D Shaped Surface

Normalized coordinates

$$
\begin{aligned}
&R = R_0 + ax(\alpha) \\
&Z = ay(\alpha) \\
&0 \le \alpha \le 2\pi
\end{aligned}
\tag{1}
$$

Model for the surface

$$
\begin{aligned}
&x(\alpha) = c_0 + c_1 \cos(\alpha) + c_2 \cos(2\alpha) + c_3 \cos(3\alpha) \\
&y(\alpha) = \kappa \sin(\alpha)
\end{aligned}
\tag{2}
$$

Constraints determining the c_j

$$
\begin{aligned}
&x(0) = 1 \\
&x(\pi) = -1 \\
&x(\pi\ /\ 2) = -\delta \\
&x_{\alpha\alpha}(\pi) = 0.3(1 - \delta^2)
\end{aligned}
\tag{3}
$$

Where did the last constraint come from?

To make sure the surface is always convex we can specify the curvature at $\alpha = \pi$. A trial and error empirical fit resulted in the choice $x_{\alpha\alpha}(\pi) = 0.3(1 - \delta^2)$. Maybe you can find a better choice.

The constraint equations

Applying the constraints leads to

$$
\begin{aligned}
&x(0) = 1 && c_0 + c_1 + c_2 + c_3 = 1 \\
&x(\pi) = -1 && c_0 - c_1 + c_2 - c_3 = -1 \\
&x(\pi / 2) = -\delta && c_0 - c_2 = -\delta \\
&x_{\alpha\alpha}(\pi) = 0.3(1-\delta^2) && c_1 - 4c_2 + 9c_3 = x_{\alpha\alpha} = 0.3(1-\delta^2)
\end{aligned}
\tag{4}
$$

Solution

Solve simultaneously to obtain

$$
\begin{aligned}
c_0 &= -\frac{\delta}{2} \\
c_2 &= \frac{\delta}{2} \\
c_1 &= \frac{1}{8}\left[9 - 2\delta - x_{\alpha\alpha}(\pi)\right] \\
c_3 &= \frac{1}{8}\left[-1 + 2\delta + x_{\alpha\alpha}(\pi)\right]
\end{aligned}
\tag{5}
$$

Area and Volume

$$
\begin{aligned}
A &= \int\int dRdZ = a^2 \int\int dxdy = a^2 \int_0^{2\pi} x \frac{dy}{d\alpha} d\alpha = \pi a^2 \kappa \left[\frac{9 - 2\delta - x_{\alpha\alpha}(\pi)}{8}\right] \\
V &= \int\int\int RdRdZd\phi = 2\pi a^2 \int\int Rdxdy = 2\pi a^2 R_0 \int_0^{2\pi} \left(x + \varepsilon \frac{x^2}{2}\right) \frac{dy}{d\alpha} d\alpha \\
&\approx 2\pi a^2 R_0 \int_0^{2\pi} x \frac{dy}{d\alpha} d\alpha = 2\pi^2 R_0 a^2 \kappa \left[\frac{9 - 2\delta - x_{\alpha\alpha}(\pi)}{8}\right]
\end{aligned}
\tag{6}
$$

Model flux surfaces

Equation (2) is a relatively formula describing the shape of the outer plasma surface. We can next modify the model so that it gives a plausible description of the interior flux surfaces as well as only the outer surface. The idea is to introduce a normalized flux label, which is radial like in behavior. This label is denoted by ρ and $0 \leq \rho \leq 1$ with $\rho = 1$ the outer plasma surface and $\rho = 0$ the magnetic axis. After some more trial and error I came up with the following representation for the flux surfaces,

$$\begin{aligned} x(\rho,\alpha) &= \sigma(1-\rho^2) + c_0\rho^4 + c_1\rho\cos(\alpha) + c_2\rho^2\cos(2\alpha) + c_3\rho^3\cos(3\alpha) \\ y(\rho,\alpha) &= \kappa\rho\sin(\alpha) \end{aligned} \tag{7}$$

with σ being the sift of the magnetic axis. Typically, $\sigma \sim 0.1$ for a high field tokamak.

Plot the results

If you get a chance please plot a set of flux contours using some typical parameters that we might expect on our general reactor.

Volume and cross sectional area integrals

In the course of the work we will need to integrate functions of ρ over the volume and over the cross sectional area of the plasma. Specifically we will need to evaluate

$$\begin{aligned} Q_V &= \int\int\int Q(\rho) R\,dR\,dZ\,d\phi \approx 2\pi R_0 a^2 \int\int Q(\rho)\,dx\,dy \\ Q_A &= \int\int Q(\rho)\,dR\,dZ = a^2 \int\int Q(\rho)\,dx\,dy \end{aligned} \tag{8}$$

Here, $Q(\rho)$ is an arbitrary function of ρ such as pressure or current density. In the large aspect ratio limit both integrals require the evaluation of the same quantity

$$K = \int\int Q(\rho)\,dx\,dy \tag{9}$$

To evaluate this integral we need to convert from x, y coordinates to ρ, α coordinates. Using the Jacobian of the transformation leads to

$$K = \int\int Q(\rho)(x_\rho y_\alpha - x_\alpha y_\rho) d\rho d\alpha \tag{10}$$

Here,

$$\begin{aligned} x_\rho y_\alpha - x_\alpha y_\rho = {} & \kappa\rho\cos(\alpha)\left[-2\rho\sigma + 4\rho^3 c_0 + c_1\cos(\alpha) + 2c_2\rho\cos(2\alpha) + 3c_3\rho^2\cos(3\alpha)\right] \\ & +\kappa\sin(\alpha)\left[c_1\rho\sin(\alpha) + 2c_2\rho^2\sin(2\alpha) + 3c_3\rho^3\sin(3\alpha)\right] \end{aligned} \tag{11}$$

Since Q is only a function of ρ the α integral can be carried out analytically. The only terms that survives the averaging are the ones containing c_1. A simple integration over α then yields the desired results

$$\begin{aligned} Q_V &= 4\pi^2 R_0 a^2 \kappa c_1 \int_0^1 Q(\rho)\rho d\rho \\ Q_A &= 2\pi a^2 \kappa c_1 \int_0^1 Q(\rho)\rho d\rho \end{aligned} \tag{12}$$

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```
% J. Freidberg D Shaped Surface
```

constants

average value of 7 machines (see google spreasheet) @ 95 % flux surface

```
delta = 0.42;  % triangularity
kappa = 1.76; % elongation

a = 2;    % meters
R0 =5;  % meters
sigma = 0.1;
Xaa = 0.3 * ( 1- delta^2);  % constraint from J.F. so surface always
 convex
c0 = -delta/2;
c2 = delta/2 ;
c1 = (1/8) * (9 - 2*delta - Xaa);
c3 = (1/8) * (-1 + 2*delta + Xaa);
```

model of surface

```
xS = @(alpha) c0 + c1*cos(alpha) + c2*cos(2 * alpha) +
 c3*cos(3*alpha);
yS = @(alpha) kappa * sin(alpha);
```

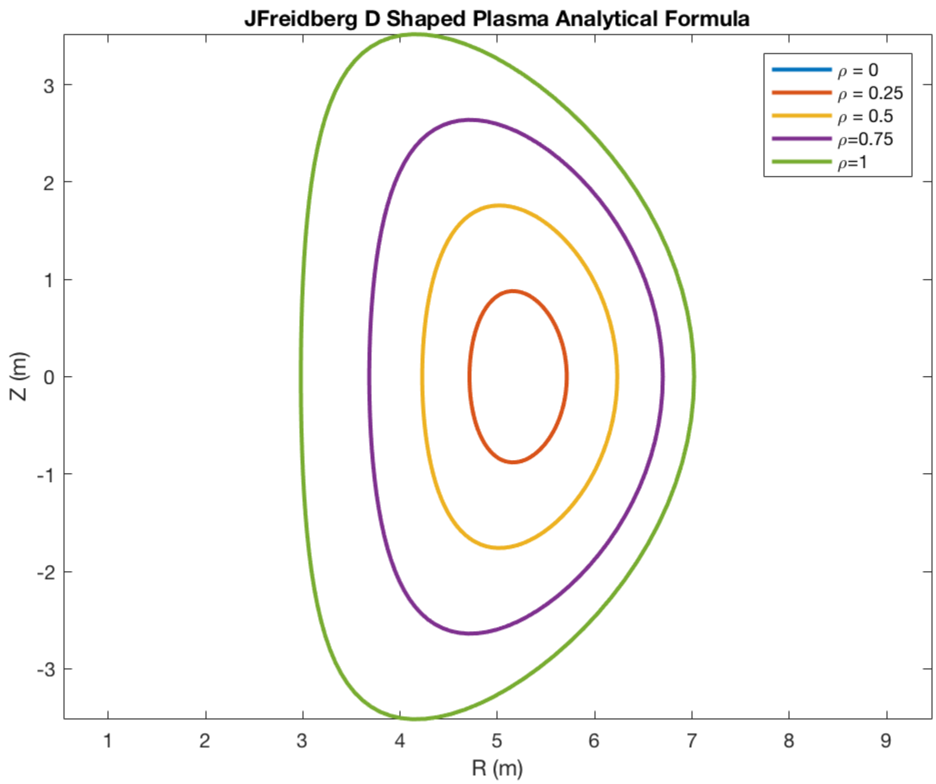
model for flux surfaces

```
x = @(alpha, rho) sigma*(1 - rho.^2) + c0 * rho.^4 +  rho .*
 cos(alpha) + ...
    c2 * rho.^2 .* cos(2 * alpha) + c3 .* rho.^3 .* cos(3 .* alpha);
y = @(alpha, rho) kappa .* rho .* sin(alpha) ;
```

set up coordinate system

```
alpha = linspace(0, 2*pi, 100);
rho = linspace(0,1, 100);
R = R0 + a* xS(alpha);
Z = a * yS(alpha);
```

```
plot(R0 + a * x(alpha, 0), a * y(alpha, 0), 'Linewidth', 2);
hold on
plot(R0 + a * x(alpha, 0.25), a * y(alpha, 0.25), 'Linewidth', 2);
plot(R0 + a * x(alpha, 0.5), a * y(alpha,0.5), 'Linewidth', 2);
plot(R0 + a * x(alpha, 0.75), a * y(alpha,0.75), 'Linewidth', 2);
plot(R0 + a * x(alpha, 1), a * y(alpha, 1), 'Linewidth', 2);
title('JFreidberg D Shaped Plasma Analytical Formula');
legend('\rho = 0', '\rho = 0.25', '\rho = 0.5', '\rho=0.75','\rho=1');
xlabel('R (m)');
ylabel('Z (m)');
axis equal
```



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