frac{1}{T}\,\mathrm{AgentUtility}(A^T, B^*(A^T,\theta),\theta,T) \\
&\ge 0\,,
\end{aligned}$$

where the inequality follows because a utility-maximizing agent can guarantee the total utility of at least 0 by not participating. Hence, S constructed above is a single-round direct IC/IR mechanism.

By construction, we have for any θ,

$$\begin{aligned}
\mathrm{PrincipalUtility}(A^T, B^*(A^T,\theta),\theta,T) &= T\cdot\frac{1}{T}\mathbb{E}\left[\sum_{t=1}^{T} u(\theta,\omega_{\theta,t})\right] \\
&= T\cdot\mathbb{E}_{\omega\sim S_{\theta}}[u(\theta,\omega)] \\
&= T\cdot\mathrm{PrincipalUtility}(S,\sigma^{\mathrm{TR}},\theta,1)\,.
\end{aligned}$$

□

Proof of Lemma 17. Fix an arbitrary single-round direct IC/IR mechanism S. For any agent's distribution F,

$$\begin{aligned}
\widehat{\mathrm{Regret}}(S,F) &= \int_{\Theta}\mathrm{OPT}(\theta,1)\mathrm{d}F(\theta) - \int_{\Theta}\int_{\Omega} u(\theta,\omega)\mathrm{d}S_{\theta}(\omega)\mathrm{d}F(\theta) \\
&= \int_{\Theta}\left(\mathrm{OPT}(\theta,1) - \int_{\Omega} u(\theta,\omega)\mathrm{d}S_{\theta}(\omega)\right)\mathrm{d}F(\theta) \\
&= \int_{\Theta}\widehat{\mathrm{Regret}}(S,\theta)dF(\theta) \\
&\le \sup_{\theta}\widehat{\mathrm{Regret}}(S,\theta)\,.
\end{aligned}$$

As F was arbitrary, it follows that $\sup_{\theta}\widehat{\mathrm{Regret}}(S,\theta) \ge \sup_{F}\widehat{\mathrm{Regret}}(S,F)$.

On the other hand, we observe that point-mass distributions are a subset of the general distributions and, therefore, $\sup_\theta \widehat{\text{Regret}}(S, \theta) \leq \sup_F \widehat{\text{Regret}}(S, F)$. $\square$

Proof of Lemma 18. Recall that a utility-maximizing strategy for the agent is one that maximizes the agent utility and, if there are multiple such strategies, also maximizes the principal utility among such strategies. Note the lemma statement and following proof are with respect to the multi-round problem for general T and $T = 1$; the $\widehat{\text{Regret}}$ notion and the single-round minimax regret problem are not relevant here.

Fix an arbitrary single-round mechanism $A^1 = (\mathcal{M}, \pi)$ with the specified report space and decision rule for one round of reporting and an arbitrary distribution $F \in \mathcal{F}$ for the agent. Let $B^*((A^1)^{\times T}, F)$ be the agent's utility-maximizing strategy against the T repetitions of A^1 under which he reports $\mathsf{CONTINUE}$ or QUIT in Round 0.

Without loss in terms of the principal and agent utilities (i.e., maintaining the same quantities), we assume $B^*((A^1)^{\times T}, F)$ deterministically reports $\mathsf{CONTINUE}$ or QUIT in Round 0. If $B^*((A^1)^{\times T}, F)$ randomizes between reporting $\mathsf{CONTINUE}$ and reporting QUIT, then the principal and agent utilities must be 0 under the event that the agent reports $\mathsf{CONTINUE}$ in Round 0. It means the utility-maximizing strategy chosen in the principal's favor conditioned on the agent participating leads to the same principal and agent utilities as the no-participation strategy that reports QUIT in Round 0. Then, we can assume $B^*((A^1)^{\times T}, F)$ reports QUIT in Round 0. Otherwise, we assume it reports $\mathsf{CONTINUE}$ in Round 0. We assume similarly for $B^*(A^1, F)$. Given the above discussion, if $B^*((A^1)^{\times T}, F)$ reports $\mathsf{CONTINUE}$ in Round 0, we represent the strategy as $B^*((A^1)^{\times T}, F) = \{\sigma_t\}_{1:T}$ with the subsequent per-round strategies. If $B^*((A^1)^{\times T}, F)$ reports QUIT, we represent the no-participation strategy as $B^*((A^1)^{\times T}, F) = \mathsf{QUIT}$. For $B^*(A^1, F)$, we represent similarly as $B^*(A^1, F) = \sigma$ for the single-round reporting strategy σ or $B^*(A^1, F) = \mathsf{QUIT}$.

We define per-round expected agent utility V_t and principal utility U_t when the principal implements $(A^1)^{\times T}$ and the agent plays $B^*((A^1)^{\times T}, F)$ as

$$\begin{aligned} V_t &= \mathbb{E}[v(\theta_t, \pi(\sigma_t(\theta_t, h_t^+)))] \,, \text{ and} \\ U_t &= \mathbb{E}[u(\theta_t, \pi(\sigma_t(\theta_t, h_t^+)))] \,, \end{aligned}$$

for Rounds $t \in [T]$. If $B^*((A^1)^{\times T}, F) = \mathsf{QUIT}$ and the agent does not participate, $V_t = U_t = 0$ for all t. Note the principal's mechanism has no dependence on histories while the agent's utility-maximizing strategy may depend on the history h_t^+.

Claim 1. *For any t, there is a single-round agent strategy against A^1 that achieves the expected agent utility and principal utility equal to V_t and U_t, respectively.*

Proof. Assume $B^*((A^1)^{\times T}, F)$ reports $\mathsf{CONTINUE}$ in Round 0. Fix arbitrary $t \in [T]$. The agent can implement the t-th round strategy σ_t as a standalone single-round strategy by internally choosing randomness w and y and simulating the history $h_t^+ = (F, \theta_{1:t-1}, m_{1:t-1}, \omega_{1:t-1})$. Note there is an independent copy of randomness w for each repetition of A^1 that precedes Round t. This is possible from the knowledge of A^1 and $\{\sigma_{t'}\}_{1:t-1}$. By construction, when the agent reports $\mathsf{CONTINUE}$ in Round 0 and then implements σ_t as described above against A^1, the expected agent utility and principal utility equal V_t and U_t, respectively.

If $B^*((A^1)^{\times T}, F) = \mathsf{QUIT}$ and the agent does not participate, the no-participation strategy considered as a single-round strategy against A^1 yields the agent and principal utilities of 0 which are equal to V_t and U_t, respectively, for all t. □

In the multi-round problem with T rounds, it must be that the per-round expected agent utility is the same across all T rounds. If the maximum and minimum of the expected quantity are different, consider the alternative strategy for the agent where he reports $\mathsf{CONTINUE}$ in Round 0 and then repeats σ_t that yields the maximum per-round quantity for T rounds. By the above claim, this strategy achieves T times the maximum per-round expected agent utility and the agent can strictly improve the overall expected agent utility. This would contradict the choice of $B^*((A^1)^{\times T}, F)$. Similarly, the per-round principal utility must be the same across all T rounds given the per-round agent utilities are the same. If $B^*((A^1)^{\times T}, F) = \mathsf{QUIT}$, $V_t = U_t = 0$ for all t and these same observations clearly hold.

From these observations and, again, the claim, there is a single-round agent strategy whose T repetitions against $(A^1)^{\times T}$ yields the same expected overall agent utility and principal utility as $B^*((A^1)^{\times T}, F)$. This strategy, as a standalone strategy against the single-round mechanism A^1, must be also a utility-maximizing strategy that, if multiple such strategies exists, also maximizes the principal utility among utility-maximizing strategies. Otherwise, we can strictly increase the agent utility or strictly increase the principal utility while holding the agent utility constant by repeating the single-round strategy $B^*(A^1, F)$. If $B^*(A^1, F) = \sigma$ (i.e., the agent reports $\mathsf{CONTINUE}$ in Round 0 and follows by the single-round strategy σ), then repeating it would mean reporting $\mathsf{CONTINUE}$ in Round 0 and then repeating σ T times. If $B^*(A^1, F) = \mathsf{QUIT}$, repeating it would mean reporting QUIT in Round 0 and not participating. It follows that T repetitions of $B^*(A^1, F)$ achieves the same agent utility and principal utility over T rounds as $B^*((A^1)^{\times T}, F)$. If $B^*((A^1)^{\times T}, F) = \mathsf{QUIT}$, the same reasoning applies. Hence, the lemma follows. □

A.4 Missing Proofs from Section 3.3

Proof of Proposition 6. Part 1): Fix an arbitrary distribution $F \in \mathcal{F}$. Assume the principal commits to a dynamic mechanism A^T and the agent plays the strategy $B^*(A^T, F)$. Let $\{\omega_t\}_{t=1}^T$ be the resulting random sequence of realized outcomes. For each $\theta \in \Theta$, we define measure $\mu_\theta(Q) = \frac{1}{T}\sum_{t=1}^T \Pr(\omega_t \in Q \mid \theta_t = \theta)$ for any $Q \subseteq \Omega$ and let S_θ be the corresponding distribution over Ω such that $\omega \sim S_\theta$ means an outcome ω is realized with probability $\mu_\theta(\omega)$. Consider a single-round direct mechanism

$S = \{S_\theta\}_{\theta \in \Theta}$ that given a report θ returns an outcome $\omega \sim S_\theta$. We note that

$$\begin{aligned}
\text{PrincipalUtility}(A^T, B^*(A^T, F), F, T) &= \mathbb{E}\left[\sum_{t=1}^{T} u(\theta_t, \omega_t)\right] \\
&= \sum_{t=1}^{T} \mathbb{E}_{\theta_t}\left[\mathbb{E}[u(\theta_t, \omega_t)|\theta_t]\right] \\
&= \sum_{t=1}^{T} \mathbb{E}_{\theta \sim F}\left[\mathbb{E}[u(\theta_t, \omega_t)|\theta_t = \theta]\right] \\
&= T \cdot \mathbb{E}_{\theta \sim F}\left[\frac{1}{T}\sum_{t=1}^{T} \mathbb{E}_{\omega_t|\theta_t=\theta}[u(\theta, \omega_t)]\right] \\
&= T \cdot \mathbb{E}_{\theta \sim F}\left[\mathbb{E}_{\omega \sim S_\theta}[u(\theta, \omega)]\right] \,,
\end{aligned}$$

where the second equality follows from the linearity of expectations and the tower rule, the third from that the idiosyncratic shocks are drawn independently and identically, and the last from the construction of S. Hence, we have

$$\text{PrincipalUtility}(A^T, B^*(A^T, F), F, T) = T \cdot \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \,.$$

Similarly, we have

$$\text{AgentUtility}(A^T, B^*(A^T, F), F, T) = T \cdot \int_\Theta \int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \,.$$

Since the agent's strategy $B^*(A^T, F)$ is a utility-maximizing strategy and the agent can achieve the aggregate utility of 0 by not participating, it must be that $\text{AgentUtility}(A^T, B^*(A^T, F), F, T) \geq 0$. It follows that S is a feasible solution for $\bar{u}(F)$ and achieves the objective of $\frac{1}{T} \cdot \text{PrincipalUtility}(A^T, B^*(A^T, F), F, T)$. As the dynamic mechanism A^T was arbitrary, the first part of the lemma follows.

Part 2): Since we have the first part, it suffices to show $\text{OPT}(\theta, T) \geq \bar{u}(\theta)$ for any $\theta \in \Theta$. Fix an arbitrary point-mass distribution $\theta \in \Theta$. Note $\bar{u}(\theta)$ is equivalently

$$\begin{aligned}
\bar{u}(\theta) := \sup_{G \in \Delta(\Omega)} &\int_\Omega u(\theta, \omega) \mathrm{d}G(\omega) \\
\text{s.t.} &\int_\Omega v(\theta, \omega) \mathrm{d}G(\omega) \geq 0 \,,
\end{aligned}$$

where G is an outcome distribution over Ω. For an arbitrary $\epsilon > 0$, let G_ϵ be an outcome distribution that satisfies the IR constraint in the above optimization problem and

$$\int_\Omega u(\theta, \omega) \mathrm{d}G_\epsilon(\omega) \geq \bar{u}(\theta) - \epsilon \,.$$

Consider the corresponding dynamic mechanism A_ϵ^T that repeatedly determines an outcome according to G_ϵ in each round. Since G_ϵ satisfies the IR constraint, participating is a utility-maximizing strategy and the agent accepts the outcomes being drawn independently and identically from G_ϵ. The agent's only other option is to not participate which leads to the aggregate utility of 0. Hence, the aggregate utility of the principal under A_ϵ^T is at least $T \cdot \bar{u}(\theta) - \epsilon \cdot T$. As ϵ was arbitrary, this implies $\mathrm{OPT}(\theta, T) \geq T \cdot \bar{u}(\theta)$. Combined with the first part, $\mathrm{OPT}(\theta, T) = T \cdot \bar{u}(\theta)$.

Part 3): Recall that

$$\begin{aligned} \bar{u}(F) := \sup_{S \in \mathcal{S}} & \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \\ \text{s.t.} & \int_\Theta \int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \geq 0\,. \end{aligned} \tag{5}$$

For $\mathbb{E}_{\theta \sim F}[\bar{u}(\theta)]$, we can equivalently write

$$\begin{aligned} \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)] = \sup_{S \in \mathcal{S}} & \int_\Theta \int_\Omega u(\theta, \omega) \mathrm{d}S_\theta(\omega) \mathrm{d}F(\theta) \\ \text{s.t.} & \int_\Omega v(\theta, \omega) \mathrm{d}S_\theta(\omega) \geq 0\,, \quad \forall \theta \in \Theta\,. \end{aligned} \tag{6}$$

Note we always have $\bar{u}(F) \geq \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)]$ for all $F \in \mathcal{F}$ unconditionally. This is because a feasible solution in the optimization problem in (6) is a feasible solution in the optimization problem (5) and obtains the same objective value. □

Proof of Proposition 7. The first part of Proposition 6 says that $\mathrm{OPT}(F, T) \leq T \cdot \bar{u}(F)$ for any $F \in \mathcal{F}$. Taking supremum of both sides over $F \in \mathcal{F}$, we obtain

$$\sup_{F \in \mathcal{F}} \mathrm{OPT}(F, T) \leq T \cdot \sup_{F \in \mathcal{F}} \bar{u}(F) < \infty\,.$$

For the second statement, for any distribution $F \in \mathcal{F}$, we have

$$\mathrm{OPT}(F, T) \leq T \cdot \bar{u}(F) = \mathbb{E}_{\theta \sim F}[T \cdot \bar{u}(\theta)] = \mathbb{E}_{\theta \sim F}[\mathrm{OPT}(\theta, T)]\,,$$

where the first step is by the first part of Proposition 6, the second by the linearity assumption on $\bar{u}(F)$, and the third by the second part of Proposition 6. □

Proof of Proposition 8. In what follows, we show $\bar{u}(F) \leq \mathbb{E}_{\theta \sim F}[\bar{u}(\theta)]$ when the stated conditions hold. By Part 3 of Proposition 6, this suffices. Consequently, it would follow that the second part of Assumption 1 holds by Proposition 7.

Part 1): In words, the game is such that the payment is part of the outcome and enters linearly with coefficients of opposing signs into the utility functions of the principal and agent. Separating out the payment, the outcome space can be represented as $\Omega = \Omega^0 \times \mathbb{R}$ where Ω^0 is the space of non-payment component of the outcomes and an outcome $\hat{\omega}$ is a pair $(\hat{\omega}^0, \hat{p})$ where $\hat{\omega}^0$ is the non-payment component and $\hat{p}$ is the payment. We use superscript 0 to denote the non-payment parts of the outcome and outcome space. Since the payment enters linearly into the utility functions of the principal and agent, we can represent $u(\theta, (\hat{\omega}^0, \hat{p})) = u^0(\theta, \hat{\omega}^0) + \alpha \cdot \hat{p}$ for some function $u^0 : \Theta \times \Omega^0 \to \mathbb{R}$ and scalar $\alpha \geq 0$ and, similarly, $v(\theta, (\hat{\omega}^0, \hat{p})) = v^0(\theta, \hat{\omega}^0) - \beta \cdot \hat{p}$ for some function $v^0 : \Theta \times \Omega^0 \to \mathbb{R}$ and scalar $\beta > 0$. Note we interpret a payment as a monetary transfer from the agent to the principal and this fixes the signs in front of α and β.

Fix an arbitrary distribution $F \in \mathcal{F}$. Let S be an arbitrary feasible solution for the optimization problem (5) defined for $\bar{u}(F)$. We define the payment offset $q_\theta = \frac{1}{\beta}\int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega)$ for all $\theta \in \Theta$. Now, consider a single-round direct mechanism S' where S'_θ is the outcome distribution S_θ modified with the fixed offset q_θ such that to realize an outcome $\hat{\omega} \sim S'_\theta$, we draw $(\hat{\omega}^0, \hat{p}) \sim S_\theta$ and set $\hat{\omega} = (\hat{\omega}^0, \hat{p} + q_\theta)$.

We show that S' is a feasible solution to the optimization problem (6) defining $\mathbb{E}_{\theta\sim F}[\bar{u}(\theta)]$ and that S' obtains the objective value in (6) that is at least that obtained by S in (5). As S was arbitrary, it would follow that $\bar{u}(F) \leq \mathbb{E}_{\theta\sim F}[\bar{u}(\theta)]$ and, as F was arbitrary, the proposition statement would follow.

For any $\theta \in \Theta$,

$$\begin{aligned}
\int_\Omega v(\theta,\omega)\mathrm{d}S'_\theta(\omega) &= \int_{\Omega^0\times\mathbb{R}} v(\theta,(\omega^0, p+q_\theta))\mathrm{d}S_\theta((\omega^0,p)) \\
&= \int_{\Omega^0\times\mathbb{R}} \left(v^0(\theta,\omega^0) - \beta(p+q_\theta)\right)\mathrm{d}S_\theta((\omega^0,p)) \\
&= \int_{\Omega^0\times\mathbb{R}} \left(v(\theta,(\omega^0,p)) - \beta q_\theta\right)\mathrm{d}S_\theta((\omega^0,p)) \\
&= \int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega) - \beta q_\theta \\
&= 0\,,
\end{aligned}$$

where the last step follows from how the payment offset is defined. Hence, S' is a feasible solution to (6).

Similarly, for any $\theta \in \Theta$,

$$\int_\Omega u(\theta,\omega)\mathrm{d}S'_\theta(\omega) = \int_{\Omega^0\times\mathbb{R}} u(\theta,(\omega^0,p+q_\theta))\mathrm{d}S_\theta((\omega^0,p)) = \int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega) + \alpha q_\theta\,.$$

Integrating the first and last expressions over Θ, we obtain

$$\begin{aligned}
\int_\Theta\int_\Omega u(\theta,\omega)\mathrm{d}S'_\theta(\omega)\mathrm{d}F(\theta) &= \int_\Theta\int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta) + \alpha\int_\Theta q_\theta \mathrm{d}F(\theta) \\
&= \int_\Theta\int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta) + \frac{\alpha}{\beta}\int_\Theta\int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta) \\
&\geq \int_\Theta\int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta)\,,
\end{aligned}$$

where the second-to-last step follows from the definition of the payment offset q_θ and the last step follows since S is a feasible solution to (5) and $\int_\Theta\int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta) \geq 0$. Therefore, S' obtains the objective value in (6) that is at least that obtained by S in (5). As S was arbitrary, $\bar{u}(F) \leq \mathbb{E}_{\theta\sim F}[\bar{u}(\theta)]$.

Part 2): By the assumption on the game, we have $v(\theta,\omega) \geq 0$ for all $\theta \in \Theta$ and $\omega \in \Omega$. Then, for any single-round direct mechanism S and shock $\theta \in \Theta$,

$$\int_\Omega v(\theta,\omega)\mathrm{d}S_\theta(\omega) \geq 0\,.$$

Clearly, for any $F \in \mathcal{F}$, any feasible solution to the optimization problem (5) is a feasible solution to the optimization problem (6) and obtains the same objective. The proposition follows. $\square$

Proof of Theorem 9. We first derive a lower bound on the multi-round minimax regret $\text{Regret}(T)$ via the same reasoning used in the proof of Lemma 3. Note that

$$\begin{aligned}
\text{Regret}(T) &= \inf_{A^T \in \mathcal{A}^T} \sup_{F \in \mathcal{F}} \text{Regret}(A^T, F, T) \\
&\geq \inf_{A^T \in \mathcal{A}^T} \sup_{\theta \in \Theta} \text{Regret}(A^T, \theta, T) \\
&= \inf_{A^T \in \mathcal{A}^T} \sup_{\theta \in \Theta} \left\{ \text{OPT}(\theta, T) - \text{PrincipalUtility}(A^T, B^*(A^T, \theta), \theta, T) \right\} \\
&= T \cdot \inf_{A^T \in \mathcal{A}^T} \sup_{\theta \in \Theta} \left\{ \bar{u}(\theta) - \text{PrincipalUtility}(S(A^T), \sigma^{\text{TR}}, \theta, 1) \right\} \\
&= T \cdot \inf_{A^T \in \mathcal{A}^T} \sup_{\theta \in \Theta} \widehat{\text{Regret}}(S(A^T), \theta) \\
&\geq T \cdot \inf_{\substack{S' \in \mathcal{S}: \\ \text{(IC),(IR)}}} \sup_{\theta \in \Theta} \widehat{\text{Regret}}(S', \theta) \\
&= T \cdot \inf_{\substack{S' \in \mathcal{S}: \\ \text{(IC),(IR)}}} \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S', F) \,,
\end{aligned}$$

where the first inequality follows since point-mass distributions are a subset of general distributions $\mathcal{F}$ in the inner maximization expression; the third equality is by Proposition 2 and Lemma 16 where $S(A^T)$ is the single-round direct IC/IR mechanism derived from A^T as described in the proof of Lemma 16; the last inequality follows because the single-round direct IC/IR mechanisms in $\mathcal{S}$ are a superset of those mechanisms $S(A^T)$ derived from dynamic mechanisms, i.e., $\{S(A^T) \mid A^T \in \mathcal{A}^T\}$; the last equality is by Lemma 17; and other steps are by appropriate definitions.

Now, define $\delta = \sup_{F \in \mathcal{F}} \{\text{OPT}(F, T) - \mathbb{E}_{\theta \sim F}[\text{OPT}(\theta, T)]\}$. Let $\epsilon \geq 0$ be arbitrary and consider a single-round direct IC/IR mechanism S satisfying

$$\sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F) \leq \inf_{\substack{S' \in \mathcal{S}: \\ \text{(IC),(IR)}}} \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S', F) + \frac{\epsilon}{T} \,.$$

Then,

$$\begin{aligned}
&\sup_{F \in \mathcal{F}} \text{Regret}(S^{\times T}, F, T) \\
&\qquad = \sup_{F \in \mathcal{F}} \left\{ \text{OPT}(F, T) - \text{PrincipalUtility}(S^{\times T}, B^*(S^{\times T}, F), F, T) \right\} \\
&\qquad = \sup_{F \in \mathcal{F}} \{ \text{OPT}(F, T) - T \cdot \text{PrincipalUtility}(S, B^*(S, F), F, 1) \} \\
&\qquad \leq \sup_{F \in \mathcal{F}} \{ \mathbb{E}_{\theta \sim F}[\text{OPT}(\theta, T)] - T \cdot \text{PrincipalUtility}(S, B^*(S, F), F, 1) \} + \delta \\
&\qquad = T \cdot \sup_{F \in \mathcal{F}} \{ \mathbb{E}_{\theta \sim F}[\text{OPT}(\theta, 1)] - \text{PrincipalUtility}(S, B^*(S, F), F, 1) \} + \delta \\
&\qquad \leq T \cdot \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F) + \delta \,,
\end{aligned}$$

where the first step is by the definition of Regret notion; the second step is by Lemma 18; the third step is because the supremum operator is sublinear; the second-to-last step is by Proposition 2; and the last step follows since truthful reporting is a utility-maximizing strategy for the agent but may not be one that also maximizes the principal's utility among utility-maximizing strategies.

By the above bounds and the property of the single-round direct IC/IR mechanism S, it follows that

$$\begin{aligned}\sup_{F\in\mathcal{F}} \mathrm{Regret}(S^{\times T}, F, T) &\leq T\cdot \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F) + \delta \\ &\leq T\cdot \inf_{\substack{S'\in\mathcal{S}:\\ \text{(IC),(IR)}}} \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S', F) + \delta + \epsilon \\ &\leq \mathrm{Regret}(T) + \delta + \epsilon\,.\end{aligned}$$

To see the second bound in terms of $\bar{u}(F)$, we note for any $F\in\mathcal{F}$ that $\mathrm{OPT}(F,T)\leq T\cdot\bar{u}(F)$ and that $\mathbb{E}_{\theta\sim F}[\mathrm{OPT}(\theta,T)] = T\cdot\mathbb{E}_{\theta\sim F}[\bar{u}(\theta)]$ by Proposition 6. Then,

$$\delta \leq T\cdot \sup_{F\in\mathcal{F}} \left\{\bar{u}(F) - \mathbb{E}_{\theta\sim F}[\bar{u}(\theta)]\right\}\,,$$

and the theorem follows. □

A.5 Proof of Theorem 10

We first prove that the single-round saddle-point property implies the multi-round saddle-point property and then prove the converse. Then, we prove the statements about the saddle points. Following the discussion in Section 3.5, we write the multi-round saddle-point property in terms of both $\inf_{A\in\mathcal{A}}\sup_{F\in\mathcal{F}}\mathrm{Regret}(A,F,T)$ and $\inf_{A\in\mathcal{A}}\sup_{G\in\Delta(\mathcal{F})}\mathbb{E}_{F\sim G}\left[\mathrm{Regret}(A,F,T)\right]$.

Only-If Direction for Saddle-Point Properties. We prove the result by showing first that the minimax regret is at least the maximin regret and then that the minimax regret is at most the maximin regret in the multi-round problem. For any mechanism $A\in\mathcal{A}$ and distribution over distributions $G\in\Delta(\mathcal{F})$, we have

$$\sup_{F\in\mathcal{F}}\mathrm{Regret}(A,F,T) = \mathbb{E}_{F\sim G}\left[\sup_{F'\in\mathcal{F}}\mathrm{Regret}(A,F',T)\right] \geq \mathbb{E}_{F\sim G}\left[\mathrm{Regret}(A,F,T)\right]\,,$$

because $\sup_{F'\in\mathcal{F}}\mathrm{Regret}(A,F',T)\geq\mathrm{Regret}(A,F,T)$. Taking the infimum over $A\in\mathcal{A}$ on both sides, we obtain

$$\inf_{A\in\mathcal{A}}\sup_{F\in\mathcal{F}}\mathrm{Regret}(A,F,T)\geq\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G}\left[\mathrm{Regret}(A,F,T)\right]\,.$$

Taking the supremum over $G\in\Delta(\mathcal{F})$ on the right-hand side, we obtain

$$\inf_{A\in\mathcal{A}}\sup_{F\in\mathcal{F}}\mathrm{Regret}(A,F,T)\geq\sup_{G\in\Delta(\mathcal{F})}\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G}\left[\mathrm{Regret}(A,F,T)\right]\,.$$

We next prove that the minimax regret is at most the maximin regret. Fix an arbitrary distribution $F'\in\mathcal{F}$ and take G' to be the corresponding distribution over atomic distributions (i.e., point-mass distributions) induced by F' as in the statement of the theorem. That is, for every measurable set

$E \subseteq \mathcal{F}$, we have $G'(E) = F'(\{\theta \in \Theta : \delta_\theta \in E\})$ where $\delta_\theta \in \mathcal{F}$ is the point-mass distribution that assigns probability 1 to θ. We have

$$\begin{aligned}
&\sup_{G\in\Delta(\mathcal{F})} \inf_{A\in\mathcal{A}} \mathbb{E}_{F\sim G}\left[\text{Regret}(A,F,T)\right] \\
&\qquad \geq \inf_{A\in\mathcal{A}} \mathbb{E}_{F\sim G'}\left[\text{Regret}(A,F,T)\right] \\
&\qquad = \inf_{A\in\mathcal{A}} \mathbb{E}_{\theta\sim F'}\left[\text{Regret}(A,\theta,T)\right] \\
&\qquad = \inf_{A\in\mathcal{A}} \mathbb{E}_{\theta\sim F'}\left\{\text{OPT}(\theta,T) - \text{PrincipalUtility}(A,B^*(A,\theta),\theta,T)\right\} \\
&\qquad = \underbrace{\mathbb{E}_{\theta\sim F'}\left[\text{OPT}(\theta,T)\right]}_{(I)} - \underbrace{\sup_{A\in\mathcal{A}} \mathbb{E}_{\theta\sim F'}\left[\text{PrincipalUtility}(A,B^*(A,\theta),\theta,T)\right]}_{(II)},
\end{aligned}$$

where the first inequality follows because G' is feasible; the first equality from the definition of G'; and the last equality from extracting the constant term (independent of A) from the infimum over $A \in \mathcal{A}$ and flipping the direction of the optimization. For the first term in the last expression, we have

$$(I) = \mathbb{E}_{\theta\sim F'}\left[\text{OPT}(\theta,T)\right] = T \cdot \mathbb{E}_{\theta\sim F'}\left[\text{OPT}(\theta,1)\right],$$

where we used Proposition 2. The second term corresponds to a Bayesian mechanism design problem in which the agent's shock is constant throughout the rounds and the realization of the shock is private and drawn according to F'. Note Lemma 16 implies that, when the agent's shock is constant, any dynamic mechanism can be reduced to a single-round direct IC/IR mechanism. Therefore, we obtain

$$\begin{aligned}
(II) &= \sup_{A\in\mathcal{A}} \mathbb{E}_{\theta\sim F'}\left[\text{PrincipalUtility}(A,B^*(A,\theta),\theta,T)\right] \\
&= \sup_{A\in\mathcal{A}} \mathbb{E}_{\theta\sim F'}\left[T\cdot\text{PrincipalUtility}(S(A),\sigma^{\text{TR}},\theta,1)\right] \\
&\leq \sup_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \mathbb{E}_{\theta\sim F'}\left[T\cdot\text{PrincipalUtility}(S,\sigma^{\text{TR}},\theta,1)\right] \\
&= T\cdot \sup_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \int_\Theta\int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F'(\theta),
\end{aligned}$$

where the second equality follows from Lemma 16 and $S(A)$ is the single-round direct IC/IR mechanism corresponding to a dynamic mechanism A; the second-to-last step follows because the set of direct IC/IR mechanisms is a superset of $\{S(A) \mid A \in \mathcal{A}\}$; and the last step follows from explicitly writing the principal's utility and using that S is direct and incentive compatible. Putting these two expressions together, we obtain

$$
\begin{aligned}
\sup_{G\in\Delta(\mathcal{F})}\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G}\left[\text{Regret}(A,F,T)\right] &\geq (I)-(II)\\
&\geq T\cdot\mathbb{E}_{\theta\sim F'}[\text{OPT}(\theta,1)] - T\cdot \sup_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}}\int_\Theta\int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F'(\theta)\\
&= T\cdot \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}}\left\{\mathbb{E}_{\theta\sim F'}[\text{OPT}(\theta,1)] - \int_\Theta\int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F'(\theta)\right\}\\
&= T\cdot \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}}\widehat{\text{Regret}}(S,F')\,.
\end{aligned}
$$

Note $F'\in\mathcal{F}$ was an arbitrary distribution. Taking a supremum over $\mathcal{F}$, we conclude that

$$
\begin{aligned}
\sup_{G\in\Delta(\mathcal{F})}\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G}\left[\text{Regret}(A,F,T)\right] &\geq T\cdot\sup_{F\in\mathcal{F}}\inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}}\widehat{\text{Regret}}(S,F)\\
&= T\cdot \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}}\sup_{F\in\mathcal{F}}\widehat{\text{Regret}}(S,F)\\
&= \inf_{A\in\mathcal{A}}\sup_{F\in\mathcal{F}}\text{Regret}(A,F,T)\,,
\end{aligned}
$$

where the first equality follows from the saddle-point property for the single-round problem and the second equality from Theorem 1, which holds since Assumption 1 holds.

If Direction for Saddle-Point Properties. The max-min inequality implies that the minimax regret is at least the maximin regret in the single-round problem and it remains to show that the minimax regret is at most the maximin regret. To do so, it suffices to show that for every $\epsilon>0$, there exists some $F\in\mathcal{F}$ such that $\widehat{\text{Regret}}\leq\epsilon+\inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}}\widehat{\text{Regret}}(S,F)$ where $\widehat{\text{Regret}}=\inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}}\sup_{F\in\mathcal{F}}\widehat{\text{Regret}}(S,F)$.

Fix $\epsilon>0$. From the multi-round saddle-point property, that is,

$$
\text{Regret}(T)=\inf_{A\in\mathcal{A}}\sup_{F\in\mathcal{F}}\text{Regret}(A,F,T)=\sup_{G\in\Delta(\mathcal{F})}\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G}\left[\text{Regret}(A,F,T)\right]\,,
$$

we know that there exists some $G\in\Delta(\mathcal{F})$ such that

$$
\text{Regret}(T)\leq T\cdot\epsilon+\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G}\left[\text{Regret}(A,F,T)\right]\,. \tag{7}
$$

Consider the distribution $\hat{F}\in\mathcal{F}$ obtained from marginalizing over G, i.e., $\hat{F}(E)=\mathbb{E}_{F\sim G}\left[F(E)\right]$ for any measurable set $E\subseteq\Theta$. We have that

$$
\mathbb{E}_{F\sim G}\left[\text{OPT}(F,T)\right]\leq\mathbb{E}_{F\sim G,\theta\sim F}[\text{OPT}(\theta,T)]=\mathbb{E}_{\theta\sim\hat{F}}[\text{OPT}(\theta,T)]=T\cdot\mathbb{E}_{\theta\sim\hat{F}}[\text{OPT}(\theta,1)]\,, \tag{8}
$$

where the inequality follows from Assumption 1 and the law of total expectation; the first equality from the definition of the compounded distribution $\hat{F}$; and the last equality from Proposition 2. For any single-round direct IC/IR mechanism $S\in\mathcal{S}$, consider the mechanism $(S)^{\times T}$ obtained from T

repetitions of S. In turn, the principal's utility satisfies

$$\begin{aligned}\mathbb{E}_{F\sim G}\left[\text{PrincipalUtility}((S)^{\times T}, B^*((S)^{\times T}, F), F, T)\right] &= T\cdot \mathbb{E}_{F\sim G}\left[\text{PrincipalUtility}(S, B^*(S,F), F, 1)\right]\\ &\geq T\cdot \mathbb{E}_{F\sim G}\left[\text{PrincipalUtility}(S, \sigma^{\text{TR}}, F, 1)\right]\\ &= T\cdot \text{PrincipalUtility}(S, \sigma^{\text{TR}}, \hat{F}, 1)\,, \end{aligned} \tag{9}$$

where the first step follows from Lemma 18, the second step is because the truthful reporting strategy σ^{TR} is a utility-maximizing strategy for the agent that may not necessarily maximize the principal's utility, and the last step follows from that the mechanism S is static and does not screen the agent for his distribution and from the definition of $\hat{F}$. Theorem 1 implies $\text{Regret}(T) = T\cdot\widehat{\text{Regret}}$ and, consequently,

$$\begin{aligned}T\cdot\widehat{\text{Regret}} &\leq T\cdot\epsilon + \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \mathbb{E}_{F\sim G}\left[\text{Regret}((S)^{\times T}, F, T)\right]\\ &= T\cdot\epsilon + \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \mathbb{E}_{F\sim G}\left[\text{OPT}(F,T) - \text{PrincipalUtility}((S)^{\times T}, B^*((S)^{\times T}, F), F, T)\right]\\ &\leq T\cdot\epsilon + T\cdot \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \left\{\mathbb{E}_{\theta\sim\hat{F}}[\text{OPT}(\theta,1)] - \text{PrincipalUtility}(S, \sigma^{\text{TR}}, \hat{F}, 1)\right\}\\ &= T\cdot\epsilon + T\cdot \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S, \hat{F})\,,\end{aligned}$$

where the first inequality follows from (7) and using that the space of static mechanisms repeating a single-round direct IC/IR mechanism is a subset of all possible multi-round mechanisms $\mathcal{A}$; the first equality from the definition of Regret; the second inequality from (8) and (9); and the last equality from the definition of $\widehat{\text{Regret}}$. The result follows from dividing by $T\geq 1$.

First Statement on Saddle Points. Let (S^*, F^*) be a saddle point for the single-round problem such that

$$\widehat{\text{Regret}}(S^*, F) \leq \widehat{\text{Regret}}(S^*, F^*) \leq \widehat{\text{Regret}}(S, F^*)$$

for any $S\in\mathcal{S}$ satisfying the IC/IR constraints and $F\in\mathcal{F}$. Note that the existence of the saddle point implies that $\widehat{\text{Regret}} = \widehat{\text{Regret}}(S^*, F^*)$ and the single-round saddle-point property holds and that S^* and F^* are optimal solutions in respective single-round problems achieving the objective value of $\widehat{\text{Regret}}$. To see this, the existence of the saddle point implies

$$\begin{aligned}\inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \sup_{F\in\mathcal{F}} \widehat{\text{Regret}}(S,F) &\leq \sup_{F\in\mathcal{F}} \widehat{\text{Regret}}(S^*,F) \leq \widehat{\text{Regret}}(S^*,F^*)\\ &\leq \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S,F^*) \leq \sup_{F\in\mathcal{F}} \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S,F)\,,\end{aligned}$$

and the max-min inequality implies all the above relations are equalities. By the first statement of the theorem which we proved above, the multi-round saddle-point property holds. By Theorem 1, $(S^*)^{\times T}$ is an optimal solution to the multi-round minimax regret problem since S^* is an optimal solution to the single-round problem.

To show that $((S^*)^{\times T}, G^*)$, for G^* constructed as in the theorem statement, is a saddle point in the multi-round problem, it suffices to show that G^* is an optimal solution to the multi-round maximin regret problem, i.e., $\sup_{G\in\Delta(\mathcal{F})}\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G}[\mathrm{Regret}(A,F,T)] = \inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G^*}[\mathrm{Regret}(A,F,T)]$. Then, we would have

$$\begin{aligned}\mathbb{E}_{F\sim G^*}\left[\mathrm{Regret}((S^*)^{\times T},F,T)\right] &\geq \inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G^*}[\mathrm{Regret}(A,F,T)]\\ &= \sup_{G\in\Delta(\mathcal{F})}\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G}[\mathrm{Regret}(A,F,T)]\\ &= \inf_{A\in\mathcal{A}}\sup_{G\in\Delta(\mathcal{F})}\mathbb{E}_{F\sim G}[\mathrm{Regret}(A,F,T)]\\ &= \sup_{G\in\Delta(\mathcal{F})}\mathbb{E}_{F\sim G}\left[\mathrm{Regret}((S^*)^{\times T},F,T)\right]\\ &\geq \mathbb{E}_{F\sim G^*}\left[\mathrm{Regret}((S^*)^{\times T},F,T)\right]\ ,\end{aligned}$$

where the first equality would follow if G^* is an optimal solution to the multi-round maximin regret problem; the second equality is from the multi-round saddle-point property; and the last equality is from that $(S^*)^{\times T}$ is an optimal solution to the multi-round minimax regret problem. Since the first and last expressions are the same, all the above relations would be equalities and we would have

$$\sup_{G\in\Delta(\mathcal{F})}\mathbb{E}_{F\sim G}\left[\mathrm{Regret}((S^*)^{\times T},F,T)\right] = \mathbb{E}_{F\sim G^*}\left[\mathrm{Regret}((S^*)^{\times T},F,T)\right] = \inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G^*}[\mathrm{Regret}(A,F,T)]\ ,$$

which would imply that $((S^*)^{\times T},G^*)$ is a saddle point in the multi-round problem, as desired.

We now show that G^* is an optimal solution to the multi-round maximin regret problem. Let $A\in\mathcal{A}$ be an arbitrary dynamic mechanism. Then, we have

$$\begin{aligned}\mathbb{E}_{F\sim G^*}[\mathrm{Regret}(A,F,T)] &= \mathbb{E}_{\theta\sim F^*}[\mathrm{Regret}(A,\theta,T)]\\ &= \mathbb{E}_{\theta\sim F^*}[\mathrm{OPT}(\theta,T) - \mathrm{PrincipalUtility}(A,B^*(A,\theta),\theta,T)]\\ &= \mathbb{E}_{\theta\sim F^*}[T\cdot\mathrm{OPT}(\theta,1) - T\cdot\mathrm{PrincipalUtility}(S(A),\sigma^{\mathrm{TR}},\theta,1)]\\ &= T\cdot\widehat{\mathrm{Regret}}(S(A),F^*)\\ &\geq T\cdot\widehat{\mathrm{Regret}}\\ &= \sup_{G\in\Delta(\mathcal{F})}\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G}[\mathrm{Regret}(A,F,T)]\ ,\end{aligned}$$

where the first equality is by the construction of G^*; the second by the definition of Regret; the third by Proposition 2 and Lemma 16 where $S(A)$ is the single-round direct IC/IR mechanism corresponding to the dynamic mechanism A; the fourth by the definition of $\widehat{\mathrm{Regret}}$; the second-to-last step is from that (S^*,F^*) is a saddle point in the single-round problem and $\widehat{\mathrm{Regret}} = \widehat{\mathrm{Regret}}(S^*,F^*)$; and the last step is because Theorem 1 and the multi-round saddle-point property implies

$$T\cdot\widehat{\mathrm{Regret}} = \mathrm{Regret}(T) = \inf_{A\in\mathcal{A}}\sup_{F\in\mathcal{F}}\mathrm{Regret}(A,F,T) = \sup_{G\in\Delta(\mathcal{F})}\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G}[\mathrm{Regret}(A,F,T)]\ .$$

As $A\in\mathcal{A}$ was arbitrary,

$$\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G^*}[\mathrm{Regret}(A,F,T)] \geq \sup_{G\in\Delta(\mathcal{F})}\inf_{A\in\mathcal{A}}\mathbb{E}_{F\sim G}[\mathrm{Regret}(A,F,T)]\ ,$$

and G^* is an optimal solution to the multi-round maximin regret problem. This completes the proof.

Second Statement on Saddle Points. Let the pair (A^*, G^*) of a dynamic mechanism $A^* \in \mathcal{A}$ and a distribution over distributions $G^* \in \Delta(\mathcal{F})$ be a saddle point for the multi-round problem. As in the proof for the first statement on saddle points, the existence of the saddle point implies that the saddle-point property holds in the multi-round problem and, furthermore, that A^* is an optimal dynamic mechanism in the multi-round minimax regret problem and G^* is an optimal solution in the multi-round maximin regret problem, both with the objective value of $\mathrm{Regret}(T)$. By the first part of the theorem, the single-round saddle-point property holds. Given this, it suffices to show that the single-round minimax regret problem admits an optimal solution S^* and that the single-round maximin regret problem admits an optimal solution F^*, or equivalently, the worst-case distribution. Then, the pair (S^*, F^*) would form a saddle-point in the single-round problem, as desired. This is because if S^* and F^* are optimal solutions in respective single-round problems, we would have

$$\begin{aligned}\inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S, F) = \sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S^*, F) \geq \widehat{\mathrm{Regret}}(S^*, F^*) \\ \geq \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\mathrm{Regret}}(S, F^*) = \sup_{F\in\mathcal{F}} \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\mathrm{Regret}}(S, F)\,,\end{aligned}$$

where the equalities would follow from the optimality of S^* and F^*. The single-round saddle-point property would imply that all the relations in the above sequence are equalities and $\widehat{\mathrm{Regret}} = \widehat{\mathrm{Regret}}(S^*, F^*)$. In particular,

$$\sup_{F\in\mathcal{F}} \widehat{\mathrm{Regret}}(S^*, F) = \widehat{\mathrm{Regret}}(S^*, F^*) = \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\mathrm{Regret}}(S, F^*)\,,$$

and this would imply that (S^*, F^*) is a saddle point.

The existence of an optimal single-round direct IC/IR mechanism S^* in the single-round problem follows from the existence of an optimal dynamic mechanism, in particular, A^*, by Theorem 1. We can construct such an optimal mechanism S^* from A^* as discussed in the proof of Theorem 1.

We now show there exists an optimal distribution F^* in the single-round maximin regret problem. Consider the distribution F^* constructed from G^* such that $F^*(Q) = \Pr(\theta \in Q \mid F \sim G^*, \theta \sim F)$ for any $Q \subseteq \Theta$ and, in particular, $\Pr(\hat{\theta} = \theta \mid \hat{\theta} \sim F^*) = \Pr(\hat{\theta} = \theta \mid F \sim G^*, \hat{\theta} \sim F)$ for any $\theta \in \Theta$. Note

$$\begin{aligned}\mathrm{Regret}(T) &\leq \inf_{A\in\mathcal{A}} \mathbb{E}_{F\sim G^*}[\mathrm{Regret}(A, F, T)] \\ &\leq \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \mathbb{E}_{F\sim G^*}[\mathrm{Regret}(S^{\times T}, F, T)] \\ &\leq T \cdot \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \mathbb{E}_{F\sim G^*}[\widehat{\mathrm{Regret}}(S, F)]\,,\end{aligned}$$

where the first inequality is by the optimality of G^*, the second is because the set of static mechanisms that repeat a single-round direct IC/IR mechanism is a subset of dynamic mechanisms $\mathcal{A}$, and the third is by Lemma 4 which holds since Assumption 1 holds. In the last expression, we can equivalently

Algorithm 1. *Mechanism $A^*(F,T)$*

1. Round 0: The report space is $\mathcal{M} = \{\mathsf{CONTINUE}, \mathsf{QUIT}\}$. If the report is QUIT, the outcome is $\emptyset$ for all rounds. If the report is $\mathsf{CONTINUE}$, we continue as follows.
2. Rounds 1 - T: The report space is $\mathcal{M} = \{\theta^1, \theta^2\}$. If the report is θ^1, the outcome is ω^1 with probability q and $\emptyset$ with probability $1-q$. If the report is θ^2, the outcome is ω^2 with probability 1. Note $q = \begin{cases} 1 & \text{, if } f_1 < \frac{1}{2} \\ \frac{1-f_1}{f_1} & \text{, if } f_1 \geq \frac{1}{2} \end{cases}$.

write the expected quantity inside the infimum as, for any single-round direct IC/IR mechanism S,

$$\begin{aligned}\mathbb{E}_{F\sim G^*}[\widehat{\text{Regret}}(S,F)] &= \mathbb{E}_{F\sim G^*}\left[\int_\Theta \text{OPT}(\theta,1)\mathrm{d}F(\theta) - \int_\Theta\int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta)\right] \\ &= \mathbb{E}_{F\sim G^*,\theta\sim F}\left[\text{OPT}(\theta,1) - \int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\right] \\ &= \mathbb{E}_{\theta\sim F^*}\left[\text{OPT}(\theta,1) - \int_\Omega u(\theta,\omega)\mathrm{d}S_\theta(\omega)\right] \\ &= \widehat{\text{Regret}}(S,F^*)\,,\end{aligned}$$

where the first step is by the definition of $\widehat{\text{Regret}}$ notion; the second step is by the total law of expectation; the third step is by the construction of F^*; and the last step is by the definition of $\widehat{\text{Regret}}$ notion. Hence, it follows that

$$\text{Regret}(T) \leq T \cdot \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S,F^*)\,.$$

Since $\text{Regret}(T) = T\cdot\widehat{\text{Regret}}$ by Theorem 1, which holds since Assumption 1 holds, and the single-round saddle-point property holds, we have

$$\widehat{\text{Regret}} = \sup_{F\in\mathcal{F}} \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S,F) \leq \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S,F^*)\,.$$

Then, F^* is an optimal solution in the single-round maximin regret problem. This completes the proof.

B Additional Materials for Section 4

Recall that the game has the outcome space $\Omega = \{\emptyset, \omega^1, \omega^2\}$, the shock space $\Theta = \{\theta^1, \theta^2\}$, and the utility functions of the principal and agent as follows:

$u(\cdot,\cdot)$	$\emptyset$	ω^1	ω^2
θ^1	0	1	0
θ^2	0	1	0

$v(\cdot,\cdot)$	$\emptyset$	ω^1	ω^2
θ^1	0	-1	$-\infty$
θ^2	0	$-\infty$	1

Note $\text{OPT}(F,T) \leq T\cdot\bar{u}(F)$. For the above game, we show $\text{OPT}(F,T) \geq T\cdot\bar{u}(F)$ and it would follow that

$$\mathrm{OPT}(F,T) = T \cdot \bar{u}(F)\,.$$

Consider $A^*(F,T)$ in Algorithm 1. We show reporting $\mathsf{CONTINUE}$ in Round 0 and then reporting truthfully in Rounds 1 - T is optimal for the agent. Given the agent participates in a round (i.e., Rounds 1 - T), truthful reporting is optimal on the per-round basis. If the shock is θ^1, reporting θ^1 yields $-q$ and reporting θ^2 yields $-\infty$. If the shock is θ^2, reporting θ^1 yields $-\infty$ (or 0 if $q = 0$) and reporting θ^2 yields 1. Hence, truthful reporting is optimal in each round. If the agent reports $\mathsf{CONTINUE}$ in Round 0 and participates in all the remaining rounds, the overall utility is $T \cdot (-q \cdot f_1 + f_2)$. The overall utility is $T \cdot (-f_1 + f_2)$ if $f_1 < \frac{1}{2}$ and 0 if $f_1 \geq \frac{1}{2}$, which is at least 0 for any distribution F. If the agent reports QUIT in Round 0 and does not participate in the remaining rounds, the utility is 0. Hence, reporting $\mathsf{CONTINUE}$ followed by truthful reporting is optimal over the entire horizon.

We use σ^{TR} to denote the agent's utility-maximizing strategy of reporting $\mathsf{CONTINUE}$ in Round 0 and then reporting truthfully in Rounds 1 - T. Given that the agent plays σ^{TR}, the principal's utility in each round (i.e., Rounds 1 - T) is $q \cdot f_1 = \bar{u}(F)$ and

$$\mathrm{PrincipalUtility}(A^*(F,T), \sigma^{\mathrm{TR}}, F, T) = T \cdot \bar{u}(F)\,.$$

Then, by that $B^*(A^*(F,T))$ is a utility-maximizing strategy chosen in the principal's favor,

$$\begin{aligned}
\mathrm{OPT}(F,T) &= \sup_{A^T \in \mathcal{A}^T} \mathrm{PrincipalUtility}(A^T, B^*(A^T, F), F, T) \\
&\geq \mathrm{PrincipalUtility}(A^*(F,T), B^*(A^*(F,T), F), F, T) \\
&\geq \mathrm{PrincipalUtility}(A^*(F,T), \sigma^{\mathrm{TR}}, F, T) \\
&= T \cdot \bar{u}(F)\,.
\end{aligned}$$

We now prove Proposition 11. We prove the first part in Appendix B.1 and the second part in Appendix B.2. Proofs of additional results such as lemmas and claims used in the proof of Proposition 11 will be provided in Appendix B.3.

For the proof of Proposition 12 for the modified game with payments, see Appendix B.4.

B.1 First Part of Proposition 11

First, we consider the minimax regret of the multi-round problem when restricted to static mechanisms that repeat a single-round mechanism and show the lower bound: $\inf_{A^1 \in \mathcal{A}^1} \sup_{F \in \mathcal{F}} \mathrm{Regret}((A^1)^{\times T}, F, T) \geq \left(1 - \frac{\sqrt{2}}{2}\right) \cdot T$. Then, we show a static mechanism of the form $(A^1)^{\times T}$ has the regret at most $\left(1 - \frac{\sqrt{2}}{2}\right) T$ and this would imply the upper bound: $\inf_{A^1 \in \mathcal{A}^1} \sup_{F \in \mathcal{F}} \mathrm{Regret}((A^1)^{\times T}, F, T) \leq \left(1 - \frac{\sqrt{2}}{2}\right) \cdot T$. Combining with the lower bound, we would have $\inf_{A^1 \in \mathcal{A}^1} \sup_{F \in \mathcal{F}} \mathrm{Regret}((A^1)^{\times T}, F, T) = \left(1 - \frac{\sqrt{2}}{2}\right) \cdot T$.

We consider static mechanisms that do not screen the agent for his distribution in Round 0. The report space of a static mechanism in Round 0 is $\mathcal{M}_0 = \{\mathsf{CONTINUE}, \mathsf{QUIT}\}$ and the agent can choose to participate or not after learning his distribution. Note those mechanisms that screen the agent for his distribution are referred as "dynamic" in the sequential screening literature, even when

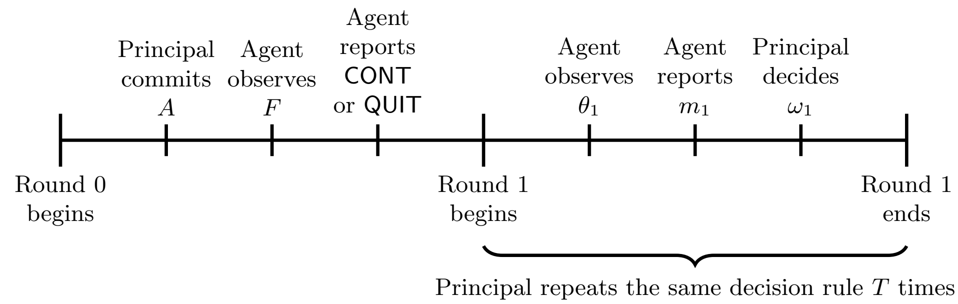

Figure 2: The order of events over the time horizon for a static mechanism, where CONTINUE is shortened to CONT.

$T = 1$ (see, e.g., Krähmer and Strausz 2015; Bergemann et al. 2017). Figure 2 summarizes the order of events for static mechanisms.

Lower bound. By Lemma 18, for any arbitrary $A^1 \in \mathcal{F}^1$ and $F \in \mathcal{F}$,

$$
\begin{aligned}
&\mathrm{Regret}((A^1)^{\times T}, B^*((A^1)^{\times T}, F), F, T) \\
&\qquad = \mathrm{OPT}(F, T) - \mathrm{PrincipalUtility}((A^1)^{\times T}, B^*((A^1)^{\times T}, F), F, T) \\
&\qquad = \mathrm{OPT}(F, T) - T \cdot \mathrm{PrincipalUtility}(A^1, B^*(A^1, F), F, 1)\,.
\end{aligned}
$$

Since $\mathrm{OPT}(F, T) = T \cdot \bar{u}(F) = T \cdot \mathrm{OPT}(F, 1)$,

$$
\begin{aligned}
\mathrm{Regret}((A^1)^{\times T}, B^*((A^1)^{\times T}, F), F, T) &= T \cdot \big(\mathrm{OPT}(F, 1) - \mathrm{PrincipalUtility}(A^1, B^*(A^1, F), F, 1)\big) \\
&= T \cdot \mathrm{Regret}(A^1, B^*(A^1, F), F, 1)\,.
\end{aligned}
$$

It suffices to show that for any $A^1 \in \mathcal{A}^1$,

$$
\sup_{F \in \mathcal{F}} \mathrm{Regret}(A^1, B^*(A^1, F), F, 1) \geq 1 - \frac{\sqrt{2}}{2}\,. \tag{10}
$$

Fix an arbitrary single-round mechanism $A^1 \in \mathcal{A}^1$. By the revelation principle, for any F and $B^*(A^1, F)$, there exists a single-round direct mechanism $S^F = \{S^F_{\theta^1}, S^F_{\theta^2}\}$ that is outcome-equivalent and satisfies the ex-ante IR constraint, where $S^F_{\theta^i}$ is the outcome distribution over $\{\emptyset, \omega^1, \omega^2\}$ for report θ^i. That is, it is outcome-equivalent in the sense that

$$
\begin{aligned}
&\mathrm{PrincipalUtility}(A^1, B^*(A^1, F), F, 1) = \mathrm{PrincipalUtility}(S^F, \sigma^{\mathrm{TR}}, F, 1) \quad \text{and} \\
&\mathrm{AgentUtility}(A^1, B^*(A^1, F), F, 1) = \mathrm{AgentUtility}(S^F, \sigma^{\mathrm{TR}}, F, 1)\,,
\end{aligned}
$$

and satisfies

$$
\mathrm{AgentUtility}(S^F, \sigma^{\mathrm{TR}}, F, 1) \geq 0\,,
$$

where σ^{TR} is the strategy that reports CONTINUE in Round 0 and then truthfully reports the shock in Round 1. We assume that for all distributions F, the agent's strategy $B^*(A^1, F)$ deterministically reports either CONTINUE or QUIT in Round 0. If $B^*(A^1, F)$ reports CONTINUE with some probability, the agent utility conditioned on reporting CONTINUE in Round 0 must be at least 0 which is the agent utility conditioned on reporting QUIT in Round 0. Since $B^*(A^1, F)$ is a utility-maximizing strategy chosen in the principal's favor and it is better if the agent to continue than to quit from the principal's perspective, we can assume $B^*(A^1, F)$ reports CONTINUE with probability 1 in Round 0. If $B^*(A^1, F)$ reports QUIT with probability 1 in Round 0, the maximum utility achievable by the agent after reporting CONTINUE in Round 0 is at most 0.

We represent the outcome distribution $S^F_{\theta^1}$ with probabilities $(\alpha^F_0, \alpha^F_1, \alpha^F_2)$ where $\sum_i \alpha^F_i = 1$ such that the outcome is $\emptyset$ with probability α^F_0, ω^1 with probability α^F_1 and ω^2 with probability α^F_2. We similarly represent $S^F_{\theta^2}$ with probabilities $(\beta^F_0, \beta^F_1, \beta^F_2)$ where $\sum_i \beta^F_i = 1$. When convenient, we use the vectors α^F and β^F to denote these probabilities. For distributions F for which $B^*(A^1, F)$ reports CONTINUE in Round 0, the agent utility is

$$\begin{aligned}\text{AgentUtility}(A^1, B^*(A^1, F), F, 1) &= f_1 \cdot (0 \cdot \alpha^F_0 - 1 \cdot \alpha^F_1 - \infty \cdot \alpha^F_2) + f_2 \cdot \left(0 \cdot \beta^F_0 - \infty \cdot \beta^F_1 + 1 \cdot \beta^F_2\right) \\ &\geq 0\,,\end{aligned}$$

and the principal utility is

$$\text{PrincipalUtility}(A^1, B^*(A^1, F), F, 1) = f_1 \cdot \alpha^F_1 + f_2 \cdot \beta^F_1\,.$$

Since the agent utility is at least 0, it must be that $\alpha^F_2 = 0$ if $f_1 > 0$ and $\beta^F_1 = 0$ if $f_2 > 0$. For distributions F for which $B^*(A^1, F)$ reports QUIT in Round 0, $\alpha^F = \beta^F = 0$ and both the agent and principal utilities are 0.

Without loss of generality, we assume $B^*(A^1, F)$ reports QUIT in Round 0 for $F = (1, 0)$. If $f_1 = 1$, there is no way to make the agent utility to be strictly positive. Hence, the maximum utility achievable for the agent is 0 which is achieved by, in particular, reporting QUIT in Round 0. For all possible outcome distributions that lead to the agent utility of 0, we have the principal utility to be equal to 0.

If there exists no distribution F such that $B^*(A^1, F)$ reports CONTINUE in Round 0, then

$$\text{PrincipalUtility}(A^1, B^*(A^1, F), F, 1) = 0\,,$$

for all $F \in \mathcal{F}$. Then, it follows that

$$\sup_{F \in \mathcal{F}} \text{Regret}(A^1, B^*(A^1, F), F, 1) = \sup_{F \in \mathcal{F}} \text{OPT}(F, 1) = \frac{1}{2}\,,$$

which implies (10). Hence, we assume there exists at least one distribution F with $f_1 \in [0, 1)$ such that $B^*(A^1, F)$ reports CONTINUE in Round 0. Note we rule out $f_1 = 1$ because of the above assumption that $B^*(A^1, (1, 0))$ reports QUIT in Round 0.

The following lemma simplifies our analysis. We defer its proof to Appendix B.3:

Lemma 19. *For distributions F for which $B^*(A^1, F)$ reports* CONTINUE *in Round 0, we can assume the same probabilities α and β where $\alpha_2 = \beta_1 = 0$ without loss of generality.*

By the above lemma, we assume the same probabilities $(\alpha_0, \alpha_1, \alpha_2)$ and $(\beta_0, \beta_1, \beta_2)$, with $\alpha_2 = \beta_1 = 0$, for these distributions F for which $B^*(A^1, F)$ reports CONTINUE in Round 0. In particular, for these distributions $F = (f_1, f_2)$, it must be that

$$f_1 \cdot (0 \cdot \alpha_0 - 1 \cdot \alpha_1 - \infty \cdot \alpha_2) + f_2 \cdot (0 \cdot \beta_0 - \infty \cdot \beta_1 + 1 \cdot \beta_2) = f_1 \cdot -\alpha_1 + f_2 \cdot \beta_2 \geq 0\,.$$

If $\alpha_1 = 0$, the principal utility is

$$\text{PrincipalUtility}(A^1, B^*(A^1, F), F, 1) = 0\,,$$

for all distributions F whether or not $B^*(A^1, F)$ reports $\mathsf{CONTINUE}$ in Round 0. This means

$$\sup_{F \in \mathcal{F}} \text{Regret}(A^1, B^*(A^1, F), F, 1) = \sup_{F \in \mathcal{F}} \text{OPT}(F, 1) = \frac{1}{2}\,,$$

which implies (10). Therefore, we assume $\alpha_1 > 0$ in what follows.

The following claim is useful and is proved in Appendix B.3:

Claim 2. *Assume $\alpha_1 > 0$ and let $r(\alpha, \beta) := \frac{\beta_2}{\alpha_1 + \beta_2} \in [0, 1]$. Then, $f_1 \leq r(\alpha, \beta)$ for all distributions $F = (f_1, f_2)$ for which $B^*(A^1, F)$ reports* $\mathsf{CONTINUE}$ *in Round 0.*

Given the claim, we assume without loss in terms of the principal and agent utilities that for all distributions $F = (f_1, f_2)$, $f_1 \leq r(\alpha, \beta)$ if and only if $B^*(A^1, F)$ reports $\mathsf{CONTINUE}$ in Round 0. The claim shows the sufficiency (if) direction. For the necessity (only-if) direction, assume a distribution F with $f_1 \leq r(\alpha, \beta)$ and $B^*(A^1, F)$ reporting QUIT in Round 0. Multiplying both sides of $f_1 \leq r(\alpha, \beta)$ by the denominator of $r(\alpha, \beta)$ and rearranging terms, we obtain

$$f_1 \cdot -\alpha_1 + f_2 \cdot \beta_2 \geq 0\,.$$

The left-hand side is the agent utility when the agent reports $\mathsf{CONTINUE}$ in Round 0 and implements $B^*(A^1, F')$ for some distribution F' that reports $\mathsf{CONTINUE}$ in Round 0. This is at least 0 which is the agent utility when the agent reports QUIT in Round 0. Note if the agent reports QUIT in Round 0, the principal utility will always be 0. Since $B^*(A^1, F)$ is the agent's utility-maximizing strategy chosen in the principal's favor, we can assume $B^*(A^1, F)$ reports $\mathsf{CONTINUE}$ in Round 0. From the principal's perspective, it is better for the agent to continue than to quit.

Now, we consider the following cases.

Case 1) $r(\alpha, \beta) < \frac{1}{2}$

For $F^* = (\frac{1}{2}, \frac{1}{2})$, $B^*(A^1, F^*)$ reports QUIT in Round 0 and the principal utility is 0. Then,

$$\begin{aligned}
&\sup_{F \in \mathcal{F}} \text{Regret}(A^1, B^*(A^1, F), F, 1) \\
&\qquad \geq \text{OPT}(F^*, 1) - \text{PrincipalUtility}(A^1, B^*(A^1, F^*), F^*, 1) \\
&\qquad = \frac{1}{2}\,,
\end{aligned}$$

which implies (10).

Case 2) $r(\alpha, \beta) \geq \frac{1}{2}$

We consider two subcases. For F with $f_1 > r(\alpha, \beta)$, $B^*(A^1, F)$ reports QUIT in Round 0 and the principal utility is 0. Then,

$$\begin{aligned}\sup_{F: f_1 > r(\alpha,\beta)} \text{Regret}(A^1, B^*(A^1, F), F, 1) &= \sup_{F: f_1 > r(\alpha,\beta)} \text{OPT}(F, 1) \\ &= \sup_{F: f_1 > r(\alpha,\beta)} \bar{u}(F) \\ &= \sup_{F: f_1 > r(\alpha,\beta)} 1 - f_1 \\ &= 1 - r(\alpha, \beta) \\ &= \frac{\alpha_1}{\alpha_1 + \beta_2},\end{aligned}$$

where the second step is because $\text{OPT}(F, 1) = \bar{u}(F)$ and the third is because $\bar{u}(F) = 1 - f_1$ for $f_1 > r(\alpha, \beta) \geq \frac{1}{2}$. Note the denominator of the last expression is strictly positive.

For F with $f_1 \leq r(\alpha, \beta)$, $B^*(A^1, F)$ reports $\mathsf{CONTINUE}$ in Round 0 and the principal utility is $f_1 \cdot \alpha_1$. Then,

$$\begin{aligned}&\sup_{F: f_1 \leq r(\alpha,\beta)} \text{Regret}(A^1, B^*(A^1, F), F, 1) \\ &\qquad= \sup_{F: f_1 \leq r(\alpha,\beta)} \text{OPT}(F, 1) - \text{PrincipalUtility}(A^1, B^*(A^1, F), F, 1) \\ &\qquad= \sup_{F: f_1 \leq r(\alpha,\beta)} \bar{u}(F) - f_1 \cdot \alpha_1 \\ &\qquad\geq \frac{1}{2} - \frac{1}{2}\alpha_1,\end{aligned}$$

where the last step follows because $F = (\frac{1}{2}, \frac{1}{2})$ is one such distribution with $f_1 \leq r(\alpha, \beta)$.

Combining the two subcases,

$$\begin{aligned}\sup_{F \in \mathcal{F}} \text{Regret}(A^1, B^*(A^1, F), F, 1) &\geq \max\left\{\frac{1}{2} - \frac{1}{2}\alpha_1 \ , \ \frac{\alpha_1}{\alpha_1 + \beta_2}\right\} \\ &\geq \max\left\{\frac{1}{2} - \frac{1}{2}\alpha_1 \ , \ \frac{\alpha_1}{\alpha_1 + 1}\right\},\end{aligned}$$

where the last inequality is because $\beta_2 \leq 1$. Then,

$$\begin{aligned}\sup_{F \in \mathcal{F}} \text{Regret}(A^1, B^*(A^1, F), F, 1) &\geq \max\left\{\frac{1}{2} - \frac{1}{2}\alpha_1 \ , \ \frac{\alpha_1}{\alpha_1 + 1}\right\} \\ &\geq 1 - \frac{\sqrt{2}}{2},\end{aligned} \tag{11}$$

because the right-hand side is minimized when $\alpha_1 = -1 + \sqrt{2}$ which is a root of $x^2 + 2x - 1 = 0$. For this particular value of α_1, the arguments in the max operator are equal and the right-hand side evaluates to $1 - \frac{\sqrt{2}}{2} \approx 0.292893$. It follows that (10) holds.

In all cases, (10) holds and this completes the lower bound.

Algorithm 2. *Static mechanism A^T*

1. Round 0: The report space is $\mathcal{M} = \{\mathsf{CONTINUE}, \mathsf{QUIT}\}$. If the report is QUIT, the outcome is $\emptyset$ for all rounds. If the report is CONTINUE, we continue as follows.
2. Rounds 1 - T: The report space is $\mathcal{M} = \{\theta^1, \theta^2\}$. If the report is θ^1, the outcome is ω^1 with probability q and $\emptyset$ with probability $1 - q$ where $q = \sqrt{2} - 1$. If the report is θ^2, the outcome is ω^2 with probability 1.

Upper bound. Consider the static mechanism A^T in Algorithm 2. This is essentially $A^*(F, T)$ in Algorithm 1 except that the probability q is not tailored to the agent's distribution and is fixed to be $\sqrt{2} - 1$ which is the value of α_1 that realizes the lower bound in (11).

Let σ^{TR} denote the agent's strategy in which he reports CONTINUE in Round 0 and then reports truthfully in the subsequent rounds. Note that σ^{TR} is a utility-maximizing strategy for the agent when his distribution $F = (f_1, f_2)$ satisfies $f_1 \leq \frac{1}{1+q}$. Given that the agent participates in a round, truthful reporting is optimal on the per-round basis for any values of q. If the shock is θ^1, reporting θ^1 yields $-q$ and reporting θ^2 yields $-\infty$. If the shock is θ^2, reporting θ^1 yields $-\infty$ (or 0 if $q = 0$) and reporting θ^2 yields 1. Then, σ^{TR} leads to the agent utility of $T \cdot (-q \cdot f_1 + 1 \cdot f_2)$ which is greater than or equal to 0 if and only if $f_1 \leq \frac{1}{1+q}$. Not participating leads to the agent utility of 0 and, hence, σ^{TR} is a utility-maximizing strategy when $f_1 \leq \frac{1}{1+q}$. When the agent implements σ^{TR}, the principal utility will be $T \cdot q \cdot f_1$. When $f_1 > \frac{1}{1+q}$, the best the agent could do by participating leads to the agent utility of strictly less than 0 and not participating is a utility-maximizing strategy.

We upper bound the regret $\mathrm{Regret}(A^T, F)$ in the following cases depending on $F = (f_1, f_2)$:

Case 1) $f_1 \in [0, \frac{1}{2})$

We have $\mathrm{OPT}(F, T) = T \cdot \bar{u}(F) = T \cdot f_1$ and $\mathrm{PrincipalUtility}(A^T, \sigma^{\mathrm{TR}}, F, T) = T \cdot q \cdot f_1$. Then,

$$\begin{aligned}
\mathrm{Regret}(A^T, F) &\leq \mathrm{OPT}(F, T) - \mathrm{PrincipalUtility}(A^T, \sigma^{\mathrm{TR}}, F, T) \\
&= T f_1 (1 - q) \\
&< T \frac{1 - q}{2} \\
&= T \cdot \left(1 - \frac{\sqrt{2}}{2}\right),
\end{aligned}$$

where the first inequality is because σ^{TR} is a utility-maximizing strategy for the agent that might be different from $B^*(A^T, F)$ and the second is because $f_1 < \frac{1}{2}$.

Case 2) $f_1 \in [\frac{1}{2}, \frac{1}{1+q}]$

Since $f_1 \geq \frac{1}{2}$, $\mathrm{OPT}(F, T) = T \cdot \bar{u}(F) = T \cdot (1 - f_1)$. As in the above case, $\mathrm{PrincipalUtility}(A^T, \sigma^{\mathrm{TR}}, F, T) =$

Algorithm 3. *Dynamic mechanism A^T parametrized in terms of T_1, T_2, δ, and q_0.*

1. Round 0: The report space is $\mathcal{M} = \{\mathsf{CONTINUE}, \mathsf{QUIT}\}$. If the report is QUIT, the outcome is $\emptyset$ for all rounds. If the report is $\mathsf{CONTINUE}$, we continue.
2. Phase 1 (T_1 rounds): The report space is $\mathcal{M} = \{\theta^1, \theta^2\}$. If the report is θ^1, the outcome is ω^1 with probability q_0 and $\emptyset$ with probability $1 - q_0$. If the report is θ^2, the outcome is ω^2 with probability 1.
3. Round $T_1 + 1$: The report space is $\mathcal{M} = \{\mathsf{CONTINUE}, \mathsf{QUIT}\}$. If the report is QUIT, the outcome is $\emptyset$ for the current and all remaining rounds. If the report is $\mathsf{CONTINUE}$, the outcome is $\emptyset$ for the current round and we continue.
4. Phase 2 (T_2 rounds):
   - From Phase 1, compute the fraction $\hat{f}_1$ of reports of θ^1. Let $\tilde{f}_1 = \hat{f}_1 + \delta$ and $\tilde{q} = \begin{cases} 1 & \text{, if } \tilde{f}_1 < \frac{1}{2} \\ \frac{1-\tilde{f}_1}{\tilde{f}_1} & \text{, if } \tilde{f}_1 \geq \frac{1}{2} \end{cases}$.
   - The report space is $\mathcal{M} = \{\theta^1, \theta^2\}$. If the report is θ^1, the outcome is ω^1 with probability $\tilde{q}$ and $\emptyset$ with probability $1 - \tilde{q}$. If the report is θ^2, the outcome is ω^2 with probability 1.

$T \cdot q \cdot f_1$. It follows that

$$\begin{aligned} \text{Regret}(A^T, F) &\leq \text{OPT}(F, T) - \text{PrincipalUtility}(A^T, \sigma^{\text{TR}}, F, T) \\ &= T(1 - f_1 - f_1 q) \\ &\leq T\frac{1-q}{2} \\ &= T \cdot \left(1 - \frac{\sqrt{2}}{2}\right), \end{aligned}$$

where the first inequality is because σ^{TR} is not necessarily a utility-maximizing strategy chosen in the principal's favor and the second inequality is because the expression is maximized when f_1 is the smallest possible, i.e., $f_1 = \frac{1}{2}$.

Case 3) $f_1 \in (\frac{1}{1+q}, 1]$

In this case, the agent does not participate and $\text{PrincipalUtility}(A^T, B^*(A^T, F), F, T) = 0$. Since $f_1 \geq \frac{1}{2}$, $\text{OPT}(F, T) = T \cdot \bar{u}(F) = T \cdot (1 - f_1)$. Then,

$$\text{Regret}(A^T, F) = \text{OPT}(F, T) = T \cdot (1 - f_1) < T \cdot \left(1 - \frac{1}{1+q}\right) = T \cdot \left(1 - \frac{\sqrt{2}}{2}\right),$$

where we used $f_1 > \frac{1}{1+q}$.

Combining all the above cases, $\sup_{F \in \mathcal{F}} \text{Regret}(A^T, F) \leq \left(1 - \frac{\sqrt{2}}{2}\right) \cdot T$, which gives the desired upper bound of $\inf_{A^1 \in \mathcal{A}^1} \sup_{F \in \mathcal{F}} \text{Regret}((A^1)^{\times T}, F, T) \leq \left(1 - \frac{\sqrt{2}}{2}\right) \cdot T$.

B.2 Second Part of Proposition 11

Consider A^T in Algorithm 3 which is parametrized in terms of T_1, T_2, δ and q_0. We choose $T_1 = T^{2/3}$, $T_2 = T - T_1 - 1$, $\delta = \sqrt{\frac{\ln T_1}{4T_1}}$ and $q_0 = \frac{1}{\sqrt{T_1}}$. For ease of presentation, we mostly use T_1, T_2, δ and q_0 as parameters and use their values when necessary. We prove the result in three steps. First, we show that truthful reporting is a utility-maximizing strategy for the agent when his distribution $F = (f_1, f_2)$ satisfies $f_1 \in [0, \frac{1}{1+q_0}]$. Second, we lower bound the principal's utility when the agent reports truthfully. Finally, we analyze the regret of the dynamic mechanism.

Step 1. Assume the agent's distribution $F = (f_1, f_2)$ satisfies $f_1 \in [0, \frac{1}{1+q_0}]$. Let σ^{TR} denote the strategy that reports $\mathsf{CONTINUE}$ in Round 0, reports truthfully during Phase 1, reports $\mathsf{CONTINUE}$ if $-\tilde{q} \cdot f_1 + 1 \cdot f_2 \geq 0$ in Round $T_1 + 1$, and then reports truthfully during Phase 2 if the game continues to Phase 2. We note that the agent utility is at least 0 under σ^{TR}. In Phase 1, truthful reporting leads to the utility of $T_1 \cdot (-q_0 \cdot f_1 + 1 \cdot f_2) \geq 0$, since $f_1 \leq \frac{1}{1+q_0}$ implies $-q_0 \cdot f_1 + 1 \cdot (1 - f_1) = 1 - (1 + q_0) \cdot f_1 \geq 1 - (1 + q_0) \cdot \frac{1}{1+q_0} = 0$. If the game continues to Phase 2, it must be that $-\tilde{q} \cdot f_1 + 1 \cdot f_2 \geq 0$ and σ^{TR} leads to the utility of $T_2 \cdot (-\tilde{q} \cdot f_1 + 1 \cdot f_2) \geq 0$ in Phase 2. If the game does not continue, then it leads to the utility of 0 in Phase 2. In expectation, σ^{TR} leads to the utility of at least 0 in Phase 2. Hence, the overall utility is at least 0.

In fact, σ^{TR} is a utility-maximizing strategy for the agent. To see this, we first note truthful reporting is optimal on the per-round basis in each round in Phase 1 and in Phase 2 (for any value of $\tilde{q}$). That is, given the agent participates in a round, truthful reporting is a utility-maximizing strategy for the agent in that round. In each round in Phase 1, if the shock is θ^1, reporting θ^1 yields $-q_0$ and reporting θ^2 yields $-\infty$. If the shock is θ^2, reporting θ^2 yields 1 and reporting θ^1 yields $-\infty$. In each round in Phase 2, for any value of $\tilde{q}$, truthful reporting is optimal for the agent. If the shock is θ^1, reporting θ^1 yields $-\tilde{q}$ and reporting θ^2 yields $-\infty$. If the shock is θ^2, reporting θ^2 yields 1 and reporting θ^1 yields $-\infty$ (or 0 if $\tilde{q} = 0$).

Then, we note the only way for the agent to influence the principal's mechanism A^T is through reports in Phase 1 which determine the probability $\tilde{q}$ in Phase 2. Intuitively, the agent may consider some non-truthful reporting strategy in Phase 1 and continue to Phase 2 with $\tilde{q}$ determined favorably to benefit himself. Non-truthful reporting can only lead to lower per-round utilities during Phase 1 and each misreport of the shock costs $-\infty$. The agent needs to continue to Phase 2 in order to gain from such non-truthful reporting, but the cost overwhelms the potential gain from Phase 2. It follows that it is optimal for the agent to truthfully report during Phase 1 and given this observation, reporting $\mathsf{CONTINUE}$ in Round T_1+1 if $-\tilde{q} \cdot f_1 + 1 \cdot f_2 \geq 0$ can only benefit the agent because truthfully reporting is optimal on the per-round basis in Phase 2 and the utility from Phase 2 is $T_2 \cdot (-\tilde{q} \cdot f_1 + 1 \cdot f_2)$ given the game continues to Phase 2. Note the agent cannot influence the principal's mechanism during Phase 2. Hence, it is not possible to realize a greater utility overall than that achieved under σ^{TR}.

Step 2. Assume the agent's distribution $F = (f_1, f_2)$ satisfies $f_1 \in [0, \frac{1}{1+q_0}]$. Let U_1^{TR} and U_2^{TR} be the principal utility from Phases 1 and 2, respectively, when the principal's mechanism is A^T and the agent's utility-maximizing strategy is σ^{TR}, such that

$$\mathrm{PrincipalUtility}(A^T, \sigma^{\mathrm{TR}}, F, T) = U_1^{\mathrm{TR}} + U_2^{\mathrm{TR}}.$$

Note

$$U_1^{\mathrm{TR}} = T_1 \cdot q_0 \cdot f_1 \geq 0\,. \tag{12}$$

We now consider U_2^{TR}. Let $\mathcal{E}$ be the event that the game continues to Phase 2 under σ^{TR}, or equivalently, $-\tilde{q}\cdot f_1 + 1\cdot f_2 \geq 0$ and $\mathbf{1}_{\mathcal{E}}$ be the indicator that equals to 1 if the event occurs, and 0 otherwise; so, $\mathbf{1}_{\mathcal{E}} = \mathbf{1}\{-\tilde{q}\cdot f_1 + 1\cdot f_2 \geq 0\}$. We have $U_2^{\mathrm{TR}} = T_2 \cdot \mathbb{E}[\tilde{q}\cdot f_1 \cdot \mathbf{1}_{\mathcal{E}}]$. Note that $\frac{1}{1+q_0} \geq \frac{1}{2}$ because $q_0 \leq 1$. We consider the following cases depending on whether f_1 is above or below $\frac{1}{2}$.

If $f_1 \leq 1/2$, then the event $\mathcal{E}$ always occurs because $f_2 = 1 - f_1 \geq f_1$ and $\tilde{q} \in [0,1]$. Therefore,

$$U_2^{\mathrm{TR}} = f_1 \cdot T_2 \cdot \mathbb{E}\left[\min\left\{1, \frac{1-\tilde{f}_1}{\tilde{f}_1}\right\}\right]\,, \tag{13}$$

where $\tilde{f}_1$ is always strictly positive because $\tilde{f}_1 = \hat{f}_1 + \delta \geq \delta > 0$.

If $f_1 > 1/2$, whenever the event $\mathcal{E}$ occurs we have $\tilde{f}_1 \geq 1/2$ and, consequently, $\tilde{q} = \frac{1-\tilde{f}_1}{\tilde{f}_1}$. To see this, note that if $\tilde{f}_1 < 1/2$, then $\tilde{q} = 1$, which implies that $-\tilde{q}\cdot f_1 + 1\cdot f_2 = -f_1 + f_2 < 0$ and event $\mathcal{E}$ does not occur. Then, $-\tilde{q}\cdot f_1 + 1 \cdot f_2 \geq 0$ is equivalent to $\frac{1-f_1}{f_1} \geq \tilde{q} = \frac{1-\tilde{f}_1}{\tilde{f}_1}$. Since the transformation $x \mapsto \frac{1-x}{x}$ is decreasing, the event $\mathcal{E}$ can be equivalently written as $\mathcal{E} = \{\tilde{f}_1 \geq f_1\}$. This implies that

$$\begin{aligned}
U_2^{\mathrm{TR}} &= f_1 \cdot T_2 \cdot \mathbb{E}\left[\mathbf{1}\{\tilde{f}_1 \geq f_1\} \cdot \frac{1-\tilde{f}_1}{\tilde{f}_1}\right] \\
&= T_2 \cdot \left((1-f_1)\cdot \Pr(\tilde{f}_1 \geq f_1) - \mathbb{E}\left[\max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\}\right]\right)\,,
\end{aligned} \tag{14}$$

where we used that $\frac{f_1(1-\tilde{f}_1)}{\tilde{f}_1} = (1-f_1) - \left(1 - \frac{f_1}{\tilde{f}_1}\right)$ and $\mathbf{1}\{\tilde{f}_1 \geq f_1\}\cdot\left(1 - \frac{f_1}{\tilde{f}_1}\right) = \max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\}$.

The following result will be used in bounding the principal's utility. The proof is provided in Appendix B.3.

Lemma 20. *The following hold:*

1. *If* $f_1 \leq 1/2$, *then* $\mathbb{E}\left[\min\left\{1, \frac{1-\tilde{f}_1}{\tilde{f}_1}\right\}\right] \geq 1 - 4\delta - \frac{2}{\sqrt{T_1}}$.
2. *If* $f_1 > 1/2$, *then* $\mathbb{E}\left[\max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\}\right] \leq 2\delta + \frac{1}{\sqrt{T_1}}$.
3. $\Pr(\tilde{f}_1 \geq f_1) \geq 1 - e^{-2\delta^2 T_1}$.

Step 3. We next bound the regret of the dynamic mechanism A^T. Note

$$\begin{aligned}
\mathrm{Regret}(A^T, F, T) &= \mathrm{Regret}(A^T, B^*(A^T,F), F, T) \\
&\leq \mathrm{Regret}(A^T, \sigma^{\mathrm{TR}}, F, T) \\
&= \mathrm{OPT}(F,T) - \mathrm{PrincipalUtility}(A^T, \sigma^{\mathrm{TR}}, F, T) \\
&= T \cdot \bar{u}(F) - \mathrm{PrincipalUtility}(A^T, \sigma^{\mathrm{TR}}, F, T)\,,
\end{aligned}$$

where the first inequality is because $B^*(A^T, F)$ is a utility-maximizing strategy chosen in the principal's favor. We upper bound the regret in three separate cases depending on the agent's distribution $F = (f_1, f_2)$.

If $f_1 \in [0, \frac{1}{2}]$, then $\bar{u}(F) = f_1$. From (12) and (13), we have

$$\text{PrincipalUtility}(A^T, \sigma^{\text{TR}}, F, T) \ge f_1 \cdot T_2 \cdot \mathbb{E}\left[\min\left\{1, \frac{1-\tilde{f}_1}{\tilde{f}_1}\right\}\right].$$

Then,

$$\begin{aligned}
\frac{1}{T}\text{Regret}(A^T, F, T) &\le f_1\left(1 - \frac{T_2}{T}\cdot \mathbb{E}\left[\min\left(1, \frac{1-\tilde{f}_1}{\tilde{f}_1}\right)\right]\right) \\
&\le f_1\left(1 - \left(1 - \frac{T_1+1}{T}\right)\cdot\left(1 - 4\delta - \frac{2}{\sqrt{T_1}}\right)\right) \\
&= f_1\left(1 - \left(1 - \frac{T_1+1}{T} - 4\delta - \frac{2}{\sqrt{T_1}} + \frac{T_1+1}{T}\cdot\left(4\delta + \frac{2}{\sqrt{T_1}}\right)\right)\right) \\
&\le f_1\left(\frac{T_1+1}{T} + 4\delta + \frac{2}{\sqrt{T_1}}\right) \\
&\le 2\delta + \frac{T_1}{T} + \frac{1}{\sqrt{T_1}}, \qquad (15)
\end{aligned}$$

where the second inequality follows from $T_2 = T - T_1 - 1$ and Part 1 of Lemma 20; the second-to-last inequality follows from dropping the negative term in the resulting expression in the parentheses; and the last inequality follows from $f_1 \le \frac{1}{2}$ and $T_1 \ge 1$ which implies $T_1 + 1 \le 2T_1$.

If $f_1 \in (\frac{1}{2}, \frac{1}{1+q_0}]$, then $\bar{u}(F) = 1 - f_1$. From (12) and (14), we have

$$\text{PrincipalUtility}(A^T, \sigma^{\text{TR}}, F, T) \ge T_2 \cdot \left((1-f_1)\cdot \Pr(\tilde{f}_1 \ge f_1) - \mathbb{E}\left[\max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\}\right]\right).$$

Consequently, we obtain

$$\begin{aligned}
\frac{1}{T}\text{Regret}(A^T, F, T) &\le (1-f_1)\left(1 - \frac{T_2}{T}\cdot\Pr(\tilde{f}_1 \ge f_1)\right) + \frac{T_2}{T}\mathbb{E}\left[\max\left(1 - \frac{f_1}{\tilde{f}_1}, 0\right)\right] \\
&\le (1-f_1)\left(1 - \left(1 - \frac{T_1+1}{T}\right)\cdot\left(1 - e^{-2\delta^2 T_1}\right)\right) + \frac{T_2}{T}\cdot\left(2\delta + \frac{1}{\sqrt{T_1}}\right) \\
&= (1-f_1)\left(1 - \left(1 - \frac{T_1+1}{T} - e^{-2\delta^2 T_1} + \frac{T_1+1}{T}\cdot e^{-2\delta^2 T_1}\right)\right) + \frac{T_2}{T}\cdot\left(2\delta + \frac{1}{\sqrt{T_1}}\right) \\
&\le (1-f_1)\left(\frac{T_1+1}{T} + e^{-2\delta^2 T_1}\right) + \frac{T_2}{T}\cdot\left(2\delta + \frac{1}{\sqrt{T_1}}\right) \\
&\le \frac{1}{2}e^{-2\delta^2 T_1} + 2\delta + \frac{T_1}{T} + \frac{1}{\sqrt{T_1}}, \qquad (16)
\end{aligned}$$

where the second inequality follows from $T_2 = T - T_1 - 1$ and Parts 2 and 3 of Lemma 20; the second-to-last inequality follows from dropping the product term $\frac{T_1+1}{T}\cdot e^{-2\delta^2 T_1}$; and the last inequality follows because $1 - f_1 \le \frac{1}{2}$, $T_2 \le T$, and $T_1 \ge 1$ which implies $T_1 + 1 \le 2T_1$.

If $f_1 \in (\frac{1}{1+q_0}, 1]$, then $\bar{u}(F) = 1 - f_1$. Using PrincipalUtility$(A^T, \sigma^{\mathrm{TR}}, F, T) \geq 0$ we obtain

$$\frac{1}{T}\,\mathrm{Regret}(A^T, F, T) \leq 1 - f_1 \leq 1 - \frac{1}{1+q_0} = \frac{q_0}{1+q_0} \leq q_0\,, \tag{17}$$

where the last inequality follows because $q_0 \geq 0$.

Combining the upper bounds on the regret in above three cases, (15)–(17), and using that $q_0 = \frac{1}{\sqrt{T_1}}$, we obtain

$$\begin{aligned}
\frac{1}{T}\sup_{F\in\mathcal{F}} \mathrm{Regret}(A^T, F, T) &\leq 2\delta + \frac{1}{2}e^{-2\delta^2 T_1} + \frac{T_1}{T} + \frac{1}{\sqrt{T_1}} \\
&\leq \frac{(\ln T)^{1/2}}{T^{1/3}} + \frac{1}{2T^{1/3}} + \frac{1}{T^{1/3}} + \frac{1}{T^{1/3}} \\
&= \frac{(\ln T)^{1/2}}{T^{1/3}} + \frac{5}{2T^{1/3}}\,.
\end{aligned}$$

where the second inequality follows from our choices for δ and T_1.

B.3 Missing Proofs from Appendices B.1 and B.2

Proof of Lemma 19. We assume there exists at least one distribution F for which $B^*(A^1, F)$ reports CONTINUE in Round 0. Otherwise, the lemma statement is vacuous. Also, recall that $\alpha_2^F = 0$ if $f_1 > 0$ and $\beta_1^F = 0$ if $f_2 > 0$ for these distributions F.

First, we show the claim for the probabilities α^F. We restrict to those distributions F with $B^*(A^1, F)$ reporting CONTINUE in Round 0 and $f_1 > 0$ and show the probabilities α^F can be assumed to be the same for these distributions. This suffices because if $B^*(A^1, (0, 1))$ reports CONTINUE in Round 0, we can also assume the same α^F and both the principal and agent utilities stay unchanged since $f_1 = 0$.

If there is no such distribution after restricting, then $F = (0, 1)$ is the only distribution for which $B^*(A^1, F)$ reports CONTINUE in Round 0 and the claim holds trivially after setting $\alpha_2 = 0$ without loss. Hence, we assume otherwise. Note it must be that the quantity $-1 \cdot \alpha_1^F - \infty \cdot \alpha_2^F = -\alpha_1^F$ is the same for these distributions. This is because if there exist $F = (f_1, f_2)$ and $F' = (f_1', f_2')$ such that $-\alpha_1^F > -\alpha_1^{F'}$, then $B^*(A^1, F')$ can be modified such that when the shock is θ^1, it implements what $B^*(A^1, F)$ does when the shock is θ^1 and when the shock is θ^2, it implements the same strategy as before. The modified strategy would strictly improve the agent utility since $f_1' > 0$, contradicting the choice of $B^*(A^1, F')$. Hence, the probability α_1^F must be the same for these distributions. Since $\alpha_2^F = 0$, $\alpha_0^F = 1 - \alpha_1^F$ and α_1^F's are the same, it follows that the probabilities α^F are the same for these distributions.

Next, we argue about the probabilities β^F. Similarly, we restrict to those distributions F with $B^*(A^1, F)$ reporting CONTINUE in Round 0 and $f_1 < 1$ and show the probabilities β^F can be assumed to be the same for these distributions. If $B^*(A^1, (1, 0))$ reports CONTINUE in Round 0, we can assume the same β^F without affecting the principal utility and the agent utility because $f_2 = 0$. If there is no such distribution after restricting, then $F = (1, 0)$ is the only distribution for which $B^*(A^1, F)$ reports CONTINUE in Round 0 and the claim holds trivially after setting $\beta_1 = 0$ without loss. Hence, we assume otherwise. For these distributions, $f_2 = 1 - f_1 > 0$ since $f_1 < 1$ and the quantity $-\infty \cdot \beta_1^F + 1 \cdot \beta_2^F = \beta_2^F$ must be the same. If there exist F and F' for which the quantity is different, say, $\beta_2^F > \beta_2^{F'}$, we can arrive at a contradiction by modifying $B^*(A^1, F')$ when the shock is θ^2 similarly as in the above argument. Then, $\beta_1^F = 0$, $\beta_0^F = 1 - \beta_2^F$ and β_2^F's are the same, and we have that the probabilities β^F are the same for all these distributions. □

Proof of Claim 2. Recall we want to prove the claim assuming the same probabilities α and β (with $\alpha_2 = \beta_1 = 0$) for all distributions F for which $B^*(A^1, F)$ reports $\mathsf{CONTINUE}$ in Round 0, according to Lemma 19. Note the denominator of $r(\alpha, \beta)$ is strictly positive since $\alpha_1 > 0$ and, thus, $r(\alpha, \beta)$ is well-defined and is in the range $[0, 1]$.

Assume any arbitrary distribution $F = (f_1, f_2)$ such that $B^*(A^1, F)$ reports $\mathsf{CONTINUE}$ in Round 0. The agent utility must be at least 0 or, equivalently,

$$f_1 \cdot -\alpha_1 + f_2 \cdot \beta_2 \geq 0\,.$$

Substituting $f_2 = 1 - f_1$ and collecting terms with f_1 to the right-hand side, we obtain

$$\beta_2 \geq f_1 \cdot (\alpha_1 + \beta_2)\ .$$

Since the denominator of $r(\alpha, \beta)$, which appears on the right-hand side, is strictly positive, we divide both sides by it and obtain

$$r(\alpha, \beta) \geq f_1\,.$$

□

Proof of Lemma 20. We prove each part at a time. For Part 1, note that the function $x \mapsto \frac{1}{x}$ is convex and a first-order expansion around $\frac{1}{2}$ yields the lower bound $\frac{1}{\tilde{f}_1} \geq 2 - 4\left(\tilde{f}_1 - \frac{1}{2}\right)$. Therefore,

$$\min\left\{1, \frac{1 - \tilde{f}_1}{\tilde{f}_1}\right\} \geq \min\left\{1, 1 - 4\left(\tilde{f}_1 - \frac{1}{2}\right)\right\} = 1 - 4\max\left\{\tilde{f}_1 - \frac{1}{2}, 0\right\}$$

Because $\tilde{f}_1 = \hat{f}_1 + \delta$, $\delta \geq 0$, and $f_1 \leq \frac{1}{2}$, we have that

$$\max\left\{\tilde{f}_1 - \frac{1}{2}, 0\right\} \leq \delta + \max\left\{\hat{f}_1 - f_1, 0\right\} \leq \delta + |\hat{f}_1 - f_1|\,,$$

where the last inequality follows because $\max\{x, 0\} \leq |x|$ for all $x \in \mathbb{R}$. Note that $T_1 \cdot \hat{f}_1$ is binomially distributed with T_1 trials and the success probability of f_1. Jensen's inequality and that $\mathbb{E}[\hat{f}_1] = f_1$ imply that

$$\mathbb{E}\left[|\hat{f}_1 - f_1|\right] \leq \sqrt{\mathrm{Var}(\hat{f}_1)} = \sqrt{\frac{f_1(1 - f_1)}{T_1}} \leq \frac{1}{2\sqrt{T_1}}\,, \tag{18}$$

where the equality follows from the variance formula for a binomially distributed random variable and the last inequality follows because $f_1(1-f_1) \le \frac{1}{4}$ for $f_1 \in [0,1]$. Putting everything together,

$$\begin{aligned}
\mathbb{E}\left[\min\left\{1, \frac{1-\tilde{f}_1}{\tilde{f}_1}\right\}\right] &\ge 1 - 4\mathbb{E}\left[\max\left\{\tilde{f}_1 - \frac{1}{2}, 0\right\}\right] \\
&\ge 1 - 4\delta - 4\mathbb{E}\left[|\hat{f}_1 - f_1|\right] \\
&\ge 1 - 4\delta - \frac{2}{\sqrt{T_1}}\,.
\end{aligned}$$

For Part 2, we use, again, that the function $x \mapsto \frac{1}{x}$ is convex to obtain that a first-order expansion around f_1 yields the lower bound $\frac{1}{\tilde{f}_1} \ge \frac{1}{f_1} - \frac{1}{f_1^2}(\tilde{f}_1 - f_1)$. Therefore,

$$\begin{aligned}
\max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\} &\le \frac{1}{f_1}\max\left\{\tilde{f}_1 - f_1, 0\right\} \\
&\le 2\delta + 2\max\left\{\hat{f}_1 - f_1, 0\right\} \\
&\le 2\delta + 2|\hat{f}_1 - f_1|\,,
\end{aligned}$$

where the first inequality follows from the above lower bound; the second inequality follows from $f_1 > \frac{1}{2}$, $\tilde{f}_1 = \hat{f}_1 + \delta$, and $\delta \ge 0$; and the last is because $\max\{x, 0\} \le |x|$ for all $x \in \mathbb{R}$. Taking expectations and using (18), we obtain

$$\mathbb{E}\left[\max\left\{1 - \frac{f_1}{\tilde{f}_1}, 0\right\}\right] \le 2\delta + 2\mathbb{E}\left[|\hat{f}_1 - f_1|\right] \le 2\delta + \frac{1}{\sqrt{T_1}}\,.$$

For Part 3, we use that $\tilde{f}_1 = \hat{f}_1 + \delta$ to obtain

$$\Pr(\tilde{f}_1 \ge f_1) = \Pr(\hat{f}_1 \ge f_1 - \delta) = 1 - \Pr(\hat{f}_1 < f_1 - \delta) \ge 1 - e^{-2\delta^2 T_1}\,,$$

where the last inequality follows from Hoeffding's inequality because $T_1 \cdot \hat{f}_1$ is binomially distributed with T_1 trials and success probability f_1. □

B.4 Proof of Proposition 12

Since Assumption 1 holds and Theorem 1 applies, it suffices to show $\widehat{\text{Regret}} = 0$ and the direct mechanism S that determines the outcome $(\omega^1, -1)$ with probability 1 for the report θ^1 and the outcome $(\omega^2, 1)$ with probability 1 for the report θ^2 is an optimal solution in the single-round minimax regret problem. To see that S satisfies the IC/IR constraints, note that truthful reporting yields the agent utility of 0 and non-truthful reporting yields the agent utility of $-\infty$ for both when the shock is θ^1 and when it is θ^2. Then, for any distribution $F = (f_1, f_2)$,

$$\int_\Theta \int_\Omega u'(\theta, \omega)\mathrm{d}S_\theta(\omega)\mathrm{d}F(\theta) = f_1 \cdot 0 + f_2 \cdot 1 = \mathbb{E}_{\theta\sim F}[\bar{u}(\theta)] = \mathbb{E}_{\theta\sim F}[\text{OPT}(\theta, 1)]\,.$$

Hence, $\widehat{\text{Regret}}(S, F) = 0$ for any distribution $F \in \mathcal{F}$. Since $\widehat{\text{Regret}} \ge 0$ by the definition of $\mathbb{E}_{\theta\sim F}[\text{OPT}(\theta, 1)]$ being the first-best performance without the IC constraint and the above argument shows $\sup_{F\in\mathcal{F}} \widehat{\text{Regret}}(S,F) = 0$, it follows that $\widehat{\text{Regret}} = 0$ and S is an optimal single-round direct IC/IR mechanism in the single-round problem.

C Additional Materials for Section 5

We present missing materials from Section 5 on revenue maximization in the single-good case. As discussed already, Theorem 13 follows if we show that the single-round IC/IR mechanism S^* is an optimal solution to the single-round problem and $\widehat{\text{Regret}} = \frac{1}{e}$. We do so by proving the following saddle-point result, which is closely related to a similar result due to Bergemann and Schlag (2008).

Theorem 21. *Let S^* be the randomized posted pricing strategy given in Theorem 13, which is a single-round direct IC/IR mechanism, and the agent's distribution F^* be given by*

$$F^*(\theta) = \begin{cases} 0, & \text{if } \theta \in [0, \frac{1}{e}) \\ 1 - \frac{1}{e\theta}, & \text{if } \theta \in [\frac{1}{e}, 1) \\ 1, & \text{if } \theta = 1 \end{cases} .$$

Then, $\widehat{\text{Regret}} = \widehat{\text{Regret}}(S^*, F^*) = \frac{1}{e}$ *and*

$$\widehat{\text{Regret}}(S^*, F) \leq \widehat{\text{Regret}}(S^*, F^*) \leq \widehat{\text{Regret}}(S, F^*)\,,$$

for any $S \in \mathcal{S}$ satisfying the IC/IR constraints and $F \in \mathcal{F}$.

Proof. When the principal implements a randomized posted pricing strategy, it is best for the agent to truthfully respond, that is, buy the item if the price is lower than his value, and, therefore, the randomized posted pricing strategy satisfies the IC and IR constraints.

First, we prove $\widehat{\text{Regret}}(S^*, F) \leq \widehat{\text{Regret}}(S^*, F^*)$ for any agent's distribution F. Note we can represent any single-round direct IC/IR mechanism S with the corresponding interim rules (x, p) where, by standard arguments in mechanism design, the allocation rule x is monotonically non-decreasing and the payment rule p satisfies

$$p(\theta) = p(0) + x(\theta)\cdot\theta - \int_0^\theta x(t)dt\,, \quad \forall \theta \in [0,1]$$

and $p(0) \leq 0$. In particular, let (x^*, p^*) be the interim allocation and payment rules of S^* where $x^*(\theta) = 0$ for $\theta \in [0, \frac{1}{e})$ and $x^*(\theta) = 1 + \ln\theta$ for $\theta \in [\frac{1}{e}, 1]$ and $p^*(\theta) = x^*(\theta)\cdot\theta - \int_0^\theta x^*(t)dt$ for all θ.

It suffices to show that F^* is a solution to the following optimization problem:

$$\max_{F\in\mathcal{F}} \left\{ \int_\Theta \theta - p^*(\theta) dF(\theta) \right\}\,,$$

where F is over any buyer's distribution. Given x^*, we can simplify p^* as $p^*(\theta) = 0$ for $\theta \in [0, \frac{1}{e})$ and

$$p^*(\theta) = (1 + \ln\theta)\theta - \int_{\frac{1}{e}}^\theta 1 + \ln t dt = \theta - \frac{1}{e}\,,$$

for $\theta \in [\frac{1}{e}, 1]$. Then, the integrand in the objective function is, equivalently, $\theta\mathbf{1}\{\theta < \frac{1}{e}\} + \frac{1}{e}\mathbf{1}\{\theta \geq \frac{1}{e}\}$ and the optimization problem becomes

$$\max_{F \in \mathcal{F}} \left\{ \Pr_{\theta \sim F}\left(\theta < \frac{1}{e}\right) \cdot \mathbb{E}_{\theta \sim F}\left[\theta \mid \theta < \frac{1}{e}\right] + \Pr_{\theta \sim F}\left(\theta \geq \frac{1}{e}\right) \cdot \frac{1}{e} \right\}.$$

It follows that any distribution with its support contained in $[\frac{1}{e}, 1]$ is an optimal solution and F^* is one such distribution. Furthermore, we see that the optimization problem has the value of $\frac{1}{e}$ and, so, $\widehat{\text{Regret}} = \frac{1}{e}$.

Next, we show $\widehat{\text{Regret}}(S^*, F^*) \leq \widehat{\text{Regret}}(S, F^*)$ for any single-round direct IC/IR mechanism S. Similar to the above argument, we show that S^* is a solution to:

$$\min_{(x,p)} \left\{ \int_\Theta \theta - p(\theta) dF^*(\theta) \ \text{ s. t. } \ (\text{IC}), (\text{IR}) \right\},$$

where (x, p) are over all possible interim rules satisfying the IC/IR constraints. By standard arguments, the payment rule p satisfies $p(\theta) = p(0) + x(\theta) \cdot \theta - \int_0^\theta x(t)dt$ for $\theta \in [0, 1]$ and $p(0) \leq 0$ and the allocation rule x is monotonically non-decreasing. Then, the above optimization problem becomes

$$\min_{\text{non-decreasing } x, p(0) \leq 0} \left\{ -p(0) + \int_0^1 \left(\theta - x(\theta) \cdot \theta + \int_0^\theta x(t)dt \right) f^*(\theta)d\theta \right\},$$

where f^* is the probability density function for F^* with $f^*(\theta) = 0$ for $\theta \in [0, \frac{1}{e})$, $f^*(\theta) = \frac{1}{e\theta^2}$ for $\theta \in [\frac{1}{e}, 1)$ and a point-mass of $\frac{1}{e}$ at $\theta = 1$. By changing the ordering of the integrals, $\int_0^1 \int_0^\theta x(t) f^*(\theta) dt d\theta = \int_0^1 \int_t^1 x(t) f^*(\theta) d\theta dt = \int_0^1 (1 - F^*(t)) x(t) dt$, and the optimization problem is equivalently

$$\min_{\text{non-decreasing } x, p(0) \leq 0} \left\{ -p(0) + \mathbb{E}_{\theta \sim F^*}[\theta] + \int_0^1 \left(-\theta \cdot f^*(\theta) + (1 - F^*(\theta)) \right) x(\theta) d\theta \right\}.$$

In the integral, the expression $\phi(\theta) := -\theta \cdot f^*(\theta) + (1 - F^*(\theta))$ can be further simplified as 1 if $\theta \in [0, \frac{1}{e})$, 0 if $\theta \in [\frac{1}{e}, 1)$ and a point-mass of $-\frac{1}{e}$ if $\theta = 1$. Then, the objective function is equal to

$$-p(0) + \mathbb{E}_{\theta \sim F^*}[\theta] + \int_0^{\frac{1}{e}} 1 \cdot x(\theta) d\theta + \int_{\frac{1}{e}}^1 0 \cdot x(\theta) d\theta - \frac{1}{e} \cdot x(1).$$

It follows that an optimal solution has $p(0) = 0$ and a non-decreasing $x(\cdot)$ such that $x(\theta) = 0$ for $\theta \in [0, \frac{1}{e})$, $x(\theta) \geq 0$ for $\theta \in [\frac{1}{e}, 1)$ and $x(\theta) = 1$ for $\theta = 1$ (in the almost everywhere sense for $\theta < 1$). Clearly, S^* satisfies these conditions and is, therefore, an optimal solution. □

D Additional Materials for Section 6

D.1 Single-Round Problem

We provide further details on the single-round direct IC/IR mechanisms and a justification for the restriction to those with deterministic contracts for the single-round problem.

We can think of a single-round direct IC/IR mechanism as a collection of distributions S_θ on $\mathbb{R}^+ \times \mathbb{R}$ indexed by $\theta \in [\underline{\theta}, \bar{\theta}]$ such that when the agent reports θ, the outcome is determined by drawing from S_θ, i.e., $(\hat{q}, \hat{z}) \sim S_\theta$ for production level $\hat{q}$ and payment $\hat{z}$. Abusing notations, let $q(\theta) = \mathbb{E}_{(\hat{q},\hat{z})\sim S_\theta}[\hat{q}]$ and $z(\theta) = \mathbb{E}_{(\hat{q},\hat{z})\sim S_\theta}[\hat{z}]$ be the interim allocation and payment rules, respectively. Then, the IC and IR constraints can be expressed as follows:

$$z(\theta) - \theta \cdot q(\theta) \geq z(\theta') - \theta \cdot q(\theta')\,, \quad \forall \theta, \theta' \in [\underline{\theta}, \bar{\theta}] \tag{IC}$$

$$z(\theta) - \theta \cdot q(\theta) \geq 0\,, \quad \forall \theta \in [\underline{\theta}, \bar{\theta}] \tag{IR}$$

Fix an arbitrary single-round direct IC/IR mechanism. Let $V(\theta) = z(\theta) - \theta \cdot q(\theta)$ for $\theta \in [\underline{\theta}, \bar{\theta}]$. Note $V(\theta)$ is convex and by standard arguments (similar to the auction case, e.g., in Chapter 5 in Krishna (2009)), $q(\theta)$ is non-increasing and V is absolutely continuous and $V'(\theta) = -q(\theta)$ where the derivative exists. As $q(\theta)$ is nonnegative, $V(\theta)$ is non-increasing. Furthermore, we can write

$$z(\theta) = V(\bar{\theta}) + \theta \cdot q(\theta) + \int_\theta^{\bar{\theta}} q(x)dx\,, \text{ for } \theta \in [\underline{\theta}, \bar{\theta}]\,.$$

As defined in Section 2, we have

$$\widehat{\text{Regret}}(S, F) := \mathbb{E}_{\theta\sim F}[\bar{R}(\theta)] - \mathbb{E}_{\theta\sim F,(\hat{q},\hat{z})\sim S_\theta}[R(\hat{q}) - \hat{z}]\,,$$

and

$$\widehat{\text{Regret}} := \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \sup_{F\in\mathcal{F}} \widehat{\text{Regret}}(S, F)\,.$$

Since the revenue function $R(x)$ is concave, $\mathbb{E}_{(\hat{q},\hat{z})\sim S_\theta}[R(\hat{q})] \leq R(\mathbb{E}_{(\hat{q},\hat{z})\sim S_\theta}[\hat{q}])$ for any θ. Given any single-round direct IC/IR mechanism S, we can potentially improve (but not hurt) its performance by modifying S_θ to always return a deterministic production level $\hat{q}$ that is the average $q(\theta)$:

$$\mathbb{E}_{(\hat{q},\hat{z})\sim S_\theta}[R(\hat{q}) - \hat{z}] \leq R(\mathbb{E}_{(\hat{q},\hat{z})\sim S_\theta}[\hat{q}]) - \mathbb{E}_{(\hat{q},\hat{z})\sim S_\theta}[\hat{z}] = R(q(\theta)) - z(\theta)\,.$$

Note the IC/IR constraints are still satisfied. Without loss in the minimax regret objective, we can restrict to those single-round IC/IR mechanisms that can be described in terms of a menu of deterministic contracts $(q(\theta), z(\theta))$ for $\theta \in [\underline{\theta}, \bar{\theta}]$. Still using $\mathcal{S}$ to denote this restricted set of single-round mechanisms, the minimax regret for the single-round problem is equal to:

$$\widehat{\text{Regret}} := \inf_{\substack{(q,z)\in\mathcal{S}:\\ \text{(IC),(IR)}}} \sup_{F\in\mathcal{F}} \int_\Theta \bar{R}(\theta) - (R(q(\theta)) - z(\theta))\, dF(\theta)\,,$$

and for $S = (q, z) \in \mathcal{S}$ and $F \in \mathcal{F}$,

$$\widehat{\text{Regret}}(S, F) := \int_\Theta \bar{R}(\theta) - (R(q(\theta)) - z(\theta))\, dF(\theta)\,.$$

D.2 Proof of Theorem 14

By Theorem 1, to prove Theorem 14, it suffices that we prove the stated single-round direct IC/IR mechanism is an optimal solution to the single-round problem via the following saddle-point result.

Theorem 22. *Let S^* be the single-round direct IC/IR mechanism corresponding to the menu of deterministic contracts $\{(q^*(\theta), z^*(\theta)\}_{\theta\in\Theta}$ given in Theorem 14 and the agent's distribution be given by a point-mass of $F^*(\underline{\theta})$, which can be 0, at $\theta = \underline{\theta}$ and a density $\frac{d}{d\theta}F^*(\theta)$ for $\theta \in (\underline{\theta}, \kappa]$ where*

$$F^*(\theta) = e^{-\int_\theta^\kappa \frac{1}{R'(q^*(x))-x}dx} ,$$

for the same κ in the definition of S^*. Then, S^* and F^* are well-defined and the minimax regret is $\widehat{\text{Regret}} = \widehat{\text{Regret}}(S^*, F^*) = \int_{\underline{\theta}}^{\bar{\theta}} q^*(x)dx$ and is strictly positive, and

$$\widehat{\text{Regret}}(S^*, F) \leq \widehat{\text{Regret}}(S^*, F^*) \leq \widehat{\text{Regret}}(S, F^*) ,$$

for any $S \in \mathcal{S}$ satisfying the IC/IR constraints and $F \in \mathcal{F}$.

We first show that single-round direct IC/IR mechanism S^* and distribution F^* in the statement of Theorem 22 are well-defined with the claimed characterizations.

Lemma 23. *The single-round direct IC/IR mechanism S^* and distribution F^* in Theorems 14 and 22 are well-defined. Furthermore, q^* is continuous over the shock space and, in particular, strictly decreases over $[\underline{\theta}, \kappa]$.*

Proof. We proceed in two steps showing S^* and then F^* are well-defined.

(Single-Round Direct IC/IR Mechanism S^*): It suffices to show that a strictly decreasing solution q^* exists to the ordinary differential equation

$$\begin{aligned} &\frac{dq}{d\theta}(\theta) = \frac{-\bar{q}(\theta)}{R'(q(\theta)) - \theta} , \text{ for } \theta \in (\underline{\theta}, \kappa) \\ &\lim_{\theta\to\underline{\theta}^+} q(\theta) = \bar{q}(\underline{\theta}) , \end{aligned} \tag{19}$$

for some κ to be determined and then complete it for $\theta = \underline{\theta}$ and $\theta > \kappa$ accordingly. We follow similar reasoning steps as in the proof of Lemma 2 in Carrasco et al. (2018b). We equivalently solve the following differential system, with the roles of q and θ interchanged:

$$\begin{aligned} &\frac{d\theta}{dq}(q) = \frac{R'(q) - \theta(q)}{-\bar{q}(\theta(q))} , \text{ for } q \leq \bar{q}(\underline{\theta}) \\ &\theta(\bar{q}(\underline{\theta})) = \underline{\theta} . \end{aligned} \tag{20}$$

As we show, this system has a solution θ^* that is strictly decreasing over a suitable interval and we can invert the relationship between θ and q to obtain a solution q^* to the original differential system with a well-defined κ.

Recall a solution to an ordinary differential equation (ODE) is a continuously differentiable function defined on some interval satisfying the specified relations. Let $\psi(q, \theta) := \frac{R'(q)-\theta}{-\bar{q}(\theta)}$ defined on the domain $D = (0, \bar{q}(\underline{\theta}) + \epsilon] \times [\underline{\theta} - \epsilon, \bar{\theta} + \epsilon]$ for arbitrarily small $\epsilon > 0$; ϵ is there to make the domain an open set. By the assumptions on R, ψ is continuous on the domain. Furthermore, it is continuously differentiable on any closed set of the domain and, hence, locally Lipschitz with respect to θ. For any initial value point in D, there exists a unique solution to the differential equation $\frac{d\theta}{dq}(q) = \psi(q, \theta)$ in a neighborhood of the initial value point (e.g., Theorem 3.1 in Hale (1969)). In particular, the above system of differential equation (20) has a unique solution θ^* in a neighborhood of the point $(\bar{q}(\underline{\theta}), \underline{\theta})$.

Let $(\underline{q}, \bar{q}(\underline{\theta})]$ be the left maximal interval of definition of the ordinary differential equation (20). We show θ^* is strictly decreasing in this interval. Note if a solution $\theta(\cdot)$ has $\theta'(q) = 0$, then

$$\frac{d^2\theta}{dq^2}(q) = \psi_1(q, \theta) + \psi_2(q, \theta) \cdot \frac{d\theta}{dq} = -\frac{R''(q)}{\bar{q}(\theta)} > 0,$$

where ψ_i denotes the partial derivative with respect to the i-th parameter. Since $(\theta^*)'(\bar{q}(\underline{\theta})) = 0$, θ^* is strictly convex at $q = \bar{q}(\underline{\theta})$ and decreases over $[\bar{q}(\underline{\theta}) - \epsilon, \bar{q}(\underline{\theta})]$ for sufficiently small $\epsilon > 0$. Fix an arbitrary $q \in (\underline{q}, \bar{q}(\underline{\theta}))$ and assume θ^* achieves the maximum at some $x \in [q, \bar{q}(\underline{\theta})]$. Note x cannot be in the interior because the first-order condition $(\theta^*)'(x) = 0$ is satisfied and it would mean θ^* is strictly convex and is increasing to the left or right of x. By the above observation, x cannot be $\bar{q}(\underline{\theta})$. Hence, the maximum is achieved at the left-end $x = q$. As q was arbitrary, the argument extends and it implies θ^* is strictly decreasing over $(\underline{q}, \bar{q}(\underline{\theta})]$.

Now, we invert θ^* to obtain q^* that is a solution to the original differential system (19) that we want to solve. Let $\theta^*(\underline{q}) = \sup_{q \in (\underline{q}, \bar{q}(\underline{\theta})]} \theta^*(q)$ which may be ∞. If $\theta^*(\underline{q}) < \bar{\theta}$, then $(\theta^*)'(\underline{q}) = \lim_{q \to \underline{q}^+} \psi(q, \theta^*(q))$ would be equal to $\frac{R'(\underline{q}) - \theta^*(\underline{q})}{-\bar{q}(\theta^*(\underline{q}))}$ which is not defined, more specially, $R'(\underline{q})$ in the numerator, if $\underline{q} = 0$ but defined if $\underline{q} > 0$. Since we chose the left maximal interval of definition of the ODE, it must be that $\underline{q} = 0$. Then, we let $\kappa = \theta^*(\underline{q})$ and truncate the solution θ^* so that its range is exactly $[\underline{\theta}, \kappa)$. We let q^* be the inverted curve of the truncated solution on $[\underline{\theta}, \kappa)$ which strictly decreases and converges to 0 over the interval and extend $q^*(\theta) = 0$ for $\theta \in [\kappa, \bar{\theta}]$.

In the other case when $\theta^*(\underline{q}) \geq \bar{\theta}$, we truncate the solution θ^* such that its range is exactly $[\underline{\theta}, \bar{\theta}]$ and consider q^* to be the corresponding inverted solution over the interval $[\underline{\theta}, \bar{\theta}]$. By construction, q^* satisfies the desired differential system and stays positive over the whole interval. We choose $\kappa = \bar{\theta}$.

In both cases, since θ^* is continuous at $q = \bar{q}(\underline{\theta})$ with $\theta^*(\bar{q}(\underline{\theta})) = \underline{\theta}$, we have $q^*(\underline{\theta}) = \bar{q}(\underline{\theta})$ and $\lim_{\theta \to \underline{\theta}^+} q^*(\theta) = \bar{q}(\underline{\theta})$. Also, by our choice of κ, $\lim_{\theta \to \kappa^-} q^*(\theta) = q^*(\kappa)$. That is, q^* is continuous over the whole interval $[\underline{\theta}, \bar{\theta}]$.

(Distribution F^*): Given that we have a solution q^* that is strictly decreasing over $[\underline{\theta}, \kappa]$ and continuously differentiable over $(\underline{\theta}, \kappa)$, the fraction $\frac{1}{R'(q^*(\theta)) - \theta}$ is well-defined and positive for $\theta \in (\underline{\theta}, \kappa)$. We argue that the integral $\int_\theta^\kappa \frac{1}{R'(q^*(x)) - x} dx$ exists over the same interval. If $\kappa = \bar{\theta}$ and q^* stays positive, the integrand is well-defined and continuous over the compact set. Hence, the integral exists. If $\kappa < \bar{\theta}$, then the integrand goes to 0 as x approaches κ and thus bounded. In this case, again, the integral exists.

As θ approaches $\underline{\theta}$, the integral $\int_\theta^\kappa \frac{1}{R'(q^*(x)) - x} dx$ can potentially grow to ∞. But, $F^*(\theta) = e^{-\int_\theta^\kappa \frac{1}{R'(q^*(x)) - x} dx}$ is absolutely continuous over $(\underline{\theta}, \kappa]$ and the distribution F^* can be described with a point-mass of $\lim_{\theta' \to \underline{\theta}^+} F^*(\theta')$ at $\theta = \underline{\theta}$, which can be 0, and the absolute continuous part with density $f^*(\theta) = \frac{d}{d\theta} F^*(\theta) = F^*(\theta) \cdot \frac{1}{R'(q^*(\theta)) - \theta}$.

$\square$

We now prove Theorem 22.

Proof of Theorem 22. We restrict, without loss, to those single-round IC/IR mechanisms that can be described in terms of a menu of deterministic contracts $(q(\theta), z(\theta))$ for $\theta \in [\underline{\theta}, \bar{\theta}]$ in our analysis. We still use $\mathcal{S}$ to denote this set of single-round mechanisms. By well-definedness, we mean both S^* and F^* exist with the stated characterizations. In particular, it would mean that q^* is a continuous monotone function over $[\underline{\theta}, \bar{\theta}]$ and is integrable. The well-definedness of S^* and F^* has been proved in Lemma 23.

For the first part of the saddle-point result, we show that F^* is an optimal solution to $\max_F \widehat{\text{Regret}}(S^*, F)$ which is equivalent to:

$$\max_{F \in \mathcal{F}} \left\{ \int_\Theta \bar{R}(\theta) - (R(q^*(\theta)) - z^*(\theta))\, dF(\theta) \right\}.$$

By the definition of z^*, the optimization problem is equivalent to

$$\max_{F \in \mathcal{F}} \left\{ \int_\Theta \left(\bar{R}(\theta) - R(q^*(\theta)) + \theta \cdot q^*(\theta) + \int_\theta^{\bar{\theta}} q^*(x) dx \right) dF(\theta) \right\}.$$

The integrand is continuous and its derivative with respect to θ is

$$-\bar{q}(\theta) - (R'(q^*(\theta)) - \theta) \cdot (q^*)'(\theta)\,,$$

where we used $\bar{R}'(\theta) = -\bar{q}(\theta)$. Since $(q^*)'(\theta) = \frac{-\bar{q}(\theta)}{R'(q^*(\theta)) - \theta}$ for $\theta \in (\underline{\theta}, \kappa)$, the derivative is equal to 0 over the same interval. For $\theta \in [\kappa, \bar{\theta}]$, the integrand is equal to $\bar{R}(\theta)$ and the derivative is equal to $-\bar{q}(\theta)$, which is negative. Since q^* is continuous, it follows that the integrand stays constant for $\theta \in [\underline{\theta}, \kappa]$ and then decreases for $\theta \in [\kappa, \bar{\theta}]$. Since F^* has support equal to exactly $[\underline{\theta}, \kappa]$, it maximizes the objective and is an optimal solution, as desired.

Similarly, for the second part, we show that S^* is an optimal solution to

$$\min_{\substack{S \in \mathcal{S}: \\ \text{(IC),(IR)}}} \left\{ \int_\Theta \bar{R}(\theta) - (R(q(\theta)) - z(\theta))\, dF^*(\theta) \right\}.$$

By the standard arguments (see Appendix D.1), it suffices to show that S^* is an optimal solution to the following equivalent problem:

$$\min_{\text{non-increasing } q, V(\bar{\theta}) \geq 0} \left\{ \int_\Theta \left(\bar{R}(\theta) - R(q(\theta)) + V(\bar{\theta}) + \theta \cdot q(\theta) + \int_\theta^{\bar{\theta}} q(x) dx \right) dF^*(\theta) \right\},$$

where $V(\theta) = z(\theta) - \theta \cdot q(\theta)$ for $\theta \in [\underline{\theta}, \bar{\theta}]$.

Note F^* has a point-mass of $\lim_{\theta' \to \underline{\theta}^+} F^*(\theta')$ which we, for notational convenience, equate to $F^*(\underline{\theta})$ at $\theta = \underline{\theta}$. But it is otherwise absolutely continuous and has a corresponding density function. We denote the cumulative function without the point-mass at $\theta = \underline{\theta}$ by F^*_- with corresponding density $f^*_-(\theta) = \frac{d}{d\theta} F^*(\theta)$ such that $F^*(\theta) = F^*(\underline{\theta}) + F^*_-(\theta)$ for $\theta \in [\underline{\theta}, \bar{\theta}]$.

Then, we can rewrite the objective function, denoted OBJ, as follows. Note that

$$\begin{aligned}\text{OBJ} = \mathbb{E}_{\theta\sim F^*}[\bar{R}(\theta)] + V(\bar{\theta}) - F^*(\underline{\theta})\left(R(q(\underline{\theta})) - \underline{\theta}\cdot q(\underline{\theta}) - \int_{\underline{\theta}}^{\bar{\theta}} q(x)dx\right) \\ - \int_{\underline{\theta}}^{\bar{\theta}} f_-^*(\theta)\left(R(q(\theta)) - \theta\cdot q(\theta) - \int_{\theta}^{\bar{\theta}} q(x)dx\right)d\theta\,.\end{aligned}$$

The last term is equivalently $-\int_{\underline{\theta}}^{\bar{\theta}} f_-^*(\theta)\left(R(q(\theta)) - \theta\cdot q(\theta)\right)d\theta + \int_{\underline{\theta}}^{\bar{\theta}} F_-^*(\theta)q(\theta)d\theta$ where we changed the order of integrals. Then,

$$\begin{aligned}\text{OBJ} = \mathbb{E}_{\theta\sim F^*}[\bar{R}(\theta)] + V(\bar{\theta}) - F^*(\underline{\theta})\left(R(q(\underline{\theta})) - \underline{\theta}\cdot q(\underline{\theta})\right) \\ - \int_{\underline{\theta}}^{\bar{\theta}} f_-^*(\theta)\left(R(q(\theta)) - \left(\theta + \frac{F^*(\theta)}{f_-^*(\theta)}\right)\cdot q(\theta)\right)d\theta\,.\end{aligned}$$

We now show that S^* minimizes OBJ pointwise. Since $q^*(\underline{\theta}) = \bar{q}(\theta)$, the third term is minimized. Note $\frac{\partial}{\partial\theta}\ln F^*(\theta) = \frac{f_-^*(\theta)}{F^*(\theta)} = \frac{1}{R'(q^*(\theta))-\theta}$ for $\theta\in[\underline{\theta},\kappa]$. Then, $R'(q^*(\theta)) = \theta + \frac{F^*(\theta)}{f_-^*(\theta)}$ for $\theta\in[\underline{\theta},\kappa]$ which is exactly the support of f_-^* and where q^* is nonnegative. It follows that the integrand in the fourth term is minimized pointwise and, thus, the fourth term is minimized. Finally, note that $V(\underline{\theta}) = 0$ for S^* and that the second term is minimized. Overall, the objective is minimized and S^* is an optimal solution. This completes the proof that S^* and F^* form a saddle point.

To compute the minimax regret, we recall from the analysis for the first part of the saddle-point result that the integrand, equivalently, $\widehat{\text{Regret}}(S^*,\theta)$, is constant for $\theta\in[\underline{\theta},\kappa]$ and decreases for $\theta>\kappa$. It follows that the minimax regret is equal to the integrand evaluated at $\theta=\underline{\theta}$ which is $\int_{\underline{\theta}}^{\bar{\theta}} q^*(x)dx$. The minimax regret is clearly nonnegative because q^* is nonnegative. We argue it is strictly greater than 0. For the sake of contradiction, assume that $\int_{\underline{\theta}}^{\bar{\theta}} q^*(x)dx = 0$. As q^* is non-increasing and nonnegative, it would follow that $q^*(\theta) = 0$ for $\theta > \underline{\theta}$. Together with $q^*(\underline{\theta}) = \bar{q}(\theta) > 0$, it would contradict the continuity of q^* which is implied by the well-definedness of S^*. □

E Missing Proofs from Section 7

We prove Theorem 15. By Theorem 1, it suffices to show the following:

Theorem 24. *Let S^* be the single-round direct IC/IR mechanism with the allocation rule given in Theorem 15. Then, $\widehat{\text{Regret}} = c\cdot(1-c)$ and S^* is an optimal solution to the single-round minimax problem, i.e.,*

$$\widehat{\text{Regret}} = \sup_{F\in\mathcal{F}} \widehat{\text{Regret}}(S^*,F)\,.$$

Furthermore, there exist no saddle points but the following asymmetric saddle-point result holds:

$$\sup_{F\in\mathcal{F}} \widehat{\text{Regret}}(S^*,F) = \sup_{F\in\mathcal{F}} \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S,F)\,.$$

Before proving Theorem 24, we will argue that the IC/IR constraints restrict the allocation rule x to be constant on $(0, 1]$ and $x(0)$ to be at most the constant value. This reduces the single-round problem to optimizing over two decision variables and Theorem 24 would follow from a case-by-case analysis. For intuition on the non-existence of a worst-case distribution in the supremum-infimum problem $\sup_{F\in\mathcal{F}} \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S, F)$, we note that the IC constraint allows for allocation probabilities that are discontinuous at $\theta = 0$. In order to achieve a regret of $\widehat{\text{Regret}} = c(1 - c)$, nature would like the agent's distribution F to be $\theta = 1$ with probability c and $\theta = \epsilon$ with probability $1 - c$ for some small value $\epsilon \in (0, c)$. In this case, the single-round benchmark would coincide with $c(1 - c)$ and the principal's maximum utility would be $\mathbb{E}_{\theta\sim F}[\theta - c] = (1 - c)c + (\epsilon - c)(1 - c) = \epsilon(1 - c)$, which is achieved by always allocating, and the corresponding regret would be $(c - \epsilon)(1 - c)$. Letting $\epsilon \downarrow 0$ would yield the optimal regret $\widehat{\text{Regret}}$. The regret, however, has a discontinuity at $\epsilon = 0$ because in this case, the principal can respond by not allocating when $\theta = 0$ to obtain an expected utility of $\mathbb{E}[(\theta - c)\mathbf{1}\{\theta > 0\}] = c(1 - c)$ and, hence, a regret of 0 at $\epsilon = 0$.

Recall we represent single-round IC/IR mechanisms with the interim allocation rule $x : \Theta \to [0, 1]$ with the understanding that when the agent reports θ, the probability of allocation is $x(\theta)$. Note $\bar{u}(\theta) = \begin{cases} \theta - c & , \text{ if } \theta \geq c \\ 0 & , \text{ otherwise} \end{cases}$ for any $\theta \in \Theta$ and for any distribution F,

$$\bar{u}(F) = \mathbb{E}_{\theta\sim F}[\bar{u}(\theta)] = \mathbb{E}_{\theta\sim F}[\mathbf{1}\{\theta \geq c\} \cdot (\theta - c)] \,,$$

where $\mathbf{1}\{\theta \geq c\}$ is the indicator that is 1 if $\theta \geq c$, and 0 otherwise. Then, the single-round minimax regret problem is:

$$\widehat{\text{Regret}} = \inf_{x:\text{(IC),(IR)}} \sup_{F\in\mathcal{F}} \left\{\mathbb{E}_{\theta\sim F}[\mathbf{1}\{\theta \geq c\} \cdot (\theta - c)] - \mathbb{E}_{\theta\sim F}[(\theta - c) \cdot x(\theta)]\right\} \,,$$

and the IC/IR constraints for single-round direct mechanisms are:

$$\theta \cdot x(\theta) \geq \theta \cdot x(\theta') \,, \quad \forall \theta, \theta' \in \Theta \tag{IC}$$

$$\theta \cdot x(\theta) \geq 0 \,, \quad \forall \theta \in \Theta \,. \tag{IR}$$

The IR constraint is always satisfied. From the IC constraint, we show x is constant on $(0, 1]$ and $x(0)$ is at most the constant value. For θ and θ' arbitrarily chosen in $(0, 1]$, the IC constraint implies $\theta \cdot x(\theta) \geq \theta \cdot x(\theta')$ and, dividing θ on both sides, $x(\theta) \geq x(\theta')$. Changing the roles of θ and θ', we also have $x(\theta) \leq x(\theta')$. Hence, x is constant on $(0, 1]$. Assume $\theta \in (0, 1]$ and $\theta' = 0$. Then, the IC constraint implies $\theta \cdot x(\theta) \geq \theta \cdot x(0)$ and, consequently, $x(\theta) \geq x(0)$. It follows that $x(0)$ is at most the constant value. Note the IC constraint is always satisfied when $\theta = 0$. Hence, in what follows, we parametrize single-round direct IC/IR mechanisms in terms of $0 \leq x_0 \leq x_1 \leq 1$ such that $x(0) = x_0$ and $x(\theta) = x_1$ for $\theta \in (0, 1]$.

We now prove Theorem 24. For ease of presentation, we present the proof for the first part below and the second part on the nonexistence of saddle points and the third part on the asymmetric saddle-point result in Appendices E.1 and E.2, respectively.

Proof of the First Part of Theorem 24. Let $F = F^- + F(0)$ where $F(0)$ is the point-mass, which may have the zero mass, at $\theta = 0$ and F^- is the rest of the distribution over $(0, 1]$. Then, the objective of the single-round minimax regret problem is

$$\begin{aligned}\widehat{\text{Regret}}((x_0,x_1),F) &= \mathbb{E}_{\theta\sim F^-}[\mathbf{1}\{\theta\geq c\}\cdot(\theta-c)] - \mathbb{E}_{\theta\sim F^-}[(\theta-c)\cdot x_1] + c\cdot x_0\cdot F(0)\\ &= \mathbb{E}_{\theta\sim F^-}[\mathbf{1}\{\theta\geq c\}\cdot(\theta-c)-\theta\cdot x_1] + c\cdot x_1\cdot(1-F(0)) + c\cdot x_0\cdot F(0)\,.\end{aligned}$$

Let $g(\theta)=\mathbf{1}\{\theta\geq c\}\cdot(\theta-c)-\theta\cdot x_1$ be the expression inside the expectation with c and x_1 fixed. For $\theta\in(0,c)$, $g(\theta)=-\theta\cdot x_1$. For $\theta\in[c,1]$, $g(\theta)=\theta\cdot(1-x_1)-c$.

For each possible pair (x_0,x_1) such that $0\leq x_0\leq x_1\leq 1$, we compute $\sup_{F\in\mathcal{F}}\widehat{\text{Regret}}((x_0,x_1),F)$ and find the corresponding worst-case distributions.

Case 1) $1-x_1-c\geq 0$ (Equivalently, $1-c\geq x_1$.)

Whether $x_1=0$ or $x_1>0$, $g(1)>g(\theta)$ for $\theta\in(0,1)$. That is, $g(\theta)$ achieves the unique maximum at $\theta=1$ over the interval $(0,1]$. If $x_1=0$, $g(\theta)=0$ for $\theta\in(0,c)$ and $g(\theta)=\theta-c$ for $\theta\in[c,1]$. If $x_1>0$, $g(\theta)=-\theta\cdot x_1<0$ for $\theta\in(0,c)$ and $g(\theta)=\theta\cdot(1-x_1)-c$ for $\theta\in[c,1]$, which is maximized at $\theta=1$ and at least 0 at that point.

Then, the worst-case partial distribution F^- over $(0,1]$ given $F(0)$ is the probability mass of $1-F(0)$ at $\theta=1$. It follows that

$$\begin{aligned}&\sup_{F\in\mathcal{F}}\widehat{\text{Regret}}((x_0,x_1),F)\\ &\qquad=\sup_{F(0)\in[0,1]}\left\{(1-x_1-c)\cdot(1-F(0))+c\cdot x_1\cdot(1-F(0))+c\cdot x_0\cdot F(0)\right\}\\ &\qquad=\sup_{F(0)\in[0,1]}\left\{(1-x_1-c+c\cdot x_1)\cdot(1-F(0))+c\cdot x_0\cdot F(0)\right\}\,.\end{aligned}$$

We further analyze the simplified expression in the following cases. In all cases, we show

$$\sup_{F\in\mathcal{F}}\widehat{\text{Regret}}((x_0,x_1),F)=1-c-x_1+c\cdot x_1.$$

Case a) $1-x_1-c>0$

Note $1-x_1-c+c\cdot x_1>c\cdot x_0$ since $c\cdot x_1\geq c\cdot x_0$. The worst-case distribution F is uniquely determined to be the point-mass of 1 at $\theta=1$, i.e., $F(0)=0$. Then,

$$\sup_{F\in\mathcal{F}}\widehat{\text{Regret}}((x_0,x_1),F)=1-c-(1-c)\cdot x_1\,.$$

Case b) $1-x_1-c=0$ and $x_0<x_1$

Since $1-x_1-c=0$, the maximization problem further simplifies to

$$\sup_{F(0)\in[0,1]}\left\{c\cdot x_1\cdot(1-F(0))+c\cdot x_0\cdot F(0)\right\}\,.$$

The worst-case distribution F is uniquely determined to be the point-mass of 1 at $\theta=1$, i.e., $F(0)=0$. Then,

$$\sup_{F\in\mathcal{F}}\widehat{\text{Regret}}((x_0,x_1),F)=c\cdot x_1\,.$$

Case c) $1-x_1-c=0$ and $x_0=x_1$

As in Case 1b), the maximization problem becomes

$$\sup_{F(0)\in[0,1]} \{c \cdot x_1 \cdot (1 - F(0)) + c \cdot x_0 \cdot F(0)\} .$$

The worst-case distribution F is not uniquely determined. Any split between $\theta = 0$ and $\theta = 1$ is a worst-case, i.e., any $F(0) \in [0,1]$. Then,

$$\sup_{F\in\mathcal{F}} \widehat{\text{Regret}}((x_0, x_1), F) = c \cdot x_1 = c \cdot x_0 .$$

Case 2) $1 - x_1 - c < 0$ (Equivalently, $1 - c < x_1$.)

Whether $x_1 = 1$ or $x_1 < 1$, $g(\theta) < 0$ for all $\theta \in (0, 1]$. For $\theta \in (0, c)$, $g(\theta) = -\theta \cdot x_1 < -\theta \cdot (1-c) < 0$. For $\theta \in [c, 1]$, $g(\theta) = \theta \cdot (1 - x_1) - c \leq 1 - x_1 - c < 0$. Note $\widehat{\text{Regret}}((x_0, x_1), F) \leq c \cdot x_1 \cdot (1 - F(0)) + c \cdot x_0 \cdot F(0)$ for any distribution F. Then,

$$\begin{aligned}\sup_{F\in\mathcal{F}} \widehat{\text{Regret}}((x_0, x_1), F) &\leq \sup_{F(0)\in[0,1]} \{c \cdot x_1 \cdot (1 - F(0)) + c \cdot x_0 \cdot F(0)\} \\ &= c \cdot x_1 .\end{aligned}$$

In fact, $\sup_{F\in\mathcal{F}} \widehat{\text{Regret}}((x_0, x_1), F) = c \cdot x_1$. For any arbitrarily small $\epsilon \in (0, c)$, let F^ϵ be the point-mass distribution such that $\theta = \epsilon$ with probability 1. Then, $\widehat{\text{Regret}}((x_0, x_1), F_\epsilon) = (c - \epsilon) \cdot x_1$. As ϵ was arbitrary, we indeed have

$$\sup_{F\in\mathcal{F}} \widehat{\text{Regret}}((x_0, x_1), F) = c \cdot x_1 .$$

In the following cases, we determine corresponding worst-case distributions.

Case a) $x_0 < x_1$

There exists no worst-case distribution that achieves the supremum exactly, because while the expression $c \cdot x_1 \cdot (1 - F(0)) + c \cdot x_0 \cdot F(0)$ in the regret objective is maximized by putting probability mass over $(0, 1]$, the remainder $\mathbb{E}_{\theta\sim F^-}[\mathbf{1}\{\theta \geq c\} \cdot (\theta - c) - \theta \cdot x_1]$ is strictly negative. We can achieve the supremum arbitrarily closely by point-mass distribution F_ϵ for which $\theta = \epsilon$ with probability 1 for arbitrarily small $\epsilon \in (0, c)$.

Case b) $x_0 = x_1$

The worst-case distribution F is uniquely determined to be the point-mass of 1 at $\theta = 0$, i.e., $F(0) = 1$. To see this, we note the regret objective reduces to

$$\widehat{\text{Regret}}((x_0, x_1), F) = \mathbb{E}_{\theta\sim F^-}[\mathbf{1}\{\theta \geq c\} \cdot (\theta - c) - \theta \cdot x_1] + c \cdot x_0 .$$

Since $\mathbb{E}_{\theta\sim F^-}[\mathbf{1}\{\theta \geq c\} \cdot (\theta - c) - \theta \cdot x_1]$ is strictly negative for any probability mass placed over $(0, 1]$, the maximum regret is realized for $F(0) = 1$.

Then, we choose $x_1^* = 1 - c$ such that $1 - x_1 - c + c \cdot x_1 = c \cdot x_1$ and let x_0^* be any number in $[0, x_1^*]$. Any mechanism of this form is optimal and achieves the minimax regret of $\widehat{\text{Regret}} = c \cdot (1 - c)$. Hence, the first part of the theorem follows. Note x_1^* is uniquely determined. If $x_1 < x_1^*$, we have

$$\sup_{F\in\mathcal{F}} \widehat{\text{Regret}}((x_0, x_1), F) = 1 - c - (1 - c) \cdot x_1 > 1 - c - (1 - c) \cdot x_1^* = c \cdot (1 - c) .$$

If $x > x_1^*$, then

$$\sup_{F \in \mathcal{F}} \widehat{\text{Regret}}((x_0, x_1), F) = c \cdot x_1 > c \cdot x_1^* = c \cdot (1 - c) . \qquad \square$$

E.1 Second Part of Theorem 24: Nonexistence of Saddle Points

First, we consider the implications from the existence of a saddle point. Note it is always true that

$$\inf_{\substack{S \in \mathcal{S}: \\ \text{(IC),(IR)}}} \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F) \geq \sup_{F \in \mathcal{F}} \inf_{\substack{S \in \mathcal{S}: \\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S, F) ,$$

which is the max-min inequality. If there exists a saddle point (S^*, F^*) such that

$$\widehat{\text{Regret}}(S^*, F) \leq \widehat{\text{Regret}}(S^*, F^*) \leq \widehat{\text{Regret}}(S, F^*)$$

for any single-round direct IC/IR mechanism S and distribution F, it follows that

$$\begin{aligned} \inf_{\substack{S \in \mathcal{S}: \\ \text{(IC),(IR)}}} \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F) &\leq \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S^*, F) \\ &\leq \widehat{\text{Regret}}(S^*, F^*) \\ &\leq \inf_{\substack{S \in \mathcal{S}: \\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S, F^*) \\ &\leq \sup_{F \in \mathcal{F}} \inf_{\substack{S \in \mathcal{S}: \\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S, F) . \end{aligned}$$

Combining with the max-min inequality, it follows that all the inequalities are actually equalities. In particular, the existence of a saddle point (S^*, F^*) implies that 1) S^* is an optimal solution to the infimum-supremum problem (i.e., $\inf_{\substack{S \in \mathcal{S}: \\ \text{(IC),(IR)}}} \sup_{F \in \mathcal{F}} \widehat{\text{Regret}}(S, F)$) and achieves the objective value of $\widehat{\text{Regret}}(S^*, F^*)$ and F^* is a worst-case distribution for S^*; and 2) F^* is an optimal solution to the supremum-infimum problem (i.e., $\sup_{F \in \mathcal{F}} \inf_{\substack{S \in \mathcal{S}: \\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S, F)$) and achieves the objective value of $\widehat{\text{Regret}}(S^*, F^*)$ and S^* is an optimal mechanism against F^*.

Given the above discussion, it suffices we show that there exists no optimal solution F^* to the infimum-supremum problem that achieves the objective value of $\widehat{\text{Regret}} = c(1 - c)$, or, mathematically,

$$\inf_{\substack{S \in \mathcal{S}: \\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S, F^*) = \widehat{\text{Regret}} = c(1 - c) . \tag{21}$$

We prove it by contradiction. Let F^* be such distribution that satisfies (21). We have

$$\begin{aligned}
\inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S,F^*) &= \inf_{x:\text{(IC),(IR)}} \{\bar{u}(F^*) - \mathbb{E}_{\theta\sim F^*}[(\theta-c)\cdot x(\theta)]\} \\
&= \bar{u}(F^*) - \sup_{x:\text{(IC),(IR)}} \mathbb{E}_{\theta\sim F^*}[(\theta-c)\cdot x(\theta)] \\
&= \bar{u}(F^*) - \sup_{0\le x_0\le x_1\le 1} \{x_1\mathbb{E}_{\theta\sim F^*}[(\theta-c)\mathbf{1}\{\theta>0\}] - cx_0F^*(0)\} \\
&= \bar{u}(F^*) - \max\{\underbrace{\mathbb{E}_{\theta\sim F^*}[(\theta-c)\mathbf{1}\{\theta>0\}]}_{(*)},0\}\,, \qquad (22)
\end{aligned}$$

where the second equality follows from extracting the constant term $\bar{u}(F)$ and flipping the direction of the infimum; the third equality because single-round direct IC/IR mechanisms can be parametrized in terms of $0 \le x_0 \le x_1 \le 1$ such that $x(0) = x_0$ and $x(\theta) = x_1$ for $\theta \in (0,1]$; and the last equality because it is optimal to set $x_0 = 0$ and $x_1 = 1$ if $\mathbb{E}_{\theta\sim F^*}[(\theta-c)\mathbf{1}\{\theta>0\}] \ge 0$ and $x_1 = 0$ otherwise.

We claim that F^* satisfies $(*) = 0$. First, note that the single-round benchmark satisfies

$$\begin{aligned}
\bar{u}(F^*) &= \mathbb{E}_{\theta\sim F^*}[\max\{\theta-c,0\}] \\
&= \mathbb{E}_{\theta\sim F^*}[\max\{\theta-c,0\}\cdot\mathbf{1}(\theta>0)] \\
&\le (1-c)\mathbb{E}_{\theta\sim F^*}[\theta\mathbf{1}\{\theta>0\}]\,, \qquad (23)
\end{aligned}$$

where the second equality follows because $c \in (0,1)$, and the inequality because $\max\{\theta-c,0\} \le (1-c)\theta$ since $\theta \in [0,1]$ and $c \in (0,1)$. Suppose $(*) < 0$. Combining (21), (22) and (23), we obtain

$$c(1-c) \le (1-c)\mathbb{E}_{\theta\sim F^*}[\theta\mathbf{1}\{\theta>0\}] < c(1-c)(1-F^*(0)) \le c(1-c)\,,$$

where the strict inequality follows because $c \in (0,1)$ and $(*) < 0$ implies $\mathbb{E}_{\theta\sim F^*}[\theta\mathbf{1}\{\theta>0\}] < \mathbb{E}_{\theta\sim F^*}[c\mathbf{1}\{\theta>0\}] = c(1-F^*(0))$, and the last inequality because $F^*(0) \in [0,1]$. This is a contradiction.

Similarly, suppose $(*) > 0$. Combining (21), (22), and (23), we obtain

$$c(1-c) \le c(1-F^*(0)) - c\mathbb{E}_{\theta\sim F^*}[\theta\mathbf{1}(\theta>0)] < c(1-c)(1-F^*(0)) \le c(1-c)\,,$$

where the strict inequality follows because $c \in (0,1)$ and $(*) > 0$ implies $\mathbb{E}_{\theta\sim F^*}[\theta\mathbf{1}\{\theta>0\}] > \mathbb{E}_{\theta\sim F^*}[c\mathbf{1}\{\theta>0\}] = c(1-F^*(0))$, and the last inequality because $F^*(0) \in [0,1]$. Again, a contradiction. Hence, we have $(*) = 0$.

We now argue that $(*) = 0$ implies that $F^*(0) = 1$. Combining (21), (22), and (23) together with $(*) = 0$ implies that $c(1-c) \le c(1-c)(1-F^*(0)) \le c(1-c)$; to see this, we follow the same argument above under the assumption $(*) < 0$ where the strict inequality becomes an equality. That is, $c(1-c)(1-F^*(0)) = c(1-c)$. Because $c \in (0,1)$, we can divide both sides by $c(1-c)$ and obtain that $F^*(0) = 1$. Hence, the only possible candidate distribution F^* is the point-mass distribution under which the shock is 0 with probability 1. For this particular distribution F^*, $\bar{u}(F^*) = 0$ and the IC/IR mechanism that always does not allocate (i.e., $x_0 = x_1 = 0$) is such that $\mathbb{E}_{\theta\sim F^*}[(\theta-c)\cdot x(\theta)] = 0$. This means $\inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S,F^*)$ can be at most 0 and cannot be equal to $\widehat{\text{Regret}} = c(1-c) > 0$. We, thus, conclude there exists no such distribution F^* satisfying (21).

E.2 Third Part of Theorem 24: an Asymmetric Result

While saddle points do not exist, we can still show an asymmetric saddle-point result, that is, the saddle-point property holds and the single-round minimax regret problem admits an optimal solution. In Part 1, we showed that S^* is an optimal solution to the single-round minimax regret problem and achieves $\widehat{\text{Regret}}$. In what follows, we show the saddle-point property holds:

$$\inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \sup_{F\in\mathcal{F}} \widehat{\text{Regret}}(S,F) = \sup_{F\in\mathcal{F}} \inf_{\substack{S\in\mathcal{S}:\\ \text{(IC),(IR)}}} \widehat{\text{Regret}}(S,F)\,.$$

From Part 1, we know the left-hand side is equal to $\widehat{\text{Regret}} = c(1-c)$.

For an arbitrary $\epsilon \in (0,c)$, consider the distribution F_ϵ under which $\theta = \epsilon$ with probability $1-c$ and $\theta = 1$ with probability c. Then,

$$\begin{aligned}
\widehat{\text{Regret}}((x_0,x_1),F) &= \mathbb{E}_{\theta\sim F^-}[\mathbf{1}\{\theta \geq c\}\cdot(\theta - c) - \theta\cdot x_1] + c\cdot x_1\cdot(1-F(0)) + c\cdot x_0\cdot F(0)\\
&= -\epsilon\cdot x_1\cdot(1-c) + (1-c-x_1)\cdot c + c\cdot x_1\\
&= -\epsilon\cdot(1-c)\cdot x_1 + c\cdot(1-c)\,.
\end{aligned}$$

It follows that

$$\inf_{x:\text{(IC),(IR)}} \widehat{\text{Regret}}((x_0,x_1),F_\epsilon) = -\epsilon\cdot(1-c) + c\cdot(1-c)\,,$$

where the infimum is achieved by the single-round mechanism with $x_1 = 1$ and $x_0 \in [0,1]$. As ϵ was arbitrary,

$$\sup_{F\in\mathcal{F}} \inf_{x:\text{(IC),(IR)}} \widehat{\text{Regret}}((x_0,x_1),F_\epsilon) = c\cdot(1-c)\,,$$

as desired.