LHCD Overview

Goal

To design our reactor we need to calculate η_{CD} for lower hybrid current drive. Recall that the driven lower hybrid current I_{CD} is related to the lower hybrid RF power delivered to the plasma P_{LH} by the relation

$$I_{CD} = \eta_{CD} \frac{P_{LH}}{n_{20} R} \tag{1}$$

Here, $P_{LH} = \eta_{RF} P_H$ with P_H equal to the total wall power used for heating plus current drive and $\eta_{RF} \approx 0.5$ is the conversion efficiency from wall power to RF delivered power. Also, $n_{20} = n_{20}(\rho_J)$ and $R = R(\rho_J, \theta)$ are the density and major radius evaluated at the minor radius $\rho = \rho_J$ and launch angle θ corresponding to the location of the peak driven current density: $J_{\max} = J_{CD}(\rho_J, \theta)$.

The quantity η_{CD} is related to a normalized value $\tilde{\eta}$, the efficiency usually calculated in the literature, by a series of connecting formulas. The inter-relations start with

$$\eta_I = \frac{\int_A J_{CD} dA}{\int_V S_{KL} dV} \approx \frac{1}{2\pi} \left[\frac{J_{CD}}{R S_{KL}} \right]_{\rho_J, \theta} = \frac{\eta_{LH}}{2\pi R} \left[\frac{J_{CD}}{R S_{LH}} \right]_{\rho_J, \theta} \tag{2}$$

where $\eta_I = I_{CD} / P_{KL}$ A/W is the overall current drive efficiency measuring how many delivered watts of klystron RF power are required to drive one ampere of current. Also, S_{KL} is the total klystron power density delivered to the plasma. Due to various losses, only a fraction of this power $\eta_{LH} \approx 0.75$ actually drives current. Therefore, the power density S_{LH} driving lower hybrid current is related to the klystron power density by $S_{LH} = \eta_{LH} S_{KL}$.

Now, the efficiency $\tilde{\eta}$ usually calculated in the literature is defined by

$$
\tilde{\eta}(\rho_J,\theta) = \left[\frac{J_{CD} / env_{Te}}{S_{LH} / m_e n \nu_0 v_{Te}^2}\right]_{\rho_J,\theta}
$$

$$
v_{Te}(\rho_J) = \left(\frac{2T_e}{m_e}\right)^{1/2}
$$

$$
\nu_0(\rho_J) = \frac{\omega_{pe}^4 \ln\Lambda}{2\pi n_e v_{Te}^3} \tag{3}
$$

It then follows that

$$
\eta_I = \frac{\eta_{LH}}{2\pi}\left[\frac{e}{Rm_e \nu_0 v_{Te}}\right]_{\rho_J,\theta} \tilde{\eta}(\rho_J,\theta) \tag{4}
$$

From Eq. (1) we see that $\eta_{CD} = \eta_I [n_{20} R]_{\rho_J,\theta}$ which leads to the desired conversion relation

$$
\eta_{CD} = \frac{\eta_{LH}}{2\pi}\left[\frac{en_{20}}{m_e \nu_0 v_{Te}}\right]_{\rho_J} \tilde{\eta}(\rho_J,\theta) = 0.06108\frac{\eta_{LH}}{\ln\Lambda} T_k \tilde{\eta} \tag{5}
$$

All quantities in the parentheses are assumed to be evaluated at $\rho = \rho_J$. What is needed for the design code is an expression for $\tilde{\eta}(\rho_J,\theta)$.

An expression for $\tilde{\eta}(\rho,\theta)$

An expression for $\tilde{\eta}(\rho,\theta)$ for arbitrary ρ has been determined by Ehst and Karney [Nuclear Fusion, 31, p1933, (1991)] based on a sophisticated theoretical analysis combined with extensive numerical results. Once ρ_J is determined we set $\rho = \rho_J$ in the expression for $\tilde{\eta}(\rho,\theta)$. Ehst and Karney find that a good fit for $\tilde{\eta}(\rho,\theta)$ can be written as

$$
\tilde{\eta} = CMR\eta_0 \tag{6}
$$

$$M = 1$$

For LHCD the parameters appearing in Eq. (6) have the form

$$R(\rho,\theta) = 1 - \frac{\varepsilon^n \rho^n (x_r^2 + w^2)^{1/2}}{\varepsilon^n \rho^n x_r + w} \qquad n = 0.77 \qquad x_r = 3.5 / 2^{1/2}$$

$$C(\rho,\theta) = 1 - \exp(-c^m x_t^{2m}) \qquad m = 1.38 \qquad c = 0.389 \times 2$$

$$\eta_0(\rho,\theta) = \frac{K}{w} + D + \frac{8w^2}{5 + Z_{eff}} \qquad K = \frac{3.0 / 2^{1/2}}{Z_{eff}} \qquad D = \frac{3.83}{Z_{eff}^{0.707}} \tag{7}$$

All quantities have been defined except for $x_t^2(\rho,\theta)$ and $w(\rho,\theta)$. The quantity w is a normalized form of the resonant particle velocity which absorbs energy and momentum from the lower hybrid wave,

$$w(\rho,\theta) = \frac{\omega}{k_\parallel v_{Te}} = \frac{c}{v_{Te}} \frac{1}{n_\parallel} \tag{8}$$

The value of $n_\parallel(\rho,\theta)$ will be discussed shortly.

The quantity x_t^2 is a toroidal correction associated with the fact that trapped particles cannot contribute to toroidal current flow. It can be expressed in terms of the local mirror ratio by

$$x_t^2(\rho,\theta) = w^2 \left(\frac{B}{B_M - B} \right) \tag{9}$$

where in the circular flux surface approximation

$$B_M = \frac{B_0}{1-\varepsilon\rho}$$

$$B = \frac{B_0}{1+\varepsilon\rho\cos\theta} \tag{10}$$

As an aside we point out that in the limit of large aspect ratio $\varepsilon \to 0$ and high resonant velocity $w \gg 1$, reasonable approximations for LHCD in a reactor, the formulas simplify significantly: $R \approx 1$, $C \approx 1$, and $\eta_0 \approx 8w^2 / (5+Z_{eff})$. In this limit $\tilde{\eta} \approx \eta_0$ leading to

$$\eta_I \approx \left(\frac{\eta_{LH}\varepsilon_0^2 m_e c^2}{e^3 \ln\Lambda}\right)\left(\frac{8}{5+Z_{eff}}\right)\frac{1}{nRn_\parallel^2} = \left(\frac{15.61\eta_{LH}}{\ln\Lambda}\right)\left(\frac{8}{5+Z_{eff}}\right)\frac{1}{n_{20}Rn_\parallel^2} = \frac{1.095}{n_{20}Rn_\parallel^2} \tag{11}$$

The numerical values correspond to $Z_{eff} = 1$, $\ln\Lambda = 19$, and $\eta_{LH} = 1$. It is very close to the value derived in my textbook using a much simpler model.

Calculation of $n_\parallel^2(\rho,\theta)$

The next step in the evaluation of η_{CD} is the calculation of $n_\parallel^2(\rho,\theta)$. Its value is determined by the requirements for accessibility from the plasma edge into the absorption layer. The relevant physics follows from an analysis of the cold plasma dispersion relation given by

$$n_\perp^2(\rho,\theta) = -\frac{K_\parallel}{2K_\perp}\left\{n_\parallel^2 - K_\perp + \frac{K_A^2}{K_\parallel} \pm \left[\left(n_\parallel^2 - K_\perp + \frac{K_A^2}{K_\parallel}\right)^2 + \frac{4K_\perp K_A^2}{K_\parallel}\right]^{1/2}\right\} \tag{1}$$

The plus sign corresponds to the desired root and is often referred to as the slow wave.

In the lower hybrid regime the relevant ordering of parameters is

$$\omega_{pe} / \Omega_e \sim \omega_{pi} / \omega \sim n_\parallel \sim 1$$

$$\omega_{pi} / \Omega_i \sim \omega / \Omega_i \sim \Omega_e / \omega \sim n_\perp \sim \left(m_i / m_e\right)^{1/2} \gg 1 \tag{2}$$

leading to the following simple forms for the elements of the dielectric tensor

$$K_\perp(\rho,\theta) = 1 + \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2} \sim 1$$

$$K_A(\rho,\theta) = \frac{\omega_{pe}^2}{\omega \Omega_e} \sim \left(m_i / m_e\right)^{1/2}$$

$$K_\parallel(\rho) = -\frac{\omega_{pe}^2}{\omega^2} \sim m_i / m_e \tag{3}$$

The first requirement for accessibility is that the function under the square root be positive. When this function passes through zero there is a double root for $n_\perp^2$ causing a mode conversion from the slow wave to the fast wave. The fast wave does not propagate into the plasma. It is reflected back out through the plasma surface, obviously an undesirable result. Avoiding mode conversion requires a sufficiently large value of $n_\parallel^2$ to keep the function under the square root positive. This value must satisfy

$$n_\parallel^2(\rho,\theta) \geq \left[K_\perp^{1/2} + \left(-\frac{K_A^2}{K_\parallel}\right)^{1/2}\right]^2 \tag{4}$$

Since $\eta_{CD} \propto 1 / n_\parallel^2$ we see that current drive efficiency is maximized when $\eta_\parallel^2(\rho,\theta)$ is minimized – the inequality in Eq. (4) must be set to equality.

At this point there is an important subtlety that must be taken into account. The issue is that the wavelength spectrum of the applied klystron source is not a delta function – it has a finite width $\Delta n_\parallel \approx 0.2$ plus a negative lobe as shown in Fig. 1a. For simplicity, we model the spectrum as rectangular and ignore the negative lobe, as shown in Fig. 1b. Actually, the negative lobe is accounted for through the value of η_{LH} since this power obviously does not drive current in the desired direction. Now, Eq. (4) is an inequality and we want to minimize $n_\parallel^2(\rho,\theta)$ over all ρ for the given θ where the power is absorbed. Therefore, we must use the equality sign in Eq. (4) for the strictest case $n_\parallel(\hat\rho_J,\theta) = n_\parallel(\rho_J,\theta) - \Delta n_\parallel$ where $\hat\rho_J$ and ρ_J (both as yet undetermined) are the corresponding strictest and average radii where power is absorbed.

With this in mind, after substituting the simplified expressions for the elements of the dielectric tensor we obtain

$$
\begin{aligned}
n_\parallel^2(\hat\rho_J,\theta) &= \left[\left(1 - \frac{1-\hat\omega^2}{\hat\omega^2}X\right)^{1/2} + X^{1/2}\right]^2 \\
X(\hat\rho_J,\theta) &= \omega_{pe}^2(\hat\rho_J) \,/\, \Omega_e^2(\hat\rho_J,\theta) \\
\hat\omega^2(\hat\rho_J,\theta) &= \omega^2 \,/\, \Omega_e(\hat\rho_J,\theta)\Omega_i(\hat\rho_J,\theta)
\end{aligned}
\tag{5}
$$

where $\hat x_j = \hat\rho_J \cos\theta$.

The question now is how do we choose the frequency $\hat\omega$? There are actually four constraints on the frequency and we must choose the strictest one to determine $\hat\omega^2$. The constraints are as follows:

$$
\begin{array}{ll}
\omega^2 > \omega_{LH}^2(\hat\rho_J,\theta) & \text{Avoid a wave resonance before reaching } \hat\rho_J,\theta \\
\omega^2 > 4\omega_{LH}^2(\hat\rho_J,\theta) & \text{Avoid the PDI before reacing } \hat\rho_J,\theta \\
\omega^2 \,/\, k_\perp^2(\hat\rho_J,\theta) > v_\alpha^2 & \text{Avoid coupling to } \alpha \text{ particles before reaching } \hat\rho_J,\theta \\
\sqrt{D(\hat\rho_J,\theta)} > 0 & \text{Avoid mode conversion before reaching } \hat\rho_J,\theta
\end{array}
\tag{6}
$$

Here, $\omega_{LH}^2(\hat\rho_J,\theta) = \omega_{pi}^2 \,/\, (1+\omega_{pe}^2 \,/\, \Omega_e^2)$ is the square of the lower hybrid frequency and $v_\alpha = (2E_\alpha \,/\, m_\alpha)^{1/2}$ is the alpha particle speed. The second and third constraints are approximate values, used here for simplicity.

Each of these constraints is substituted into the expression for $n_\parallel^2$, which can then be plotted as a function of X. We find that in the regime of interest the α particle coupling requirement is the strictest. We thus choose the frequency to satisfy $\omega \,/\, k_\perp = V_\alpha$, or in normalized units

$$n_\perp^2(\hat{\rho}_J,\theta) = \frac{c^2}{v_\alpha^2} \tag{7}$$

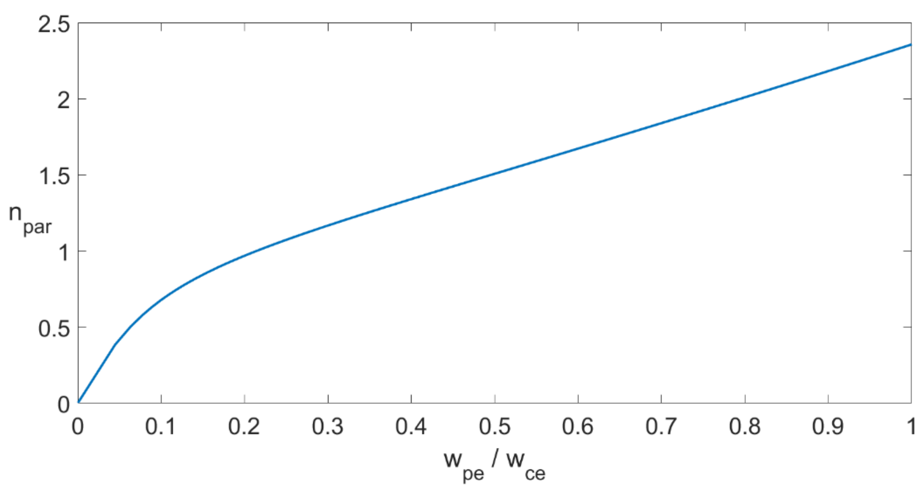
This expression is simplified by evaluating $n_\perp^2$ using Eq. (1) coupled with $n_\parallel^2$ given by Eq. (4)

$$n_\perp^2(\hat{\rho}_J,\theta) = -\frac{K_\parallel}{K_\perp^{1/2}}\left(-\frac{K_A^2}{K_\parallel}\right)^{1/2} = \frac{m_i}{m_e}\frac{X^{3/2}}{\hat{\omega}[\hat{\omega}^2(1+X)-X]^{1/2}} \tag{8}$$

Equation (8) is a quadratic equation for $\hat{\omega}^2$ which can be easily solved, yielding

$$\hat{\omega}^2(\hat{\rho}_J,\theta) = \frac{1}{2}\frac{X}{1+X} + \frac{1}{2}\left[\frac{X^2}{(1+X)^2} + 4\gamma^2\frac{X^3}{1+X}\right]^{1/2}$$
$$\gamma = \frac{m_i}{m_e}\frac{1}{n_\perp^2} = \frac{2m_iE_\alpha}{m_em_\alpha c^2} = 8.562 \tag{9}$$

This value of $\hat{\omega}^2$ is substituted into Eq. (5) to obtain the desired expression for $n_\parallel^2 = n_\parallel^2(X)$. The result is plotted on the curve below.



Calculation of ρ_J

The calculation of $\hat{\rho}_J$ requires a very lengthy analysis of Landau damping. We can bypass this complication by making use of a simple rule of thumb that is reasonably accurate. This rule states that lower hybrid power and driven current are produced in a somewhat narrow layer of the plasma whose location is determined by the requirement that the parallel phase velocity be approximately equal to three times the electron thermal speed,

$$\frac{\omega}{k_\parallel} \approx 3v_T \tag{10}$$

Observe that absorption takes place on the tail of the distribution function.

The equation can be rewritten in terms of $\hat{n}_\parallel$ leading to a transcendental algebraic equation for $\hat{\rho}_J$,

$$(1+\nu_T)(1-\hat{\rho}_J^2)^{\nu_T} n_\parallel^2(\hat{\rho}_J,\theta) = \frac{m_e c^2}{18\bar{T}} = \frac{28.39}{\bar{T}_k} \tag{11}$$

This is a simple equation to solve numerically.

Calculation of ρ_J

The last step in the analysis is to map the results at the strictest absorption location $\hat{\rho}_J,\theta$ to the center of the absorption layer ρ_J,θ where the current drive efficiency is defined. This is easily done by noting that power is always absorbed in at the local radius where $\omega / k_\parallel = 3v_{Te}$. Consequently, the relations at ρ_J are related to those at $\hat{\rho}_J$ by

$$\begin{aligned}(1+\nu_T)(1-\hat{\rho}_J^2)^{\nu_T} n_\parallel^2(\hat{\rho}_J,\theta) &= \frac{28.39}{\bar{T}_k}\\ (1+\nu_T)(1-\rho_J^2)^{\nu_T} n_\parallel^2(\rho_J,\theta) &= \frac{28.39}{\bar{T}_k}\end{aligned} \tag{12}$$

Since $n_\parallel(\hat{\rho}_J,\theta) = n_\parallel(\rho_J,\theta) - \Delta n_\parallel$ it follows that $\hat{\rho}_J$ and ρ_J are related by

$$\frac{(1-\rho_J^2)^{\nu_T}}{(1-\hat{\rho}_J^2)^{\nu_T}} = \left[1 - \frac{\Delta n_\parallel}{n_\parallel(\rho_J,\theta)}\right]^2 \quad \rightarrow \quad \rho_J^2 = 1 - (1-\hat{\rho}_J^2)\left[1 - \frac{\Delta n_\parallel}{n_\parallel(\rho_J,\theta)}\right]^{2/\nu_T} \tag{13}$$

Note that in general $\rho_J > \hat{\rho}_J$. The strictest location determining $n_\parallel(\hat{\rho}_J,\theta)$ is the innermost radial point on the temperature profile where power is absorbed.

Summary

Assume the following quantities are given as inputs: $B_0, \theta, \overline{n}_{20}, \overline{T}_k, \varepsilon, \Delta n_\parallel, \eta_{LH}$. Carry out the following steps:

1. Solve the equations below simultaneously to determine $n_\parallel^2(\hat{\rho}_J,\theta)$, $\hat{\omega}^2(\hat{\rho}_J,\theta)$, and $\hat{\rho}_J$,

$$n_\parallel^2(\hat{\rho}_J,\theta) = \left[\left(1 - \frac{1-\hat{\omega}^2}{\hat{\omega}^2}X\right)^{1/2} + X^{1/2}\right]^2$$

$$\hat{\omega}^2(\hat{\rho}_J,\theta) = \frac{1}{2}\frac{X}{1+X} + \frac{1}{2}\left[\frac{X^2}{(1+X)^2} + 4\gamma^2\frac{X^3}{1+X}\right]^{1/2}$$

$$(1+\nu_T)(1-\hat{\rho}_J^2)^{\nu_T} n_\parallel^2(\hat{\rho}_J,\theta) = \frac{m_e c^2}{2\overline{T}} = \frac{28.39}{\overline{T}_k} \tag{14}$$

2. Solve for $\tilde{\eta}(\hat{\rho}_J,\theta)$

$$\tilde{\eta}(\hat{\rho}_J,\theta) = CMR\eta_0 \tag{15}$$

3. Solve for $n_\parallel(\rho_J,\theta)$

$$n_{\parallel}(\rho_J,\theta) = n_{\parallel}(\hat{\rho}_J,\theta) + \Delta n_{\parallel} \tag{16}$$

4. Solve for ρ_J

$$\rho_J^2 = 1 - (1-\hat{\rho}_J^2)\left[1 - \frac{\Delta n_{\parallel}}{n_{\parallel}(\rho_J,\theta)}\right]^{2/\nu_T} \tag{17}$$

5. Re-evaluate $\tilde{\eta}(\rho_J,\theta)$ by substituting the values of $\rho_J, n_{\parallel}(\rho_J,\theta)$ into Eq. (15)
6. Solve for η_{CD}

$$\eta_{CD} = \frac{1}{2\pi}\left(\frac{en_{20}}{m_e\nu_0 v_{Te}}\right)\tilde{\eta} = 0.06108\frac{\eta_{LH}}{\ln\Lambda}(1+\nu_T)\overline{T}_k(1-\rho_J^2)^{\nu_T}\,\tilde{\eta}(\rho_J,\theta) \tag{18}$$

In the end there will have to be some iteration with the rest of the analysis to make sure the values of $\overline{n}_{20}$ and $\overline{T}_k$ are self consistent.