A Levelized Comparison of Pulsed and Steady-State Tokamaks

by

Daniel Joseph Segal

Submitted to the Department of Nuclear Science and Engineering
in partial fulfillment of the requirements for the degree of

Master of Science in Nuclear Science and Engineering

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

September 2018

Author . . . .
Department of Nuclear Science and Engineering
August 31, 2018

Certified by . . . .
Jeffrey P. Freidberg
KEPCO Professor Emeritus
Thesis Supervisor

Accepted by . . . .
Anne E. White
Cecil and Ida Green Associate Professor

A Levelized Comparison of Pulsed and Steady-State Tokamaks

by

Daniel Joseph Segal

Submitted to the Department of Nuclear Science and Engineering
on August 31, 2018, in partial fulfillment of the
requirements for the degree of
Master of Science in Nuclear Science and Engineering

Abstract

The goal of fusion energy research is to build a profitable reactor. This thesis develops a cost estimate of fusion reactors from a physicist's perspective. It then applies this cost metric to the two main operation schemes for a tokamak reactor: pulsed and steady-state. In the end, an apples-to-apples comparison is developed, which is used to explain: the relative advantages of the two modes of operation, as well as, the design parameters tha t provide the highest leverage in lowering machine costs.

Thesis Supervisor: Jeffrey P. Freidberg
Title: KEPCO Professor Emeritus

Contents

List of Figures

List of Tables

Chapter 1

Introducing Fusion Reactors

The goal of fusion energy research is to make a profitable nuclear reactor. It has long been joked though that fusion power will always be 20-50 years away. This paper lays a framework for exploring reactor space for functional, efficient designs based on the last half-century of the world's experiments. Based on the speed and simplicity of the model, hundreds of reactors can be explored in minutes – *outpacing the domestic program slightly.*

With this proposed model, interesting reactors can be pinpointed long before engineers hit the blueprints. This should help shorten the time till a profitable reactor, as well as illuminate ways to improve modern plasma theory.

1.1 Treating Fusion as a Science

When people talk about fusion, they usual talk about plasma physics, and when people talk about plasma physics, they often talk about things like: the sun, lighting, and the aurora borealis. Of these three, the sun is the only nuclear reactor. The sun can stay on all day because the massive gravity of its fuel source helps keep it self-contained in space. On Earth, this is not possible – the plasma fuel[1] needs to be

[1] Plasmas are the fourth state of matter after: solids, liquids, and gases. Fundamentally they are gaseous fluids that respond to electric and magnetic fields.

contained by other means (i.e. with magnets).

A tokamak is one of the leading candidates for a profitable fusion reactor. It shares the shape of a doughnut, using magnets to keep a hoop of plasma twirling inside it. The difficulty of keeping this plasma twirling though, is that it does not enjoy being spun too fast or squeezed to hard. Conversely, the tokamak housing the plasma does not like taking too much of a beating or being scaled to T-Rex sized proportions.

This sets the stage for tokamak reactor design – building on the various plasma physics and nuclear engineering constraints of the day. One of the most contentious points of building a tokamak, however, is whether it will be run in: a pulsed (the European approach) or steady-state (the United States approach) mode of operation. Here, pulsed operation refers to how a reactor is turned on and off periodically – around ten times a day. Whereas, steady state machines are meant to be left on nearly the entirety of their 50-year campaigns.

These two modes of operation, *pulsed* and *steady-state*, greatly influence the design through the current balance equation (derived later). What this means practically is tokamaks need current to spin their plasma hoops at some required speed and this current has to come from somewhere. Luckily, the plasma naturally enjoys spinning and provides some assistance through the bootstrap current. The remaining current must then be produced by external means.

The source of external current drive is what distinguishes pulsed from steady-state devices. Steady-state devices provide the required current assistance either through lasers or particle beams – this paper's model focusing on a type of laser assistance called lower-hybrid current drive (LHCD). Pulsed machines, on the other hand, rely on inductive sources – which by definition require cycles of charging and discharging several times a day.

The goal of this document is to show that pulsed and steady-state operation are really two sides of the same coin. Moreover, if desired, designs can be found which can be run in both modes of operation – yielding practical physics and engineering implications. For physicists, it would provide at least twice as much data as a machine that can only be run in one mode of operation. For engineers, it allows a machine to not be out of commission when a component needed for pulsed operation is under maintenance.

1.2 Treating Fusion as a Business

Plasmas may be neat, but that is not why countries build billion dollar research experiments. The ultimate goal of fusion research is to develop an energy resource that competes with coal and other base-load power sources. The problem is plasmas are chaotic and hard to contain, while tokamaks are expensive and slow to build. This perfect match has long put the field's project timeline to that of "fusion never."

The major problem with containing a plasma in a reactor is that plasmas do not want to be contained. Since the early days of fusion research, plasmas have often found escape mechanisms. When presented with a magnetic bottle, they found their way out the top. In a tokamak, they like falling to the ground (i.e. the vertical instability). Fusion energy has remained a Tantalizing effort – remaining within arms reach, but precluded by a shroud of instabilities.

The truth is plasmas are extremely chaotic: they show nonlinear behavior in almost everything they do. As of now, no theory or supercomputer-backed code can predict even something so fundamental to design as the movement of energy and particles within a tokamak. As such, the field has adopted several rules of thumb and empirical scalings – based on the last half century of experiments – which help one navigate around a plasma's finicky behavior.

The two most widely used rules of thumb within the fusion design community are: the Greenwald density limit and the ELMy H-Mode confinement time scaling law. As such, the model in this document heavily utilized the two to make a quick running code. These two relations are also why this model – which happens to be zero-dimensional – can reproduce with high fidelity the answers from three-dimensional codes, which can take days, weeks, or even months to run!

The use of the ELMy H-Mode scaling law also brings up another subtlety in the field. To measure the movement of energy within a plasma, scaling relations are needed that correlate to specific modes of plasma behavior – i.e. ones that can be robustly be found by device technicians. Currently, people like H-Mode scalings over L-Mode (because H stands for high and L stands for low). However, people often seek out other modes that can reliably be found on other machines. These go by names like: I-Mode (Intermediate), Enhanced H-Mode, and Reversed Shear mode.

Without going into too much detail, these alternate modes can be extremely valuable, as they are patentable and can lead to cheaper reactor designs (than the ones made under H-Mode scaling). The problem, however, is often not finding a better performing mode on a machine, it is robustly finding it on other ones. This is important, because finding a mode on multiple machines is what allows new scaling relations to be produced and refined.

1.3 Pricing a Fusion Reactor

To compare tokamaks used as fusion reactors the obvious metric to use is cost. ITER – the second most expensive experiment today (only behind the LHC) – has a history rich in countries backing out for high price tags and rejoining only when they are finally lowered. The problem is $20B is a lot of money and 20 years is a long time. Moreover, approximating a true cost becomes even trickier due to the need to neglect or project economies-of-scale for the expensive components in a design, e.g. the magnets.

As such, this paper adopts stand-ins for the conventional absolute cost and cost-per-watt metrics. This is done for simplicity – for both modeling reasons as well as conveying the two metrics to physicists. To start, the relevant approximation for absolute cost in a fusion reactor is the magnetic energy.

$$W_M \propto R^3 B^2 \tag{1.1}$$

Here, the tokamak's major radius – R – is involved in a volumetric term and B is the strength (in Teslas) of the hooped shape magnetic field that lays nested within the plasma hoop's shell (at its core). This quantity simply states that the two surefire ways to make a machine more expensive are: to make it larger and to use a stronger magnet.

The next metric, the cost-per-watt, is defined by dividing the previously given cost (i.e. the magnetic energy) by the fusion power (i.e. the watts). The reason fusion power is used is because it is the dominant source of revenue for a reactor – fusion power relies on light elements (i.e. two Hydrogens) fusing into a heavier one (i.e. one Helium) and releasing enough energy to offset the expense of causing it to happen. Although fusion power will not be defined till later, it does highlight the fact that this measure of cost-per-watt actually has units of time! (As energy per watt is time.)

The final piece of the costing puzzle is a duty factor that levelizes the comparison of pulsed and steady-state tokamaks. As pulsed machines may be off 20% of the time, their fusion power output should be reduced by that percentage. This is accounted for in the duty factor, which is simply the ratio of the flattop (the time when pulsed machines are approximately held at steady-state) to the entire length of a pulsed. In pulsed machines, the entire pulse includes charging the inductive sources as well as flushing out the tokamak between runs. These non-flattop portions of time can last around thirty minutes – where the reactor makes no money. As steady-state machines lack these non-flattop portions, their duty factors are rightfully one.

Summarizing, the cost-per-watt coupled with the duty factor gives an ad hoc pricing metric, hereafter referred to as the Gray Cost:

$$Cost_{Gray} = \frac{W_M}{f_{Duty} \cdot P_F} \tag{1.2}$$

It serves as a cornerstone for comparing the entire landscape of tokamak reactors – whether they run in pulsed or steady-state operation. Although not a true cost metric (i.e. in dollars per watt), it does provide an obvious physics meaning (as well as have the valuable units of time).

Chapter 2

Designing a Steady-State Tokamak

This chapter explores a simple model for designing steady-state tokamaks. In the next couple chapters, the model is formalized and then generalized to handle pulsed operation. These derivations highlight that the only difference between the two modes is how they generate their auxiliary plasma current: LHCD for steady-state operation and inductive sources when purely pulsed.

Along the way, equations will be derived that get rather complicated. To remedy the situation, a distinction between floating and fixed values is now given, which will allow splitting most equations into fixed and floating parts. Fixed values – i.e. the tokamak's major radius (R_0) and magnet strength (B_0), as well as the plasma's current (I_P), temperature ($\overline{T}$), and density ($\overline{n}$) – are first-class variables in this model. Everything is derived to relate them. Fixed values, then, are the input of the code, which remain constant throughout an entire scan of reactors. These include the geometric and profile parameters introduced next section.

2.1 Defining Plasma Parameters

As mentioned previously, the zero-dimensional model derived here can closely approximate solutions from three-dimensional codes that might take weeks to run. The essence of boiling down three-dimensional behaviors to one dimensional profiles and zero-dimensional averaged values begins with defining the the most important plasma parameters. These are the: current (J), the temperature (T), and the density (n) of the plasma.

Solving this problem most generally usually involves decoupling the geometry of the plasma from the shaping of nearly parabolic radial-profiles – both of which will be explained shortly.

2.1.1 Understanding Tokamak Geometry

The first thing people see when they look at a tokamak is its geometry. How big is it? Is it stretched out like a tire or smooshed together like a bagel? If it were torn in two, would the halves look like circles, ovals, or triangles?

These questions lend themselves to the three important geometry variables – the inverse aspect ratio (ϵ), the elongation (κ), and the triangularity (δ). The inverse aspect ratio is a measure of how stretched out the device is, or formulaically:

$$a = \epsilon \cdot R_0 \tag{2.1}$$

This says that the minor radius (a), measured in meters, is related to the major radius of the machine (R_0). More tangibly, the minor radius is related to the two circles that come from tearing a bagel in two. The major radius is related to the overall circle of the bagel when viewing it from the top.

The remaining two geometric parameters – κ and δ – are related to the shape of torn sides. As the name hints, elongation (κ) is a measure of how stretched out the tokamak is vertically – is the cross-section a circle or an oval? The triangularity (δ) is then how much the cross-sections point outward from the center of the device. All three's effects can be seen in Fig. N.

These geometric factors allow the volumetric and surface integrals for current, density, and temperature to be condensed to simple radial ones. The only remaining step is to define the radial profiles for these three quantities.

2.1.2 Prescribing Plasma Profiles

The first step in defining radial profiles is realizing that all three quantities are basically parabolas – i.e. the temperature, density, and current are peaked at some radius (usually the center) and then decay to zero somewhere before the walls of the tokamak enclosure.

The Density Profile

To begin, density has the simplest profile. This is because it is relatively flat, remaining near the average value – $\overline{n}$ – throughout the body of the plasma until quickly decaying to zero near the edge of the plasma. For this reason, a parabolic profile with a very low peaking factor – ν_n – is well suited.

$$n(\rho) = \overline{n} \cdot (1 + \nu_n) \cdot (1 - \rho^2)^{\nu_n} \tag{2.2}$$

The Temperature Profile

The use of a parabolic profile for the plasma temperature is slightly more dubious. This is because H-Mode plasmas are actually highly peaked at the center, decaying to a non-zero pedestal temperature near the edge before finally sharply dropping to zero. This model chooses to forego this pedestal representation for a simple parabolic one. Although the pedestal approach is discussed in the Appendix. Here again, $\overline{T}$ is the average value and ν_T is the peaking parameter.

$$T(\rho) = \overline{T} \cdot (1 + \nu_T) \cdot (1 - \rho^2)^{\nu_T} \tag{2.3}$$

The Current Profile

The plasma current is the third profile and cannot safely be represented as a simple parabola. This is because having an adequate bootstrap current heavily relies on a profile peaked at some radius – not at the center. This hollow profile can be modeled with the commonly given plasma internal inductance (l_i).

Concretely, the current's hollow profile is described by:

$$J(\rho) = \bar{J} \cdot \frac{\gamma^2 \cdot (1 - \rho^2) \cdot e^{\gamma \rho^2}}{e^\gamma - 1 - \gamma} \tag{2.4}$$

Where γ can be numerically solved from the plasma internal inductance through the following relations – with b_p representing the normalized poloidal magnetic field.

$$l_i = \frac{4\kappa}{1 + \kappa^2} \int_0^1 b_p^2 \, \frac{d\rho}{\rho} \tag{2.5}$$

$$b_p(\rho) = \frac{-e^{\gamma \rho^2} (\gamma \rho^2 - 1 - \gamma) - 1 - \gamma}{\rho \, (e^\gamma - 1 - \gamma)} \tag{2.6}$$

Combined, these three geometric parameters and profiles lay the foundation for this zero-dimensional fusion systems model.

2.2 Solving the Steady Current

As suggested, one of the most important equations in a fusion reactor is current balance. In steady-state operation, all of a plasma's current (I_P) must come from a combination of its own bootstrap current (I_{BS}), as well as auxiliary current drive (I_{CD}). This can be represented mathematically as:

$$I_P = I_{BS} + I_{CD} \tag{2.31}$$

The goal then is to write equations for bootstrap current and driven current. This will make heavy use of the Greenwald density limit. Without spoiling too much, the steady current is shown to be only a function of temperature (i.e. independent of a tokamak's geometry and magnet strength). As will be point out then, a subtlety arises that will then bring a tokamak's configuration back into the picture – self-consistency in the current drive efficiency (η_{CD}).

2.2.1 Enforcing the Greenwald Density Limit

The Greenwald density limit is ubiquitous in the fusion energy field. It sets a hard limit on the density and how it scales with current and reactor size. Although currently lacking a true first-principles theoretical explanation, it does have true meaning in the design context. Operate at too low a density and run the risk of never entering H-Mode. Run the density too low, and cause the tokamak's plasma to disrupt catastrophically!

As no theoretical backing exists, the Greenwald density limit can simply be written (with citation) as:

$$\hat{n} = N_G \frac{I_P}{\pi a^2} \tag{2.7}$$

Here, $\hat{n}$ has units of $10^{20}\ \frac{\text{particles}}{\text{m}^3}$, N_G is the Greenwald density fraction, I_P is again the plasma current (in mega-amps) and π has its usual meaning (3.141592653...). The final variable is the minor radius – a – which was previously defined through:

$$a = \epsilon \cdot R_0 \tag{2.1}$$

The next step is transforming the *line-averaged* density ($\hat{n}$) into the *volume-averaged* version ($\overline{n}$) used in this model. Harnessing the simplicity of the density's parabolic profile allows this relation to be written in a closed form as:

$$\hat{n} = \frac{\sqrt{\pi}}{2} \cdot \left(\frac{\Gamma\left(\nu_n + 2\right)}{\Gamma\left(\nu_n + \frac{3}{2}\right)} \right) \cdot \overline{n} \tag{2.8}$$

Where $\Gamma(\cdots)$ represents the gamma function – the non-integer analogue of the factorial function.

Combining these pieces allows the averaged density to be written in standardized units (i.e. the ones we use) as:

$$\overline{n} = K_n \cdot \left(\frac{I_P}{R_0^2} \right) \tag{2.9}$$

$$K_n = \frac{2N_G}{\epsilon^2\, \pi^{3/2}} \cdot \left(\frac{\Gamma\left(\nu_n + \frac{3}{2}\right)}{\Gamma\left(\nu_n + 2\right)} \right) \tag{2.10}$$

The format of the above equation pair will be used throughout the remainder of the paper. The top equation relates floating variables (i.e. $\overline{n}$, I_P, and R_0), while the fixed-value coefficient (K_n) lumps together: N_G, ϵ, 2, π, and ν_n.

2.2.2 Declaring the Bootstrap Current

The first term to define in current balance is the bootstrap current. This bootstrap current is a mechanism of tokamak plasmas that helps supply some of the current needed to keep a plasma stable. From a hand-waving perspective, it involves particles stuck in banana-shaped orbits on the outer edges of a tokamak behaving like racing-game style speed boosts that accelerate charged particles along their hooped-shaped race tracks.

To get an equation for bootstrap current, we must first introduce the surface integral – made possible from our previous choice of geometric parameters:

$$Q_S = 2\pi a^2 \kappa g \int_0^1 Q(\rho)\rho\, d\rho \tag{2.11}$$

Here, Q is an arbitrary function of the normalized radius (ρ) and g is a geometric factor (of order 1):

$$g = \frac{1}{8} \cdot \left(9 - 2\delta - 0.3\left(1 - \delta^2\right)\right) \tag{2.12}$$

This allows the bootstrap current (I_{BS}) to be written in terms of the temperature and density profiles:

$$I_{BS} = 2\pi a^2 \kappa g \int_0^1 J_{BS}\, \rho\, d\rho \tag{2.13}$$

$$J_{BS} = f\left(n, T, \frac{dn}{d\rho}, \frac{dT}{d\rho}\right) \tag{2.14}$$

For a more formal look into the J_{BS} function, check the appendix section on pedestal temperatures. The point to make now is that J_{BS} is a function of the profile's derivatives, leading to on major discrepancy in the model. As show later in the results, bootstrap fractions are often under-predicted by this model. This is due to parabolic profiles (i.e. for temperature) having much less steep declines near the edge (i.e. their derivatives) This implies that the area most positively impacted by a pedestal profile for temperature would be the plasma current.

Getting back on track – and without completeness – the bootstrap current can now be written in proportionality form as:

$$I_{BS} \propto \overline{T} \cdot \overline{n} \cdot \left(\frac{R_0^2}{I_P}\right) \tag{2.15}$$

Recognizing that the last term is basically the inverse of the Greenwald density (see Eq. 2.9), allows the proportionality to be written in the following form. Note that this implies the bootstrap current is only a function of temperature!

$$I_{BS} \propto K_n \cdot \overline{T} \tag{2.16}$$

In standardized units, this proportionality can be written as a concrete relation of the form:

$$I_{BS} = K_{BS} \cdot \overline{T} \tag{2.17}$$

$$K_{BS} = 4.879 \cdot K_n \cdot \left(\frac{1 + \kappa^2}{2} \right) \cdot \epsilon^{5/2} \cdot H_{BS} \tag{2.18}$$

$$H_{BS} = (1 + \nu_n)(1 + \nu_T)(\nu_n + 0.054\nu_T) \int_0^1 \frac{\rho^{5/2} \left(1 - \rho^2 \right)^{\nu_n + \nu_T - 1}}{b_p} \, d\rho \tag{2.19}$$

Quickly noting, this H_{BS} term serves as the analogue of fixed-value coefficients (i.e. $K_\square$) when they contain an integral. And b_p represents the poloidal magnet strength given by Eq. 2.6.

2.2.3 Deriving the Fusion Power

The next segue on our journey to solving the steady current is deriving the fusion power (P_F). This requires a more first-principles approach than those used up till now. As such, a quick background is given to motivate the parameters it adds – i.e. the dilution factor (f_D) and the Bosch-Hale fusion reactivity (σv).

The natural place to start talking about fusion is the binding-energy per nucleon plot (see fig. N). As can be seen, the function reaches a maximum at binding-energy at the element, Iron (A=56). What this means practically is: every element lighter than iron can *fuse* into a heavier one (i.e. hydrogens into helium), whereas every element heavier than iron can *fission* into lighter ones (e.g. uranium into krypton and barium). This is what differentiates fission (uranium-fueled) reactors from fusion (hydrogen-fueled) ones. For fusion reactors, the most common reaction in a first-generation tokamak will be:

$$^2H + {}^3H \rightarrow {}^4He + {}^1n + E_F \tag{2.20}$$

$$E_F = 17.6 \text{ MeV} \tag{2.21}$$

What this reaction describes is two isotopes of hydrogen – i.e. deuterium and tritium – fusion into a heavier element, helium, while simultaneously ejecting a neutron. The entire energy of the fusion reaction (E_F) is 17.6 MeV, which is then divvied up 80-20 between the neutron and helium, respectively. Quantitatively, the helium (hereafter referred to as an alpha particle) receives 3.5 MeV.

The final point to make before returning to the fusion power derivation is the main difference between the two fusion products: helium (i.e. the alpha particle) and the neutron. First, neutrons lack a charge – they are neutral. This means they cannot be confined with magnetic fields. As such, They simply move in straight lines until they collide with other particles. As the tokamak is solid metal, the neutron is much more likely to collide there than the gaseous plasma, which is orders of magnitude less dense. Conversely, alpha particles are charged – when stripped of their electrons – and can therefore be kept within the plasma. What this means practically is that of the 17.6 MeV that comes from every fusion reaction, only 3.5 MeV remains inside the plasma (within the helium particle species).

Returning to the problem at hand, the fusion power – in megawatts – is given in Jeff Freidberg's book as:

$$P_F = \int E_F \, n_D \, n_T \, \langle \sigma v \rangle \, d\vec{\mathbf{r}} \tag{2.22}$$

The n_D and n_T in this equation represent the density of the deuterium and tritium ions, respectively. Assuming a 50-50 mix of the two, they can be related to the electron density – i.e. the one used in this model – through the dilution factor:

$$n_D = n_T = f_D \cdot \left(\frac{n}{2}\right) \tag{2.23}$$

The fusion reactivity, $\langle \sigma v \rangle$, is then a nonlinear function of the temperature, T, which the model approximates using the Bosch-Hale tabulation (described in the appendix). As this tabulated value appears inside an integral, it seems important to point out that the temperature is now the most difficult floating variable to handle – over R_0, B_0, $\overline{n}$, and I_P. This will come into play when the model is formalized next chapter.

The next step in the derivation of fusion power is transforming the three-dimensional volume integral (see Eq. 2.22) into a one-dimensional radial one. First, the volume analogue of the previously given surface-area integral integral is:

$$Q_V = 4\pi^2 R_0 a^2 \kappa g \int_0^1 Q(\rho)\rho \, d\rho \tag{2.24}$$

Where again, Q is an arbitrary function of ρ and g is a geometric factor approximately equal to one. The fusion power can now be written as:

$$P_F = \pi^2 E_F f_D^2 R_0 a^2 \kappa g \int_0^1 n^2 \langle \sigma v \rangle \rho \, d\rho \tag{2.25}$$

In standardized units, this becomes:

$$P_F = K_F \cdot \overline{n}^2 \cdot R_0^3 \cdot (\sigma v) \tag{2.26}$$

$$K_F = 278.3 \cdot f_D^2 \cdot (\epsilon^2 \kappa g) \tag{2.27}$$

Where the fusion reactivity is now,

$$(\sigma v) = 10^{21} \, (1 + \nu_n)^2 \int_0^1 (1 - \rho^2)^{2\nu_n} \langle \sigma v \rangle \, \rho \, d\rho \tag{2.28}$$

As mentioned before, this fusion power is divvied up 80-20 between the neutron and alpha particle. These relations will come up shortly. For now, they can be described mathematically as:

$$P_\alpha = 0.2 \cdot P_F \tag{2.29}$$

$$P_n = 0.8 \cdot P_F \tag{2.30}$$

At this point, the current drive needed for steady-state can now be defined.

2.2.4 Using Current Drive

As may have been lost along the way, the current mission is to define a formula for steady current – from the current balance equation for steady-state tokamaks in H-Mode:

$$I_P = I_{BS} + I_{CD} \tag{2.31}$$

In standardized units, the equation for the current drive is often given in the literature as:

$$I_{CD} = \eta_{CD} \cdot \left(\frac{P_H}{\overline{n}R_0}\right) \tag{2.32}$$

Here, η_{CD} is the current drive efficiency with units $\left(\frac{\text{MA}}{\text{MW-m}^2}\right)$ and P_H is the heating power in megawatts driven by LHCD (and absorbed by the plasma).

Let it be known that driving current in a plasma is hard. Pulsed reactor designers (i.e. the European effort) think it is so difficult, the may even choose to ignore it completely – focusing on only inductive sources that require downtime. A common current drive efficiency (η_{CD}) seen in many designs is 0.3 ± 0.1 in the standard units.

It is however inherently a function of all the plasma parameters – this subtlety is put off until the discussion of self-consistency. For now it assumed to have some constant value.

The remaining step is deriving an equation for driven current (I_{CD}) is a formula for the heating power (P_H). The way fusion systems models – like this one – handle the heating power is through the physics gain factor, Q. Sometimes referred to as the big Q, the value represents how many times over the heating power (P_H) is amplified as it is transformed into fusion power (P_F):

$$P_H = \frac{P_F}{Q} \tag{2.33}$$

Now, utilizing the previously defined Greenwald density and fusion power:

$$\overline{n} = K_n \cdot \left(\frac{I_P}{R_0^2} \right) \tag{2.9}$$

$$P_F = K_F \cdot \overline{n}^2 \cdot R_0^3 \cdot (\sigma v) \tag{2.26}$$

The current from LHCD can be written as:

$$I_{CD} = K_{CD} \cdot I_P \cdot (\sigma v) \tag{2.34}$$

$$K_{CD} = (K_F K_n) \cdot \frac{\eta_{CD}}{Q} \tag{2.35}$$

As η_{CD} and Q appear within a fixed coefficient, it is implied they are both input values that remain constant during a solve. This subtlety is lifted when handling η_{CD} self-consistently, which will be discussed shortly. However, even in that context, it proves beneficial to still think of it as a fixed variable.

2.2.5 Completing the Steady Current

As hinted along the way, the goal of this section has been to derive a simple formula for steady current (I_P). The problem started with current balance in a steady-state reactor:

$$I_P = I_{BS} + I_{CD} \tag{2.31}$$

Two equations were then found for the bootstrap (I_{BS}) and driven (I_{CD}) current:

$$I_{BS} = K_{BS} \cdot \overline{T} \tag{2.17}$$

$$I_{CD} = K_{CD} \cdot I_P \cdot (\sigma v) \tag{2.34}$$

Combining these three equations and solving for the total plasma current (I_P) – in mega-amps – yields:

$$I_P = \frac{K_{BS}\overline{T}}{1 - K_{CD}(\sigma v)} \tag{2.36}$$

This is the answer we have been seeking!

As mentioned before, this simple formula appears to only depend on temperature! Apparently, the plasma should have the same current at some temperature (i.e. $\overline{T} =$ 15 keV), regardless of the size of the machine or the strength of its magnets. This has the important corollary that each temperature maps to only one current value. As has become a mantra, the subtlety of this behavior lies in the self-consistency of the current-drive efficiency – η_{CD}.

2.3 Handling Current Drive Self-Consistently

Although a thorough description of the wave theory behind lower-hybrid current drive (LHCD) is well outside the scope of this text, it does motivate the solving of a tokamak's major radius (R_0) and field strength (B_0). It also shows how what was apparently a simple problem has now transformed into a rather complex one – a common occurrence when working with plasmas.

The logic behind finding a self-consistent current-drive efficiency is starting at some plausible value (i.e. $\eta_{CD} = 0.3$), solving for the steady current – $I_P = f(\overline{T})$ – and then somehow iteratively creeping towards a value deemed self-consistent. What this means is in addition to the solver described in the last section, there needs to be a black-box function the solution is piped through to get a better guess at η_{CD}. The black-box function we use is a variation of the Ehst-Karney model.

As mentioned, a self-consistent η_{CD} is found once a trip through the Ehst-Karney black-box results in the same η_{CD} value as was piped in – to some allowed level of error. As will be tackled in a later chapter, there are sometimes two (or more) values of η_{CD} that are physically allowed. These physically allowed values are called the roots of η_{CD}. How to robustly find all roots and distinguish them (i.e. as high-drive and low-drive) will be put off till then.

The final point to make about the steady current is that the Ehst-Karney model includes dependence on the tokamak configuration. Mathematically,

$$\tilde{\eta}_{CD} = f(R_0, B_0, \overline{n}, \overline{T}, I_P) \tag{2.37}$$

As such, to recalculate it after every solution of the steady current requires a value for both B_0 and R_0 – the targets of this model's primary and secondary constraints. These will be the highlight of the next chapter.

Chapter 3

Formalizing the Fusion Systems Model

The goal of this chapter is to step back from the steady current and see the larger picture behind reactor design. As such, a more in-depth description of fixed and floating variables is given. The discussion of floating variables will then lend itself to describing the framework for this fusion systems model. As such, we will derive formulas for the radius and magnetic field of the tokamak. The current will remain a connecting piece while switching gears to pulsed tokamaks and comparing each operation's solving algorithm.

3.1 Explaining Fixed Variables

In this model, fixed variables are ones that remain constant while solving for a reactor. These include geometric scalings (i.e. ϵ, δ, κ), peaking parameters (i.e. ν_n, ν_T, l_i), and a slew of physics constants related to pulsed and steady-state design (e.g. Q, N_G, f_D). For a complete list of fixed variables, consult the appendix. Within this model, fixed variables are second-class variables. As such they often reside in fixed coefficients – $K_\square$ – which are treated as constants.

3.2 Connecting Floating Variables

Floating variables – $\overline{T}$, $\overline{n}$, I_P, R_0, B_0 – are first-class variables of this fusion systems model. They represent the fundamental properties of the plasma and tokamak (that constitute a fusion reactor). As such, they will be reintroduced one at a time, explaining how they fit into the model – and which equation is capable of representing it.

Bluntly, this fusion systems model is just a simple algebra problem: solve five equations with five unknowns (i.e. $\overline{T}$, $\overline{n}$, I_P, R_0, B_0). Although this naive approach would work, we can do a little better by wrangling these five equations down to just one. This was already done while deriving the steady current. It just happened that the current was not directly dependent on the tokamak size or magnet strength.

This will prove more challenging for the generalized current needed for pulsed operation. Even so, this equation will still be boiled down to one equation with a single unknown – I_P. A solution to which can be solved much faster than the naive 5 equation approach. This is one reason the model is so fast.

The Plasma Temperature – $\overline{T}$

The plasma temperature, measured in keV (kilo-electron-volts), is one of the most finicky variables in the fusion systems model. It first proved troublesome when it was shown that a pedestal profile – not a parabolic one used here – would be needed for an accurate calculation of bootstrap current. The unusual tabulation for reactivity – (σv) – used in fusion power only further exposed this nonlinearity.

Acknowledging that temperature is the most difficult to handle parameter prompts its use as the scanned variable. What this means practically is scanning temperatures produces curves of reactors. By example, a scan may be run over the average temperatures ($\overline{T}$): 10, 15, 20, 25, and 30 keV – each corresponding to its own reactor. In equation form, this becomes:

$$\overline{T} = const. \tag{3.1}$$

Where the constant is 10 in one run, 15 in the next, and 30 for the fifth one.

The Plasma Density – $\overline{n}$

The cornerstone of this fusion systems model has always been the application of the Greenwald density limit from square one. It is for this reason – as well as being a good approximation – that a parabolic profile was rationalized over a pedestal (H-Mode) one. Repeated, the Greenwald density limit is:

$$\overline{n} = K_n \cdot \frac{I_P}{R_0^2} \tag{3.2}$$

This is an exceptionally simple relationship and why it guided the model. Unlike the next three variables, it is actually used in their derivations. Therefore, any reactor found through this model is considered a *Greenwaldian Reactor* – one in H-Mode at the Greenwald density limit.

The Plasma Current – I_P

The plasma current is what separates steady-state from pulsed operation. From before, the steady current was found to be:

$$I_P = \frac{K_{BS}\overline{T}}{1 - K_{CD}(\sigma v)} \tag{3.3}$$

This was found by setting the total current equal to the two sources of current: bootstrap and current drive. In fractional form,

$$I_P = I_{BS} + I_{CD} \quad \rightarrow \quad 1 = f_{BS} + f_{CD} \tag{3.4}$$

This says that the current fractions of bootstrap and current drive must sum to one. As shown next chapter, inductive sources can be included in this flux balance:

$$1 = f_{BS} + f_{CD} + f_{IN} \tag{3.5}$$

This equation shows how steady-state and pulsed operation can coexist. The final point to make is reducing the model to being purely pulsed – i.e. neglecting the current drive:

$$1 = f_{BS} + f_{IN} \tag{3.6}$$

Therefore next chapter will generalize the steady current to allow pulsed operation. As steady current faced self-consistency issues with η_{CD}, this current will involve its own root solving problem – the description of which will be given in the following two chapters.

The Tokamak Magnet Strength – B_0

The tokamak magnet strength has no obvious equation to eliminate it. With foresight, the one this model chooses to use is power balance in a reactor. Similar to current balance, power balance is what separates a reactor from a toaster. As such, it is referred throughout this document as: the primary constraint. It will be derived later this chapter.

The Tokamak Major Radius – R_0

Much like the magnet strength, the major radius has no obvious relation to express it. This is convenient, because the model still has yet to resolve one of its most pressing issues: physical and engineering-based constraints. This laundry list of requirements further restricts reactor space to the curves show in the results section. Collectively, these are referred to as the secondary constraints – discussed later this chapter. By miracle, these constraints all just happen to depend on the size of the reactor.

3.3 Enforcing Power Balance

What separates a reactor from a toaster is power balance. It accounts for how the power going into a plasma's core exactly matches the power coming out of it. To approximate this conservation equation, two pairs of powers will be introduced: the sources and the sinks. The sources have already been introduced – they include the alpha power (P_α) and the heating power (P_H). The remaining two powers - the sinks – appear as the radiation and heat conduction losses. Both of which are defined shortly. In equation form, power balance becomes:

$$\sum_{sources} P = \sum_{sinks} P \tag{3.7}$$

or expanded to fit this model:

$$P_\alpha + P_H = P_{BR} + P_\kappa \tag{3.8}$$

For clarity, the left-hand side of this equality are the sources. The remaining two are the sinks: the Bremsstrahlung radiation (P_{BR}) and the heat conduction losses (P_κ).

3.3.1 Collecting Power Sources

As suggested, the two sources of power in a tokamak are: alpha power (P_α) and auxiliary heating (P_H). From earlier, it was determined that alpha particles (i.e. Helium) carried 20% of the total fusion power; or as we put it mathematically:

$$P_\alpha = \frac{P_F}{5} \tag{3.9}$$

Additionally, it was determined that heating power is what eventually amplified into fusion power, or in equation form:

$$P_H = \frac{P_F}{Q} \tag{3.10}$$

This allows the sources to be combined together with a new fixed coefficient:

$$P_{src} = P_\alpha + P_H = K_P \cdot P_F \tag{3.11}$$

$$K_P = \frac{5 + Q}{5 \times Q} \tag{3.12}$$

The two sink terms are described next.

3.3.2 Approximating Radiation Losses

All nuclear reactors emit radiation. From a power balance perspective, this means some power has to always be reserved to recoup from its losses – measured in megawatts. In a fusion reactor, the three most important types of radiation are: Bremsstrahlung radiation, line radiation, and synchrotron radiation. Without going into too much detail, this model chooses to only model Bremsstrahlung radiation – as it usually is the most dominant term in the plasma's core.

This radiation power term is therefore one area where the physics model could be greatly improved. Derived shortly, the Bremsstrahlung radiation is almost always the dominant source. However, adding the line-radiation would better handle impurities (i.e. when chunks of the tokamak fall into the plasma) and implementing synchrotron radiation would drive the values closer to real-world experiments (although a description of which is outside the scope of this text).

Bremsstrahlung – or breaking – radiation is what occurs when a charged particle (e.g. an electron) is decelerated (e.g. by changing angle). In a tokamak, this happens all the time as charged particles are spun around and around the machine. As given in Jeff Freidberg's book, this value can be written as:

$$P_{BR} = \int S_{BR}\, d\vec{r} \tag{3.13}$$

For clarity, this is a volume integral where the radiation power density (S_{BR}) is given by:

$$S_{BR} = \left(\frac{\sqrt{2}}{3\sqrt{\pi^5}} \cdot \frac{e^6}{\epsilon_0^2 c^3 h m_e^{3/2}} \right) \cdot \left(Z_{eff}\, n^2\, T^{1/2} \right) \tag{3.14}$$

The constants in the left set of parentheses all have their usual physics meanings (i.e. c is the speed of light and m_e is the mass of an electron). What is new is the effective charge: Z_{eff}.

The effective charge is a scheme for collapsing the charge that each particle has to a collective value. Fundamental charge, here, is what: neutrons lack, electrons and hydrogen have one of, and helium has two. As such, a plasma with a purely deuterium and tritium fuel would have an effective charge of one. This value would then quickly rise if a Tungsten tile – with 74 units of fundamental charge – were to fall from the inside of the tokamak into the plasma core.

Using the volume integral – seen in the derivation of fusion power – allows the Bremsstrahlung power to be written as standardized units as:

$$P_{BR} = K_{BR}\, \overline{n}^2\, \overline{T}^{1/2} R_0^3 \tag{3.15}$$

$$K_{BR} = 0.1056\, \frac{(1+\nu_n)^2\,(1+\nu_T)^{1/2}}{1 + 2\,\nu_n + 0.5\,\nu_T}\, Z_{eff}\, \epsilon^2\, \kappa\, g \tag{3.16}$$

This power term represents the radiation power losses in power balance. All that is needed now is heat conduction losses – the hardest plasma behavior to model to date.

3.3.3 Estimating Heat Conduction Losses

Heat is energy that lacks direction on a microscopic level. Macroscopically, it generally moves from hotter areas to colder ones. As hinted by the plasma profile for temperature, heat emanates from the center of the plasma – and migrates towards the inner surface of the tokamak. It therefore seems an important quantity to calculate when balancing power in a plasma's core.

The difficulty of estimating heat conduction, though, lies in the chaotic nature of plasmas – no theory or computation today can properly model it. As such reactor designers have turned to experimentalists for empirical scaling laws based on the $\sim$ 15 strongest tokamaks in the world. These collective are called confinement time scalings, i.e. the ELMy H-Mode Scaling Law.

The derivation of the heat conduction losses (P_κ) starts in a manner similar to the Bremsstrahlung radiation. To begin, an equation for P_κ is given in Jeff Freidberg's book as:

$$P_\kappa = \frac{1}{\tau_E} \int U d\vec{r} \tag{3.17}$$

This volume integral includes two new terms: the confinement time (τ_E) and the internal energy (U). Before explaining these terms, a qualitative description is in order. As mentioned previously, the heat – or microscopically random – energy is captured by the internal energy (U). Then the confinement time (τ_E) is how long it would take the heat to completely leave the device if the system were immediately turned off.

A formula for confinement time will be delayed to the end of this section, when it is needed to solve for the magnetic field (B_0). Moving forward, the internal energy, U, has its typical physics meaning – where all three species are assumed to be at the electron temperature (T):

$$U = \frac{3}{2}\,(n + n_D + n_T)\,T \tag{3.18}$$

Here again, n_D and n_T – the density of deuterium and tritium, respectively – are related to the electron density (used in this model) through the dilution factor, assuming a 50-50 mix of D-T fuel:

$$n_D = n_T = f_D \cdot \left(\frac{n}{2}\right) \tag{3.19}$$

Foregoing the mathematical rigor of previous sections, the equations here can be combined to form an equation for P_κ – the heat conduction losses – in standardized units:

$$P_\kappa = K_\kappa\,\frac{R_0^3\,\overline{n}\,\overline{T}}{\tau_E} \tag{3.20}$$

$$K_\kappa = 0.4744\,(1 + f_D)\,\frac{(1+\nu_n)\,(1+\nu_T)}{1+\nu_n+\nu_T}\,(\,\epsilon^2\,\kappa\,g\,) \tag{3.21}$$

Now that all four terms have been defined in power balance, the next step is expanding it and solving for the tokamak's toroidal magnetic field strength: B_0.

3.3.4 Writing the Lawson Criterion

Before locking in the primary constraint – i.e. the magnet strength (B_0) equation from power balance – it seems appropriate to take a detour and explain an intermediate solution: the Lawson Criterion. Within the fusion community, the Lawson Criterion is the cornerstone in any argument on the possibility of a design being used as a reactor (and not just some glorified toaster).

An equation for the Lawson Criterion is easily found in the literature as:

$$n\,\tau_E = \frac{60}{E_F} \cdot \frac{T}{\langle \sigma v \rangle} \tag{3.22}$$

Similar to the steady current derived earlier, the right-hand side is only dependent on temperature. Further, as the left-hand side is a measure of difficult to achieve values, the goal is to minimize both sides. This occurs when the plasma temperature is around 15 keV – a fact memorized by many fusion engineers. As will be seen, this is a simplified result of our mode. This is why $\overline{T} = 15$ keV is not always the optimum temperature – but usually is in the right neighborhood for reasonable reactor designs.

As all the terms in power balance have already been defined, the starting point will be simply repeating the standardized equations for all four powers.

$$P_\alpha = \frac{P_F}{5} \tag{3.9}$$

$$P_H = \frac{P_F}{Q} \tag{3.10}$$

$$P_{BR} = K_{BR}\,\overline{n}^2\,\overline{T}^{1/2} R_0^3 \tag{3.15}$$

$$P_\kappa = K_\kappa \frac{R_0^3\,\overline{n}\,\overline{T}}{\tau_E} \tag{3.20}$$

These can then be substituted into power balance:

$$P_\alpha + P_H = P_{BR} + P_\kappa \tag{3.23}$$

After a couple lines of algebra, power balance can be rewritten in a form analogous to the Lawson Criterion:

$$\overline{n}\,\tau_E = \frac{K_{law}\,\overline{T}}{(\sigma v) - K_{rad}\,\overline{T}^{1/2}} \tag{3.24}$$

$$K_{law} = \frac{K_\kappa}{K_{PF}} \tag{3.25}$$

$$K_{rad} = \frac{K_{BR}}{K_{PF}} \tag{3.26}$$

$$K_{PF} = K_P K_F \tag{3.27}$$

As can be seen, this is remarkably similar to the simple Lawson Criterion:

$$n\,\tau_E = \frac{60}{E_F} \cdot \frac{T}{\langle \sigma v \rangle} \tag{3.22}$$

The main difference is this model does not ignore radiation losses completely. The inclusion of which sets a minimum temperature for a reactor to be physically realizable.[1] With this intermediate relation in place, the goal is now to give a formula for the confinement time and solve it for the magnet strength (B_0) – the Primary Constraint.

3.3.5 Finalizing the Primary Constraint

The goal now is to transform the Lawson Criterion into an equation for magnet strength (B_0). This choice to solve the equation for B_0 was completely arbitrary, only motivated by the foresight of how it fits into the fusion systems model. To solve the primary constraint, the confinement time scaling law will need to be introduced. At the end, a messy – albeit highly useful – relation will be the reward.

[1] The denominator of Eq 3.24 has a discontinuity when the $K_{rad}\overline{T}^{1/2}$ term equals (σv). This temperature sets a barrier between real reactors (for temperatures higher than it) and impossible reactors – at or below it.

The energy confinement time – τ_E – is one of the most elusive terms in all of fusion energy. It is an attempt to boil all the chaotic nature of plasmas into a simple measure of how fast its internal energy would be ejected from the tokamak if the device was instantaneously shut off. As such, reactor designers have turned toward experimentalists for empirical scalings based on the world's tokamaks. These all share the form:

$$\tau_E = K_\tau \, H \, \frac{I_M^{\alpha_I} \, R_0^{\alpha_R} \, a^{\alpha_a} \, \kappa^{\alpha_\kappa} \, \overline{n}_{20}^{\alpha_n} \, B_0^{\alpha_B} \, A^{\alpha_A}}{(P_{src})^{\alpha_P}} \tag{3.28}$$

This mouthful of a formula is how the field actually designs machines. Let it be known, though, that this fit does a remarkable job having an error of less than 20% on interpolated data. The new terms in this equation are: K_τ, H, A, and the $\alpha_\square$ factors.

First, K_τ is simply a constant fit-makers use in their scalings. Next, H is the (H-Mode) scaling factor – the analogue of K_τ used by reactor designers – which can artificially boost the performance of machines (i.e. it helps with cheating). Continuing, A is the average mass number of the fuel source, in atomic mass units. For 50-50 D-T fuel, this is 2.5, as deuterium weighs two amus and tritium weights three. Lastly, the alpha factors (e.g. α_n, α_a, α_P) are fitting parameters that represent each variable's relative importance.

For ELMy H-Mode this confinement scaling can be written as:

$$\tau_E = 0.145 \, H \, \frac{I_M^{0.93} \, R_0^{1.39} \, a^{0.58} \, \kappa^{0.78} \, \overline{n}_{20}^{0.41} \, B_0^{0.15} \, A^{0.19}}{(P_{src})^{0.69}} \tag{3.29}$$

One final remark to make before moving on is that even these fits have subtleties. The value of κ, for example, may have a slightly different geometric meaning. And the exact definition of source power – P_{src} – introduces an even larger area of discrepancy.

Returning to the problem at hand, this model's Lawson Criterion (eq. 3.24) can be simplified after expanding the left-hand side – $\overline{n}\tau_E$ using the Greenwald density and a confinement time scaling law. Albeit a little cumbersome, this can be wrangled into an equation for B_0!

$$B_0 = \left(\frac{G_{PB}}{K_{PB}} \cdot \left(I_P^{\alpha_I^*} \, R_0^{\alpha_R^*} \right)^{-1} \right)^{\frac{1}{\alpha_B}} \tag{3.30}$$

$$G_{PB} = \frac{\overline{T} \cdot (\sigma v)^{\alpha_P}}{(\sigma v) - K_{rad} \, \overline{T}^{1/2}} \tag{3.31}$$

$$K_{PB} = H \cdot \left(\frac{K_\tau}{K_{law}} \right) \cdot \frac{(K_G)^{\alpha_n^*}}{(K_{PF})^{\alpha_P}} \cdot \left(\epsilon^{\,\alpha_a} \, \kappa^{\,\alpha_\kappa} \, A^{\,\alpha_A} \right) \tag{3.32}$$

Where we have added new starred alpha values for the density, current, and major radius:

$$\alpha_n^* = 1 + \alpha_n - 2\alpha_P \tag{3.33}$$

$$\alpha_I^* = \alpha_I + \alpha_n^* \tag{3.34}$$

$$\alpha_R^* = \alpha_R + \alpha_a + \alpha_P - 2(1 + \alpha_n) \tag{3.35}$$

This equation for B_0 – derived from power balance – is the primary constraint of our reactor designs. It is the first step in connecting the plasma (i.e. $\overline{n}$, $\overline{T}$, and I_P) with the tokamak (i.e. B_0 and R_0). The remaining step is finding an equation – or in this case, equations – for the major radius of the device. These equations will collectively be referred to as: the Secondary Constraints.

3.4 Collecting Secondary Constraints

As of now, the only missing equation within our list of fixed variables – i.e. R_0, B_0, $\overline{T}$, $\overline{n}$, and I_P – is for the major radius of the tokamak. This equation will come from around five potential limits, either physical or engineering-based. Each limit will correspond to its own curve in reactor space. As will be shown, many of these reactors will be invalid (as they violate each other's secondary constraints).

Before tackling teh subject of finding reactors that exist on the fine line of satysfying all secondary constraints, it is essential to collect them one-by-one. These are: the Troyon Beta Limit, the Kink Safety Factor, the Wall Loading Limit, the Power Cap Constraint, and the Heat Loading Limit. The goal of this section is to solve each of these constraints for the major radius. As with the primary constraint, this choice of solving for R_0 was completely arbitrary. It just so happens that each limit described here depends on the size of a reactor.

3.4.1 Introducing the Beta Limit

The Beta Limit is the most important secondary constraint – especially for steady-state reactors. It sets a maximum on the amount of pressure a plasma is willing to tolerate. As with future secondary constraints, literature-based equations will be transformed into formulas for R_0, each with some limiting quantity that can be handled as a fixed variable – as β_N will be used shortly.

The starting point for the beta limit is to define the important plasma physics quantity: β – the plasma beta. This value is a ratio between a plasma's internal pressure and the pressure exerted on it by the tokamak's magnetic configuration. Mathematically,

$$\beta = \frac{\text{plasma pressure}}{\text{magnetic pressure}} = \frac{\overline{p}}{\left(\frac{B_0^2}{2\mu_0}\right)} \tag{3.36}$$

Using this model's temperature and density profiles, the volume-averaged pressure ($\overline{p}$) can be written in units of atmospheres (i.e. atm) as:

$$\overline{p} = 0.1581\,(1 + f_D)\,\frac{(1+\nu_n)\,(1+\nu_T)}{1+\nu_n+\nu_T}\,\overline{n}\,\overline{T} \tag{3.37}$$

Moving forward, the final step is plugging this defniiton for plasma beta into the physics-based Troyon Beta Limit. Although outside the scope of this text, it is a stability limit set by treating plasmas as a charge-carrying fluid. This equation can be written in the following form, where β_N is the normalized plasma beta – i.e. a fixed variable in this model usually set between 2% and 4%.

$$\beta = \beta_N \frac{I_P}{aB_0} \tag{3.38}$$

Substituting the plasma β from eq. 3.36, into this relation results in the model's first equation for tokamak radius:

$$R_0 B_0 = K_{TB}\overline{T} \tag{3.39}$$

$$K_{TB} = 4.027e{-}2\,(K_G)\left(\frac{\epsilon}{\beta_N}\right)(1+f_D)\,\frac{(1+\nu_n)\,(1+\nu_T)}{1+\nu_n+\nu_T} \tag{3.40}$$

As mentioned, this is often the dominating constraint in a steady-state reactor. The dominating constraint for pulsed designs – the kink safety factor – will be the focus of the next subsection.

3.4.2 Giving the Kink Safety Factor

Just like how the Troyon Beta Limit set a fluids-based maximum on plasma pressure, the Kink Safety Factor sets one for a plasma's current. This constraint usually only appears in pulsed designs, as it is assumed that getting to this current in steady-state (with only LHCD) would prove extremely unpractical.

The starting point again is an equation from the literature for the kink condition:

$$q_{95} = 5\epsilon^2 f_q \cdot \frac{R_0 B_0}{I_P} \tag{3.41}$$

Here the safety factor – q_{95} – is subscripted by 95, an identifier that this value is taken at the 95% flux surface (i.e. near the statistically drawn edge of the plasma). It typically has values around 3. Next, the f_q variable is a geometric scaling factor:

$$f_q = \frac{1.17 - 0.65\epsilon}{2(1-\epsilon^2)^2} \cdot \left(1 + \kappa^2 * (1 + 2\delta^2 - 1.2\delta^3)\right) \tag{3.42}$$

Combined, the kink safety factor can now be written in standardized units as:

$$R_0 = \frac{K_{SF} I_P}{B_0} \tag{3.43}$$

$$K_{SF} = \frac{q_{95}}{5\epsilon^2 f_q} \tag{3.44}$$

This relation is the secondary constraint important for most pulsed reactor designs. As with the Beta Limit, the two are derived by plasma physics alone. The remaining secondary constraints, however, are engineering-based in origin – these include: the Wall Loading Limit, the Power Cap Constraint, and the Heat Loading Limit. Each will be defined shortly.

3.4.3 Working under the Wall Loading Limit

The first engineering-based secondary constraint – the wall loading limit – will prove to be an important quantity when determining the magnet strength at which reactor cost first starts to increase. As hinted, its definition is nuclear engineering in origin: it is a measure of the maximum neutron damage a tokamak's walls can take over the lifetime of the machine.

The first step in deriving a secondary constraint for wall loading is a description of the problem it models. In a reactor, fusion reactions typically make high-energy neutrons – at 14.1 MeV – that continually blast the inner wall of the tokamak. Therefore a quick-and-dirty metric would be limiting the amount of neutron power that can be unloaded on the surface area of a tokamak. This can be written as:

$$P_W = \frac{P_n}{S_P} \tag{3.45}$$

Here, S_P is the surface area of the tokamak's inner wall and P_n is the neutron power derived in the subsection on fusion power. The quantity, P_W, then served a role analogous to β_N for the beta limit and q_{95} for the kink safety factor – it is a fixed variable representing the maximum allowed wall loading. For fusion reactors, P_W is assumed to be around 2-4 $\frac{\mathrm{MW}}{\mathrm{m}^2}$. It will be shown that the wall loading limit is important in any tokamak – regardless of operating mode (i.e. steady-state or pulsed).

$$S_P = 4\pi^2 a_P R_0 \cdot \frac{\left(1 + \frac{2}{\pi}\left(\kappa_P^2 - 1\right)\right)}{\kappa_P} \tag{3.46}$$

With the surface area dimensions being subscripted by P's.

$$a_P = 1.04\, a \tag{3.47}$$

$$\kappa_P = 1.3\, \kappa \tag{3.48}$$

$$\epsilon_P = \frac{a_P}{R_0} \tag{3.49}$$

Finishing this secondary constraint, the Wall Loading limit can be written in standardized units as:

$$R_0 = K_{WL} \cdot I_P^{\frac{2}{3}} \cdot (\sigma v)^{\frac{1}{3}} \tag{3.50}$$

$$K_{WL} = \left(\frac{K_F K_n^2}{5\pi^2 P_W} \cdot \frac{\kappa_P}{\epsilon_P} \cdot \frac{1}{1 + \frac{2}{\pi} \cdot (\kappa_P^2 - 1)} \right)^{\frac{1}{3}} \tag{3.51}$$

3.4.4 Setting a Maximum Power Cap

As opposed to the previous three secondary constraints, the maximum power cap is more a rule of thumb. Because no reactor – coal, solar, or otherwise – has a 2500 MW reactor, neither should fusion. It makes sense from a practical position after realizing the long history of tokamaks being delayed, underfunded, or completely canceled. Mathematically, this has the simple form:

$$P_E \leq P_{CAP} \tag{3.52}$$

Here, P_{CAP} is the maximum allowed power output of the reactor. Similar to the other limiting quantities, P_{CAP} is treated as a fixed variable (i.e. set to 2500 MW). The electrical power output of the reactor (P_E) is then related to the fusion power through:

$$P_E = 1.273\, \eta_T \cdot P_F \tag{3.53}$$

The constant in front (1.273) represents some extra power the reactor makes as more fuel is bred when the fusion neutrons pass through a tokamak (inside its blanket). The variable η_T is the thermal efficiency of the reactor – usually around 40%.

Next, substituting in for fusion power and solving for the tokamak's major radius results in:

$$R_0 = K_{PC} \cdot I_P^2 \cdot (\sigma v) \tag{3.54}$$

$$K_{PC} = K_F K_n^2 \cdot \left(\frac{1.273\,\eta_T}{P_{max}} \right) \tag{3.55}$$

This secondary constraint can be used to create curves of reactors, although it is mainly used as a stopping point for designs – i.e. if you get to the power-cap regime, you have gone too far. This is different then the next constraint, which is basically a glorified warning sign in the contemporary tokamak paradigm.

3.4.5 Listing the Heat Loading Limit

Plasmas are hot. The commonly given fact is one electron volt is around 20,000 °F. Although a bit deceptive, melting a tokamak is an all too real concern. The problem is there is currently no solution to this problem. Although researchers have explored various types of heat divertors, none have been shown to withstand the gigawatts of heat emitted from a reactor-size tokamak. Further, as it is not as glamorous as plasma physics, attempts to tackle the problem head-on have often gone unfunded.

As such, this model takes the approach that we are no worse than the rest of the field. We almost completely ignore the heat load limit and just refer to it at the end, saying "and then the magical divertor will have to deal with solar corona levels of heat." After which, discussion will quickly be redirect to happier concerns.

For thoroughness though, a secondary constraint will still be derived. The first step is giving the heat load limit commonly found in the literature:

$$q_{DV} = K_{DV} \frac{P_F\, I_P^{1.2}}{R_0^{2.2}} \tag{3.56}$$

$$K_{DV} = \frac{18.31e{-3}}{\epsilon^{1.2}} \cdot K_P \cdot \left(\frac{2}{1+\kappa^2} \right)^{0.6} \tag{3.57}$$

After a simple rearrangement and substitution for fusion power, this becomes:

$$R_0 = K_{DH} \cdot I_P \cdot (\sigma v)^{\frac{1}{3.2}} \tag{3.58}$$

$$K_{DH} = \left(\frac{K_{DV} K_F K_n^2}{q_{DV}} \right)^{\frac{1}{3.2}} \tag{3.59}$$

At this point all the secondary constraints have been defined. The next step is taking a step back and motivating the derivation of a current equation suitable for pulsed tokamaks.

3.5 Summarizing the Fusion Systems Model

This chapter focused on the bigger picture behind designing a zero-dimensional fusion systems model. It started with a description of various design parameters and then segued into explaining the five relations needed to close the model – i.e. for $\overline{T}$, $\overline{n}$, I_P, B_0, and R_0.

Before moving onto generalizing the steady current to model pulsed reactors, a quick recap of the equations will prove beneficial. The first variable tackled was temperature – i.e. scan five evenly-space $\overline{T}$ values between 10 and 30 keV. This was then quickly followed by the Greenwald density limit – the cornerstone of this framework. Through equations, these two can be written as:

$$\overline{T} = const. \tag{3.1}$$

$$\overline{n} = K_n \cdot \frac{I_P}{R_0^2} \tag{3.2}$$

The next variable handled was the steady current:

$$I_P = \frac{K_{BS}\overline{T}}{1 - K_{CD}(\sigma v)} \tag{3.3}$$

As was mentioned before, this only directly depends on temperature, but is strongly affected by a tokamak's configuration – R_0 and B_0 - through the current drive efficiency (η_{CD}). For pulsed reactors, this equation proves too simple as it ignores inductive current. To remedy this situation, current balance will be revisited next chapter. The main point to make now is the R_0 and B_0 dependence will now be explicit.

Moving on, the remaining equations were the primary and secondary constraints for B_0 and R_0, respectively. It was through these relations that a tokamak's configuration was brought into the fold. The choice of solving the two constraints for their respective variables was completely arbitrary – motivated only by foresight of how they fit into the model. Without repeating them now, they served as the proper vehicles for closing the system of equations. The next step now is to learn how to generalize the current formula and design a pulsed tokamak.

Chapter 4

Designing a Pulsed Tokamak

Pulsed tokamaks are the flagship of the European fusion effort. As such, this paper's model will now be generalized to accommodate this mode of operation. Fundamentally, this involves transforming current balance into flux balance – adding inductive (pulsed) sources to stand aside the LHCD (steady-state) ones.

The first step in generalizing current balance will be understanding the problem from a basic electrical engineering perspective – i.e. with circuit analysis. The resulting equation will then be transformed into flux balance as seen in other models from the literature. All that will need to be done then is solving the problem for plasma current (I_P) and simplifying it for various situations – e.g. steady-state operation.

4.1 Modeling Plasmas as Circuits

Although it may have been lost along the way, what makes plasmas so interesting and versatile – in comparison to gases – is their ability to respond to electric and magnetic fields. It seems natural then to model plasma current from a circuit perspective (i.e. with resistors, voltage sources, and inductors). By name, this circuit is referred to as a transformer where: the plasma is the secondary and the yet-to-be discussed central solenoid (of the tokamak) is the primary.

The first step in deriving a current equation is determine the circuit equations governing pulsed operation in a tokamak. This will be done in two steps. First, drawing the circuit diagram and writing the equations that describe them. Next, a schematic of how the current in the transformer changes with time will allow us to boil these differential equations into simple algebra ones – as is the hallmark of this model.

4.1.1 Drawing the Circuit Diagram

Understanding a circuit always starts with a circuit diagram (see fig. N). This shows the transformer governing reactor pulses. The left sub-circuit is the transformer's primary – the central solenoid component of the tokamak that provides most of the inductive current. This central solenoid is a slinky-shaped metal coil that fits within the inner ring of the doughnut. The right sub-circuit is then the plasma acting as the transformer's secondary. For now, every other source of flux (besides the central solenoid) is neglected.

Hopefully without scaring the reader too much, the circuit equations – when only modeling voltage sources, resistors, and inductors – is:

$$V_i = \sum_j^n \frac{d}{dt}\left(M_{ij} I_j\right) + I_i R_i \ , \quad \forall\, i = 1, 2, .., n \tag{4.1}$$

Without going into the inductances (M) and resistances (R), the variable n is the number of sub-circuits, here being 2. Whereas, the variables i and j are the indices of sub-circuits (i.e. 1 for the primary, 2 for the secondary). For illustrative purposes, this would boil down to the following relation for a battery attached to a lightbulb:

$$V = IR \tag{4.2}$$

Back to the transformer diagram, the equations for the two can be expanded and greatly simplified. Besides ignoring every inductive source other than the central solenoid, the next powerful assumption is treating the solenoid as a superconductor (i.e. with negligible resistant). Lastly, the inductances between components and themselves are held constant – independent of time. This allows the coupled transformer equations to be written as:

$$V_1 = L_1\dot{I}_1 - M\dot{I}_2 \tag{4.3}$$

$$-I_2R_2 = L_2\dot{I}_2 - M\dot{I}_1 \tag{4.4}$$

With I_1 and I_2 going in opposite directions. Note that the subscripts on M have been dropped, as there are only two components. This was done in conjunction to adding internal (self-)inductance terms. Mathematically, the mapping between variables is:

$$M = M_{12} = M_{21} \tag{4.5}$$

$$L_1 = M_{11} \tag{4.6}$$

$$L_2 = M_{22} \tag{4.7}$$

Repeated, the one subscript represents the primary – the central solenoid – and the two stands for the plasma as the transformer's secondary. Exact definitions for the inductances will be put off till the end of the next subsection.

4.1.2 Plotting Pulse Profiles

Up till now, little has been discussed that has a time dependence. For steady-state tokamaks, this did not occur because it is an extreme case where pulses basically last the duration of the machine's lifespan (i.e. around 50 years). By definition, pulsed machines have pulses – with nearly ten schedules per day. For this reason, a fusion pulse is now investigated in detail.

Transformer pulses between the central solenoid and the plasma occur on the timescale of hours. For clarity, each pulse is subdivided into four phases: ramp-up, flattop, ramp-down, and dwell. Pictorially represented in fig. N, these divisions allow a simple scheme for transforming the coupled circuit differential equations – from **????** – into simple algebraic formulas.

Along the way, we will approximate derivatives with linear piecewise functions. Using t_i to represent the initial time and t_f the final one, this can be written as:

$$\dot{I} = \frac{I(t_f) - I(t_i)}{t_f - t_i} \tag{4.8}$$

The Ramp-Up Phase – RU

The first phase in every plasma pulse is the ramp-up. During ramp-up, the central solenoid starts discharging from its fully charged values, as the plasma is brought to a quasi-steady-state. As this occurs on the timescale of minutes – not hours – resistive effects of the plasma can safely be ignored. This results in the ramp-up equations becoming:

$$V_{max} = \frac{1}{\tau_{RU}} \cdot \Big(L_1 \cdot (I_{max} - \tilde{I}) - M \cdot I_{ID} \Big) \tag{4.9}$$

$$0 = \frac{1}{\tau_{RU}} \cdot \Big(M \cdot (I_{max} - \tilde{I}) - L_2 \cdot I_{ID} \Big) \tag{4.10}$$

Simplifying these equations will be done shortly, for now the new terms are what is important. The maximum voltage of the solenoid is V_{max}. Then, I_{max} is the solenoid's current at the beginning of ramp-up, whereas $\tilde{I}$ is the magnitude of its current once the plasma is at its flattop inductive-drive current – I_{ID}. Next, the τ_{RU} quantity is the duration of time it takes to ramp-up (i.e. RU). Again, L_1 and L_2 are the internal inductances of the solenoid and plasma, respectively, and M is the mutual inductance between them.

The last step in discussing ramp-up is giving the two important formulas that come from it:

$$\tilde{I} = I_{max} - I_{ID} \cdot \left(\frac{L_2}{M} \right) \tag{4.11}$$

$$\tau_{RU} = \frac{I_{ID}}{V_{max}} \cdot \left(\frac{L_1 L_2 - M^2}{M} \right) \tag{4.12}$$

The Flattop Phase – FT

The most important phase in any reactor's pulse is flattop – the quasi-steady-state time when the tokamak is making electricity (and money). Flattops are assumed to last a couple of hours for a profitable machine, during which the central solenoid completely discharges to overcome a plasma's resistive losses – keeping it in a quasi-steady-state mode of operation. In a steady-state reactor, this phases constitutes the entirety of the pulse.

Although the resistance cannot be safely neglected for flattop – as it was for ramp-up – the plasma's inductive current (I_{ID}) is assumed constant. This leads to its derivative in equations cancelling out! Mathematically,

$$\tilde{V} = \frac{L_1}{\tau_{FT}} \cdot \left(I_{max} + \tilde{I} \right) \tag{4.13}$$

$$I_{ID} R_2 = \frac{M}{\tau_{FT}} \cdot \left(I_{max} + \tilde{I} \right) \tag{4.14}$$

As with ramp-up, the simplifications will be given shortly. The new terms here, however, are the plasma's resistance (R_2), an intermediate voltage for the central solenoid ($\tilde{V}$), and the duration of the flattop (τ_{FT}). Solutions can then be found by substituting $\tilde{I}$ into the flattop equations:

$$\tilde{V} = I_{ID} R_2 \cdot \left(\frac{L_1}{M} \right) \tag{4.15}$$

$$\tau_{FT} = \frac{I_{max} \cdot 2M - I_{ID} \cdot L_2}{I_{ID} R_2} \tag{4.16}$$

The Ramp-Down Phase – RD

Due to the simplicity – and symmetry – of the reactor pulse, ramp-down is the exact mirror of ramp-up. It takes the same amount of time and results in the same algebraic equations. For brevity, this will just be represented as:

$$\tau_{RD} = \tau_{RU} \tag{4.17}$$

The Dwell Phase – DW

Where the first three phases had little ambiguity, the dwell phase changes definition from model to model. For now, it is assumed to be the time it takes the central solenoid to reset after a plasma has been completely ramped-down to an off-mode. Later, to get a realistic duty factor for cost estimates, it will include an evacuation time which can last around thirty minutes. During this evacuation, a plasma is vacuumed out of a device as it undergoes some inter-pulse maintenance.

Ignoring evacuation for now, the dwell phase involves resetting the central solenoid when the plasma's current is negligible. This fundamentally means the secondary of the transformer is nonexistent – the central solenoid is the entire circuit. In equation form,

$$V_{max} = \frac{L_1}{\tau_{DW}} \cdot \left(I_{max} + \tilde{I} \right) \tag{4.18}$$

Or substituting in $\tilde{I}$ and solving for τ_{DW},

$$\tau_{DW} = \frac{L_1}{M} \cdot \frac{(I_{max} \cdot 2M - I_{ID} \cdot L_2)}{V_{max}} \tag{4.19}$$

4.1.3 Specifying Circuit Variables

The goal now is to collect the results from the four phases and introduce the inductance, resistance, voltage, and current terms relevant to this model. This will motivate recasting the problem as flux balance in a reactor – the form commonly used in the literature (discussed next section).

First, collecting the phase durations in one place:

$$\tau_{RU} = \frac{I_{ID}}{V_{max}} \cdot \left(\frac{L_1 L_2 - M^2}{M}\right) \tag{4.12}$$

$$\tau_{FT} = \frac{I_{max} \cdot 2M - I_{ID} \cdot L_2}{I_{ID} R_2} \tag{4.16}$$

$$\tau_{RD} = \tau_{RU} \tag{4.17}$$

$$\tau_{DW} = \frac{L_1}{M} \cdot \frac{(I_{max} \cdot 2M - I_{ID} \cdot L_2)}{V_{max}} \tag{4.19}$$

These can be used in the definition of the duty-factor: the percentage of time a reactor is putting electricity on the grid. Formulaically,

$$f_{duty} = \frac{\tau_{FT}}{\tau_{RU} + \tau_{FT} + \tau_{RD} + \tau_{DW}} \tag{4.20}$$

As will turn out, the solving of pulsed current actually only involves **??**. What is interesting about this, is it has no explicit dependence on ramp-down or dwell! Whereas ramp-up passes $\tilde{I}$ to the flattop phase, the other two are just involved in calculating the duty factor.

The remainder of this subsection will then be defining the following circuit variables:

I_{ID}, I_{max}, V_{max}, L_1, L_2, M, and R_2.

The Inductive Current – I_D

The inductive current is the source of current that separates pulsed from steady-state operation. Quickly fitting it into the previous definitions of current balance:

$$I_{ID} = I_P - (I_{BS} + I_{CD}) \tag{4.21}$$

As before, I_P is the total plasma current in mega-amps, I_{BS} is the bootstrap current, and I_{CD} is the current from LHCD (i.e. lower hybrid current drive).

The Central Solenoid Maximums – V_{max} and I_{max}

For this simple model, the central solenoid has two maximum values: the voltage and current. The voltage is the easier to give value. Literature values have this around:

$$V_{max} \approx 5\,\mathrm{kV} \tag{4.22}$$

The maximum current, on the other hand, can be defined through Ampere's Law on a helically-shaped central solenoid:

$$I_{max} = \frac{B_{CS} h_{CS}}{N \mu_0} \tag{4.23}$$

Here, B_{CS} is a magnetic field strength the central solenoid is assumed to operate at (i.e. 20 T), h_{CS} is the height of the solenoid, N is the number of loops, and μ_0 has its usual physics meaning $\left(\text{i.e. } 40\,\pi\,\frac{\mu\mathrm{H}}{\mathrm{m}}\right)$. As will be seen, the value of N does not affect the model, as it cancels out in the final flux balance. The height of the central solenoid will be the focus of a future section on an in-depth look at tokamak geometry.

The Central Solenoid Inductance – L_1

For a central solenoid with circular cross-sections of finite thickness (d), the inductance can be written as:

$$L_1 = G_{LT} \cdot \left(\frac{\mu_0 \pi N^2}{h_{CS}} \right) \tag{4.24}$$

$$G_{LT} = \frac{R_{CS}^2 + R_{CS} \cdot (R_{CS} + d) + (R_{CS} + d)^2}{3} \tag{4.25}$$

Note that R_{CS} is the inner radius of the central solenoid and $(R_{CS}+d)$ is the outer one. In the limit where d is negligible, this says the inductance is quadratically dependent on the radius of the solenoid, as

$$\lim_{d \to 0} G_{LT} = R_{CS}^2 \tag{4.26}$$

The formulas for both R_{CS} and d will be defined in a few sections.

The Plasma Inductance – L_2

The plasma inductance is a composite of several different terms, but overall scales with radius. Through equation,

$$L_2 = K_{LP} R_0 \tag{4.27}$$

This fixed coefficient – K_{LP} – then combines three inductive behaviors of the plasma. The first is its own self inductance (through l_i). The next is resistive component through the Ejima coefficient, C_{ejima}, usually set to around $\frac{1}{3}$. And lastly, a geometric component – involving ϵ and κ – given by the Hirshman-Neilson model. Mathematically,

$$K_{LP} = \mu_0 \cdot \left(\frac{l_i}{2} + C_{ejima} + \frac{(b_{HN} - a_{HN})\,(1-\epsilon)}{(1-\epsilon) + \kappa\, d_{HN}} \right) \tag{4.28}$$

Here the HN values come from Hirshman-Neilson:

$$a_{HN}(\epsilon) = 2.0 + 9.25\sqrt{\epsilon} - 1.21\,\epsilon \tag{4.29}$$

$$b_{HN}(\epsilon) = \ln(8/\epsilon) \cdot (1 + 1.81\sqrt{\epsilon} + 2.05\,\epsilon) \tag{4.30}$$

$$d_{HN}(\epsilon) = 0.73\sqrt{\epsilon} \cdot (1 + 2\epsilon^4 - 6\epsilon^5 + 3.7\epsilon^6) \tag{4.31}$$

The Mutual Inductance – M

The mutual inductance – M – represents the coupling between the solenoid primary and the plasma secondary. A common method for treating this mutual inductance is through a coupling coefficient, k, that links the two self-inductances. Formulaically,

$$M = k\sqrt{L_1 L_2} \tag{4.32}$$

The value of the coupling coefficient, k, is always less than (or equal to) 1, but usually has a value around one-third.

The Plasma Resistance – R_2

Unlike any material humans encounter on a daily basis, plasmas have a resistance that decreases with temperature. The fusion systems model handles this R_2 variable with the neoclassical Spitzer resistivity. Quickly noting, this is one area where theory does a remarkable job predicting experimental values. Through equation,

$$R_2 = \frac{K_{RP}}{R_0 \overline{T}^{3/2}} \tag{4.33}$$

$$K_{RP} = 5.6e{-}8 \cdot \left(\frac{Z_{eff}}{\epsilon^2 \kappa}\right) \cdot \left(\frac{1}{1 - 1.31\sqrt{\epsilon} + 0.46\epsilon}\right) \tag{4.34}$$

With all the equations defined, we are now at a position to first take a step back and rearrange terms to make flux balance, and then solve and simplify the resulting equation for various plasma current – I_P – formulas.

4.2 Salvaging Flux Balance

The goal of this section is to arrive at a conservation equation for flux balance that mirrors the ones in the literature. The fusion systems model this one attempts to follow most is the PROCESS code. In a manner similar to power balance, flux balance can be written as:

$$\sum_{sources} \Phi = \sum_{sinks} \Phi \tag{4.35}$$

4.2.1 Rearranging the Circuit Equation

The way to arrive at flux balance from the circuit equation is to rearrange the flattop phase's duration equation:

$$\tau_{FT} = \frac{I_{max} \cdot 2M - I_{ID} \cdot L_2}{I_{ID} R_2} \tag{4.16}$$

Multiplying by the right-hand side's denominator and moving the negative term over yields:

$$2MI_{max} = I_{ID} \cdot (L_2 + R_2 \tau_{FT}) \tag{4.36}$$

This equation is flux balance, where the left-hand side are the sources (e.g. the central solenoid), and the other terms are the sinks (i.e. ramp-up and flattop). The source term can currently be encapsulated in:

$$\Phi_{CS} = 2MI_{max} \tag{4.37}$$

The sinks, namely the ramp-up inductive losses (Φ_{RU}) and the flattop resistive losses (Φ_{FT}) losses, are what drain up the flux. Again, ramp-down and dwell are not included as sinks because flux balance only tracks till the end of flattop. They come into play when measuring the cost of electricity – with the duty factor from **??**.

Relabeling terms, flux balance now has the form:

$$\Phi_{CS} = \Phi_{RU} + \Phi_{FT} \tag{4.38}$$

With the ramp-up and flattop flux given respectively by:

$$\Phi_{RU} = L_2 \cdot I_{ID} \tag{4.39}$$

$$\Phi_{FT} = (R_2\tau_{FT}) \cdot I_{ID} \tag{4.40}$$

On comparing these quantities to the ones from the PROCESS team, Φ_{RU} and Φ_{FT} are exactly the same. The source terms, on the other hand, are off for two reasons – both related to the central solenoid being the only source term in flux balance. This can be partially remedied by adding the second most dominant source of flux a posteriori – i.e. the PF coils. The second, and inherently limiting factor, is the simplicity of the current model. All that can be show to that regard is the Φ_{CS} terms do reasonably predict the values from PROCESS (see the results chapter).

4.2.2 Importing Poloidal Field Coils

Adding the effect of PF coils – belts of current driving plates on the outer edges of the tokamak – leads to a second-order improvement on relying solely on the central solenoid for flux generation. From the literature, this can be modeled as so:

$$\Phi_{PF} = \pi B_V \cdot \left(R_0^2 - (R_{CS} + d)^2\right) \tag{4.41}$$

Where again R_{CS} and d are the inner radius and thickness of the central solenoid, respectively. These will be the topic of the next section.

Moving forward, the vertical field – B_V – is a magnetic field oriented up-and-down with the ground. It is needed to prevent a tokamak plasma from spinning out of the machine. From the literature, the magnitude of this vertical field is given by:

$$|B_V| = \frac{\mu_0 I_P}{4\pi R_0} \cdot \left(\ln\left(\frac{8}{\epsilon}\right) + \beta_p + \frac{l_i}{2} - \frac{3}{2} \right) \tag{4.42}$$

Analogous to the previously covered plasma beta, the poloidal beta can be represented by:

$$\beta_p = \frac{\overline{p}}{\left(\frac{\overline{B_p}^2}{2\mu_0}\right)} \tag{4.43}$$

Where the average poloidal magnetic field comes from a simple application of Ampere's law:

$$\overline{B_p} = \frac{\mu_0 I_P}{l_p} \tag{4.44}$$

The variable l_p is then the perimeter of the tokamak's cross-sectional halves:

$$l_p = 2\pi a \cdot \sqrt{g_p} \tag{4.45}$$

Here, g_p is another geometric scaling factor,

$$g_p = \frac{1 + \kappa^2(1 + 2\delta^2 - 1.2\delta^3)}{2} \tag{4.46}$$

Boiled down, this relation for the magnitude of the vertical magnetic field can be written in standardized units as:

$$|B_V| = \left(\frac{1}{10 \cdot R_0}\right) \cdot \left(K_{VI} I_P + K_{VT} \overline{T}\right) \tag{4.47}$$

$$K_{VT} = K_n \cdot (\epsilon^2 \, g_P) \cdot (1 + f_D) \, \frac{(1 + \nu_n)\,(1 + \nu_T)}{1 + \nu_n + \nu_T} \tag{4.48}$$

$$K_{VI} = \ln\left(\frac{8}{\epsilon}\right) + \frac{l_i}{2} - \frac{3}{2} \tag{4.49}$$

For clarity, this will be plugged into the new PF coil flux contribution (Φ_{PF}):

$$\Phi_{PF} = \pi B_V \cdot \left(R_0^2 - (R_{CS} + d)^2\right) \tag{4.41}$$

Which then gets plugged into a more complete flux balance:

$$\Phi_{CS} + \Phi_{PF} = \Phi_{RU} + \Phi_{FT} \tag{4.50}$$

The R_{CS} and d terms found in Φ_{PF} will now be discussed as they are needed for this more sophisticated tokamak geometry.

4.3 Improving Tokamak Geometry

From before, this fusion systems model has been said to depend on the major and minor radius – R_0 and a, respectively – and along the way, various geometric parameters have been defined (e.g. ϵ, κ, δ) to describe the geometry further. Now three more thicknesses will be added: b, c, and d. Additionally, two fundamental dimension corresponding to the solenoid will be added: the radius and height – R_{CS} and h_{CS}, respectively. These are the topics of this section.

4.3.1 Defining Central Solenoid Dimensions

The best way to conceptualize tokamak geometry is through cartoon – see fig. N. What this says is there is a gap at the very center of a tokamak. This gap extends radially outwards to R_{CS} meters where the slinky-shaped central solenoid – of thickness d – begins. Between the outer edge of the solenoid and the wall of the torus (i.e. the doughnut) are the blanket and toroidal field (TF) coils.

The blanket and TF coils have thicknesses of b and c, respectively. Before defining b, c, and d, it proves fruitful to relate all the quantities in equations for the inner radius (R_{CS}) and height (h_{CS}) of the central solenoid.

$$R_{CS} = R_0 - (a + b + c + d) \tag{4.51}$$

$$h_{CS} = 2 \cdot (\kappa a + b + c) \tag{4.52}$$

Again, this relation is pictorially represented in fig. N. The next step is defining: b, c, and d – to close the variable loop.

4.3.2 Measuring Component Thicknesses

In between the inner surface of the central solenoid and the major radius of the tokamak are four thicknesses: a, b, c, and d. This subsection will go over them one-by-one.

The Minor Radius – a

The minor radius was the first of these thicknesses we encountered. To calculate it, we introduced the inverse aspect ratio (ϵ) to relate it to the major radius (R_0):

$$a = \epsilon \cdot R_0 \tag{4.53}$$

The Blanket Thickness – b

The blanket is an area between the TF coils and the torus that is strongly composed of lithium. It serves to both: protect the superconducting magnet structures from neutron damage, as well as breed a little more tritium fuel from stray fusion neutrons. In equation form, the blanket thickness is given by:

$$b = 1.23 + 0.074 \ln P_W \tag{4.54}$$

Here, the constant term (i.e. 1.23) is basically the mean-free-path of fusion neutrons through lithium – the thickness of lithium needed to reduce the population of neutrons by $\sim 65\%$. While the second term, which includes P_W, is a correction to account for extra wall loading (as discussed in the secondary constraint section).

Moving forward, the remaining two thicknesses – c and d – are handled differently, approximating structural steel portions as well as magnetic current-carrying ones.

The Toroidal Field Coil Thickness – c

The thickness of the TF coils – c – is a little beyond the scope of this paper. It does, though, have a form that combines a structural steel component with a magnetic portion. From one of Jeff's previous models, this can be given as:

$$c = G_{CI} R_0 + G_{CO} \tag{4.55}$$

$$G_{CI} = \frac{B_0^2}{4\mu_0\sigma_{TF}} \cdot \frac{1}{(1-\epsilon_b)} \cdot \left(\frac{4\,\epsilon_b}{1+\epsilon_b} + \ \ln\left(\frac{1+\epsilon_b}{1-\epsilon_b}\right)\right) \tag{4.56}$$

$$G_{CO} = \frac{B_0}{\mu_0 J_{TF}} \cdot \frac{1}{(1-\epsilon_b)} \tag{4.57}$$

The critical stress – σ_{TF} in G_{CI} implies it depends on the structural component, whereas the maximum current density – J_{TF} – implies a magnetic predisposition in G_{CO}. The use of $G_\square$ in these quantities, instead of $K_\square$ is because they include the toroidal magnetic field strength – B_0. For this reason, they are referred to as floating coefficients. Lastly, the term ϵ_b represents the blanket inverse aspect ratio that combines the minor radius with blanket thickness:

$$\epsilon_b = \frac{a+b}{R_0} \tag{4.58}$$

The Central Solenoid Thickness – d

Finishing this discussion where we started it, the central solenoid's thickness – d – has a form similar to the TF coil's (i.e. c). In mathematical form, this can be represented as:

$$d = K_{DR}R_{CS} + K_{DO} \tag{4.59}$$

$$K_{DR} = \frac{3B_{CS}^2}{6\mu_0\sigma_{CS} - B_{CS}^2} \tag{4.60}$$

$$K_{DO} = \frac{6B_{CS}\sigma_{CS}}{6\mu_0\sigma_{CS} - B_{CS}^2} \cdot \left(\frac{1}{J_{OH}}\right) \tag{4.61}$$

Here, the use of $K_\square$ for the coefficients signifies their use as fixed coefficients. Therefore, B_{CS} must be treated as a fixed variable representing the magnetic field strength in the central solenoid. For prospective solenoids using high temperature superconducting (HTS) tape, B_{CS} may be around 20 T. The values of σ_{CS} and J_{CS} have similar meanings to the ones for TF coils. These are collected in a table below with example values representative of our model.

Before moving on, it seems important to say that although K_{DI} and K_{DO} do not depend on floating variables, R_{CS} definitely does. This is what makes the central solenoid's thickness difficult.

4.3.3 Revisiting Central Solenoid Dimensions

Now that the various thicknesses have been defined (i.e. a, b, c, and d), the equations for the solenoid's dimensions (i.e. R_{CS} and h_{CS}), can now be revisited and simplified. From before,

$$R_{CS} = R_0 - (a + b + c + d) \tag{4.51}$$

$$h_{CS} = 2 \cdot (\kappa a + b + c) \tag{4.52}$$

Utilizing the four thicknesses from before, these can now be expanded to simple formulas. Repeating the thicknesses:

$$a = \epsilon \cdot R_0 \tag{4.53}$$

$$b = 1.23 + 0.074 \ln P_W \tag{4.54}$$

$$c = G_{CI} R_0 + G_{CO} \tag{4.55}$$

$$d = K_{DR} R_{CS} + K_{DO} \tag{4.59}$$

Plugging these into the central solenoid's dimensions results in:

$$h_{CS} = 2 \cdot (R_0 \cdot (\epsilon\kappa + G_{CI}) + (b + G_{CO})) \tag{4.62}$$

$$R_{CS} = \frac{1}{1 + K_{DR}} \cdot (R_0 \cdot (1 - \epsilon - G_{CI}) - (K_{DO} + b + G_{CO})) \tag{4.63}$$

These are the complete central solenoid dimension formulas. To make them more tractable to the reader, they will now be simplified one step at a time. (The same simplification exercise will be done again after the generalized current is derived in a few sections.)

Bibliography