# List of changes

# Chapter 3

# Formalizing the Systems Model

The goal of this chapter is to take a step back from the steady current derivation and see the larger picture behind reactor design. As such, a more in-depth description of static~~fixed~~ and dynamic~~floating~~ variables is given. This discussion of dynamic~~floating~~ variables will then lend itself to a description of the framework underpinning the fusion systems model. As such, we will now need formulas for the radius and magnet strength of the tokamak. Moving forward, the current will ~~then~~ remain a connecting piece as we redirect focus~~switch gears~~ to pulsed tokamaks and compare the underlying solvers of the two schemes.~~the two schemes' underlying solvers.~~

The end result of this analysis will then be equations that allow the density ($\overline{n}$), current ($I_P$), major radius ($R_0$), and magnet strength ($B_0$) to be written as functions of the temperature ($\overline{T}$) and static variables (e.g. $\nu_n$, $N_G$, $f_D$). These formulas are the product of applying constraints required for all tokamak reactors with several other limiting constraints. The constraints relevant to all tokamak reactors are: the Greenwald limit, current balance, and power balance. Limit constraints then include: the Troyon beta limit, the kink safety factor, the wall loading limit, the maximum power constraint, and the heat loading limit.

Actual methodologies for solving for the five dynamic variables simultaneously – i.e. $\overline{T}$, $\overline{n}$, $I_P$, $R_0$, $B_0$ – are put off until Chapter 5.

## 3.1 Explaining Static~~Fixed~~ Variables

In this model, static~~fixed~~ variables are ones that remain constant while solving for a reactor. These include geometric scalings (i.e. $\epsilon$, $\delta$, $\kappa$), profile parameters (i.e. $\nu_n$, $\nu_T$, $l_i$), and a couple dozen~~slew~~ of physics constants related to pulsed and steady-state design (e.g. Q, $N_G$, $f_D$). For a complete list of static~~fixed~~ variables, consult Appendix A~~the appendix~~. The point to make now is that this model treats static~~fixed~~ variables as immutable~~second-class~~ objects. As such they often reside in static~~fixed~~ coefficients – $K_\square$ – which are treated as constants.

## 3.2 Connecting Dynamic~~Floating~~ Variables

Dynamic~~Floating~~ variables – $\overline{T}$, $\overline{n}$, $I_P$, $R_0$, $B_0$ – are the first-class variables of this fusion systems model. They represent the fundamental properties of a plasma and tokamak (which constitute a fusion reactor). As such, they will be reintroduced one at a time, explaining how they fit into the model – and which equations are~~equation is~~ capable of representing them~~it~~.

Table 3.1: Dynamic Variables

| **Symbol** | **Name** | **Units** |
|---|---|---|
| $I_P$ | Plasma Current | MA |
| $\overline{T}$ | Plasma Temperature | keV |
| $\overline{n}$ | Electron Density | $10^{20}\,\mathrm{m}^{-3}$ |
| $R_0$ | Major Radius | m |
| $B_0$ | Magnetic Field | T |

Bluntly, this fusion systems model is a simple algebra problem: solve five equations with five unknowns (i.e. $\overline{T}$, $\overline{n}$, $I_P$, $R_0$, $B_0$). Although this naive approach would work, we can do a little better by collapsing~~wrangling~~ these five equations down to just one. This was already done while deriving the steady current. It just happened that the current was not directly dependent on the tokamak size ($R_0$) or magnet strength ($B_0$).

This will prove more challenging for the generalized current needed for pulsed operation. Even so, this equation will still be reduced~~boiled down~~ to one equation with a single unknown – $I_P$. A solution to which can be solved much faster than the naive 5 equation approach. This is one reason the model is so fast.

### The Plasma Temperature – $\overline{T}$

The plasma temperature, measured in keV (kilo-electron-volts), is one of the most nonlinear~~finicky~~ variables in the fusion systems framework~~model~~. It first proved troublesome when it was shown that a pedestal profile – not a parabolic one used here – would be needed for an accurate calculation of bootstrap current. The blackbox~~unusual~~ tabulation for reactivity – $(\sigma v)$ – which appeared in fusion power only further exposed this nonlinearity.

Acknowledging that temperature is the most difficult to handle parameter prompts its use as the scanned variable. What this means practically is scanning temperatures is the most straightforward method to produce~~produces~~ curves of reactors. By example, a scan may be run over the average temperatures ($\overline{T}$): 10, 15, 20, 25, and 30 keV – where each corresponds~~corresponding~~ to its own reactor with its own field strength ($B_0$), plasma current ($I_P$), etc. In equation form, this becomes:

$$\overline{T} = const. \tag{3.1}$$

The constant value, here,~~Where the constant~~ happens to be 10 keV in one run, 15 keV for the next, and 30 keV in the fifth.

### The Plasma Density – $\overline{n}$

The Greenwald density limit is a constraint with a simple form that applies to all tokamak reactors.~~The cornerstone of this fusion systems model has always been the application of the Greenwald density limit from square one.~~ It is for this reason – as well as being a good approximation – that a parabolic profile was rationalized over a pedestal (H-Mode) one. Repeated, the Greenwald density limit is:

$$\overline{n} = K_n \cdot \frac{I_P}{R_0^2} \tag{2.11}$$

This is an exceptionally simple relationship and why it guided the model. Unlike the next three variables, it is actually used in their derivations. ~~Therefore, any reactor found through this model is considered a *Greenwaldian Reactor* – one held at the Greenwald density limit.~~

### The Plasma Current – $I_P$

The plasma current is what separates steady-state from pulsed operation. From before, the steady current was found to be:

$$I_P = \frac{K_{BS}\overline{T}}{1 - K_{CD}(\sigma v)} \tag{2.30}$$

This was derived by setting the total current equal to the two sources of current: bootstrap and current drive. Or in fractional form,

$$I_P = I_{BS} + I_{CD} \quad \rightarrow \quad 1 = f_{BS} + f_{CD} \tag{3.2}$$

This says that the current fractions of bootstrap and current drive must sum to one. As shown next chapter, inductive sources can be included into this current balance:

$$1 = f_{BS} + f_{CD} + f_{ID} \tag{3.3}$$

This equation shows how steady-state and pulsed operation can coexist (see Fig. 3-1). The final point to make is reducing the model to being purely pulsed – i.e. neglecting the current drive:

$$1 = f_{BS} + f_{ID} \tag{3.4}$$

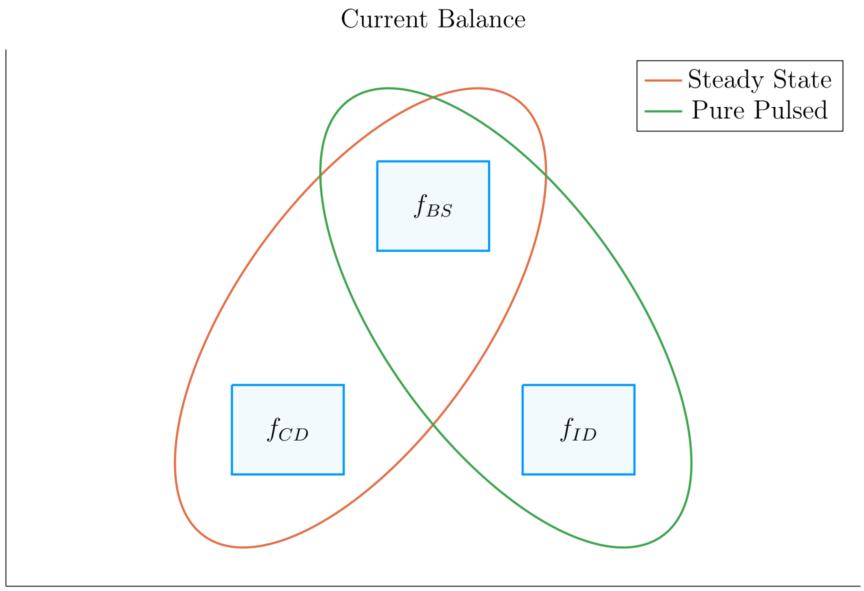



Figure 3-1: Current Balance in a Tokamak

In a tokamak, there needs to be a certain amount of current – and that current has to come from somewhere. All good reactors have an adequate bootstrap current. What provides the remaining current is what distinguishes steady state from pulsed operation.

Therefore, the next chapter will generalize the steady current to allow pulsed operation, and then simplify it to the purely pulsed case. Just as steady current faced self-consistency issues with $\eta_{CD}$, this current will also involve its own root solving conundrum – the description of which will be given in the following two chapters.

### The Tokamak Magnet Strength – $B_0$

The tokamak magnet strength has no unique~~obvious~~ equation to eliminate it. With foresight, the one this model uses~~chooses to use~~ is the power balance inherent to every~~in a~~ reactor. Similar to current balance, power balance is what separates a reactor from a device incapable of producing net electricity~~toaster~~. As such, it is referred throughout this document as: the primary constraint. It will be derived later this chapter.

### The Tokamak Major Radius – $R_0$

Much like the magnet strength, the major radius has no unique~~obvious~~ relation to express it. The model therefore uses this equation to handle a reactor's various~~This is convenient, because the model still has yet to resolve one of its most pressing issues:~~ physical and engineering-based constraints. This ~~laundry~~ list of requirements further restricts reactor space to the curves shown in the results section. Collectively, these are referred to as the limiting~~secondary~~ constraints – discussed later this chapter. These~~By miracle, these~~ constraints all just happen to depend on the size of the reactor – the reason they are chosen to represent~~substitute out~~ the radius.

## 3.3 Enforcing Power Balance

What separates a reactor from a device incapable of producing net electricity~~toaster~~ is power balance. Within a tokamak, it~~It~~ accounts for how the power going into a plasma's core exactly matches the power coming out of it. To approximate this conservation equation, two sets of power will be introduced: the sources and the sinks.

The sources have mainly been introduced at this point – they include the alpha power ($P_\alpha$) and the heating power ($P_H$), as well as a new ohmic power term ($P_\Omega$). The remaining two powers – the sinks – then appear through the radiation and heat conduction losses, which will be given shortly. In equation form, power balance becomes:

$$\sum_{sources} P = \sum_{sinks} P \tag{3.5}$$

or expanded to fit this model:

$$P_\alpha + P_H + P_\Omega = P_{BR} + P_\kappa \tag{3.6}$$

For clarity, the left-hand side of this equality are the sources. Whereas the remaining two are sinks, i.e. Bremsstrahlung radiation ($P_{BR}$) and heat conduction losses ($P_\kappa$).

### 3.3.1 Collecting Power Sources

As suggested, the two dominant sources of power in a tokamak are: alpha power ($P_\alpha$) and auxiliary heating ($P_H$). From earlier, it was determined that alpha particles (i.e. helium nuclei) carry around 20% of the total fusion power; or as we put it mathematically:

$$P_\alpha = \frac{P_F}{5} \tag{3.7}$$

Additionally, it was determined that the heating power is what was eventually amplified into fusion power – or through equation:

$$P_H = \frac{P_F}{Q} \tag{3.8}$$

The final source term then is the ohmic power ($P_\Omega$). This is identical to how copper wires in a home heat up as current runs through them. From a simple circuits picture, the power across the plasma is related to its current and resistance – in our standardized units – through:

$$P_\Omega = 10^6 \cdot I_P^2 \cdot R_P \tag{3.9}$$

~~Here, the resistance of the plasma is unlike any material humans encounter on a daily basis – actually decreasing with temperature.~~ This~~The~~ fusion systems model handles the plasma resistance ($R_P$) with the neoclassical Spitzer resistivity. Through equation,[7]

$$R_P = \frac{K_{RP}}{R_0 \overline{T}^{3/2}} \tag{3.10}$$

$$K_{RP} = 5.6e{-8} \cdot \left(\frac{Z_{eff}}{\epsilon^2 \kappa}\right) \cdot \left(\frac{1}{1 - 1.31\sqrt{\epsilon} + 0.46\epsilon}\right) \tag{3.11}$$

Combined with the Greenwald limit, ohmic power can be written more compactly as,

$$P_\Omega = K_\Omega \cdot \left( \frac{\overline{n}^2 R_0^3}{\overline{T}^{3/2}} \right) \tag{3.12}$$

$$K_\Omega = 10^6 \cdot \frac{K_{RP}}{K_n^2} \tag{3.13}$$

With the sources defined, we are now in a position to discuss the two sink terms used in this model's power balance.

### 3.3.2 Approximating Radiation Losses

All nuclear reactors emit radiation. From a power balance perspective, this means some power has to always be reserved to recoup from its losses – measured in megawatts. In a fusion reactor, the three most important types of radiation are: Bremsstrahlung radiation, line radiation, and synchrotron radiation.

Without going into too much detail, this model chooses to only model Bremsstrahlung radiation – as it usually dominates within the plasma's core. However, adding the effects of line-radiation and synchrotron radiation would drive results closer to real-world experiments. For example, line-radiation would better account for the heavy impurities that appear as pieces of a tokamak fall into the plasma.

For clarity, Bremsstrahlung – or breaking – radiation is what occurs when a charged particle (e.g. an electron) is accelerated by some means. In a tokamak, this happens all the time as charged particles are flung around and around the machine.* As given in Jeff Freidberg's book, this term is described by the volume integral:

$$P_{BR} = \int S_{BR} \, d\mathbf{r} \tag{3.14}$$

*This centripetal acceleration is akin to a child spinning a bucket as fast as they can without spilling a drop of water.

Here, the radiation power density ($S_{BR}$) is given by:

$$S_{BR} = \left( \frac{\sqrt{2}}{3\sqrt{\pi^5}} \cdot \frac{e^6}{\epsilon_0^2 c^3 h m_e^{3/2}} \right) \cdot \left( Z_{eff}\, n^2\, T^{1/2} \right) \tag{3.15}$$

The constants in the left set of parentheses all have their usual physics meanings (i.e. c is the speed of light and $m_e$ is the mass of an electron). What is new is the effective charge: $Z_{eff}$.

The effective charge is a scheme for collapsing the charge that each particle has to a collective value. Fundamental charge, here, is what: neutrons lack, electrons and hydrogen have one of, and helium has two. As such, a plasma with a purely deuterium and tritium fuel would have an effective charge of one. This value would then quickly rise if a Tungsten tile – with 74 units of charge – were to fall into the plasma core from the walls of the tokamak.

Using the volume integral – seen in the derivation of fusion power – allows the Bremsstrahlung power to be written in standardized units as:

$$P_{BR} = K_{BR}\, \overline{n}^2\, \overline{T}^{1/2} R_0^3 \tag{3.16}$$

$$K_{BR} = 0.1056\, \frac{(1+\nu_n)^2\, (1+\nu_T)^{1/2}}{1 + 2\,\nu_n + 0.5\,\nu_T}\, Z_{eff}\, \epsilon^2\, \kappa\, g \tag{3.17}$$

This power term represents the radiation power losses involved in power balance. All that is needed now is a formula for heat conduction losses – the hardest plasma behavior to model to date.

### 3.3.3 Estimating Heat Conduction Losses

Heat is energy that lacks direction on a microscopic level. Macroscopically, it generally moves from hotter areas to colder ones. As hinted by the plasma profile for temperature, heat emanates from the center of a plasma and migrates towards the walls of its tokamak enclosure. It therefore seems an important quantity to calculate when balancing power in a plasma's core.

The difficulty of estimating heat conduction, though, lies in the chaotic nature of plasmas – no theory or computation today can properly model it. As such, reactor designers have turned towards experimentalists for empirical scaling laws based on the dozen or so strongest tokamaks in the world. These are collectively referred to as confinement time scalings, i.e. the ELMy H-Mode Scaling Law.

The derivation of this heat conduction loss term ($P_\kappa$) starts in a manner similar to the previous powers. To begin, an equation for $P_\kappa$ is given in Jeff Freidberg's book as:

$$P_\kappa = \frac{1}{\tau_E} \int U d\mathbf{r} \tag{3.18}$$

This volume integral includes two new terms: the confinement time ($\tau_E$) and the internal energy (U). Before explaining these terms, a qualitative description is in order. As mentioned previously, the heat – or microscopically random – energy is captured by the internal energy (U). Then the confinement time ($\tau_E$) is how long it would take for the heat to completely leave the device if the system were suddenly turned off.

A formula for confinement time will be delayed till the end of this section, when it is needed to solve for the magnetic field ($B_0$). The internal energy (U), however, can be given now as it has its typical physics meaning. This assumes that all three plasma species are held nearly at the same temperature (T) as the electrons:

$$U = \frac{3}{2} \left(n + n_D + n_T\right) T \tag{3.19}$$

Here again, $n_D$ and $n_T$ – the density of deuterium and tritium, respectively – are related to the electron density (used in this model) through the dilution factor, which assumes a 50-50 mix of D-T fuel:

$$n_D = n_T = f_D \cdot \left(\frac{n}{2}\right) \tag{3.20}$$

Foregoing the mathematical rigor of previous sections, the equations here can be combined to form an equation for $P_\kappa$ – the heat conduction losses – in standardized units:

$$P_\kappa = K_\kappa \, \frac{R_0^3 \, \overline{n} \, \overline{T}}{\tau_E} \tag{3.21}$$

$$K_\kappa = 0.4744 \, (1 + f_D) \, \frac{(1+\nu_n)\,(1+\nu_T)}{1+\nu_n+\nu_T} \, (\, \epsilon^2 \, \kappa \, g \,) \tag{3.22}$$

Now that all five terms have been defined in power balance, the next step is expanding it and solving for the tokamak's toroidal magnetic field strength: $B_0$.

### 3.3.4 Writing the Lawson Criterion

Before locking in the primary constraint – i.e. the magnet strength ($B_0$) equation from power balance – it seems appropriate to take a detour and explain an intermediate solution: the Lawson Criterion. Within the fusion community, the Lawson Criterion is the cornerstone in any argument on the possibility of a design being used as a reactor (and not just some grandiose toaster).

An equation for the Lawson Criterion – sometimes referred to as the *triple product* – is easily found in the literature as:

$$n \cdot T \cdot \tau_E = \frac{60}{E_F} \cdot \frac{T^2}{\langle \sigma v \rangle} \tag{3.23}$$

Similar to the steady current derived earlier, the right-hand side is only dependent on temperature. Further, as the left-hand side is a measure of difficult to achieve parameters, the goal is to minimize both sides. This occurs when the plasma temperature is around 15 keV – a fact memorized by many fusion engineers. As will be seen, this is a simplified result of our model. This is why $\overline{T} = 15$ keV is not always the optimum temperature – but usually is in the right neighborhood for reasonable reactor designs.

As all the terms in power balance have already been defined, the starting point will

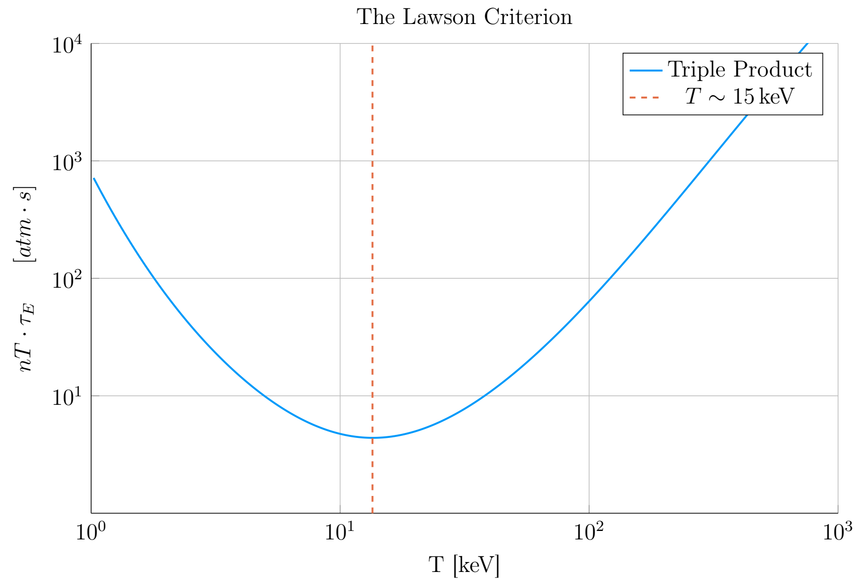



Figure 3-2: Power Balance in a Reactor

Power balance is what differentiates a reactor from a radiator. When cast as the Lawson Criterion for fusion, it explains why D-T plasmas often have a temperature around 15 keV.

be simply repeating the standardized equations for all five included powers.

$$P_\alpha = \frac{P_F}{5} \tag{3.7}$$

$$P_H = \frac{P_F}{Q} \tag{3.8}$$

$$P_\Omega = K_\Omega \cdot \left( \frac{\overline{n}^2 R_0^3}{\overline{T}^{3/2}} \right) \tag{3.12}$$

$$P_{BR} = K_{BR}\, \overline{n}^2\, \overline{T}^{1/2} R_0^3 \tag{3.16}$$

$$P_\kappa = K_\kappa \frac{R_0^3\, \overline{n}\, \overline{T}}{\tau_E} \tag{3.21}$$

With the fusion power again being,

$$P_F = K_F \cdot \overline{n}^2 \cdot R_0^3 \cdot (\sigma v) \tag{2.23}$$

These can then be substituted into power balance:

$$P_\alpha + P_H + P_\Omega = P_{BR} + P_\kappa \tag{3.6}$$

After a couple lines of algebra, power balance can be rewritten in a form analogous to the triple product:

$$\overline{n} \cdot \overline{T} \cdot \tau_E = \frac{K_\kappa \overline{T}^2}{\left(K_P \left(\sigma v\right) + K_{OH} \overline{T}^{-3/2}\right) - K_{BR} \overline{T}^{1/2}} \tag{3.24}$$

$$K_P = K_F \cdot \left(\frac{5+Q}{5 \times Q}\right) \tag{3.25}$$

As can be seen, this is remarkably similar to the simple Lawson Criterion:

$$n \cdot T \cdot \tau_E = \frac{60}{E_F} \cdot \frac{T^2}{\langle \sigma v \rangle} \tag{3.23}$$

The main difference is this model does not ignore ohmic power and radiation losses completely. The inclusion of radiation for example sometimes bars a range of temperatures from being physically realizable.* With this intermediate relation in place, the goal is now to give a formula for the confinement time and solve it for the magnetic field strength ($B_0$) – thus giving the Primary Constraint.

### 3.3.5 Finalizing the Primary Constraint

The goal now is to transform the Lawson Criterion into an equation for magnet strength ($B_0$). This choice to solve the equation for $B_0$ was completely arbitrary, only motivated by the foresight of how it fits into the fusion systems model. To solve the primary constraint, the confinement time scaling law will need to be introduced. At the end, a messy – albeit highly useful – relation will be the reward.

*The denominator of Eq 3.24 has discontinuities when the $K_{BR} \overline{T}^{1/2}$ term exactly equals the parenthesised one. Therefore, valid reactors only exist outside the discontinuities, when the entire triple product is finite and positive.

The energy confinement time – $\tau_E$ – is one of the most elusive terms in all of fusion energy. It is an attempt to boil down all the chaotic nature of plasmas into a simple measure of how fast its internal energy would be ejected from the tokamak if the device was instantaneously shut down. As such, reactor designers have turned toward experimentalists for empirical scalings based on the world's tokamaks. These all share a form similar to:

$$\tau_E = K_\tau \, H \, \frac{I_P^{\alpha_I} \, R_0^{\alpha_R} \, a^{\alpha_a} \, \kappa^{\alpha_\kappa} \, \overline{n}^{\alpha_n} \, B_0^{\alpha_B} \, A^{\alpha_A}}{P_L^{\alpha_P}} \tag{3.26}$$

This mouthful of a formula is how the field actually designs machines (i.e. ITER). Let it be known, though, that these fits often do remarkable well, having relative errors less than 20% on interpolated data. The new terms in this equation are: $P_L$, $K_\tau$, H, A, and the $\alpha_\square$ factors.

First, the loss power is a metric used in the engineering community to quantify the power being transported out of the "core" of the plasma by charged particles (i.e. not the neutrons).[3] To optimize fits, experimentalists have defined this as a combination of the source power terms:

$$P_L = P_\alpha + P_H + P_\Omega \tag{3.27}$$

However, many have argued that the term should actually be replaced by its correct physics meaning – the conductive heat loss power. As this model uses the ELMy H-Mode scaling law, which is standard in the field, this alternative definition will not be used:

$$\tilde{P}_L \approx P_\kappa = P_\alpha + P_H + P_\Omega - P_{BR} \tag{3.28}$$

Moving on, $K_\tau$ is simply a constant fit-makers use in their scalings. Whereas H is the (H-Mode) scaling factor – the analogue of $K_\tau$ used by reactor designers. This H factor can be used to artificially boost the confinement of a machine (i.e. it adds a little bit of magic). Continuing, A is the average mass number of the fuel source, in atomic mass units. For a 50-50 D-T fuel, this is 2.5, as deuterium weighs two amus and tritium weighs three. Lastly, the alpha factors (e.g. $\alpha_n$, $\alpha_a$, $\alpha_P$) are fitting parameters that represent each variable's relative importance in the scaling.

For ELMy H-Mode, this confinement scaling law can be written as:

$$\tau_E = 0.145\, H\, \frac{I_P^{0.93}\, R_0^{1.39}\, a^{0.58}\, \kappa^{0.78}\, \overline{n}^{\,0.41}\, B_0^{0.15}\, A^{0.19}}{P_L^{0.69}} \tag{3.29}$$

Where similar ones can be given for L-Mode, I-Mode, etc. One final remark to make before moving on is that even these fits have subtleties. The value of $\kappa$, for example, may have a slightly different geometric meaning from tokamak to tokamak. And the exact definition of loss power – $P_L$ – introduces an even larger area of discrepancy. Although not actually used, a better fit for our model might be one from the author:

$$\tilde{\tau}_E = 0.08\, H \frac{(R_0^{1.49} B_0^{0.3} I_P^{0.93}) \cdot (\epsilon^{0.17} A^{0.23} \kappa^{0.56})}{\tilde{P}_L^{0.54}} \tag{3.30}$$

Returning to the problem at hand, though, this model's Lawson Criterion (eq. 3.24) can be simplified after expanding the left-hand side using the Greenwald density and substituting in a confinement time scaling law. Albeit a little cumbersome, this can be wrangled into an equation for $B_0$!

$$\boxed{B_0 = \left( \frac{G_{PB}}{K_{PB}} \cdot \left( I_P^{\alpha_I^*}\, R_0^{\alpha_R^*} \right)^{-1} \right)^{\frac{1}{\alpha_B}}} \tag{3.31}$$

$$G_{PB} = \frac{\overline{T} \cdot \left( K_P \langle \sigma v \rangle + K_\Omega\, \overline{T}^{\,-3/2} \right)^{\alpha_P}}{\left( K_P \langle \sigma v \rangle + K_\Omega\, \overline{T}^{\,-3/2} - K_{BR}\, \overline{T}^{\,1/2} \right)} \tag{3.32}$$

$$K_{PB} = H \cdot \left( \frac{K_\tau K_n^{\alpha_n^*}}{K_\kappa} \right) \cdot (\epsilon^{\,\alpha_a}\, \kappa^{\,\alpha_\kappa}\, A^{\,\alpha_A}) \tag{3.33}$$

Where we have added new starred alpha values for the density, current, and major radius:

$$\alpha_n^* = 1 + \alpha_n - 2\alpha_P \tag{3.34}$$

$$\alpha_I^* = \alpha_I + \alpha_n^* \tag{3.35}$$

$$\alpha_R^* = \alpha_R + \alpha_a - 2\alpha_n^* - 3\alpha_p \tag{3.36}$$

Again, if the alternate definition for heat loss ($\tilde{P}$) were used, another definition for $G_{PB}$ would arise. Quickly reemphasizing, though, these tilded values are not actually used in the model:

$$\tilde{G}_{PB} = \frac{\overline{T}}{\left(K_P(\sigma v) + K_\Omega \overline{T}^{-3/2} - K_{BR}\overline{T}^{1/2}\right)^{(1-\alpha_P)}} \tag{3.37}$$

This equation for $B_0$ – derived from power balance – is thus the primary constraint for reactor designs. It is the first step in connecting the plasma (i.e. $\overline{n}$, $\overline{T}$, and $I_P$) to its tokamak enclosure (i.e. $B_0$ and $R_0$). The remaining step is finding an equation – or in this case, equations – for the major radius of the device. These radius equations will collectively be referred to as: the limiting constraints.~~Secondary Constraints.~~

~~Before moving onto the Secondary Constraint, it is worth noting that this power balance equation can be written in a triple product form analogous to the Lawson Criterion. For this reason, we will refer to it as the Freidberg Triple Product:~~

~~As is readily apparent, this has a shape similar to the Lawson Criterion. Again, the goal is operate when the right-hand side reaches an approximate minimum. This corresponds to when the left-hand side is also minimized – where each term represents one of the difficult to achieve quantities of a tokamak fusion reactor.~~

## 3.4 Collecting Limiting~~Secondary~~ Constraints

As of now, the only missing equation within our list of static~~fixed~~ variables – i.e. $R_0$, $B_0$, $\overline{T}$, $\overline{n}$, and $I_P$ – is for the major radius of the tokamak. This equation will come from around five potential limits, each either physical or engineering-based. These limits will then correspond to different curves through reactor space. As will be shown, many of these reactors will be invalid (as they violate at least one of the other limits).

Before tackling the subject of finding reactors that exist on the fine line of satisfying every limiting~~secondary~~ constraints, though, it is essential to collect them one-by-one. These are: the Troyon Beta Limit, the Kink Safety Factor, the Wall Loading Limit, the Power Cap Constraint, and the Heat Loading Limit.

The goal of this section is to solve for each of these constraints on the major radius. As with the primary constraint, this choice of solving for $R_0$ was completely arbitrary. It just so happens that each limit described here depends on the size of a reactor – which is not true for the magnetic field strength.

### 3.4.1 Introducing the Beta Limit

The Beta Limit is the most important limiting~~secondary~~ constraint – especially for steady-state reactors. It sets a maximum on the amount of pressure a plasma is willing to tolerate. As with future limiting~~secondary~~ constraints, literature-based equations will be transformed into formulas for $R_0$. Each will then contain some limiting quantity that can be handled by a static~~fixed~~ variable – as $\beta_N$ will be used shortly.

The starting point for the beta limit is to define the important plasma physics quantity: $\beta$ – the plasma beta. This value is a ratio between a plasma's internal pressure and the pressure exerted on it by the tokamak's magnetic configuration. Mathematically,[7]

$$\beta = \frac{\text{plasma pressure}}{\text{magnetic pressure}} = \frac{\overline{p}}{\left(\frac{B_0^2}{2\mu_0}\right)} \tag{3.38}$$

Using this model's temperature and density profiles, the volume-averaged pressure ($\overline{p}$) can be written in units of atmospheres (i.e. atm) as:

$$\overline{p} = 0.1581\,(1 + f_D)\,\frac{(1+\nu_n)\,(1+\nu_T)}{1+\nu_n+\nu_T}\,\overline{n}\,\overline{T} \tag{3.39}$$

Moving forward, the final step is plugging this definition for plasma beta into the physics-based Troyon Beta Limit. Although outside the scope of this text, it is a stability limit set by treating plasmas as charge-carrying fluids. This equation can be written in the following form, where $\beta_N$ is the normalized plasma beta – i.e. a static~~fixed~~ variable usually set between 2% and 4%.[21]

$$\beta = \beta_N \frac{I_P}{aB_0} \tag{3.40}$$

Substituting the plasma $\beta$ from eq. 3.38, into this relation results in the model's first equation for tokamak radius:

$$R_0 = \frac{K_{TB}\overline{T}}{B_0} \tag{3.41}$$

$$K_{TB} = 4.027e{-2}\left(K_n\right)\left(\frac{\epsilon}{\beta_N}\right)\left(1 + f_D\right)\frac{\left(1+\nu_n\right)\left(1+\nu_T\right)}{1+\nu_n+\nu_T} \tag{3.42}$$

As mentioned, this is often the dominating constraint in a steady-state reactor. The often dominating constraint for pulsed designs – the kink safety factor – will be the focus of the next subsection.

### 3.4.2 Giving the Kink Safety Factor

Just like how the Troyon Beta Limit set a fluids-based maximum on plasma pressure, the Kink Safety Factor sets one on the plasma's current. This constraint usually only appears in pulsed designs, as it is assumed that getting to this high a current in steady-state (with only LHCD) would prove extremely unpractical.

The starting point, again, is an equation from the literature for the kink condition:[3]

$$q_{95} = 5\epsilon^2 f_q \cdot \frac{R_0 B_0}{I_P} \tag{3.43}$$

Here the safety factor – $q_{95}$ – is subscripted by 95, an identifier that this value is taken at the 95% flux surface (i.e. near the statistically drawn edge of the plasma).

It typically has values around 3. Next, the $f_q$ variable is a geometric scaling factor:

$$f_q = \frac{1.17 - 0.65\epsilon}{2(1-\epsilon^2)^2} \cdot \left(1 + \kappa^2 * (1 + 2\delta^2 - 1.2\delta^3)\right) \tag{3.44}$$

Combined, the kink safety factor can now be written in standardized units as:

$$R_0 = \frac{K_{SF} I_P}{B_0} \tag{3.45}$$

$$K_{SF} = \frac{q_{95}}{5\epsilon^2 f_q} \tag{3.46}$$

This relation is the limiting~~secondary~~ constraint important for most pulsed reactor designs. As with the Beta Limit, the two are derived through plasma physics alone. The remaining limiting~~secondary~~ constraints, however, are engineering-based in origin – these include: the Wall Loading Limit, the Power Cap Constraint, and the Heat Loading Limit. Each will be defined shortly.

### 3.4.3 Working under the Wall Loading Limit

The first engineering-based limiting~~secondary~~ constraint – the wall loading limit – will prove to be an important quantity when determining the magnet strength at which reactor costs first start to increase. As hinted, its definition originates from nuclear engineering concerns: it is a measure of the maximum neutron damage a tokamak's walls can take over the lifetime of the machine.

The first step in deriving a limiting~~secondary~~ constraint for wall loading is a description of the problem it models. In a reactor, fusion reactions typically make high-energy neutrons – with around 14.1 MeV of kinetic energy – that continually blast the inner wall of the tokamak. Therefore a quick-and-dirty metric would be limiting the amount of neutron power that can be unloaded on the surface area of a tokamak. This can be written as:

$$P_W = \frac{P_n}{S_P} \tag{3.47}$$

Here, $S_P$ is the surface area of the tokamak's inner wall and $P_n$ is the neutron power derived in the subsection on fusion power. The quantity, $P_W$, then serves a role analogous to $\beta_N$ for the beta limit and $q_{95}$ for the kink safety factor – it is a static~~fixed~~ variable representing the maximum allowed wall loading. For fusion reactors, $P_W$ is assumed to be around 2-4 $\frac{\mathrm{MW}}{\mathrm{m}^2}$. It will be shown that the wall loading limit is important in any tokamak – regardless of operating mode (i.e. steady-state or pulsed).

For completeness, the surface area can be defined through:

$$S_P = 4\pi^2 a_P R_0 \cdot \frac{\left(1 + \frac{2}{\pi}\left(\kappa_P^2 - 1\right)\right)}{\kappa_P} \tag{3.48}$$

In this formula, the various dimensions subscripted with P's are:

$$a_P = 1.04\, a \tag{3.49}$$

$$\kappa_P = 1.3\, \kappa \tag{3.50}$$

$$\epsilon_P = \frac{a_P}{R_0} \tag{3.51}$$

Finishing this limiting~~secondary~~ constraint, the Wall Loading limit can be written in standardized units as:

$$R_0 = K_{WL} \cdot I_P^{\frac{2}{3}} \cdot (\sigma v)^{\frac{1}{3}} \tag{3.52}$$

$$K_{WL} = \left( \frac{K_F K_n^2}{5\pi^2 P_W} \cdot \frac{\kappa_P}{\epsilon_P} \cdot \frac{1}{1 + \frac{2}{\pi} \cdot \left(\kappa_P^2 - 1\right)} \right)^{\frac{1}{3}} \tag{3.53}$$

### 3.4.4 Setting a Maximum Power Cap

As opposed to the previous three limiting~~secondary~~ constraints, the maximum power cap is more of a rule of thumb. Because no reactor – coal, solar, or otherwise – has a 4000 MW reactor, neither should fusion. It makes sense from a practical position after realizing the long history of tokamaks being delayed, underfunded, or completely canceled. Mathematically, this has the simple form:

$$P_E \leq P_{CAP} \tag{3.54}$$

Here, $P_{CAP}$ is the maximum allowed power output of the reactor. Similar to the other limiting quantities, $P_{CAP}$ is treated as a static~~fixed~~ variable (i.e. set to 4000 MW). The electrical power output of the reactor ($P_E$) is then related to the fusion power through:[7]

$$P_E = 1.273\, \eta_T \cdot P_F \tag{3.55}$$

The constant in front (i.e. 1.273) represents some extra power the reactor makes as more fuel is bred when the fusion neutrons pass through a tokamak (inside its still-undiscussed blanket region). The variable $\eta_T$ is the thermal efficiency of the reactor – which is usually found to be around 40%.

Substituting in fusion power and solving for the major radius results in:

$$R_0 = K_{PC} \cdot I_P^2 \cdot (\sigma v) \tag{3.56}$$

$$K_{PC} = K_F K_n^2 \cdot \left( \frac{1.273\, \eta_T}{P_{max}} \right) \tag{3.57}$$

This limiting~~secondary~~ constraint can be used to create curves of reactors, although it is mainly used as a stopping point for designs – i.e. if you get to the power-cap regime, you have gone too far. This is different than the next constraint, which is basically a glorified warning sign in the contemporary tokamak design paradigm.

### 3.4.5 Listing the Heat Loading Limit

Plasmas are hot. The commonly given fact is one electron volt is around 20,000 °F. Although a tad deceptive, melting a tokamak is an all too real concern. The problem is there is currently no solution to the problem. Although researchers have explored various types of heat divertors, none have been shown to withstand the gigawatts of heat emitted from a reactor-size tokamak. Further, as it is not as glamorous as plasma physics, attempts to tackle the problem head-on have often gone unfunded.[22]

As such, this model takes the approach that we are no worse than the rest of the field. We almost completely ignore the heat loading limit and just refer to it at the end, saying "and then this magic divertor will have to deal with solar corona levels of heat." After which, discussion will quickly be redirected to happier concerns.

For thoroughness though, a limiting~~secondary~~ constraint will still be derived. The first step is giving the heat load limit commonly found in the literature:[23]

$$q_{DV} = \frac{K_{DV}}{K_F} \cdot \frac{P_F\, I_P^{1.2}}{R_0^{2.2}} \tag{3.58}$$

$$K_{DV} = \frac{18.31e{-3}}{\epsilon^{1.2}} \cdot K_P \cdot \left(\frac{2}{1+\kappa^2}\right)^{0.6} \tag{3.59}$$

After a simple rearrangement and substitution for fusion power, this becomes:

$$R_0 = K_{DH} \cdot I_P \cdot (\sigma v)^{\frac{1}{3.2}} \tag{3.60}$$

$$K_{DH} = \left(\frac{K_{DV} K_n^2}{q_{DV}}\right)^{\frac{1}{3.2}} \tag{3.61}$$

At this point all the limiting~~secondary~~ constraints have been defined. The next step is taking a step back and motivating the derivation of a current equation suitable for pulsed tokamaks.

## 3.5 Summarizing the Fusion Systems Model

This chapter focused on the bigger picture behind designing a zero-dimension fusion systems model. It started with a description of various design parameters and then segued into explaining the five relations needed to close the model – i.e. for $\overline{T}$, $\overline{n}$, $I_P$, $B_0$, and $R_0$.

Before moving onto generalizing the steady current to model pulsed reactors, a quick recap of the equations will prove beneficial. The first variable tackled was temperature – i.e. scan five evenly-spaced $\overline{T}$ values between 10 and 30 keV. This was then quickly followed by the Greenwald density limit – the cornerstone of this framework. Through equations, these two were written as:

$$\overline{T} = const. \tag{3.1}$$

$$\overline{n} = K_n \cdot \frac{I_P}{R_0^2} \tag{2.11}$$

The next variable handled was the steady current:

$$I_P = \frac{K_{BS}\overline{T}}{1 - K_{CD}(\sigma v)} \tag{2.30}$$

As was mentioned then, this only directly depends on temperature, but is strongly affected by a tokamak's configuration – $R_0$ and $B_0$ - through the current drive efficiency ($\eta_{CD}$). For pulsed reactors, this equation proves too simple as it ignores inductive current. To remedy the situation, current balance will be revisited next chapter. The main point to make now, though, is that the $R_0$ and $B_0$ dependence will be made explicit.

Moving on, the remaining equations were the primary and limiting~~secondary~~ constraints for $B_0$ and $R_0$, respectively. It was through these relations that a tokamak's configuration was brought back into the fold. The choice of solving the two constraints for their respective variables was completely arbitrary – motivated only by the foresight of how they fit into the model. Repeated below, they served as the proper vehicles for closing the system of equations. The next step now is to learn how to generalize the current formula and design a pulsed tokamak reactor.

$$B_0 = \left( \frac{G_{PB}}{K_{PB}} \cdot \left( I_P^{\alpha_I^*} \, R_0^{\alpha_R^*} \right)^{-1} \right)^{\frac{1}{\alpha_B}} \tag{3.31}$$

$$R_0 = \frac{K_{TB}\overline{T}}{B_0} \tag{3.41}$$

$$R_0 = \frac{K_{SF} I_P}{B_0} \tag{3.45}$$

$$R_0 = K_{WL} \cdot I_P^{\frac{2}{3}} \cdot (\sigma v)^{\frac{1}{3}} \tag{3.52}$$

$$R_0 = K_{PC} \cdot I_P^2 \cdot (\sigma v) \tag{3.56}$$

$$R_0 = K_{DH} \cdot I_P \cdot (\sigma v)^{\frac{1}{3.2}} \tag{3.60}$$

# Chapter 5

# Completing the Systems Model

As opposed to previous chapters, this one will focus on the numerics behind the fusion systems model. This will then naturally segue into a discussion of how plots are made and should be interpreted. The remaining chapters will then decouple the dissemination of results from their analytic conclusions.

## 5.1 Describing a Simple Algebra

Boiled down, the systems model used here is a simple algebra problem – given five equations, solve for five unknowns. The goal is then to pick the five equations that best represent modern fusion reactor design. This selection should also be done in such a way that actually reduces the system of equations to a simple univariate root solving equation (i.e. one equation with one unknown). As will be shown in the results, this model does remarkably well: matching year-long modeling campaigns in seconds.

The logical place to start in a discussion of this algebra problem is with the three equations fundamental to all reactor-grade tokamaks – both in steady-state and pulsed operation. These are: the Greenwald density limit, power balance, and current balance. The Greenwald density's importance was hinted early on when it was used to simplify every equation derived thereafter.

$$\overline{n} = K_n \cdot \frac{I_P}{R_0^2} \tag{2.11}$$

The two balance equations prove slightly more dubious. As was shown previously, current balance – the stability requirement for tokamaks – was most peculiar. It brought forth the notion of self-consistency for steady-state machines and a highly-coupled multi-root equation for pulsed ones. As such, this equation stands as the one everything else will be substituted into to setup for a univariate root solve.

$$I_P = \frac{\left(K_{BS} + {}^{G_{IU}}/_{G_{IP}}\right) \cdot \overline{T}}{1 - K_{CD}(\sigma v) - {}^{G_{ID}}/_{G_{IP}}} \tag{4.75}$$

Although slightly buried in Eq. (4.75), the right-hand side actually depends on all the quantities (including $I_P$ through the blanket thickness). Through equation,

$$I_P = f(I_P, \overline{T}, R_0, B_0) \tag{5.1}$$

The remaining equation common to all reactor-grade tokamaks is power balance – the relation that separates power plants from toasters. Due to the use of the ELMy H-Mode scaling law for modeling the diffusion coefficient, this had the complicated form of:

$$R_0^{\alpha_R^*} \cdot B_0^{\alpha_B} \cdot I_P^{\alpha_I^*} = \frac{G_{PB}}{K_{PB}} \tag{??}$$

Although being rather cumbersome, this equation actually remains relatively simple in that all three quantities on the left-hand side are separable. To close the system, two more equations of this form are needed. These have the following form and will be described next.

$$R_0^{\gamma_R} \cdot B_0^{\gamma_B} \cdot I_P^{\gamma_I} = G(\overline{T}) \tag{5.2}$$

## 5.2 Generalizing Previous Equations

Where the equations defined up to this point in the chapter are shared among all fusion reactors, the remaining two equations – needed to close the system – must be chosen by the user. These user-supplied equations come in three flavors: limits, intermediate~~derived~~ quantities, and dynamic~~floating~~ variables. By convention, we enforce that at least one limit must be used. The other constraint can then come from any of the three defined collections, which we will refer to as the closure equation.

Table 5.1: Main Equation Bank
To close the system of equations for potential reactors, different equations can be used to lock down tokamak designs. These include physics and engineering limits (L), as well as ways to set dynamic (D)~~floating (F)~~ or intermediate (I)~~derived (D)~~ variables to constant values.

| Variable | Category | G($\overline{T}$) | $\gamma_R$ | $\gamma_B$ | $\gamma_I$ |
|---|---|---|---|---|---|
| Power Balance | - | $G_{PB}/K_{PB}$ | $\alpha_R^*$ | $\alpha_B$ | $\alpha_I^*$ |
| Beta ($\beta_N$) | L | $K_{TB}\overline{T}$ | 1 | 1 | 0 |
| Kink ($q_{95}$) | L | $K_{KF}$ | 1 | 1 | -1 |
| Wall Loading ($P_W$) | L | $K_{WL}(\sigma v)^{1/3}$ | 1 | 0 | -2/3 |
| Power Cap ($P_E$) | L | $K_{PC}(\sigma v)$ | 1 | 0 | -2 |
| Heat Loading ($q_{DV}$) | L | $K_{DV}(\sigma v)^{1/3.2}$ | 1 | 0 | -1 |
| Major Radius ($R_0$) | D | $(R_0)_{const}$ | 1 | 0 | 0 |
| Magnet Strength ($B_0$) | D | $(B_0)_{const}$ | 0 | 1 | 0 |
| Plasma Current ($I_P$) | D | $(I_P)_{const}$ | 0 | 0 | 1 |
| Plasma Temperature ($\overline{T}$) | D | $(\overline{T})_{const}/\overline{T}$ | 0 | 0 | 0 |
| Electron Density ($\overline{n}$) | D | $(\overline{n})_{const}/K_n$ | -2 | 0 | 1 |
| Plasma Pressure ($\overline{p}$) | I | $(\overline{p})_{const}/K_n K_{nT}\overline{T}$ | -2 | 0 | 1 |
| Bootstrap Current ($f_{BS}$) | I | $(f_{BS})_{const}/K_{BS}\overline{T}$ | 0 | 0 | -1 |
| Fusion Power ($P_F$) | I | $(P_F)_{const}/K_F K_n^2(\sigma v)$ | -1 | 0 | 2 |
| Magnetic Energy ($W_M$) | I | $(W_M)_{const}/K_{WM}$ | 3 | 2 | 0 |
| Cost per Watt ($C_W$) | I | $(C_W)_{const} \cdot \left(K_F K_n^2(\sigma v)/K_{WM}\right)$ | 4 | 2 | -2 |

### 5.2.1 Rehashing the Limits

The limits category is simply limiting constraints given in Chapter 3.~~a rebranding of the secondary constraints given previously.~~ These include the physics derived limits from MHD theory – i.e. the beta limit ($\beta_N$) and the kink safety factor ($q_{95}$) – which for clarity, set maximums on the allowed plasma pressure and velocity, respectively. Additionally, there were several engineering limits also described: wall loading, heat loading, and maximum power capacity. For this paper, wall loading from neutrons ($P_W$) is assumed to be important, whereas the other two engineering limits are not allowed to explicitly guide designs.

Combined all these limits, as well as the yet to be defined dynamic~~float~~ and intermediate~~derived~~ equations, are given in Table 5.1. These share a remarkably similar form to power balance when put into a generalized, separable state. This hints at why the major radius ($R_0$), the toroidal field strength ($B_0$), and the plasma current ($I_P$) can easily be separated and substituted out of the current balance equation.

Before moving on, it proves useful to explain the two limits not used to explicitly guide reactor design – divertor heat loading and the maximum power capacity. The simpler of the two to reason is the heat loading limit. Although removing the gigawatts of heat is extremely difficult, it remains an unsolved problem worthy of its own research machine, but currently neglected financially. As such, it is only kept to provide a human-interpreted measure of difficulty. The power cap, on the other hand, is just handled informally. If a reactor surpasses it (i.e. $P_E > 4000MW$), it is considered invalid.

While the maximum power cap informally sets a maximum major radius for a machine, there also exists an implicit minimum major radius. This minimum occurs due to the hole-size constraint – i.e. at some point there is no longer enough room on the inside of the machine to store the central solenoid, blanket, and TF coils.

At this point, we can now explain how various quantities in the systems model can be set to user-given constant values. This basically allows users to treat one dynamic~~floating~~ variable as a static~~fixed~~ one (e.g. the temperature and bootstrap fraction).

### 5.2.2 Minimizing Intermediate~~Derived~~ Quantities

Whereas the limits from the previous section represented constraints with real physics and engineering repercussions, the intermediate~~derived~~ quantities here are just used to find when reactors reach certain user-supplied values. Most notable are the capital cost (through the magnetic energy – $W_M$) and the cost-per-watt ($C_W$). The model also, however, allows easily setting values for the bootstrap fraction, plasma pressure, and fusion power. As mentioned previously, they are given in Table 5.1 through a generalized representation of the form:

$$R_0^{\gamma_R} \cdot B_0^{\gamma_B} \cdot I_P^{\gamma_I} = G(\overline{T}) \tag{5.2}$$

What this collection of variables is really useful for, though, is finding minimum cost reactors – both in a capital context as well as a cost-per-watt one. Without boring the reader, this is done in a three stage process. First, some valid reactor is found: it does not matter if it is good, just valid. This of course can be found by systematically throwing darts at a dart board – see Fig. 5-1

After a valid reactor is found, its cost is recorded leading to a drill-down stage. In this step, the cost is continuously halved until a valid reactor cannot be found. Once this invalid reactor is reached, it sets a bound on the minimum cost reactor. As such, the final stage is a simple bisection step where the minimum cost is honed down to some acceptable margin of error – see Fig. 5-2.

### 5.2.3 Pinning Dynamic~~Floating~~ Variables

The remaining collection of closure equations is for the five dynamic~~floating~~ variables in the systems model: $R_0$, $B_0$, $\overline{n}$, $\overline{T}$, and $I_P$. As we are making equations of the

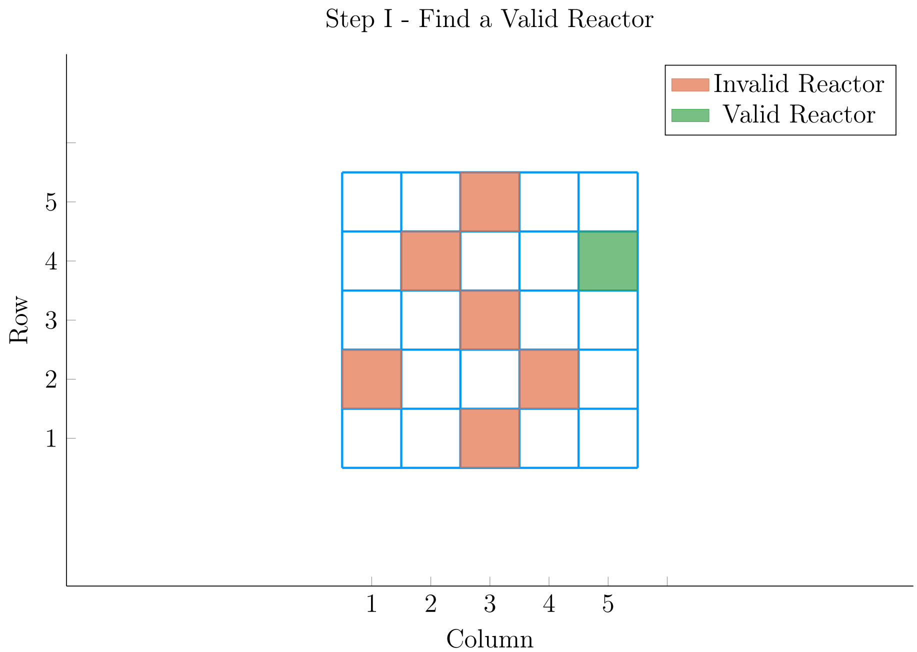



Figure 5-1: Minimize Cost Step I – Find Valid Reactor

following form, the formulas for $R_0$, $B_0$, and $I_P$ are trivial.

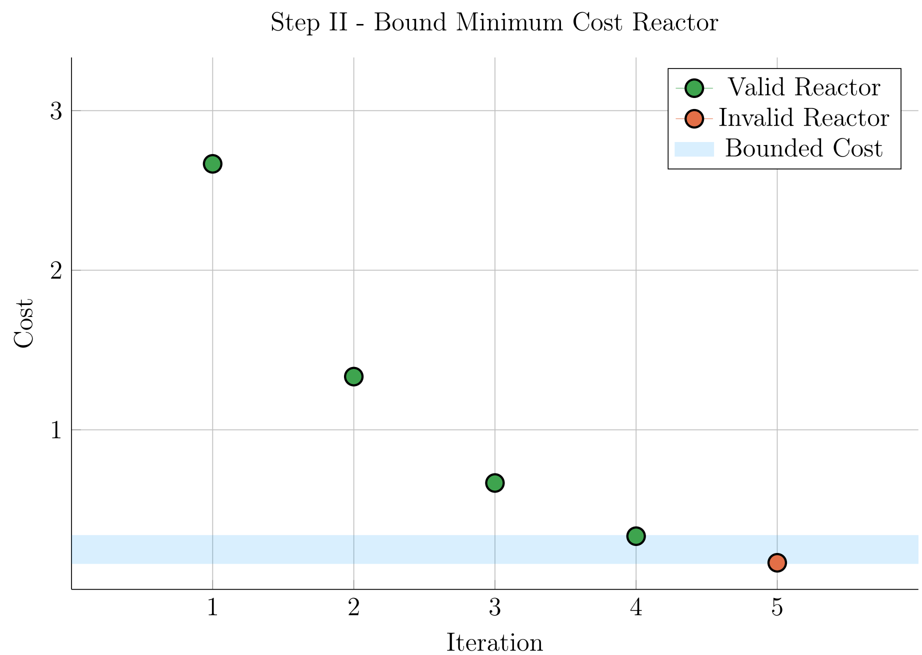
$$R_0^{\gamma_R} \cdot B_0^{\gamma_B} \cdot I_P^{\gamma_I} = G(\overline{T}) \tag{5.2}$$

Next, the equation for $\overline{n}$ – shown in Table 5.1 – is just a simple undoing of the Greenwald density limit. The remaining equation is then from the original temperature equation:

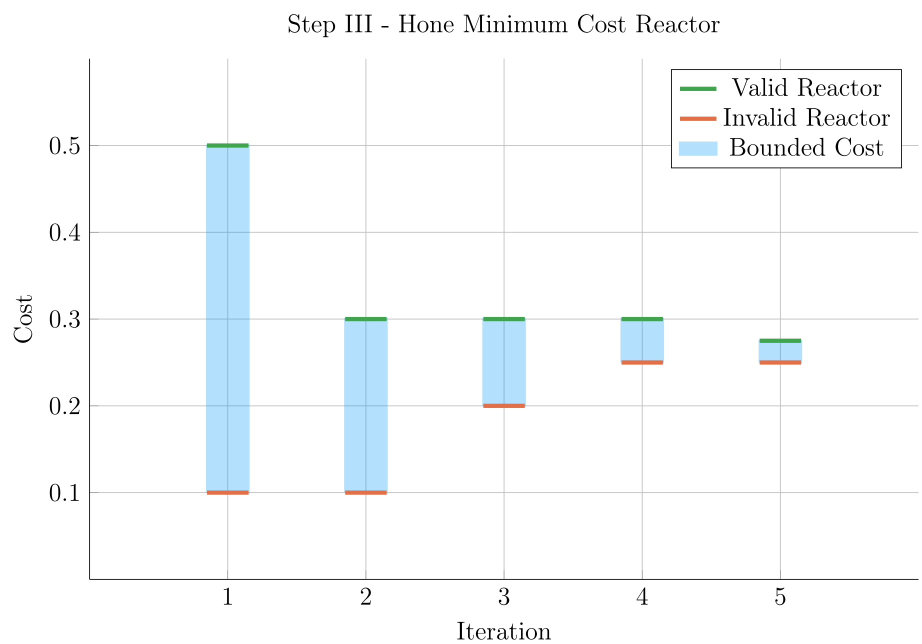
$$\overline{T} = const. \tag{3.1}$$

As was assumed earlier, this is sort of a default equation for the systems model. By this, we mean reactor curves can be created by scanning over temperatures, i.e. set $\overline{T} = 5$ keV in one run, 10 in the next, etc. This temperature equation also brings up a subtlety of the model, as it does not depend on current, radius, or magnet strength.

The algorithm that motivated this generalized equation approach most notably bifurcates in the situation where the closure equation does not depend on $R_0$, $B_0$, or $I_P$




(a) Minimize Step II




(b) Minimize Step III

Figure 5-2: Minimize Cost Step II/III – Optimize Reactor (i.e. the temperature equation). The two scenarios are given in Eqs. (5.3) to (5.9) – where at least $R_0$ and $B_0$ are substituted out of the system. In the temperature case, $I_P$ is not needed to be explicitly removed.

Concretely, the root solve for the temperature scenario is for the current, whereas it is for the temperature in all other cases. The nomenclature in the code is a *match* for Scenario I (i.e. root solving for plasma temperature), and a *solve* for Scenario II (i.e. root solving for plasma current).

**Scenario I – Match for $\overline{T}$**

$$R_0(\overline{T}) = \left(G_1^{(\gamma_{B,2}\,\gamma_{I,3} - \gamma_{B,3}\,\gamma_{I,2})} \cdot G_2^{(\gamma_{B,3}\,\gamma_{I,1} - \gamma_{B,1}\,\gamma_{I,3})} \cdot G_3^{(\gamma_{B,1}\,\gamma_{I,2} - \gamma_{B,2}\,\gamma_{I,1})}\right)^{\frac{1}{\gamma_{RBI}}} \tag{5.3}$$

$$B_0(\overline{T}) = \left(G_1^{(\gamma_{I,2}\,\gamma_{R,3} - \gamma_{I,3}\,\gamma_{R,2})} \cdot G_2^{(\gamma_{I,3}\,\gamma_{R,1} - \gamma_{I,1}\,\gamma_{R,3})} \cdot G_3^{(\gamma_{I,1}\,\gamma_{R,2} - \gamma_{I,2}\,\gamma_{R,1})}\right)^{\frac{1}{\gamma_{RBI}}} \tag{5.4}$$

$$I_P(\overline{T}) = \left(G_1^{(\gamma_{R,2}\,\gamma_{B,3} - \gamma_{R,3}\,\gamma_{B,2})} \cdot G_2^{(\gamma_{R,3}\,\gamma_{B,1} - \gamma_{R,1}\,\gamma_{B,3})} \cdot G_3^{(\gamma_{R,1}\,\gamma_{B,2} - \gamma_{R,2}\,\gamma_{B,1})}\right)^{\frac{1}{\gamma_{RBI}}} \tag{5.5}$$

$$\begin{aligned} \gamma_{RBI} = {} & (\gamma_{R,1}\,\gamma_{B,2}\,\gamma_{I,3} + \gamma_{R,2}\,\gamma_{B,3}\,\gamma_{I,1} + \gamma_{R,3}\,\gamma_{B,1}\,\gamma_{I,2}) - \\ & (\gamma_{R,1}\,\gamma_{B,3}\,\gamma_{I,2} + \gamma_{R,2}\,\gamma_{B,1}\,\gamma_{I,3} + \gamma_{R,3}\,\gamma_{B,2}\,\gamma_{I,1}) \end{aligned} \tag{5.6}$$

**Scenario II – Solve for $I_P$**

$$R_0(\overline{T}) = \left(G_1^{\gamma_{B,2}} \cdot G_2^{-\gamma_{B,1}} \cdot I_P^{(\gamma_{B,1}\,\gamma_{I,2} - \gamma_{B,2}\,\gamma_{I,1})}\right)^{\frac{1}{\gamma_{RBT}}} \tag{5.7}$$

$$B_0(\overline{T}) = \left(G_1^{-\gamma_{R,2}} \cdot G_2^{\gamma_{R,1}} \cdot I_P^{(\gamma_{I,1}\,\gamma_{R,2} - \gamma_{I,2}\,\gamma_{R,1})}\right)^{\frac{1}{\gamma_{RBT}}} \tag{5.8}$$

$$\gamma_{RBT} = \gamma_{R,1}\,\gamma_{B,2} - \gamma_{R,2}\,\gamma_{B,1} \tag{5.9}$$

## 5.3 Wrapping up the Logic

As stated at the beginning of the chapter, this systems model basically boils down to a simple 5 equation/5 unknown algebra problem. The Greenwald density was implicitly used in the initial derive to simplify the logic. The current balance was then delegated to be the root solve equation. Lastly, three equations were needed to remove the major radius and magnet strength, as well as either the current or temperature. These 16 equations were given in Table 5.1 with the generalized solution given in Eqs. (5.3) to (5.9).

This now sets the stage for the most interesting part of the document – the results. In true Dickens fashion, they will come in several forms. The first result type we will encounter will be temperature scans. These allow us to validate the model by comparing it to several designs from the literature. These will use the Scenario II solver.

Moving onto examples of the Scenario I matcher are sensitivity studies and Monte Carlo samplings. The simple one variable sensitivities will reveal local trends from sweeping various static~~fixed~~ (i.e. input) variables – namely H, $\kappa$, $B_{CS}$, etc. Whereas the samplings will highlight global trends as many static~~fixed~~/input variables are allowed to vary simultaneously.

These Scenario I flavors are further subdivided in regards to the nature of their closure equation. The first flavor comes from finding so called two limit solutions, which live at the point where the beta and kink (or wall) limits are just marginally satisfied. The second main type is then minimum cost reactors – measured in either a capital cost or cost-per-watt context. These will be used in depth next chapter.

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