Euler angles

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> *This article is about the Euler angles used in mathematics. For the use of the term in physics and aerospace engineering, see rigid body dynamics. For chained rotations, see chained rotations.*

The **Euler angles** are three angles introduced by Leonhard Euler to describe the orientation of a rigid body with respect to a fixed coordinate system.[1] They can also represent the orientation of a mobile frame of reference in physics or the orientation of a general basis in 3-dimensional linear algebra.

Any orientation can be achieved by composing three *elemental rotations*, i.e. rotations about the axes of a coordinate system. Euler angles can be defined by three of these rotations. They can also be defined by elemental geometry and the geometrical definition demonstrates that three rotations are always sufficient to reach any frame.

The three elemental rotations may be extrinsic (rotations about the axes *xyz* of the original coordinate system, which is assumed to remain motionless), or intrinsic(rotations about the axes of the rotating coordinate system *XYZ*, solidary with the moving body, which changes its orientation after each elemental rotation).

Euler angles are typically denoted as α, β, γ, or φ, θ, ψ. Different authors may use different sets of rotation axes to define Euler angles, or different names for the same angles. Therefore, any discussion employing Euler angles should always be preceded by their definition.

Without considering the possibility of using two different conventions for the definition of the rotation axes (intrinsic or extrinsic), there exist twelve possible sequences of rotation axes, divided in two groups:

- **Proper Euler angles** (*z-x-z, x-y-x, y-z-y, z-y-z, x-z-x, y-x-y*)
- **Tait–Bryan angles** (*x-y-z, y-z-x, z-x-y, x-z-y, z-y-x, y-x-z*).

Tait–Bryan angles are also called **Cardan angles**; **nautical angles**; **heading, elevation, and bank**; or **yaw, pitch, and roll**. Sometimes, both kinds of sequences are called "Euler angles". In that case, the sequences of the first group are called *proper* or *classic* Euler angles.

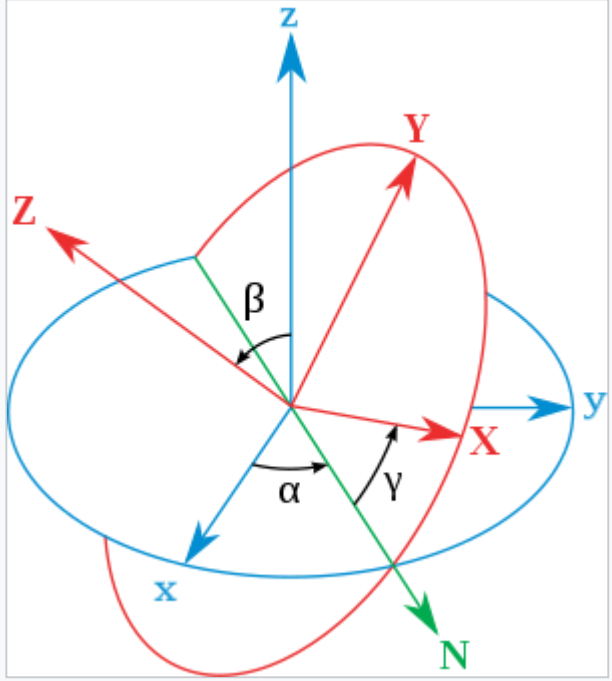
Proper Euler angles [edit]

Geometrical definition [edit]

The axes of the original frame are denoted as *x*,*y*,*z* and the axes of the rotated frame as *X*,*Y*,*Z*. The **geometrical definition** (sometimes referred to as static) begins by defining the line of nodes as the intersection of the planes *xy*and *XY* (it can also be defined as the common perpendicular to the axes *z* and *Z* and then written as the vector product $N = z \times Z$). Using it, the three Euler angles can be defined as follows:

- α (or φ) is the angle between the *x* axis and the *N* axis (*x*-convention - it could also be defined between *y* and *N*, called *y*-convention).
- β (or θ) is the angle between the *z* axis and the *Z* axis.
- γ (or ψ) is the angle between the *N* axis and the *X* axis (*x*-convention).

Euler angles between two reference frames are defined only if both frames have the same handedness.



Proper Euler angles geometrical definition. The *xyz* (fixed) system is shown in blue, the *XYZ* (rotated) system is shown in red. The line of nodes (*N*) is shown in green

Definition by extrinsic rotations [edit]

Extrinsic rotations are elemental rotations that occur about the axes of the fixed coordinate system *xyz*. The *XYZ* system rotates, while *xyz* is fixed. Starting with *XYZ*overlapping *xyz*, a composition of three extrinsic rotations can be used to reach any target orientation for *XYZ*. The Euler or Tait-Bryan angles (*α*, *β*, *γ*) are the amplitudes of these elemental rotations. For instance, the target orientation can be reached as follows:

- The *XYZ* system rotates about the *z* axis by *γ*. The *X* axis is now at angle *γ* with respect to the *x* axis.
- The *XYZ* system rotates again about the *x* axis by *β*. The *Z* axis is now at angle β with respect to the *z* axis.
- The *XYZ* system rotates a third time about the *z* axis by *α*.

In sum, the three elemental rotations occur about *z*, *x* and *z*. Indeed, this sequence is often denoted *z-x-z* (or 3-1-3). Sets of rotation axes associated with both proper Euler angles and Tait–Bryan angles are commonly named using this notation (see above for details).