Problem 1 – Written Part

$$a = \ln\left(\frac{p(x|C_1)\,p(C_1)}{p(x|C_k)\,p(C_k)}\right)$$

$$a = \underbrace{\ln\left(p(x|C_1)\,p(C_1)\right)}_{①} - \underbrace{\ln\left(p(x|C_k)\,p(C_k)\right)}_{②}$$

① $\ln\left(p(x|C_e)\,p(C_e)\right) = \ln p(x|C_e) + \ln \pi_e$

$$\ln p(x|C_e) = \ln\left[\frac{1}{(2\pi)^{p/2}\,|\Sigma|^{1/2}} \cdot e^{-\frac{1}{2}(x-\mu_e)^T \Sigma^{-1} (x-\mu_e)}\right]$$

$$= d - \frac{1}{2}(x-\mu_e)^T \Sigma^{-1}(x-\mu_e) \quad \Bigg| \quad \left[d = \ln \frac{1}{(2\pi)^{p/2}\,|\Sigma|^{1/2}}\right]$$

$$= d - \frac{1}{2}(x^T - \mu_e^T)(\Sigma^{-1}x - \Sigma^{-1}\mu_e)$$

$$= d - \frac{1}{2}\left(x^T\Sigma^{-1}x - \underbrace{x^T\Sigma^{-1}\mu_e}_{} - \underbrace{\mu_e^T\Sigma^{-1}x}_{} - \mu_e^T\Sigma^{-1}\mu_e\right)$$

Scalar values $\Rightarrow$

$$\Rightarrow \mu_e^T\Sigma^{-1}x = x^T(\Sigma^{-1})^T\mu_e = x^T\Sigma^{-1}\mu_e$$

hence

$$\boxed{= d - \frac{1}{2}x^T\Sigma^{-1}x + \mu_e^T\Sigma^{-1}x + \frac{1}{2}h_e} \quad \Bigg| \quad \left[h_e = \mu_e^T\Sigma^{-1}\mu_e\right]$$

② We use similar approach to find $\ln p(x|C_k)$

$$\ln p(x|C_k) =$$

$$= \boxed{d - \frac{1}{2}x^T\Sigma^{-1}x + \mu_k^T\Sigma^{-1}x + \frac{1}{2}h_k} \quad \Bigg| \quad \left[h_k = \mu_k^T\Sigma^{-1}\mu_k\right]$$

Using answers 1 and 2 and combining we get:

$$a = \ln\left(p(x|C_e)\, p(C_1)\right) - \ln\left(p(x|C_k)\, p(C_k)\right)$$

$$= \ln p(x|C_e) - \ln p(x|C_k) + \ln \pi_e - \ln \pi_k$$

$$= \left(\cancel{d} - \cancel{\tfrac{1}{2} x^T \Sigma^{-1} x} + \mu_e^T \Sigma^{-1} x + \tfrac{1}{2} h_e\right)$$

$$- \cancel{d} + \cancel{\tfrac{1}{2} x^T \Sigma^{-1} x} - \mu_k^T \Sigma^{-1} x - \tfrac{1}{2} h_k + \ln \frac{\pi_e}{\pi_k}$$

$$= (\mu_e - \mu_k)^T \Sigma^{-1} x + \tfrac{1}{2}(h_e - h_k) + \ln \frac{\pi_e}{\pi_k}$$

$$= \tfrac{1}{2}(h_e - h_k) + \ln \frac{\pi_e}{\pi_k} + (\mu_e - \mu_k)^T \Sigma^{-1} x$$

$$= \theta_0 + \theta^T x$$

$\Rightarrow$ a is linear in x