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Appendix F

Expanding on the Bootstrap Current

The bootstrap current fraction – f_{BS} – is an important parameter that enters in the design of tokamak reactors. It must be calculated with reasonable accuracy to determine how much external current drive is required. The value of f_{BS} thus has a strong impact on the overall fusion energy gain. Obtaining reasonable accuracy requires a moderate amount of analysis, which is presented in a following section. The results are summarized below.

F.1 Summarized Results

The analysis is based on an expression for the bootstrap current valid for arbitrary cross section assuming (1) equal temperature electrons and ions $T_e = T_i = T$, (2) large aspect ratio $\epsilon \ll 1$, and (3) negligible collisionality $\nu_* \to 0$. Under these assumptions the bootstrap current $\mathbf{J}_{BS} \approx J_{BS}\mathbf{e}_\phi$ has the form

$$J_{BS} = -3.32 f_T R_0 n T \left(\frac{1}{n}\frac{dn}{d\psi} + 0.054\,\frac{1}{T}\frac{dT}{d\psi} \right) \tag{F.1}$$

Here, $f_T \approx 1.46(r/R_0)^{1/2}$ is an approximate expression for the trapped particle fraction and ψ is the poloidal flux.

The analysis next section shows that Eq. (F.1) leads to an expression for the bootstrap fraction, assuming for simplicity elliptical flux surfaces, that can be written as:

$$f_{BS} = \frac{I_{BS}}{I} = \frac{2\pi a^2 \kappa}{I} \int_0^1 J_{BS}\, \rho \, d\rho = \frac{K_{BS}}{K_n} \frac{\overline{n}\,\overline{T} R_0^2}{I_P^2} \tag{F.2}$$

$$K_{BS} = 4.879 \cdot K_n \cdot \left(\frac{1+\kappa^2}{2} \right) \cdot \epsilon^{5/2} \cdot H_{BS} \tag{F.3}$$

$$H_{BS} = (1+\nu_n)(1+\nu_T)(\nu_n + 0.054\nu_T) \int_0^1 \frac{\rho^{5/2}\,(\,1-\rho^2\,)^{\,\nu_n+\nu_T-1}}{b_p}\, d\rho \tag{F.4}$$

$$b_p(\rho) = \frac{-e^{\gamma\rho^2}(\gamma\rho^2 - 1 - \gamma) - 1 - \gamma}{\rho\,(e^{\gamma} - 1 - \gamma)} \tag{F.5}$$

$$\overline{J}_\phi(\rho) = -\frac{I}{\pi a^2 \kappa} \left[\frac{\gamma^2 (1-\rho^2) e^{\gamma\rho^2}}{e^{\gamma} - 1 - \gamma} \right] \tag{F.6}$$

In this expression b_p is a normalized form of the poloidal magnetic field derived from a prescribed model for the *total* flux surface averaged current density profile $\overline{J}_\phi(\rho)$. The $\overline{J}_\phi(\rho)$ profile, in analogy with the density and temperature profiles, is not self-consistent but is chosen to have a plausible experimental shape characterized by the parameter γ. The profile can have either an on-axis ($\gamma < 1$) or off-axis peak ($\gamma > 1$). The normalized internal inductance l_i and radial location of the current peak ρ_m are related to the value of γ by:

$$\boxed{l_i = \frac{4\kappa}{1+\kappa^2} \int_0^1 b_p^2\, \rho \, d\rho} \tag{F.7}$$

$$\rho_m = \begin{cases} \left(\frac{\gamma}{\gamma-1}\right)^{1/2}, & \gamma > 1 \\ 0, & \gamma < 1 \end{cases} \tag{F.8}$$

F.2 Detailed Analysis

The starting point for the analysis is the general expression for the bootstrap current in a tokamak with arbitrary cross section.[43] This expression can be simplified by assuming (1) equal temperature electrons and ions $T_e = T_i = T$, (2) large aspect ratio $\epsilon \ll 1$, and (3) negligible collisionality $\nu_* \to 0$. The bootstrap current $\mathbf{J}_{BS} \approx J_{BS}\mathbf{e}_\phi$ reduces to

$$J_{BS} = -3.32 f_T R_0 n T \left(\frac{1}{n}\frac{dn}{d\psi} + 0.054\, \frac{1}{T}\frac{dT}{d\psi} \right) \tag{F.9}$$

Several values of the trapped particle fraction f_T have been given in the literature.[44] For simplicity we use a form valid for large aspect ratio. This is a slightly optimistic value but saves a large amount of detailed calculation. It can be written as,

$$f_T \approx 1.46 (r/R_0)^{1/2} = 1.46 \epsilon^{1/2} \rho^{1/2} \tag{F.10}$$

Here, as in the main text, ρ is a radial-like flux surface label that varies between $0 \leq \rho \leq 1$. In other words $\psi = \psi(\rho)$. Under these assumptions the bootstrap current reduces to:

$$J_{BS} = -4.85\, R_0 \epsilon^{1/2} \left(\frac{\rho^{1/2} n T}{{}^{d\psi}\!/_{d\rho}} \right) \left(\frac{1}{n}\frac{dn}{d\rho} + 0.054\, \frac{1}{T}\frac{dT}{d\rho} \right) \tag{F.11}$$

Since we have specified profiles for $n(\rho)$ and $T(\rho)$ all that remains in order to be able to evaluate $J_{BS}(\rho)$ is to determine $\psi' = {}^{d\psi}\!/_{d\rho}$. Keep in mind that at this point, in spite of the approximations that have been made, the expression for $J_{BS}(\rho)$ is still valid for arbitrary cross section.

The analysis that follows shows how to calculate ψ' for an arbitrary cross section including finite aspect ratio. As an example an explicit expression for large aspect ratio, finite elongation ellipse is obtained. Consider the Grad-Shafranov equation for the flux: $\Delta^* \psi = -\mu_0 R J_\psi$. We integrate this equation over the volume of an arbitrary flux surface making use of Gauss' theorem, which leads to:

$$\int_S \frac{\mathbf{n}\cdot\nabla\psi}{R^2}dS = -\mu_0\int_V \frac{J_\phi}{R}d\mathbf{r} \tag{F.12}$$

Next, assume that the coordinates of the flux surface can be expressed in terms of ρ and an angular-like parameter α with $0 \leq \alpha \leq 2\pi$. In other words, the flux surface coordinates can be written as $R = R(\rho,\alpha) = R_0 + ax(\rho,\alpha)$ and $Z = Z(\rho,\alpha) = ay(\rho,\alpha)$. The functions $R(\rho,\alpha)$ and $Z(\rho,\alpha)$ are assumed to be known. The term on the left hand side can be evaluated by noting that

$$d\mathbf{l} = dl\mathbf{t} \tag{F.13}$$

$$dl = (R_\alpha^2 + Z_\alpha^2)^{1/2}d\alpha \tag{F.14}$$

$$\mathbf{t} = \frac{R_\alpha\mathbf{e}_R + Z_\alpha\mathbf{e}_Z}{(R_\alpha^2+Z_\alpha^2)^{1/2}} \tag{F.15}$$

$$\mathbf{n} = \mathbf{e}_\phi \times \mathbf{t} = \frac{Z_\alpha\mathbf{e}_R - R_\alpha\mathbf{e}_Z}{(R_\alpha^2+Z_\alpha^2)^{1/2}} \tag{F.16}$$

$$dS = Rd\phi dl = 2\pi R(R_\alpha^2+Z_\alpha^2)^{1/2}d\alpha \tag{F.17}$$

It then follows that

$$\mathbf{n}\cdot\nabla\psi = \frac{1}{(R_\alpha^2+Z_\alpha^2)^{1/2}}\left(Z_\alpha\frac{\partial\psi}{\partial R} - R_\alpha\frac{\partial\psi}{\partial Z}\right) = \frac{1}{(R_\alpha^2+Z_\alpha^2)^{1/2}}\frac{d\psi}{d\rho}Z_\alpha\rho_R - R_\alpha\rho_Z \tag{F.18}$$

We can rewrite the last term by noting that

$$\begin{aligned} dR = R_\rho d\rho + R_\alpha d\alpha \quad &\rightarrow \quad d\rho = (Z_\alpha dR - R_\alpha dZ)\,/\,(R_\rho Z_\alpha - R_\alpha Z_\rho) \\ dZ = Z_\rho d\rho + Z_\alpha d\alpha \quad &\rightarrow \quad d\alpha = (-Z_\rho dR + R_\rho dZ)\,/\,(R_\rho Z_\alpha - R_\alpha Z_\rho) \end{aligned} \tag{F.19}$$

from which follows

$$\begin{aligned}\rho_R &= \frac{Z_\alpha}{(R_\rho Z_\alpha - R_\alpha Z_\rho)} \\ \rho_Z &= -\frac{R_\alpha}{(R_\rho Z_\alpha - R_\alpha Z_\rho)}\end{aligned} \tag{F.20}$$

the normal gradient reduces to

$$\mathbf{n}\cdot\nabla\psi = \frac{R_\alpha^2 + Z_\alpha^2}{(R_\rho Z_\alpha - R_\alpha Z_\rho)}\frac{d\psi}{d\rho} \tag{F.21}$$

Using this relation we see that the left hand side of Eq. (F.12) can now be written as:

$$\int_S \frac{\mathbf{n}\cdot\nabla\psi}{R^2}dS = 2\pi\frac{d\psi}{d\rho}\int_0^{2\pi}\frac{R_\alpha^2 + Z_\alpha^2}{(R_\rho Z_\alpha - R_\alpha Z_\rho)}\frac{d\alpha}{R} \tag{F.22}$$

Consider now the right hand side of Eq. (F.12). The critical assumption is that the current density is approximated by its flux surface averaged value, $J_\phi(\rho,\alpha) \approx \overline{J}_\phi(\rho)$. This is obviously not self-consistent with the Grad-Shafranov equation. Even so, it should suffice for present purposes where we only need to evaluate global volume integrals. Also, in the same spirit as prescribing $n(\rho)$ and $T(\rho)$ we assume that $\overline{J}_\phi(\rho)$ is also prescribed. Under these assumptions the right hand side of Eq. (F.12) simplifies to:

$$\begin{aligned}-\mu_0\int_V \frac{J_\phi}{R}d\mathbf{r} &= -2\pi\mu_0\int_A J_\phi dA \\ &= -2\pi\mu_0\int_0^\rho d\rho\int_0^{2\pi} J_\phi\left(R_\rho Z_\alpha - R_\alpha Z_\rho\right)d\alpha \\ &\approx -2\pi\mu_0\int_0^\rho d\rho\left[\overline{J}_\diamond\int_0^{2\pi}\left(R_\rho Z_\alpha - R_\alpha Z_\rho\right)d\alpha\right]\end{aligned} \tag{F.23}$$

Combining the results in Eqs. (F.22) and (F.23) leads to the required general expression for $d\psi/d\rho$,

$$\frac{d\psi}{d\rho}\int_0^{2\pi}\frac{R_\alpha^2 + Z_\alpha^2}{(R_\rho Z_\alpha - R_\alpha Z_\rho)}\frac{d\alpha}{R} = -\mu_0\int_0^\rho d\rho\left[\overline{J}_\omega\int_0^{2\pi}\left(R_\rho Z_\alpha - R_\alpha Z_\rho\right)d\alpha\right] \tag{F.24}$$

Next, to help specify a plausible choice for $\overline{J}_\phi$ it is useful to define the kink safety factor and the actual local safety factor. The kink safety factor is defined by

$$q_* = \frac{2\pi a^2 B_0}{\mu_0 R_0 I}\left(\frac{1+\kappa^2}{2}\right) \tag{F.25}$$

where

$$I = \int J_o dA = \int_0^1 d\rho \left[\overline{J}_o \int_0^{2\pi} \left(R_\rho Z_\alpha - R_a Z_\rho\right) d\alpha\right] \tag{F.26}$$

This leads to

$$\frac{1}{q_*} = \frac{\mu_0 R_0}{2\pi a^2 B_0}\left(\frac{2}{1+\kappa^2}\right)\int_0^1 d\rho \left[\overline{J}_\phi \int_0^{2\pi} \left(R_\rho Z_\alpha - R_\alpha Z_\rho\right) d\alpha\right] \tag{F.27}$$

Similarly, the local safety factor can be expressed as

$$q(\rho) = \frac{F(\rho)}{2\pi}\int \frac{dl}{RB_p} \tag{F.28}$$

Here, $F(\rho) = RB_o$. Substituting $RB_p = \mathbf{n}\cdot\nabla\psi$ then yields

$$q(\rho) = \frac{F(\rho)}{2\pi\psi'}\int_0^{2\pi} \frac{1}{R}\left(R_\rho Z_\alpha - R_\alpha Z_\rho\right) d\alpha \tag{F.29}$$

with $\psi' = d\psi/d\rho$.

For present purposes we can obtain relatively simple analytic expressions for all the quantities of interest by assuming the flux surfaces are concentric ellipses, characterized by $R = R_0 + a\rho\cos\alpha$ and $Z = \kappa a\rho\sin\alpha$. We assume low β so that $F(\rho) \approx R_0 B_0$. This model accounts for elongation but neglects the effects of triangularity and finite aspect ratio. The derivatives in Eqs. (F.24), (F.27) and (F.29) can now be easily evaluated. Also, after some trial and error we chose $\overline{J}_\phi(\rho)$ to be a plausible profile which is peaked off-axis at $\rho = \rho_m$.

$$\overline{J}_\phi(\rho) = -\frac{I}{\pi a^2 \kappa}\left[\frac{\gamma^2\left(1-\rho^2\right)e^{\gamma\rho^2}}{e^\gamma - 1 - \gamma}\right] \tag{F.30}$$

Here, $\gamma = 1/(1-\rho_m^2)$.

These profiles are substituted into Eq. (F.24) after which each of the integrals can be evaluated analytically. A straightforward calculation yields:

$$\begin{aligned} \rho\frac{d\psi}{d\rho} &= -2\mu_0 R_0 a^2\left(\frac{\kappa^2}{1+\kappa^2}\right)\int_0^\rho \overline{J}_\phi \rho d\rho \\ &= \frac{\mu_0 R_0 I}{\pi}\left(\frac{\kappa}{1+\kappa^2}\right)\frac{\left(1+\gamma-\gamma\rho^2\right)e^{\gamma\rho^2}-1-\gamma}{e^\gamma-1-\gamma} \end{aligned} \tag{F.31}$$

The safety factors are given by

$$\begin{aligned} \frac{1}{q_*} &= \frac{\psi'(1)}{\kappa a^2 B_0} \\ \frac{q(\rho)}{q_*} &= \frac{\rho\psi'(1)}{\psi'(\rho)} \end{aligned} \tag{F.32}$$

Eq. (F.31) is now substituted into the expression for the bootstrap current given by Eq. (F.11). The resulting expression can then be integrated over the plasma cross section to yield the bootstrap fraction. A straightforward calculation leads to:

$$f_{BS} = \frac{I_{BS}}{I} = \frac{2\pi a^2\kappa}{I}\int_0^1 J_{BS}\,\rho\,d\rho = \frac{K_{BS}}{K_n}\frac{\overline{n}\,\overline{T}R_0^2}{I_P^2} \tag{F.33}$$

$$K_{BS} = 4.879\cdot K_n\cdot\left(\frac{1+\kappa^2}{2}\right)\cdot\epsilon^{5/2}\cdot H_{BS} \tag{F.34}$$

$$H_{BS} = (1+\nu_n)(1+\nu_T)(\nu_n+0.054\nu_T)\int_0^1 \frac{\rho^{5/2}\left(1-\rho^2\right)^{\nu_n+\nu_T-1}}{b_p}\,d\rho \tag{F.35}$$

$$b_p(\rho) = \frac{-e^{\gamma\rho^2}(\gamma\rho^2-1-\gamma)-1-\gamma}{\rho\left(e^\gamma-1-\gamma\right)} \tag{F.36}$$

This is the desired result.

Appendix G

Elaborating on the Current Drive

The driven current fraction – f_{CD} – is an important parameter that enters in the design of steady-state tokamak reactors. It must be calculated with reasonable accuracy to determine how much bootstrap current is required. The value of f_{CD} thus has a strong impact on the overall fusion energy gain. Obtaining reasonable accuracy requires a moderate amount of analysis, which is presented in a following section. The results are summarized below.

G.1 Summarized Results

We assume that current drive is provided by lower hybrid waves because of the corresponding relatively high efficiency and naturally occurring off-axis peaking which aligns with the bootstrap current. The externally driven lower hybrid current (I_{CD}) is given in terms of the current drive efficiency, η_{CD}, defined as follows:[45]

$$I_{CD} = \eta_{CD} \frac{P_H}{\overline{n}_{20} R_0} = \frac{\eta_{CD}}{Q} \frac{P_F}{\overline{n}_{20} R_0} \tag{G.1}$$

Here, for simplicity and slightly optimistically, we assume that 100% of the klystron RF power, P_H, is absorbed in the plasma.

The current drive fraction – $f_{CD} = I_{CD}/I_P$ – can then be written as,

$$\begin{aligned} f_{CD} &= K_{CD}\tfrac{\eta_{CD}\overline{n}_{20}R_0^2(\hat{\sigma}v)}{I_M} \\ K_{CD} &= 278\tfrac{f_D^2\varepsilon^2\kappa}{Q} = 0.634 \end{aligned} \tag{G.2}$$

Typical values for η_{CD} are around 0.3.[45] However, this current drive efficiency is actually a function of: $\overline{n}$, $\overline{T}$, and B_0. This dependence must be included in the design to obtain reliable results. A self consistent calculation of $\eta_{CD} = \eta_{CD}(\overline{n}, \overline{T}, B_0)$ requires considerable analysis, the details of which are presented next section.

G.2 Detailed Analysis

To design a steady state fusion reactor, it is necessary to calculate η_{CD} for lower hybrid current drive (LHCD). Recall that the driven lower hybrid current I_{CD} is related to the lower hybrid RF klystron power absorbed by the plasma P_H by the relation:

$$I_{CD} = \eta_{CD}\frac{P_H}{n_{20}R} \tag{G.3}$$

Here, $P_H = \eta_{RF}P_{RF}$, with P_{RF} equal to the total wall power used for current drive (plus heating) and $\eta_{RF} \approx 0.5$ is the conversion efficiency from wall power to RF absorbed power. Also, $n_{20} = n_{20}(\rho_j)$ and $R = R(\rho_j, \theta)$ are the density and major radius evaluated at the minor radius $\rho = \rho_j$ and launch angle θ with ρ_j corresponding to the location of the peak driven current density: $J_{max} = J_{CD}(\rho_j, \theta)$. The angle θ is a known quantity set by the experimental configuration while ρ_j is yet to be determined.

The value of η_{CD} is related to a normalized quantity $\tilde{\eta}$, the efficiency usually calculated in the literature, by a series of connecting formulas. The inter-relations start with

$$\eta_I = \frac{\int_A J_{CD}dA}{\int_V S_H dV} \approx \frac{1}{2\pi}\left[\frac{J_{CD}}{RS_H}\right]_{\rho_j,\theta} = \frac{\eta_{LH}}{2\pi R}\left[\frac{J_{CD}}{RS_{LH}}\right]_{\rho_J,\theta} \tag{G.4}$$

where $\eta_I = I_{CD} = P_H$ A/W is the overall current drive efficiency measuring how many delivered watts of klystron RF power are required to drive one ampere of current. For simplicity and slightly optimistically all delivered power is assumed to be absorbed by the plasma. Also, $S_H(\rho,\theta)$ is the klystron power density delivered to the plasma, whose absorption is localized around $\rho = \rho_j$.

Due to various losses, only a fraction of the absorbed klystron power, $\eta_{LH} \approx 0.75$, actually drives current. These losses have to do with the fact that the power spectrum arising from a realistic waveguide array has both positive and negative lobes – it is not an ideal positive delta function. The combination of finite spectral width plus oppositely driven current from the negative lobe implies that only a portion of the total absorbed power actually drives a net positive current. The result of this discussion is that the power density, S_{LH}, driving lower hybrid current is related to the delivered klystron power density by $S_{LH} = \eta_{LH} S_H$.

Now, the efficiency, $\tilde{\eta}$ usually calculated in the literature is defined by:

$$\tilde{\eta}\left(\rho_J,\theta\right) = \left[\frac{J_{CD}/env_{T_e}}{S_{LH}/m_e n \nu_0 v_{Te}^2}\right]_{\rho_j,\theta} \tag{G.5}$$

$$v_{T_e}\left(\rho_J\right) = \left[\frac{2T_e}{m_e}\right]^{1/2}_{\rho_J} \tag{G.6}$$

$$\nu_0\left(\rho_J\right) = \left[\frac{\omega_{pe}^4 \ln\Lambda}{2\pi n_e v_{Te}^3}\right]_{\rho_J} \tag{G.7}$$

It then follows that

$$\eta_I = \frac{\eta_{LH}}{2\pi}\left[\frac{e}{R m_e \nu_0 v_{Te}}\right]_{\rho_j,\theta} \tilde{\eta}\left(\rho_J,\theta\right) \tag{G.8}$$

From Eq. (G.3), we see that $\eta_{CD} = \eta_I \left[n_{20} R\right]_{\rho_j,\theta}$, which leads to the desired conversion relation:

$$\eta_{CD} = \frac{\eta_{LH}}{2\pi}\left[\frac{e n_{20}}{m_e \nu_0 v_{T_e}}\right]_{\rho_j} \tilde{\eta}\left(\rho_J,\theta\right) = 0.06108 \frac{\eta_{LH}}{\ln\Lambda} T_k \tilde{\eta} \tag{G.9}$$

An expression for $\tilde{\eta}$

Needed for the design code is an expression for $\tilde{\eta}(\rho_j, \theta)$. Such an expression, valid for arbitrary ρ, has been determined by Ehst and Karney[36] – based on a sophisticated theoretical analysis combined with extensive numerical results. Once ρ_j is determined we set $\rho = \rho_j$ in the expression for $\tilde{\eta}(\rho, \theta)$. Ehst and Karney find that a good fit for $\tilde{\eta}(\rho, \theta)$ can be written as:

$$\tilde{\eta} = CMR\eta_0 \tag{G.10}$$

For LHCD, the parameters appearing in Eq. (G.10) have the form:

$$M = 1 \tag{G.11}$$

$$R(\rho, \theta) = 1 - \frac{\varepsilon^n \rho^n \left(x_r^2 + w^2\right)^{1/2}}{\varepsilon^n \rho^n x_r + w} \qquad n = 0.77 \qquad x_r = 2.47 \tag{G.12}$$

$$C(\rho, \theta) = 1 - \exp\left(-c^m x_t^{2m}\right) \qquad m = 1.38 \qquad c = 0.778 \tag{G.13}$$

$$\eta_0(\rho, \theta) = \frac{K}{w} + D + \frac{8w^2}{5 + Z_{eff}} \qquad K = \frac{2.12}{Z_{eff}} \qquad D = \frac{3.83}{Z_{eff}^{0.707}} \tag{G.14}$$

All quantities have been defined except for $x_t^2(\rho, \theta)$ and $w(\rho, \theta)$. The quantity w is a normalized form of the resonant particle velocity which absorbs energy and momentum from the lower hybrid wave,

$$w(\rho, \theta) = \frac{\omega}{k_\| v_{Te}} = \frac{c}{v_{T_e}} \frac{1}{n_\|} \tag{G.15}$$

with $n_\|$ the parallel index of refraction. The value of $n_\|(\rho, \theta)$ will be discussed shortly. The quantity x_t^2 is a toroidal correction associated with the fact that trapped particles cannot contribute to toroidal current flow. It can be expressed in terms of the local mirror ratio by

$$x_t^2(\rho, \theta) = w^2 \left(\frac{B}{B_M - B}\right) \tag{G.16}$$

where from simple guiding center theory assuming that $B \approx B_\phi$

$$B_M = \frac{B_0}{1 - \varepsilon\rho} \tag{G.17}$$

$$B = \frac{B_0}{1 + \varepsilon\rho\cos\theta} \tag{G.18}$$

Calculation of $n_\parallel^2(\rho, \theta)$

The next step in the evaluation of η_{CD} is the calculation of $n_\parallel^2(\rho, \theta)$. Its value is determined by the requirements for accessibility from the plasma edge into the absorption layer. The relevant physics follows from an analysis of the cold plasma dispersion relation given by

$$n_\perp^2(\rho, \theta) = -\frac{K_\parallel}{2K_\perp}\left\{n_\parallel^2 - K_\perp + \frac{K_A^2}{K_\parallel} \pm \left[\left(n_\parallel^2 - K_\perp + \frac{K_A^2}{K_\parallel}\right)^2 + \frac{4K_\perp K_A^2}{K_\parallel}\right]^{1/2}\right\} \tag{G.19}$$

The plus sign corresponds to the desired root and is often referred to as the slow wave.

In the lower hybrid regime the relevant ordering of parameters is

$$\begin{gathered} \omega_{pe}/\Omega_e \sim \omega_{pi}/\omega \sim n_\parallel \sim 1 \\ \omega_{pi}/\Omega_i \sim \omega/\Omega_i \sim \Omega_e/\omega \sim n_\perp \sim m_i/m_e^{1/2} \gg 1 \end{gathered} \tag{G.20}$$

leading to the following simple forms for the elements of the dielectric tensor

$$\begin{aligned} K_\perp(\rho, \theta) &= 1 + \frac{\omega_{pe}^2}{\Omega_e^2} - \frac{\omega_{pi}^2}{\omega^2} \sim 1 \\ K_A(\rho, \theta) &= \frac{\omega_{pe}^2}{\omega\Omega_e} \sim m_i/m_e^{1/2} \\ K_\parallel(\rho) &= -\frac{\omega_{pe}^2}{\omega^2} \sim m_i/m_e \end{aligned} \tag{G.21}$$

The first requirement for accessibility is that the function under the square root be positive. When this function passes through zero there is a double root for $n_\perp^2$ causing a mode conversion from the slow wave to the fast wave. The fast wave does not propagate into the plasma. It is reflected back out through the plasma surface, obviously an undesirable result. Avoiding mode conversion requires a sufficiently large value of $n_{\parallel}^2$ to keep to keep the function under the square root positive. This value must satisfy

$$n_{\parallel}^2(\rho,\theta) \geq \left[K_{\perp}^{1/2} + \left(-\frac{K_A^2}{K_{\parallel}}\right)^{1/2}\right]^2 \tag{G.22}$$

Since $\eta_{CD} \propto 1/n_{\parallel}^2$ we see that current drive efficiency is maximized when $n_{\parallel}^2(\rho,\theta)$ is minimized – the inequality in Eq. (G.22) must be set to equality.

At this point there is an important subtlety that must be taken into account. The issue is that the wavelength spectrum of the applied klystron source is not a delta function – it has a finite half width, $\Delta n_{\parallel} \approx 0.2$ and a negative lobe. For simplicity, we have modeled the spectrum as rectangular and ignore the negative lobe. The negative lobe is accounted for through the value of η_{LH}, since this power obviously does not drive current in the desired direction. Now, Eq. (G.22) is an inequality and we want to minimize $n_{\parallel}^2(\rho,\theta)$ over all ρ for the given θ where the power is absorbed. Therefore, we must use the equality sign in Eq. (G.22) for the strictest case – that $n_{\parallel}\left(\hat{\rho}_J,\theta\right) = n_{\parallel}\left(\rho_J,\theta\right) - \Delta n_{\parallel}$ – where $\hat{\rho}_J$ and ρ_J (both as yet undetermined) are the corresponding strictest and average radii where power is absorbed.

With this in mind, after substituting the simplified expressions for the elements of the dielectric tensor we obtain

$$\begin{gathered} n_{\parallel}^2\left(\hat{\rho}_J,\theta\right) = \left[\left(1 - \tfrac{1-\hat{\omega}^2}{\hat{\omega}^2}X\right)^{1/2} + X^{1/2}\right]^2 \\ X\left(\hat{\rho}_J,\theta\right) = \omega_{pe}^2\left(\hat{\rho}_J\right)/\Omega_e^2\left(\hat{\rho}_J,\theta\right) \\ \hat{\omega}^2\left(\hat{\rho}_J,\theta\right) = \omega^2/\Omega_e\left(\hat{\rho}_J,\theta\right)\Omega_i\left(\hat{\rho}_J,\theta\right) \end{gathered} \tag{G.23}$$

The question now is how do we choose the frequency: $\hat{\omega}$? There are actually three constraints on the frequency and we must choose the strictest one to determine $\hat{\omega}^2$.

The constraints are as follows:

$$\begin{aligned} &\omega^2 > \omega_{LH}^2\left(\hat{\rho}_J,\theta\right) && \text{Avoid mode conversion before reaching } \hat{\rho}_J,\theta \\ &\omega^2 > 4\omega_{LH}^2\left(\hat{\rho}_J,\theta\right) && \text{Avoid the PDI before reacing } \hat{\rho}_J,\theta \\ &\omega^2 > k_\perp^2\left(\hat{\rho}_J,\theta\right)v_\alpha^2 && \text{Avoid coupling to } \alpha \text{ particles before reaching } \hat{\rho}_J,\theta \end{aligned} \tag{G.24}$$

Here, $\omega_{LH}^2\left(\hat{\rho}_J,\theta\right) = \omega_{pi}^2/\left(1+\omega_{pe}^2/\Omega_e^2\right)$ is the square of the lower hybrid frequency and $v_\alpha = \left(2E_\alpha/m_\alpha\right)^{1/2}$ is the alpha particle speed. Also, PDI denotes parametric decay instability. The second and third constraints are approximate values, used here for simplicity.

Each of these constraints is substituted into the expression for $n_\parallel^2$. We find that in the regime of interest the α particle coupling requirement is the strictest. We thus choose the frequency to satisfy $\omega/k_\perp = v_\alpha$, or in normalized units:

$$n_\perp^2\left(\hat{\rho}_J,\theta\right) = \frac{c^2}{v_\alpha^2} \tag{G.25}$$

This expression is simplified by evaluating $n_\perp^2$ using Eq. (G.19) coupled with $n_\parallel^2$ given by Eq. (G.22)

$$n_\perp^2\left(\hat{\rho}_J,\theta\right) = -\frac{K_\parallel}{K_\perp^{1/2}}\left(-\frac{K_A^2}{K_\parallel}\right)^{1/2} = \frac{m_i}{m_e}\frac{X^{3/2}}{\hat{\omega}\left[\hat{\omega}^2(1+X)-X\right]^{1/2}} \tag{G.26}$$

Eq. (G.26) is a quadratic equation for $\hat{\omega}^2$, which can be easily solved, yielding:

$$\begin{gathered} \hat{\omega}^2\left(\hat{\rho}_J,\theta\right) = \tfrac{1}{2}\tfrac{X}{1+X} + \tfrac{1}{2}\left[\tfrac{X^2}{(1+X)^2} + 4\gamma^2\tfrac{X^3}{1+X}\right]^{1/2} \\ \gamma = \tfrac{m_i}{m_e}\tfrac{1}{n_\perp^2} = \tfrac{2m_iE_\alpha}{m_em_\alpha c^2} = 8.562 \end{gathered} \tag{G.27}$$

This value of $\hat{\omega}^2$ is substituted into Eq. (G.23) to obtain the desired expression for $n_\parallel^2 = n_\parallel^2(X)$.

Calculation of $\hat{\rho}_j$

The calculation of $\hat{\rho}_j$ requires a very lengthy analysis of Landau damping. We can bypass this complication by making use of a simple rule of thumb that is reasonably accurate. This rule states that lower hybrid power is absorbed and driven current produced in a somewhat narrow layer of the plasma profile whose location is determined by the requirement that the parallel phase velocity be approximately equal to three times the electron thermal speed,

$$\frac{\omega}{k_{\parallel}} \approx 3v_T \tag{G.28}$$

The equation can be rewritten in terms of $\hat{n}_{\parallel}$ leading to a transcendental algebraic equation for $\hat{\rho}_j$,

$$(1+\nu_T)\left(1-\hat{\rho}_J^2\right)^{\nu_T} n_{\parallel}^2\left(\hat{\rho}_J,\theta\right) = \frac{m_e c^2}{18\overline{T}} = \frac{28.39}{\overline{T}_k} \tag{G.29}$$

This is a simple equation to solve numerically.

Calculation of ρ_j

The last step in the analysis is to map the results at the strictest absorption location – (ρ,θ) – to the center of the absorption layer – (ρ_J,θ) where the current drive efficiency is defined. This is easily done by noting that power is always absorbed in at the local radius where $\omega/k_{\parallel} = 3v_{Te}$. Consequently, the relations at ρ_J are related to those at $\hat{\rho}_J$ by:

$$\begin{aligned}(1+\nu_T)\,(1-\hat{\rho}_J^2)^{\nu_T}\, n_{\parallel}^2\left(\hat{\rho}_J,\theta\right) &= \tfrac{28.39}{\overline{T}_k}\\ (1+\nu_T)\,(1-\rho_J^2)^{\nu_T}\, n_{\parallel}^2\left(\rho_J,\theta\right) &= \tfrac{28.39}{\overline{T}_k}\end{aligned} \tag{G.30}$$

Since $n_{\parallel}\left(\hat{\rho}_J,\theta\right) = n_{\parallel}\left(\rho_J,\theta\right) - \Delta n_{\parallel}$, it follows that $\hat{\rho}_J$ and ρ_J are related by:

$$\frac{\left(1-\rho_J^2\right)^{\nu_T}}{\left(1-\hat{\rho}_J^2\right)^{\nu_T}} = \left[1-\frac{\Delta n_{\parallel}}{n_{\parallel}\left(\rho_J,\theta\right)}\right]^2 \rightarrow \rho_J^2 = 1-\left(1-\hat{\rho}_J^2\right)\left[1-\frac{\Delta n_{\parallel}}{n_{\parallel}\left(\rho_J,\theta\right)}\right]^{2/\nu_T} \tag{G.31}$$

Note that in general: $\rho_J > \hat{\rho}_J$. The strictest location determining $n_{\parallel}(\hat{\rho}_J,\theta)$ is the innermost radial point on the temperature profile where power is absorbed.

Abridged Modus Operandi

Assume the following quantities are given as inputs: $B_0, \theta, \overline{n}_{20}, \overline{T}_k, \varepsilon, \Delta n_\parallel, \eta_{LH}$. Carry out the following steps:

1. Solve the equations below simultaneously to determine $n_\parallel^2(\hat{\rho}_J, \theta), \hat{\omega}^2(\hat{\rho}_J, \theta),$ and $\hat{\rho}_J$

$$\begin{aligned} n_\parallel^2(\hat{\rho}_J, \theta) &= \left[\left(1 - \tfrac{1-\hat{\omega}^2}{\hat{\omega}^2} X\right)^{1/2} + X^{1/2}\right]^2 \\ \hat{\omega}^2(\hat{\rho}_J, \theta) &= \tfrac{1}{2}\tfrac{X}{1+X} + \tfrac{1}{2}\left[\tfrac{X^2}{(1+X)^2} + 4\gamma^2 \tfrac{X^3}{1+X}\right]^{1/2} \\ (1+\nu_T)\left(1-\hat{\rho}_J^2\right)^{\nu_T} n_\parallel^2(\hat{\rho}_J, \theta) &= \tfrac{m_e c^2}{2\overline{T}} = \tfrac{28.39}{\overline{T}_k} \end{aligned} \tag{G.32}$$

2. Solve for $\tilde{\eta}(\hat{\rho}_J, \theta)$

$$\tilde{\eta}(\hat{\rho}_J, \theta) = CMR\eta_0 \tag{G.33}$$

3. Solve for $n_\parallel(\rho_J, \theta)$

$$n_\parallel(\rho_J, \theta) = n_\parallel(\hat{\rho}_J, \theta) + \Delta n_\parallel \tag{G.34}$$

4. Solve for ρ_J

$$\rho_J^2 = 1 - \left(1 - \hat{\rho}_J^2\right)\left[1 - \frac{\Delta n_\parallel}{n_\parallel(\rho_J, \theta)}\right]^{2/\nu_T} \tag{G.35}$$

5. Re-evaluate $\tilde{\eta}(\rho_J, \theta)$ by substituting the values of $\rho_J, n_\parallel(\rho_J, \theta)$ into Eq. (G.33).
6. Solve for η_{CD}

$$\eta_{CD} = \frac{1}{2\pi}\left(\frac{e n_{20}}{m_e \nu_0 v_{Te}}\right)\tilde{\eta} = 0.06108\frac{\eta_{LH}}{\ln\Lambda}(1+\nu_T)\overline{T}_k\left(1-\rho_J^2\right)^{\nu_r}\tilde{\eta}(\rho_J, \theta) \tag{G.36}$$

In the end there will have to be some iteration with the rest of the analysis to make sure the values of $\overline{n}_2 0$ and $\overline{T}_k$ are self-consistent.

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