Homeworks # 2

due March 21, 2016 (5:00pm Minnesota time)

1. **Two-Terminal Switches**

   This problem is based on C. E. Shannon's Masters Thesis from 1938, "A Symbolic Analysis of Relay and Switching Circuits." This classical paper was the first work that connected between logic and circuit design.

   In his seminal Master's Thesis, Claude Shannon made the connection between Boolean algebra and switching circuits. He considered two-terminal switches corresponding to electromagnetic relays. An example of a two-terminal switch is shown in the top part of Figure 1. The switch is either ON (closed) or OFF (open). A Boolean function can be implemented in terms of connectivity across a network of switches, often arranged in a series/parallel configuration. An example is shown in the bottom part of Figure 1.

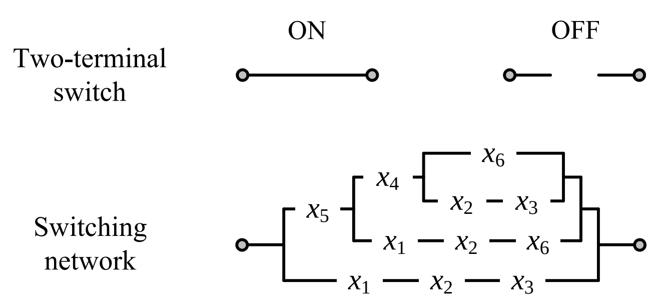


Figure 1: Two-terminal switching network implementing the Boolean function $x_1x_2x_3 + x_1x_2x_5x_6 + x_2x_3x_4x_5 + x_4x_5x_6$.

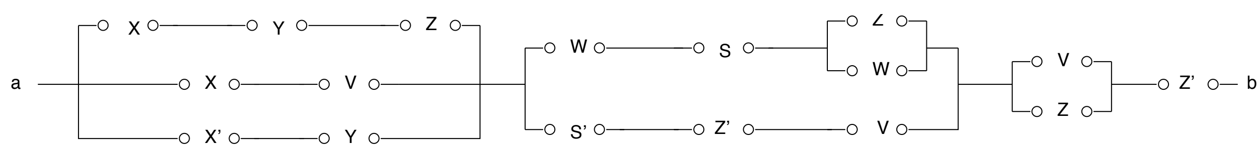
   Each switch is controlled by a Boolean variable. If the variable is 1 (0) then the corresponding switch is ON (OFF). The Boolean function for the network evaluates to 1 if there is a closed path between the left and right nodes. It can be computed by taking the sum (OR) of the product (AND) of literals along each path. The function is

$$x_1x_2x_3 + x_1x_2x_5x_6 + x_4x_5x_2x_3 + x_4x_5x_6.$$

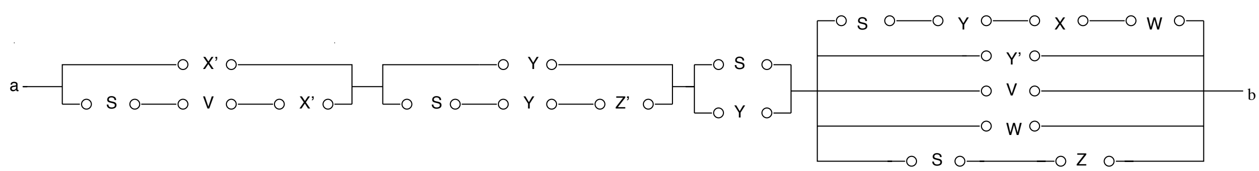
We use the modern convention: 1 denotes a closed circuit, 0 denotes an open circuit, + (addition) denotes a parallel connection and $\cdot$ (multiplication) denotes a series connection. (*Shannon used the opposite of the modern convention for everything: 1 for open, + for a series connection, etc. Forget about that.*)

Using De Morgan's Theorem and the theorems given on page 476 of Shannon's Masters Thesis, simplify the following circuits as much as possible and draw the simplified circuits. The number of switching elements achievable in the final answer is given for you.

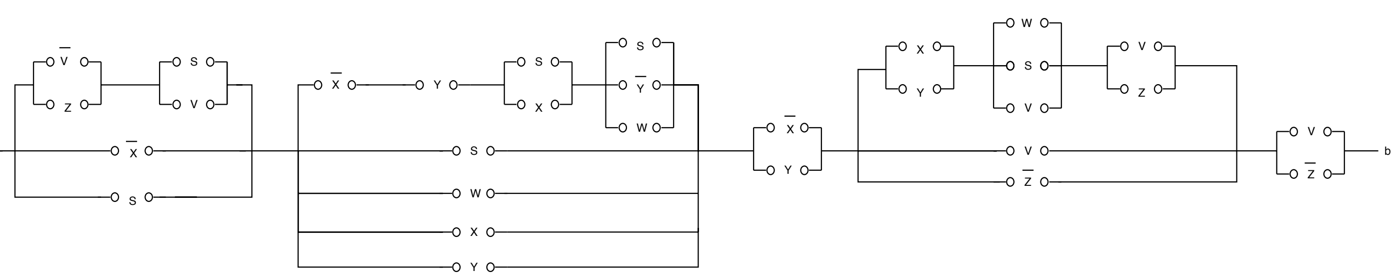
(a) This circuit can be simplified down to 6 switching elements. (Note that 4 of the switching elements shown are for negated variables.)



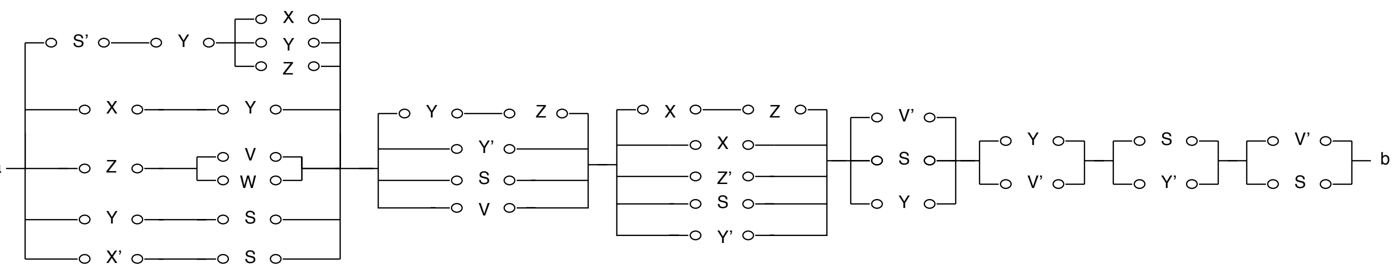
(b) This circuit can be simplified down to 6 switching elements. (Note that 4 of the switching elements shown are for negated variables.)



(c) This circuit can be simplified down to 9 switching elements. (Note that 7 of the switching elements shown are for negated variables.)

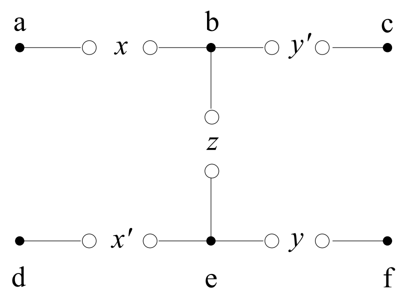


(d) This circuit can be simplified down to 8 switching elements (7 for extra credit). (Note that 9 of the switching elements shown are for negated variables.)



2. **Two-Terminal Switches, but Multiple Circuit Terminals**
(no collaboration on this one)

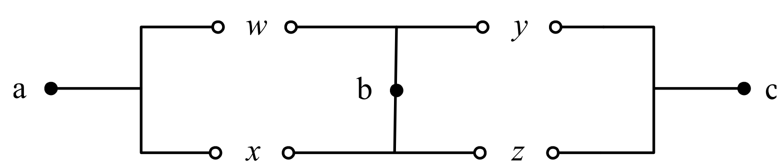
A generalization of the switching circuit model is a circuit with *multiple circuit terminals.* A function F_{ab} is 1 if there is a closed path between terminals a and b, and 0 otherwise. With multiple circuit terminals, $a, b, c, d, \ldots$, different functions can be implement between *pairs* of terminals. For instance, for the circuit



we have

$$\begin{aligned} F_{af} &= xyz \\ F_{bd} &= x'z \\ F_{cf} &= 0. \end{aligned}$$

and so on. For the circuit



we have

$$\begin{aligned} F_{ab} &= x + w \\ F_{bc} &= y + z \\ F_{ac} &= (x + w)(y + z). \end{aligned}$$

As these examples show, you can use either a variable or its complement to control a switch. (Each terminal is, in fact, any stretch of wire.)

(a) Construct a circuit with 3 switches that implements the functions

$$\begin{aligned} f_1 &= xy \\ f_2 &= x'y \end{aligned}$$

(b) Construct a circuit with 4 switches that implements the functions

$$\begin{aligned} f_1 &= xy + x'y' \\ f_2 &= x'y + xy' \end{aligned}$$

(c) Construct a circuit with 6 switches that implements the functions

$$\begin{aligned} f_1 &= x(y+z) \\ f_2 &= y(x+z) \\ f_3 &= z(x+y) \\ f_4 &= x + yz \\ f_5 &= y + xz \\ f_6 &= z + xy \end{aligned}$$

(d) Construct a circuit with as few switches as possible that implements all functions of two variables. A solution with 8 switches gets full credit. These functions are

$$\begin{aligned} f_0 &= 0 \\ f_1 &= x \\ f_2 &= x' \\ f_3 &= y \\ f_4 &= y' \\ f_5 &= xy \\ f_6 &= xy' \\ f_7 &= x'y \\ f_8 &= x'y' \\ f_8 &= x + y \\ f_{10} &= x + y' \\ f_{11} &= x' + y \\ f_{12} &= x' + y' \\ f_{13} &= xy' + x'y \\ f_{14} &= xy + x'y' \\ f_{15} &= 1 \end{aligned}$$

The constant 1 function is trivial: choose same terminal, e.g., connect a back to a. So forget about that one. For all the others, including the constant 0 function, you must select a pair of distinct terminals. As in the example above, the constant 0 function is implemented between a pair of terminals that are never connected.

3. **Coding for Lattices of Four-Terminal Switches**

   In this problem, we'll consider a new model, applicable to novel technologies such as nanowire crossbar arrays: four-terminal switches. An example is shown in the top part of Figure 2. The four terminals of the switch are all either mutually connected (ON) or disconnected (OFF). We consider networks of four-terminal switches arranged in rectangular *lattices.* An example is shown in the bottom part of Figure 2. Each switch is controlled by a Boolean literal. If the literal takes the value 1 (0) then corresponding switch is ON (OFF). The Boolean function for the lattice evaluates to 1 iff there is a closed path between the top and bottom edges of the lattice. The function is computed by taking the sum of the products of the literals along each path. These products are $x_1x_2x_3$, $x_1x_2x_5x_6$, $x_4x_5x_2x_3$, and $x_4x_5x_6$.

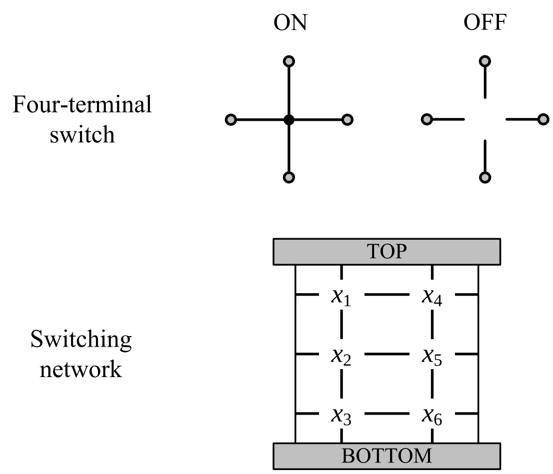


Figure 2: Four-terminal switching network implementing the Boolean function $x_1x_2x_3 + x_1x_2x_5x_6 + x_2x_3x_4x_5 + x_4x_5x_6$.

Problem

Implement the following functions in rectangular lattices of four-terminal switches. There should be a closed path from the top to bottom plates if and only if the Boolean function evaluates to 1. Use the smallest lattice possible. (See " Logic Synthesis for Switching Lattices Mustafa Altun and Marc Riedel, IEEE Transactions on Computers, 2011.)

(a) Function 1:

$$\bar{a}(b + c(\bar{d} + e))$$

(b) Function 2:

$$x_1 x_2 \bar{x}_3 + x_1 \bar{x}_2 x_3 + x_1 x_2 \bar{x}_3$$

(c) Function 3:

$$a\bar{b} + b\bar{c} + c\bar{d} + e\bar{f}$$

(d) Function 4:

$$x_1 \oplus x_2 \oplus x_3 \oplus x_4 \oplus x_5$$

Here + denotes OR; multiplication represents AND; $\oplus$ denotes Exclusive-OR; a over-bar represents NOT.

4. **Percolation**

Percolation theory is a rich mathematical topic that forms the basis of explanations of physical phenomena such as diffusion and phase changes in materials. It tells us that in media with random local connectivity, there is a critical threshold for global connectivity: below the threshold, the probability of global connectivity quickly drops to zero; above it, the probability quickly rises to one.

Broadbent and Hammersley described percolation with the following metaphorical model. Suppose that water is poured on top of a large porous rock. Will the water find its way through holes in the rock to reach the bottom? We can model the rock as a collection of small regions each of which is either a hole or not a hole. Suppose that each region is a hole with independent probability p_1 and not a hole with probability $1 - p_1$. The theory tells us that if p_1 is above a critical value p_c, the water will always reach the bottom; if p_1 is below p_c, the water will never reach the bottom. The transition in the probability of water reaching bottom as a function of increasing p_1 is extremely abrupt. For an infinite size rock, it is a step function from 0 to 1 at p_c.

In two dimensions, percolation theory can be studied with a lattice, as shown in Figure 3(a). Here each site is black with probability p_1 and white with probability $1-p_1$. Let p_2 be the probability that a connected path of black sites exists between the top and bottom plates. Figure 3(b) shows the relationship between p_1 and p_2 for different square lattice sizes. Percolation theory tells us that with increasing lattice size, the steepness of the curve increases. (In the limit, an infinite lattice produces a perfect step function.) Below the critical probability p_c, p_2 is approximately 0 and above it p_2 is approximately 1.

Suppose that each site of a percolation lattice is a four-terminal switch controlled by the same literal x_1. Also suppose that each switch is independently defective with the same probability. Defective switches are represented by white and black sites while the switch is supposed to be ON and OFF, respectively. Let's analyze the cases $x_1 = 0$ and $x_1 = 1$. If $x_1 = 0$ then each site is black with the defect probability, and the defective black sites might cause an error by forming a path between the top and bottom plates. In this case, p_1 and p_2 described in the percolation model correspond to the defect probability and the probability of an error in top-to-bottom connectivity, respectively. If $x_1 = 1$ then each site is white with the defect probability and the defective white sites might cause an error by destroying the connection between the top and bottom plates. In this case, p_1 and p_2 in the percolation model correspond to $1-$(defect probability) and $1-$(probability of an error in top-to-bottom connectivity), respectively. The

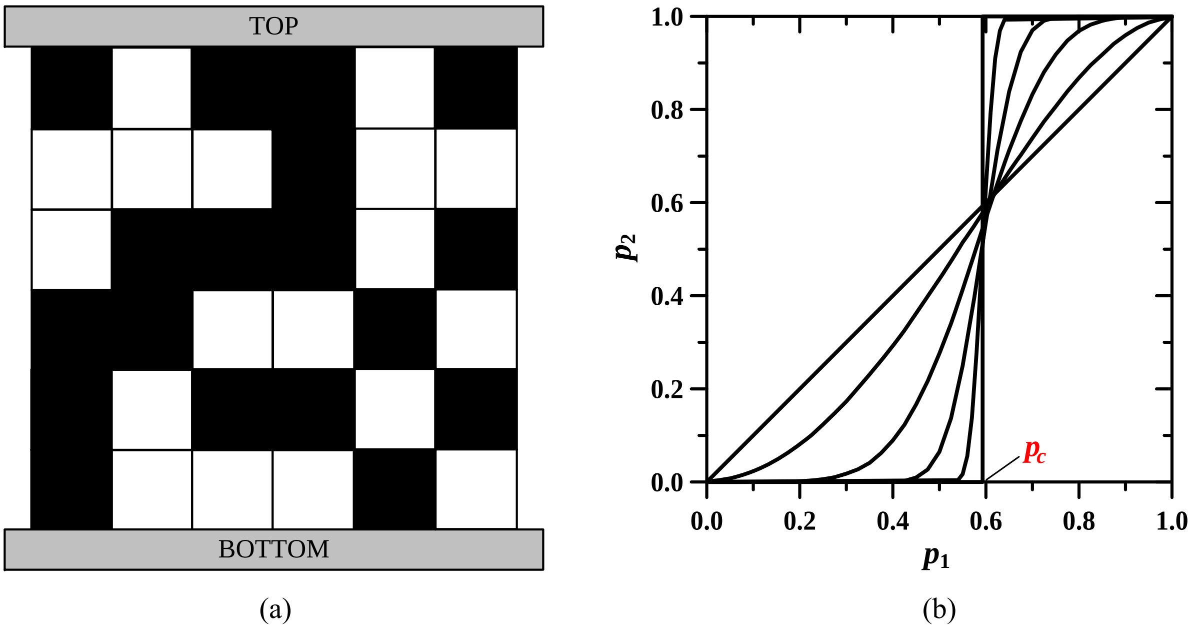


Figure 3: (a): Percolation lattice with random connections; there is a path of black sites between the top and bottom plates. (b) p_2 versus p_1 for 1×1, 2×2, 6×6, 24×24, 120×120, and infinite-size lattices.

relationship between p_1 and p_2 is shown in Figure 4.

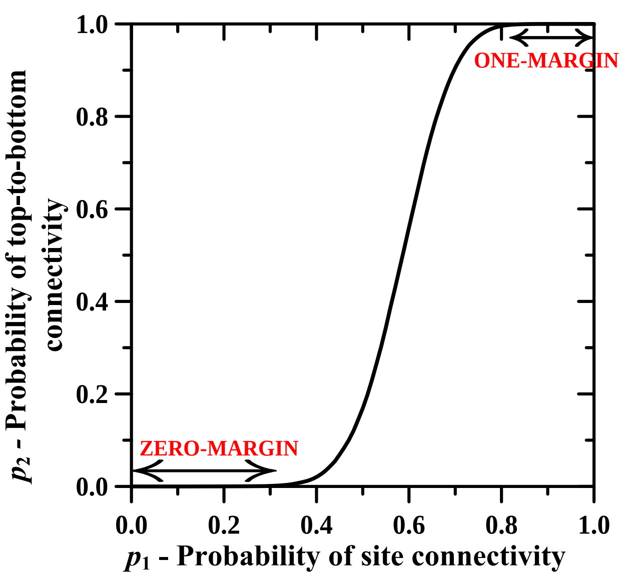


Figure 4: Non-linearity through percolation in random media.

Problem

Compute or estimate the critical thresholds for percolation in the following models.

(a) Suppose you have a graph consisting of n vertices. Each vertex has a collection of m neighbors (randomly chosen from the total set of n vertices). It is connected to each neighbor with an edge with probability p. Consider connectivity between a randomly chosen specific pair of vertices, A and B. Consider the probability that they are connected as a function of n, m and p. What is the critical threshold for percolation?

(b) Consider the following scenario for an ad-hoc peer-to-peer mobile network. There is a cell phone tower located in the center of some geographical area. There are n mobile users located at random locations with a r kilometer radius of the tower. Each user's phone can communicate with another user or with the tower if they are within q kilometers. Consider the probability that every user can communicate (directly or via a sequence of hops through other users) with the tower as a function of n, r and q. What is the critical threshold for percolation?