1. Notation & Elementary Tools

- summation symbol:

$$\sum_{k=m}^{n} a_k = a_m + a_{m+1} + \ldots + a_{n-1} + a_n \qquad (n, m \in \mathbb{Z},\ n \geq m)$$

product symbol:

$$\prod_{k=m}^{n} a_k = a_m \times a_{m+1} \times \ldots \times a_{n-1} \times a_n \qquad (n, m \in \mathbb{Z},\ n \geq m)$$

binomial coefficients:

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

- Proof by induction:

Phrase the theorem to be proven in the standard form
$A_n = 0$ for all $n \in \mathbb{N}$

(i) the basis: prove that $A_1 = 0$

(ii) induction step: prove that if $A_n = 0$ then also $A_{n+1} = 0$

- geometric series:

$$\sum_{k=0}^{n-1} x^k = \frac{1 - x^n}{1 - x}$$

exponential series:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + \frac{z}{1!} + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \ldots$$

- conventions:

$$0! = 1 \qquad 0^0 = 1$$

$\exists$: there exists $\qquad \forall$: for all $\qquad \Leftrightarrow$: if and only if

2. Complex numbers

- definitions:

  the number i: solution of the equation $z^2 + 1 = 0$

  'complex numbers': all expressions of the form $a + ib$ with $a, b \in \mathbb{R}$

  addition and multiplication: let $a, b, c, d \in \mathbb{R}$,

$$\begin{aligned} (a+bi)+(c+di) &= (a+c)+(b+d)i \\ (a+bi)(c+di) &= (ac-bd)+(ad+bc)i \end{aligned}$$

  real/imaginary parts: let $a, b \in \mathbb{R}$

$$z = a+ib: \qquad \mathrm{Re}(z) = a, \quad \mathrm{Im}(z) = b$$

  conjugate & absolute value of $z = a + ib$: $\overline{z} = a - ib$, $|z| = \sqrt{z.\overline{z}}$

- simple properties:

$$\overline{z+w} = \overline{z} + \overline{w}, \qquad \overline{zw} = \overline{z}.\overline{w}, \qquad z\overline{z} \in \mathbb{R}, \ \geq 0$$

$$|\mathrm{Re}(z)| \leq |z|, \qquad |\mathrm{Im}(z)| \leq |z|, \qquad |\overline{z}| = |z|$$

$$|z.w| = |z||w|, \qquad |z+w| \leq |z| + |w|$$

$$z = a+ib \ \ (a, b \in \mathbb{R}): \qquad |z| = \sqrt{a^2+b^2}$$

- roots (i.e. zeros) of polynomials:

  Every quadratic eqn can be solved in $\mathbb{C}$,

$$az^2 + bz + c = 0: \qquad z_\pm = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

  Every n-th order polynomial with real or complex coefficients,
  $P(z) = z^n + a_{n-1}z^{n-1} + \ldots + a_1 z + a_0$, can be factorized

$$P(z) = (z - z_1)(z - z_2)\ldots(z - z_{n-1})(z - z_n)$$

  real coefficients:
  if z is a root of $P(z)$, then also $\overline{z}$

- division:

  multiply numerator & denominator by complex conjugate of denominator

$$\frac{a+ib}{c+id} = \frac{a+ib}{c+id}\frac{c-id}{c-id} = \frac{(a+ib)(c-id)}{|c+id|^2} = \ldots$$

- 'complex plane' (Argand diagram):

  complex numbers $z = a + ib$ $(a, b \in \mathbb{R}) \Leftrightarrow$ points with coordinates (a, b)

- exponential notation:

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

exponential form of complex number:

$$z = re^{i\theta}, \quad \text{with} \;\; r, \theta \in \mathbb{R} \;\; \text{and} \;\; r = |z| \geq 0$$

consequences:

$$\overline{z} = re^{-i\theta}, \qquad 1/z = \frac{1}{r}e^{-i\theta}, \qquad 1/\overline{z} = \frac{1}{r}e^{i\theta}$$

$$re^{i\theta}.\rho e^{i\phi} = r\rho\; e^{i(\theta+\phi)} \qquad re^{i\theta}/\rho e^{i\phi} = (r/\rho)\; e^{i(\theta-\phi)}$$

$$\Big(\cos(\theta) + i\sin(\theta)\Big)^n = \cos(n\theta) + i\sin(n\theta) \qquad \text{(De Moivre)}$$

- argument of a complex number:

$\arg(z)$ = angle $\theta \in \mathbb{R}$ such that

$$z = re^{i\theta} \;\; \text{with } r, \theta \in \mathbb{R}, \;\; r \geq 0 \qquad \text{and} \qquad -\pi < \theta \leq \pi$$

natural logarithm:

$$\ln(z) = \ln(|z|) + i\; \arg(z)$$

from standard $(a+ib)$ to exponential $(re^{i\theta})$ form:
(i) calculate $r = |z|$ using $r^2 = z\overline{z}$
(ii) calculate $e^{i\theta}$ using $z/r = e^{i\theta} = \cos(\theta) + i\sin(\theta)$
(iii) determine which θ obeys $-\pi < \theta \leq \pi$: $\arg(z) = \theta$

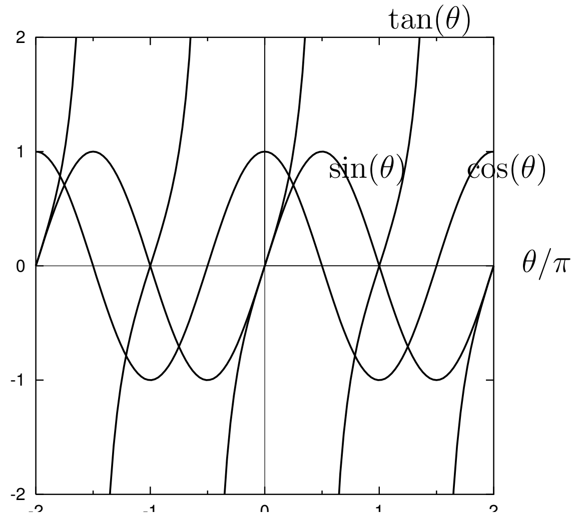
- some applications:

derivation of identities for trigonometric functions of multiple angles
the n solutions of $z^n = 1$ are

$$z = e^{2\pi i m/n} \qquad \text{for} \;\; m = 0, 1, 2, \ldots, n-1$$

complex equations: $F(z, \overline{z}) = 0$
(i) $z\overline{z} - 1 = 0$: unit circle in complex plane
(ii) $z^n - 1 = 0$: n points in complex plane
(iii) $uz + v\overline{z} + w = 0$: lines in complex plane
(iv) $|z-u| + |z-w| - R = 0$: ellipse in complex plane
(v) combinations of the above, e.g. $(z\overline{z}-1)(z^3 - 2iz^2 - 2(2i+1)z - 20 - 8i) = 0$
and much more ...

3. Trigonometric & hyperbolic functions



- Definitions of sine & cosine:

  (i) geometric (rotate around origin)

  (ii) via differential eqns:

$$\frac{d}{d\theta}\sin(\theta) = \cos(\theta)$$
$$\frac{d}{d\theta}\cos(\theta) = -\sin(\theta)$$
$$\cos(0) = 1, \quad \sin(0) = 0$$

  (iii) combine power series for e^z with

$$\cos(z) = \frac{1}{2}(e^{iz} + e^{-iz}) \qquad \sin(z) = \frac{1}{2i}(e^{iz} - e^{-iz})$$

  giving

$$\sin(z) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!} \;=\; z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots$$
$$\cos(z) \;=\; \sum_{k=0}^{\infty} \frac{(-1)^k z^{2k}}{(2k)!} \;=\; 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \frac{z^6}{6!} + \ldots$$

- special values:

  $\cos(0) = 1$, $\sin(0) = 0$

  $\cos(\pi/4) = \sin(\pi/4) = 1/\sqrt{2}$

  $\sin(\pi/6) = \frac{1}{2}$, $\cos(\pi/6) = \frac{1}{2}\sqrt{3}$

  $\sin(\pi/3) = \frac{1}{2}\sqrt{3}$, $\cos(\pi/3) = 1/2$

  $\cos(\pi/2) = 0$, $\sin(\pi/2) = 1$

- other trigonometric functions:

$$\tan(\theta) = \sin(\theta)/\cos(\theta), \qquad \cot(\theta) = \cos(\theta)/\sin(\theta)$$

$$\sec(\theta) = 1/\cos(\theta), \qquad \mathrm{cosec}(\theta) = 1/\sin(\theta)$$

- inverse trigonometric functions:

  $\arcsin(x)$:
  angle $\theta \in [-\pi/2, \pi/2]$ such that $\sin(\theta) = x$

  $$\arcsin : [-1, 1] \to [-\pi/2, \pi/2]$$

  $$\arcsin(\sin(\theta)) = \theta \quad \text{for all } \theta \in [-\pi/2, \pi/2]$$

  $$\sin(\arcsin(x)) = x \quad \text{for all } x \in [-1, 1]$$

  $\arccos(x)$:
  angle $\theta \in [0, \pi]$ such that $\cos(\theta) = x$

  $$\arccos : [-1, 1] \to [0, \pi]$$

  $$\arccos(\cos(\theta)) = \theta \quad \text{for all } \theta \in [0, \pi]$$

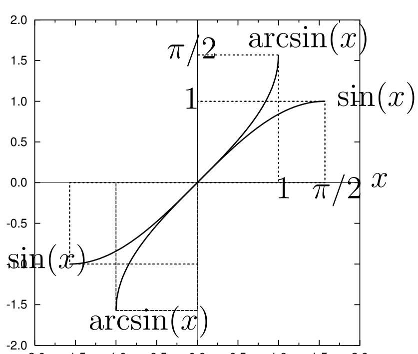
  $$\cos(\arccos(x)) = x \quad \text{for all } x \in [-1, 1]$$

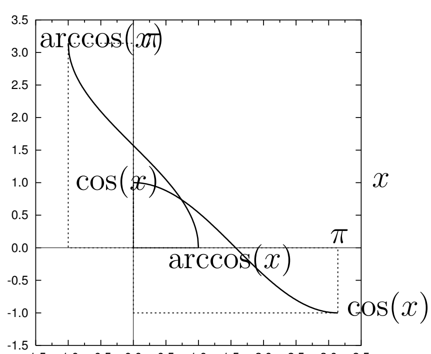
  $\arctan(x)$:
  angle $\theta \in (-\pi/2, \pi/2)$ such that $\tan(\theta) = x$

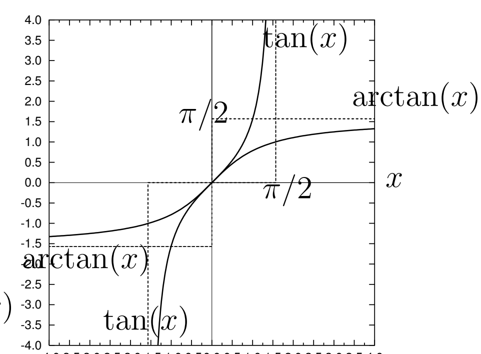
  $$\arctan : \mathbb{R} \to (-\pi/2, \pi/2)$$

  $$\arctan(\tan(\theta)) = \theta \quad \text{for all } \theta \in (-\pi/2, \pi/2)$$

  $$\tan(\arctan(x)) = x \quad \text{for all } x \in \mathbb{R}$$







- symmetry properties:

  $\sin(-\theta) = -\sin(\theta),\;\; \cos(-\theta) = \cos(\theta)$
  $\sin(\pi - \theta) = \sin(\theta),\;\; \cos(\pi - \theta) = -\cos(\theta)$
  $\sin(\theta + \pi) = -\sin(\theta),\;\; \cos(\theta + \pi) = -\cos(\theta)$
  $\sin(\pi/2 - \theta) = \cos(\theta),\;\; \cos(\pi/2 - \theta) = \sin(\theta)$

- addition formulae:

$$\begin{aligned}\sin(\theta + \phi) &= \sin(\theta)\cos(\phi) + \cos(\theta)\sin(\phi)\\ \cos(\theta + \phi) &= \cos(\theta)\cos(\phi) - \sin(\theta)\sin(\phi)\end{aligned}$$

  applications:

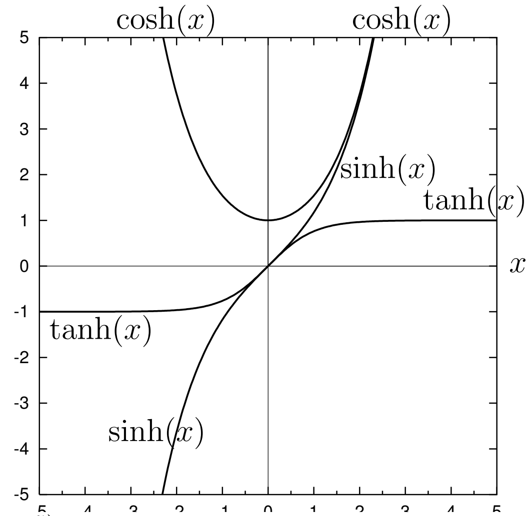
$$\begin{aligned}\cos(\theta)\cos(\phi) &= \frac{1}{2}\Big(\cos(\theta + \phi) + \cos(\theta - \phi)\Big)\\ \sin(\theta)\sin(\phi) &= \frac{1}{2}\Big(\cos(\theta - \phi) - \cos(\theta + \phi)\Big)\\ \sin(\theta)\cos(\phi) &= \frac{1}{2}\Big(\sin(\theta + \phi) + \sin(\theta - \phi)\Big)\end{aligned}$$

$$\begin{aligned}\sin(2\theta) &= 2\sin(\theta)\cos(\theta)\\ \cos(2\theta) &= \cos^2(\theta) - \sin^2(\theta)\\ \tan(2\theta) &= 2\tan(\theta)/[1 - \tan^2(\theta)]\end{aligned}$$

- linear combinations $a\cos(\theta) + b\sin(\theta)$
  can always be written in the form $c\sin(\theta + \alpha)$
  with some suitable $c, \alpha \in \mathbb{R}$ and with $c \geq 0$

- Let $t = \tan(\theta/2)$:

$$\tan(\theta) = \frac{2t}{1 - t^2} \qquad \cos(\theta) = \frac{1 - t^2}{1 + t^2} \qquad \sin(\theta) = \frac{2t}{1 + t^2}$$



- Definitions of hyperbolic sine & cosine:

  (i) via differential eqns:

$$\frac{d}{dz}\sinh(z) = \cosh(z)$$
$$\frac{d}{dz}\cosh(z) = \sinh(z)$$
$$\cosh(0) = 1, \quad \sinh(0) = 0$$

  (ii) direct (also ok for complex nrs):

$$\cosh(z) = \frac{1}{2}(e^z + e^{-z}) \qquad \sinh(z) = \frac{1}{2}(e^z - e^{-z})$$

- other trigonometric functions:

$$\tanh(z) = \sinh(z)/\cosh(z), \qquad \coth(z) = \cosh(z)/\sinh(z)$$
$$\mathrm{sech}(z) = 1/\cosh(z), \qquad \mathrm{cosech}(z) = 1/\sinh(z)$$

- special values:

  $\cosh(0) = 1, \ \sinh(0) = 0, \tanh(0) = 0$

  $\cosh(\infty) = \sinh(\infty) = \infty, \tanh(\infty) = 1$

  $\cosh(-\infty) = \infty, \sinh(-\infty) = -\infty, \tanh(-\infty) = -1$

  properties:

$$\cosh^2(z) - \sinh^2(z) = 1$$
$$\sinh(-x) = -\sinh(x), \quad \cosh(-x) = \cosh(x), \quad \tanh(-z) = -\tanh(z)$$

- connection with trigonometric functions:

$$\sin(x) = -i\sinh(ix) \qquad \sinh(x) = -i\sin(ix)$$
$$\cos(x) = \cosh(ix) \qquad \cosh(x) = \cos(ix)$$
$$\tan(x) = -i\tanh(ix) \qquad \tanh(x) = -i\tan(ix)$$

$$\sinh(\theta+\phi) = \sinh(\theta)\cosh(\phi) + \cosh(\theta)\sinh(\phi)$$
$$\cosh(\theta+\phi) = \cosh(\theta)\cosh(\phi) + \sinh(\theta)\sinh(\phi)$$

- inverse hyperbolic functions:

  $\operatorname{arcsinh}(x)$:

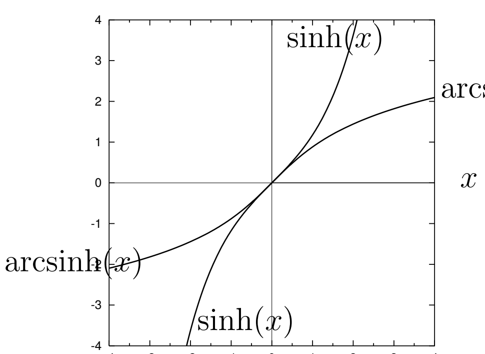
$$\operatorname{arcsinh} : \mathbb{R} \to \mathbb{R}$$
$$\operatorname{arcsinh}(\sinh(x)) = x \quad \text{for all } x \in \mathbb{R}$$
$$\sinh(\operatorname{arcsinh}(y)) = y \quad \text{for all } y \in \mathbb{R}$$
$$\operatorname{arcsinh}(y) = \ln(y + \sqrt{y^2+1}) \qquad \text{for all } \; y \in \mathbb{R}$$

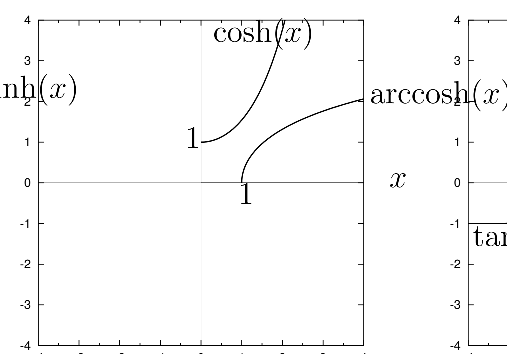
  $\operatorname{arccosh}(x)$:

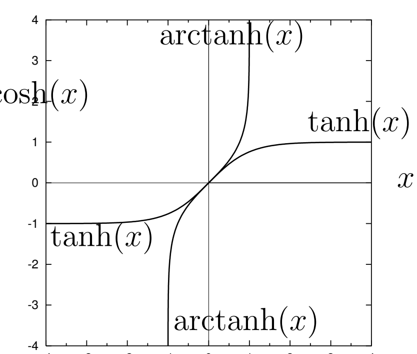
$$\operatorname{arccosh} : [1,\infty) \to [0,\infty)$$
$$\operatorname{arccosh}(\cosh(x)) = x \quad \text{for all } x \in [0,\infty)$$
$$\cosh(\operatorname{arccosh}(y)) = y \quad \text{for all } y \in [1,\infty)$$
$$\operatorname{arccosh}(y) = \ln(y + \sqrt{y^2-1}) \qquad \text{for all } \; y \in [1,\infty)$$

  $\operatorname{arctanh}(x)$:

$$\operatorname{arctanh} : (-1,1) \to \mathbb{R}$$
$$\operatorname{arctanh}(\tanh(x)) = x \quad \text{for all } x \in \mathbb{R}$$
$$\tanh(\operatorname{arctanh}(y)) = y \quad \text{for all } y \in (-1,1)$$
$$\operatorname{arctanh}(y) = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right) \qquad \text{for all } \; y \in (-1,1)$$







4. Functions, limits and differentiation

derivative $f'(x)$: slope of graph for $f(x)$ at x

- simple version (Fermat), if allowed:

$$f'(x) = \frac{f(x+h)-f(x)}{h}, \quad \text{then put } h=0$$

  more general: slope as a limit

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

- Limits:

$$\lim_{x\downarrow x_0} f(x) = L \quad \Leftrightarrow \quad (\forall \varepsilon\!>\!0)(\exists \delta\!>\!0): \ |f(x)\!-\!L| < \varepsilon \text{ whenever } x \in (x_0, x_0\!+\!\delta)$$

  'one may get $f(x)$ as close as one wishes to the value L
  simply by lowering x sufficiently close to x_0'

$$\lim_{x\uparrow x_0} f(x) = L \quad \Leftrightarrow \quad (\forall \varepsilon\!>\!0)(\exists \delta\!>\!0): \ |f(x)\!-\!L| < \varepsilon \text{ whenever } x \in (x_0\!-\!\delta, x_0)$$

  'one may get $f(x)$ as close as one wishes to the value L
  simply by raising x sufficiently close to x_0'

$$\lim_{x\to\infty} f(x) = L \quad \Leftrightarrow \quad (\forall \varepsilon\!>\!0)(\exists X\!>\!0): \ |f(x)\!-\!L| < \varepsilon \text{ whenever } x > X$$

  'one may get $f(x)$ as close as one wishes to the value L
  simply by making x larger and larger'

$$\lim_{x\to-\infty} f(x) = L \quad \Leftrightarrow \quad (\forall \varepsilon\!>\!0)(\exists X\!<\!0): \ |f(x)\!-\!L| < \varepsilon \text{ whenever } x < X$$

  'one may get $f(x)$ as close as one wishes to the value L
  simply by making x smaller and smaller'

- left/right limits exist and identical:

$$\lim_{x\to x_0} f(x) = L \quad \Leftrightarrow \quad (\forall \varepsilon\!>\!0)(\exists \delta > 0): \ |f(x)\!-\!L|\!<\!\varepsilon \text{ whenever } |x\!-\!x_0| < \delta$$

  'one may get $f(x)$ as close as one wishes to the value L
  simply by taking x sufficiently close to x_0, from either side'

  f is continuous at x_0:

$$\lim_{x\to x_0} f(x) \quad \text{exists}, \qquad \lim_{x\to x_0} f(x) = f(x_0)$$

- basic limits:

$$\lim_{h\to 0} \frac{1}{h}(\cos(h)-1) = 0 \qquad \lim_{h\to 0} \frac{1}{h}\sin(h) = 1$$

$$\lim_{h\to 0} \frac{1}{h}(a^h-1) = \ln a \qquad \lim_{x\to\infty} x^n e^{-x} = 0 \ \ (n \geq 0)$$

basic derivatives:

$$\frac{d}{dx}a^x = a^x \ln(a) \qquad \frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x)$$

- rules for composite expressions:

  If $\lim_{x\to x_0} f(x)$ and $\lim_{x\to x_0} g(x)$ exist:

$$\lim_{x\to x_0}\Big(f(x)+g(x)\Big) = \lim_{x\to x_0} f(x) + \lim_{x\to x_0} g(x)$$

$$\lim_{x\to x_0}\Big(f(x)g(x)\Big) = \Big(\lim_{x\to x_0} f(x)\Big)\Big(\lim_{x\to x_0} g(x)\Big)$$

$$\text{if } \lim_{x\to x_0} g(x) \neq 0: \quad \lim_{x\to x_0}\Big(f(x)/g(x)\Big) = \Big(\lim_{x\to x_0} f(x)\Big)/\Big(\lim_{x\to x_0} g(x)\Big)$$

$$\text{if } \lim_{x\to x_0} g(x) = 0, \;\; \lim_{x\to x_0} f(x) \neq 0: \quad \lim_{x\to x_0}\Big(f(x)/g(x)\Big) \;\text{ does not exist}$$

  If $\lim_{x\to x_0} f(x) = a$, $\lim_{y\to a} g(y) = b$, $g(a) = b$:

$$\lim_{x\to x_0} g(f(x)) = b$$

- determine limits via ‘pinching’ or ‘sandwiching’:

  Let there be a $\delta > 0$ such that

$$\left.\begin{array}{l} (\forall x,\ |x-x_0| < \delta):\ f(x) \le g(x) \le h(x) \\ \lim_{x\to x_0} f(x) = \lim_{x\to x_0} h(x) = a \end{array}\right. \quad \Rightarrow \quad \lim_{x\to x_0} g(x) = a$$

- derivatives to memorize:

$$\frac{d}{dx}\sin(x) = \cos(x) \qquad \frac{d}{dx}\cos(x) = -\sin(x) \qquad \frac{d}{dx}a^x = a^x \ln(a)$$

$$\frac{d}{dx}x^n = nx^{n-1} \qquad \frac{d}{dx}\ln(x) = 1/x \qquad \frac{d}{dx}x^a = ax^{a-1}$$

- rules for composite expressions:

$$
\begin{aligned}
y = f(x) + g(x): &\quad \frac{dy}{dx} = f'(x) + g'(x) \\
y = f(x)g(x): &\quad \frac{dy}{dx} = f'(x)g(x) + f(x)g'(x) \\
y = f(g(x)): &\quad \frac{dy}{dx} = f'(g(x))g'(x) \\
y = f(x)/g(x): &\quad \frac{dy}{dx} = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}
\end{aligned}
$$

- derivatives of implicit functions $y = f(x)$:
  (i.e. functions without explicit formula for f)

  A. functions defined as solutions of equations of the type $F(x, y) = 0$:

  (i) differentiate the equation, using chain rule
  (ii) solve the result for dy/dx

  B. functions defined as inverse of another function $f(x)$:

  (i) differentiate the equation $f(f^{-1}(x)) = x$, using chain rule
  (ii) solve the result for $\frac{d}{dx}f^{-1}(x)$

  result: $\frac{d}{dx}f^{-1}(x) = 1/f'(f^{-1}(x))$

  C. functions defined parametrically, as $x(t)$ and $y(t)$ with $t \in \mathbb{R}$:

  (i) calculate $x'(t) = dx/dt$ and $y'(t) = dy/dt$
  (ii) work out $dy/dx = (dy/dt)/(dx/dt)$
  (iii) if possible, use formulas for $x(t)$ and $y(t)$ to eliminate t

5. Integration

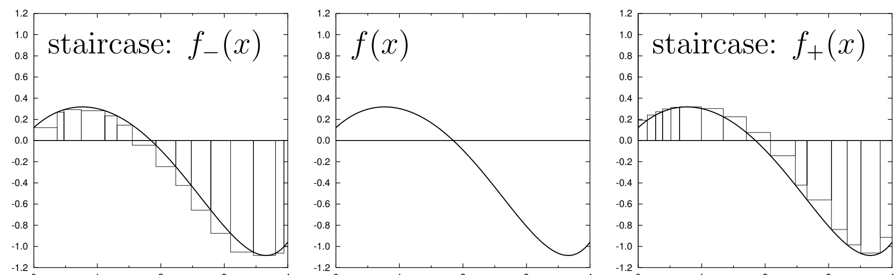
- $\int_a^b f(x)dx$: total area in (x,y) plane between graph of $y=f(x)$ and x-axis, counted positively for $f(x)>0$ and negatively for $f(x)<0$

- direct calculation of integrals:
  sandwich method using staircases

  (i) build staircase functions $f_\pm(x)$ such that
  $f_-(x)\leq f(x)\leq f_+(x)$ for all $x\in[a,b]$, e.g.

$$x\in[x_\ell,x_{\ell+1}): \quad \begin{array}{l} f_+(x)=f_\ell^+=\max_{x\in[x_\ell,x_{\ell+1})}f(x) \\ f_-(x)=f_\ell^-=\min_{x\in[x_\ell,x_{\ell+1})}f(x) \end{array}$$

  (let $x_1=a$, $x_L=b$)



  now

$$\sum_{\ell=1}^{L-1} f_\ell^-(x_{\ell+1}-x_\ell) \quad\leq\quad \int_a^b f(x)dx \quad\leq\quad \sum_{\ell=1}^{L-1} f_\ell^+(x_{\ell+1}-x_\ell)$$

  (ii) take the limit where $x_\ell - x_{\ell+1}\to 0$ for all ℓ

$$\lim_{\text{step widths}\to 0} A_- \;\leq\; \int_a^b f(x)dx \;\leq\; \lim_{\text{step widths}\to 0} A_+$$

  (iii) conclusion
  if $\lim_{\text{step widths}\to 0} A_- = \lim_{\text{step widths}\to 0} A_+ = A$: $\quad \int_a^b f(x)dx = A$

- indirect calculation of integrals:
  via fundamental theorem of calculus

  If $f:[a,b]\to\mathbb{R}$ is continuous on $[a,b]$,
  and $F:[a,b]\to\mathbb{R}$ is another function such that
  $F'(x)=f(x)$ for all $x\in[a,b]$, then

$$\int_a^b f(x)dx = F(b)-F(a)$$

- terminology:
  definite integral (a number): $A = \int_a^b f(x)dx$
  indefinite integral (a function, the 'primitive' of f): $F(x) = \int f(x)dx$

- techniques of integration via fundamental theorem:

  (i) objective: find a function $F(x)$ such that $F'(x) = f(x)$
  (ii) strategy: break up & simplify the integral until it has been reduced
  to expressions of which you know the primitives

  ten elementary integrals to memorize:

$$
\begin{aligned}
&\int x^a dx \quad (a \neq -1) &&= (a+1)^{-1}\ x^{a+1} \\
&\int x^{-1} dx &&= \ln|x| \\
&\int \ln(x) dx &&= x\ln(x) - x \\
&\int e^x dx &&= e^x \\
&\int \cos(x) dx &&= \sin(x) \\
&\int \sin(x) dx &&= -\cos(x) \\
&\int \frac{1}{\sqrt{1-x^2}} dx &&= \arcsin(x) \\
&\int \frac{1}{1+x^2} dx &&= \arctan(x) \\
&\int \frac{1}{\sqrt{x^2-1}} dx &&= \mathrm{arccosh}(x) \\
&\int \frac{1}{\sqrt{x^2+1}} dx &&= \mathrm{arcsinh}(x)
\end{aligned}
$$

  manipulation rules:

$$
\int \Big(cf(x)\Big) dx = cF(x) \qquad \int \Big(f(x)+g(x)\Big) = F(x)+G(x)
$$

$$
\int \Big(f(x)G(x)\Big) dx = F(x)G(x) - \int \Big(g(x)F(x)\Big) dx \qquad \int f(x)dx = \int \left(f(x(t)\ \frac{dx}{dt}\right) dt
$$

- further tricks: recursion formulae

  if integral of the form $I_n = \int_a^b f_n(x)dx$, with $n \in \mathbb{N}$
  e.g.

$$I_n = \int_0^\infty x^n e^{-x} dx \qquad n \in \mathbb{Z}^+$$

(i) express I_n in terms of I_{n-1}
(ii) repeat until I_n is expressed in terms of I_0 (which can be done directly)

- further tricks: differentiation with respect to a parameter

  if integral of the form $I(y) = \int_a^b f(x,y)dx$, with $y \in \mathbb{R}$
  (i) *assume* moving d/dy inside or outside the integral is allowed
  (ii) use $\frac{d}{dy}I(y) = \int \frac{d}{dy} f(x,y)dx$ to simplify the integral, if possible
  (iii) check the result by explicit differentiation

- further tricks: partial fractions

  if integral of the form

$$\int \frac{p(x)}{q(x)}\, dx \qquad p(x), q(x): \text{ polynomials}$$

$p(x)$ can always be written as $p(x) = s(x)q(x) + r(x)$
$s(x), r(x)$: other polynomials, $r(x)$ ('remainder') of lower order than $q(x)$

$$\int \frac{p(x)}{q(x)}\, dx = \int s(x)dx + \int \frac{r(x)}{q(x)} \qquad \begin{array}{l}\text{1st part : easy!} \\ \text{2nd part : doable}\end{array}$$

if $q(x)$ of following form, with $\alpha_i, \beta_j, \gamma_j \in \mathbb{R}$ (all different),

$$q(x) = \prod_{i=1}^{n} (x+\alpha_i)^{a_i} \prod_{j=1}^{m} (x^2 + \beta_j x + \gamma_j)^{b_j} \qquad \text{with} \qquad a_i, b_i \in \mathbb{Z}^+$$

and the order of $r(x)$ is less than that of $q(x)$,
then there always exists constants $A_{ik}, B_{j\ell}, C_{j\ell} \in \mathbb{R}$ such that

$$\frac{r(x)}{q(x)} = \sum_{i=1}^{n} \sum_{k=1}^{a_i} \frac{A_{ik}}{(x+\alpha_i)^k} + \sum_{j=1}^{m} \sum_{\ell=1}^{b_j} \frac{B_{j\ell}x + C_{j\ell}}{(x^2 + \beta_j x + \gamma_j)^\ell}$$

- simplest case (for partial fractions):

$$q(x) = \prod_{i=1}^{n} (x+\alpha_i) \qquad \text{with } \ \alpha_i \in \mathbb{R}$$

if order of $r(x)$ less than that of $q(x)$: there are constants $A_i \in \mathbb{R}$ such that

$$\frac{r(x)}{q(x)} = \sum_{i=1}^{n} \frac{A_i}{x+\alpha_i} \qquad \text{so} \qquad \int \frac{p(x)}{q(x)}\, dx = \int s(x)dx + \sum_{i=1}^{n} A_i \ln|x+\alpha_i|$$

- Applications of integrals: I. surface areas

  Area A of the surface between x-axis and a curve $f(x) \geq 0$, from $x = a$ to $x = b$:

  $$A = \int_a^b f(x)dx$$

  e.g. $A_{\text{circle}} = \pi R^2$, $A_{\text{ellipse}} = \pi R\sqrt{R^2 - a^2}$, etc

- Applications of integrals: II. volumes of revolution

  Volume V of solid obtained by revolving graph of $f(x) \geq 0$ *around* the x-axis, from $x = a$ to $x = b$:

  $$V = \pi \int_a^b f^2(x)dx$$

  e.g. $V_{\text{sphere}} = \frac{4}{3}\pi R^3$, $V_{\text{cigar}} = \frac{4}{3}\pi R(R^2 - a^2)$, etc

- Applications of integrals: III. length of curves

  Length L of curve in the plane described by $y = f(x)$ from $x = a$ to $x = b$:

  $$L = \int_a^b \sqrt{1 + \left(\frac{df}{dx}\right)^2}\ dx$$

  e.g. circumference of circle $L = 2\pi R$, etc

  If curve described parametrically by $x(t)$ and $y(t)$, from $t = a$ to $t = b$:

  $$L = \int_a^b \sqrt{(dx/dt)^2 + (dy/dt)^2}\ dt$$

6. Taylor's theorem and series

- terminology:

  series: expression of the form $\sum_{n=n_0}^{\infty} a_n$
  partial sum of the series: expression of the form $S_N = \sum_{n=n_0}^{N} a_n$
  convergent series: limit $S = \lim_{N\to\infty} S_N$ exists
  divergent series: limit $S = \lim_{N\to\infty} S_N$ does not exist
  if series convergent: $\lim_{n\to\infty} a_n = 0$ (but not other way around!)

- convergence criteria:

$$
\begin{array}{lllll}
(C1) & \sum\limits_n b_n \ \text{convergent}, \ \lim\limits_{n\to\infty} |a_n|/b_n \ \text{exists} & \Rightarrow & \sum\limits_n a_n & \text{convergent} \\
(C2) & \lim\limits_{n\to\infty} |a_n|^{1/n} < 1 & \Rightarrow & \sum\limits_n a_n & \text{convergent} \\
(C3) & \lim\limits_{n\to\infty} |a_n|^{1/n} > 1 & \Rightarrow & \sum\limits_n a_n & \text{divergent} \\
(C4) & \lim\limits_{n\to\infty} |a_{n+1}/a_n| < 1 & \Rightarrow & \sum\limits_n a_n & \text{convergent} \\
(C5) & \lim\limits_{n\to\infty} |a_{n+1}/a_n| > 1 & \Rightarrow & \sum\limits_n a_n & \text{divergent}
\end{array}
$$

  Examples:
  $\sum_n^{\infty} n^{-1}$ is *divergent*
  $\sum_n^{\infty} n^{-2}$ is *convergent*
  $\sum_n (-1)^n n^{-1}$ is *convergent*

- power series:

  expression of the form $S(x) = \sum_{n=n_0}^{\infty} b_n x^n$

  some functions could be written as power series $f(x) = \sum_{n=0}^{\infty} b_n x^n$
  with unique coefficients $\{b_n\}$

  questions:
  (i) which other functions can be written as power series?
  (ii) how to find the coefficients $\{b_n\}$ for any given function?
  (iii) how to know whether the power series would converge?

  convergence radius R of power series:
  number such that series converges for $|x| < R$, and diverges for $|x| > R$
  expressions for R (if the limits exist):

$$R = \lim_{n\to\infty} |b_n|^{-1/n} \qquad R = \lim_{n\to\infty} |b_n/b_{n+1}|$$

  If $R = 0$: power series diverges for all nonzero x
  If $R = \infty$: power series converges for all x

  Examples:

$e^x = \sum_{n=0}^{\infty} x^n/n! \qquad R = \infty$
$f(x) = \sum_{n=0}^{\infty} x^n \qquad R = 1$

- Taylor's theorem (gives power series for arbitrary functions):

$$f(x) = \sum_{n=0}^{N} \frac{f^{(n)}(0)}{n!} \, x^n + R_{N+1}(x)$$
$$R_{N+1}(x) = \frac{1}{N!} \int_0^x f^{(N+1)}(y)(x-y)^N dy$$

remainder term:
If $|f^{(n)}(x)| \leq C_n$ for all $x \in [-R, R]$: $|R_{N+1}(x)| \leq C_n x^{N+1}/(N+1)!$ on $[-R, R]$
(i.e. for small $|x|$ the remainder is much smaller than any term in the polynomial)

If $\lim_{N\to\infty} R_N(x) = 0$:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} \, x^n$$

- indirect methods for finding first few terms of Taylor series:
combine & manipulate series you know
(add, multiply, change signs, divide, differentiate, integrate, etc)
keep track of which powers you keep on board, for consistency