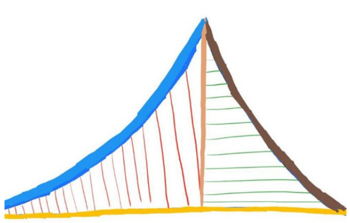
DIGITAL COMMUNICATION

Through Simulations

G. V. V. Sharma



Copyright ©2022 by G. V. V. Sharma.

https://creativecommons.org/licenses/by-sa/3.0/

and

https://www.gnu.org/licenses/fdl-1.3.en.html

Contents

Introduction

This book introduces digital communication through probability.

Chapter 1

Introduction

Chapter 2

Axioms

2.0.1 Given that the events A and B are such that $P(A) = \frac{1}{2}, P(A+B) = \frac{3}{5}$ and $P(B) = p$. Find p if they are

(a) mutually exclusive

(b) independent

Solution:

(a) In this case

$$\Pr(A+B) = \Pr(A) + \Pr(B) \tag{2.1}$$

$$\implies \frac{3}{5} = \frac{1}{2} + p \tag{2.2}$$

$$\therefore p = \frac{1}{10} \tag{2.3}$$

(b) Given A and B are independent events, then,

$$\Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(AB) \tag{2.4}$$

$$\implies \Pr(A+B) = \Pr(A) + \Pr(B) - \Pr(A)\Pr(B) \tag{2.5}$$

$$\implies \frac{3}{5} = \frac{1}{2} + p - \frac{p}{2} \tag{2.6}$$

$$\therefore p = \frac{1}{5} \tag{2.7}$$

Chapter 3

Conditional Probability

3.1 Given that E and F are events such that $P(E) = 0.6, P(F) = 0.3$ and $P(EF) = 0.2$, find $P(E \mid F)$ and $P(F \mid E)$.

Solution:

$$\Pr(E|F) = \frac{\Pr(EF)}{\Pr(F)} = \frac{0.2}{0.3} = \frac{2}{3} \tag{3.1}$$

$$\Pr(F|E) = \frac{\Pr(EF)}{\Pr(E)} = \frac{0.2}{0.6} = \frac{1}{3} \tag{3.2}$$

Chapter 4

Two Dice

4.1. Sum of Independent Random Variables

Two dice, one blue and one grey, are thrown at the same time. The event defined by the sum of the two numbers appearing on the top of the dice can have 11 possible outcomes 2, 3, 4, 5, 6, 6, 8, 9, 10, 11 and 12. A student argues that each of these outcomes has a probability $\frac{1}{11}$. Do you agree with this argument? Justify your answer.

4.1.1. The Uniform Distribution: Let $X_i \in \{1, 2, 3, 4, 5, 6\}, i = 1, 2$, be the random variables representing the outcome for each die. Assuming the dice to be fair, the probability mass function (pmf) is expressed as

$$p_{X_i}(n) = \Pr(X_i = n) = \begin{cases} \frac{1}{6} & 1 \leq n \leq 6 \\ 0 & otherwise \end{cases} \tag{4.1.1.1}$$

The desired outcome is

$$X = X_1 + X_2, \tag{4.1.1.2}$$

$$\implies X \in \{1, 2, \ldots, 12\} \tag{4.1.1.3}$$

The objective is to show that

$$p_X(n) \neq \frac{1}{11} \tag{4.1.1.4}$$

4.1.2. Convolution: From (4.1.1.2),

$$p_X(n) = \Pr\left(X_1 + X_2 = n\right) = \Pr\left(X_1 = n - X_2\right) \tag{4.1.2.1}$$

$$= \sum_k \Pr\left(X_1 = n - k | X_2 = k\right) p_{X_2}(k) \tag{4.1.2.2}$$

after unconditioning. $\because$ X_1 and X_2 are independent,

$$\Pr\left(X_1 = n - k | X_2 = k\right)$$

$$= \Pr\left(X_1 = n - k\right) = p_{X_1}(n-k) \tag{4.1.2.3}$$

From (4.1.2.2) and (4.1.2.3),

$$p_X(n) = \sum_k p_{X_1}(n-k) p_{X_2}(k) = p_{X_1}(n) * p_{X_2}(n) \tag{4.1.2.4}$$

where $*$ denotes the convolution operation. Substituting from (4.1.1.1) in (4.1.2.4),

$$p_X(n) = \frac{1}{6} \sum_{k=1}^{6} p_{X_1}(n-k) = \frac{1}{6} \sum_{k=n-6}^{n-1} p_{X_1}(k) \tag{4.1.2.5}$$

$$\because p_{X_1}(k) = 0, \quad k \leq 1, k \geq 6. \tag{4.1.2.6}$$

From (4.1.2.5),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{1}{6}\sum_{k=1}^{n-1} p_{X_1}(k) & 1 \le n-1 \le 6 \\ \frac{1}{6}\sum_{k=n-6}^{6} p_{X_1}(k) & 1 < n-6 \le 6 \\ 0 & n > 12 \end{cases} \tag{4.1.2.7}$$

Substituting from (4.1.1.1) in (4.1.2.7),

$$p_X(n) = \begin{cases} 0 & n < 1 \\ \frac{n-1}{36} & 2 \le n \le 7 \\ \frac{13-n}{36} & 7 < n \le 12 \\ 0 & n > 12 \end{cases} \tag{4.1.2.8}$$

satisfying (4.1.1.4).

4.1.3. <u>The Z-transform:</u> The Z-transform of $p_X(n)$ is defined as

$$P_X(z) = \sum_{n=-\infty}^{\infty} p_X(n) z^{-n}, \quad z \in \mathbb{C} \tag{4.1.3.1}$$

From (4.1.1.1) and (4.1.3.1),

$$P_{X_1}(z) = P_{X_2}(z) = \frac{1}{6}\sum_{n=1}^{6} z^{-n} \tag{4.1.3.2}$$

$$= \frac{z^{-1}\left(1 - z^{-6}\right)}{6\left(1 - z^{-1}\right)}, \quad |z| > 1 \tag{4.1.3.3}$$

upon summing up the geometric progression.

$$\because p_X(n) = p_{X_1}(n) * p_{X_2}(n), \tag{4.1.3.4}$$

$$P_X(z) = P_{X_1}(z)P_{X_2}(z) \tag{4.1.3.5}$$

The above property follows from Fourier analysis and is fundamental to signal processing. From (4.1.3.3) and (4.1.3.5),

$$P_X(z) = \left\{\frac{z^{-1}\left(1-z^{-6}\right)}{6\left(1-z^{-1}\right)}\right\}^2 \tag{4.1.3.6}$$

$$= \frac{1}{36}\frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{\left(1-z^{-1}\right)^2} \tag{4.1.3.7}$$

Using the fact that

$$p_X(n-k) \overset{\mathcal{H}}{\longleftrightarrow} ZP_X(z)z^{-k}, \tag{4.1.3.8}$$

$$nu(n) \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{z^{-1}}{\left(1-z^{-1}\right)^2} \tag{4.1.3.9}$$

after some algebra, it can be shown that

$$\frac{1}{36}\left[(n-1)\,u(n-1) - 2\,(n-7)\,u(n-7) + (n-13)\,u(n-13)\right] \overset{\mathcal{H}}{\longleftrightarrow} Z\frac{1}{36}\frac{z^{-2}\left(1-2z^{-6}+z^{-12}\right)}{\left(1-z^{-1}\right)^2} \tag{4.1.3.10}$$

where

$$u(n) = \begin{cases} 1 & n \geq 0 \\ \\ 0 & n < 0 \end{cases} \tag{4.1.3.11}$$

From (4.1.3.1), (4.1.3.7) and (4.1.3.10)

$$\begin{aligned} p_X(n) = \frac{1}{36} [& (n-1)\, u(n-1) \\ & -2\,(n-7)\, u(n-7) + (n-13)\, u(n-13)] \end{aligned} \tag{4.1.3.12}$$

which is the same as (4.1.2.8). Note that (4.1.2.8) can be obtained from (4.1.3.10) using contour integration as well.

4.1.4. The experiment of rolling the dice was simulated using Python for 10000 samples. These were generated using Python libraries for uniform distribution. The frequencies for each outcome were then used to compute the resulting pmf, which is plotted in Figure 4.1.4.1. The theoretical pmf obtained in (4.1.2.8) is plotted for comparison.

4.1.5. The python code is available in

```
/codes/sum/dice.py
```

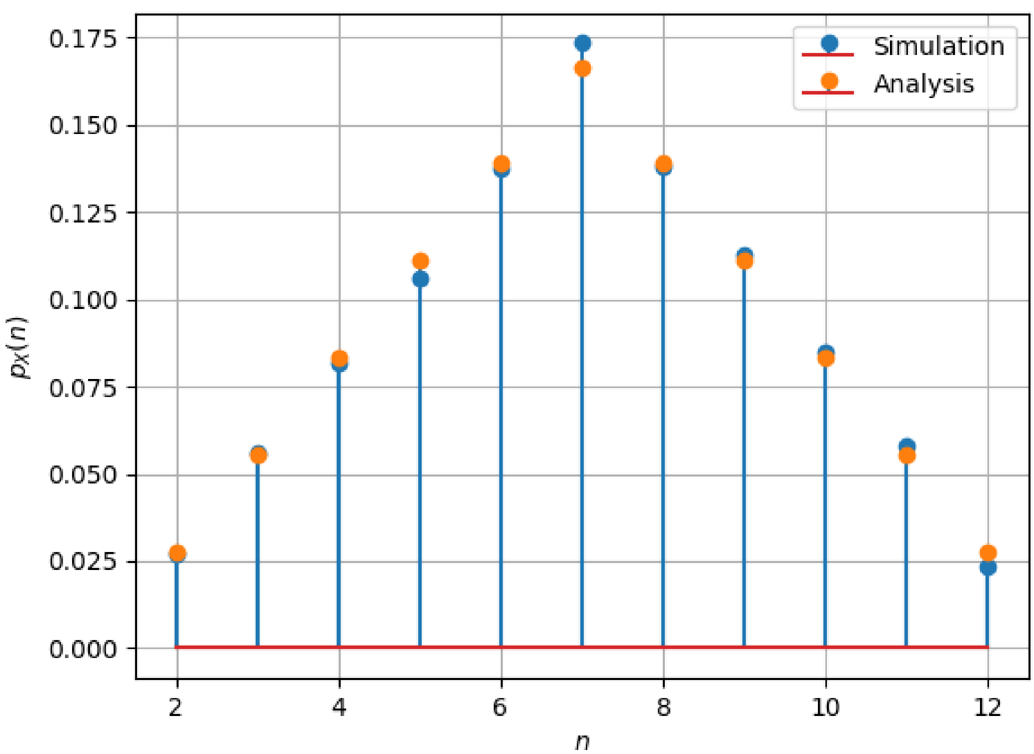


Figure 4.1.4.1: Plot of $p_X(n)$. Simulations are close to the analysis.

Chapter 5

Random Numbers

5.1. Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

5.1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .

Solution: Download the following files and execute the C program.

```
codes/exrand.c
codes/coeffs.h
```

5.1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{5.1.2.1}$$

Solution: The following code plots Fig. 5.1.2.1

```
codes/cdf_plot.py
```

5.1.3 Find a theoretical expression for $F_U(x)$.

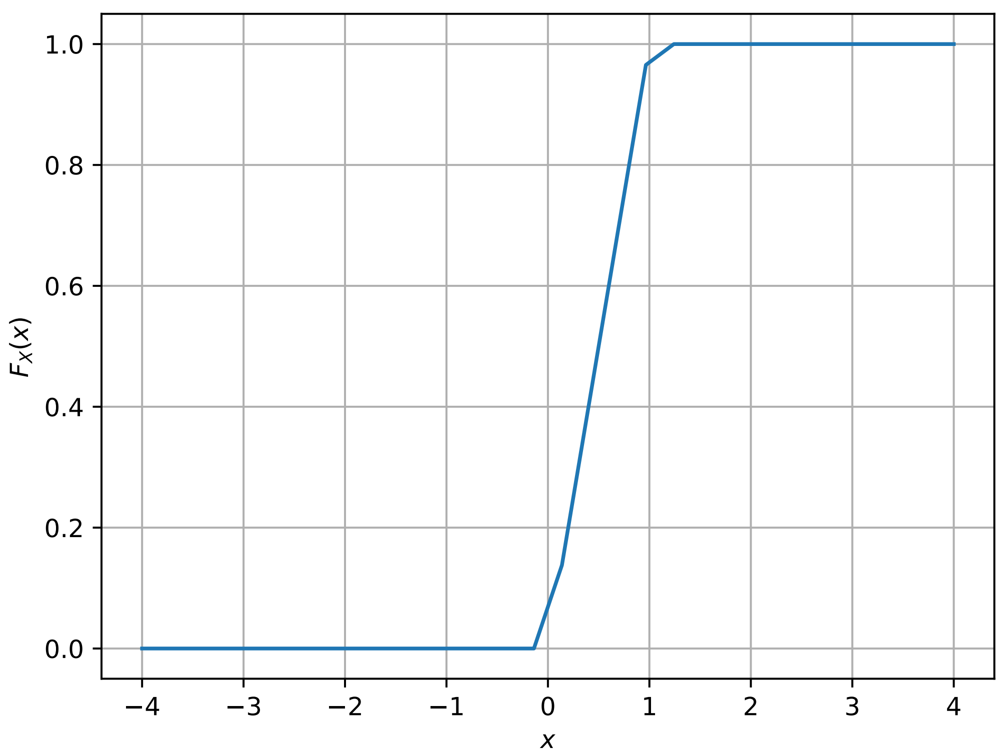


Figure 5.1.2.1: The CDF of U

5.1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{5.1.4.1}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{5.1.4.2}$$

Write a C program to find the mean and variance of U.

5.1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{5.1.5.1}$$

5.2. Central Limit Theorem

5.2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{5.2.1.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat

5.2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?

Solution: The CDF of X is plotted in Fig. 5.2.2.1

5.2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat.

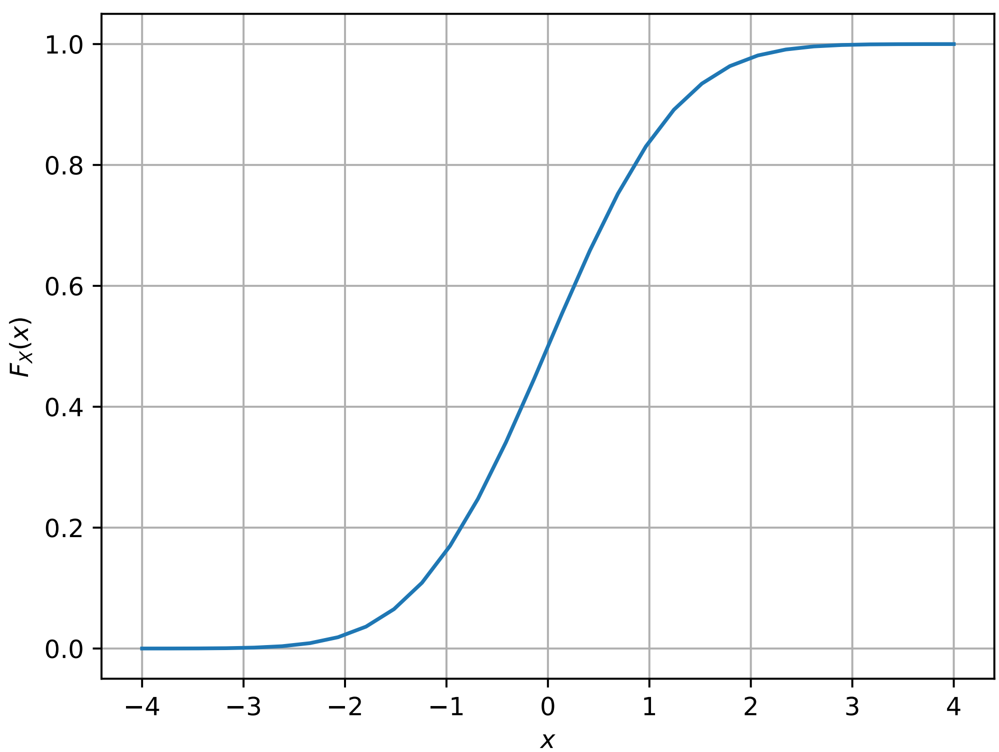


Figure 5.2.2.1: The CDF of X

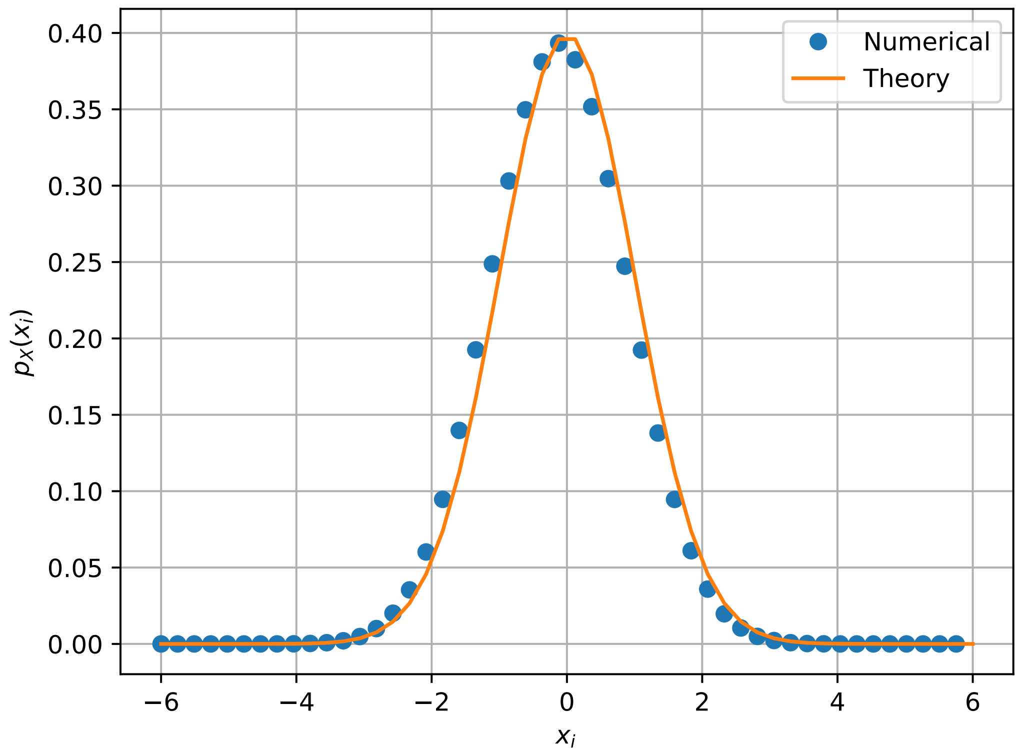


Figure 5.2.3.1: The PDF of X

The PDF of X is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \tag{5.2.3.1}$$

What properties does the PDF have?

Solution: The PDF of X is plotted in Fig. 5.2.3.1 using the code below

```
codes/pdf_plot.py
```

5.2.4 Find the mean and variance of X by writing a C program.

5.2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{5.2.5.1}$$

repeat the above exercise theoretically.

5.3. From Uniform to Other

5.3.1 Generate samples of

$$V = -2\ln(1-U) \tag{5.3.1.1}$$

and plot its CDF.

5.3.2 Find a theoretical expression for $F_V(x)$.

5.4. Triangular Distribution

5.4.1 Generate

$$T = U_1 + U_2 \tag{5.4.1.1}$$

5.4.2 Find the CDF of T.

5.4.3 Find the PDF of T.

5.4.4 Find the theoretical expressions for the PDF and CDF of T.

5.4.5 Verify your results through a plot.

Chapter 6

Maximum Likelihood Detection: BPSK

6.1. Maximum Likelihood

6.1.1 Generate equiprobable $X \in \{1, -1\}$.

6.1.2 Generate

$$Y = AX + N, \tag{6.1.2.1}$$

where $A = 5$ dB, and $N \sim \mathcal{N}(0, 1)$.

6.1.3 Plot Y using a scatter plot.

6.1.4 Guess how to estimate X from Y.

6.1.5 Find

$$P_{e|0} = \Pr\left(\hat{X} = -1 | X = 1\right) \tag{6.1.5.1}$$

and

$$P_{e|1} = \Pr\left(\hat{X} = 1 | X = -1\right) \tag{6.1.5.2}$$

6.1.6 Find P_e assuming that X has equiprobable symbols.

6.1.7 Verify by plotting the theoretical P_e with respect to A from 0 to 10 dB.

6.1.8 Now, consider a threshold δ while estimating X from Y. Find the value of δ that maximizes the theoretical P_e.

6.1.9 Repeat the above exercise when

$$p_X(0) = p \tag{6.1.9.1}$$

6.1.10 Repeat the above exercise using the MAP criterion.

Chapter 7

Transformation of Random Variables

7.1. Gaussian to Other

7.1.1 Let $X_1 \sim \mathcal{N}(0,1)$ and $X_2 \sim \mathcal{N}(0,1)$. Plot the CDF and PDF of

$$V = X_1^2 + X_2^2 \tag{7.1.1.1}$$

7.1.2 If

$$F_V(x) = \begin{cases} 1 - e^{-\alpha x} & x \geq 0 \\ 0 & x < 0, \end{cases} \tag{7.1.2.1}$$

find α.

7.1.3 Plot the CDF and PDf of

$$A = \sqrt{V} \tag{7.1.3.1}$$

7.2. Conditional Probability

7.2.1 Plot

$$P_e = \Pr\left(\hat{X} = -1|X = 1\right) \tag{7.2.1.1}$$

for

$$Y = AX + N, \tag{7.2.1.2}$$

where A is Raleigh with $E\left[A^2\right] = \gamma, N \sim \mathcal{N}\left(0, 1\right), X \in (-1, 1)$ for $0 \le \gamma \le 10$ dB.

7.2.2 Assuming that N is a constant, find an expression for P_e. Call this $P_e(N)$

7.2.3 For a function g,

$$E\left[g(X)\right] = \int_{-\infty}^{\infty} g(x) p_X(x)\, dx \tag{7.2.3.1}$$

Find $P_e = E\left[P_e(N)\right]$.

7.2.4 Plot P_e in problems 7.2.1 and 7.2.3 on the same graph w.r.t γ. Comment.

Chapter 8

Bivariate Random Variables: FSK

8.1. Two Dimensions

Let

$$\mathbf{y} = A\mathbf{x} + \mathbf{n}, \tag{4.1}$$

where

$$x \in (\mathbf{s}_0, \mathbf{s}_1), \mathbf{s}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.2}$$

$$\mathbf{n} = \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, n_1, n_2 \sim \mathcal{N}(0, 1). \tag{4.3}$$

8.1.1 Plot

$$\mathbf{y}|\mathbf{s}_0 \text{ and } \mathbf{y}|\mathbf{s}_1 \tag{8.1.1.1}$$

on the same graph using a scatter plot.

8.1.2 For the above problem, find a decision rule for detecting the symbols $\mathbf{s}_0$ and $\mathbf{s}_1$.

8.1.3 Plot

$$P_e = \Pr(\hat{\mathbf{x}} = \mathbf{s}_1 | \mathbf{x} = \mathbf{s}_0) \tag{8.1.3.1}$$

with respect to the SNR from 0 to 10 dB.

8.1.4 Obtain an expression for P_e. Verify this by comparing the theory and simulation plots on the same graph.

Chapter 9

Exercises

9.1. BPSK

1. The signal constellation diagram for BPSK is given by Fig. 1.1. The symbols s_0 and s_1 are equiprobable. $\sqrt{E_b}$ is the energy transmitted per bit. Assuming a zero mean additive white gaussian noise (AWGN) with variance $\frac{N_0}{2}$, obtain the symbols that are received.

   Solution: The possible received symbols are

$$y|s_0 = \sqrt{E_b} + n \tag{1.1}$$

$$y|s_1 = -\sqrt{E_b} + n \tag{1.2}$$

   where the AWGN $n \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

2. From Fig. 1.1 obtain a decision rule for BPSK

   Solution: The decision rule is

$$y \underset{s_1}{\overset{s_0}{\gtrless}} 0 \tag{2.1}$$

3. Repeat the previous exercise using the MAP criterion.

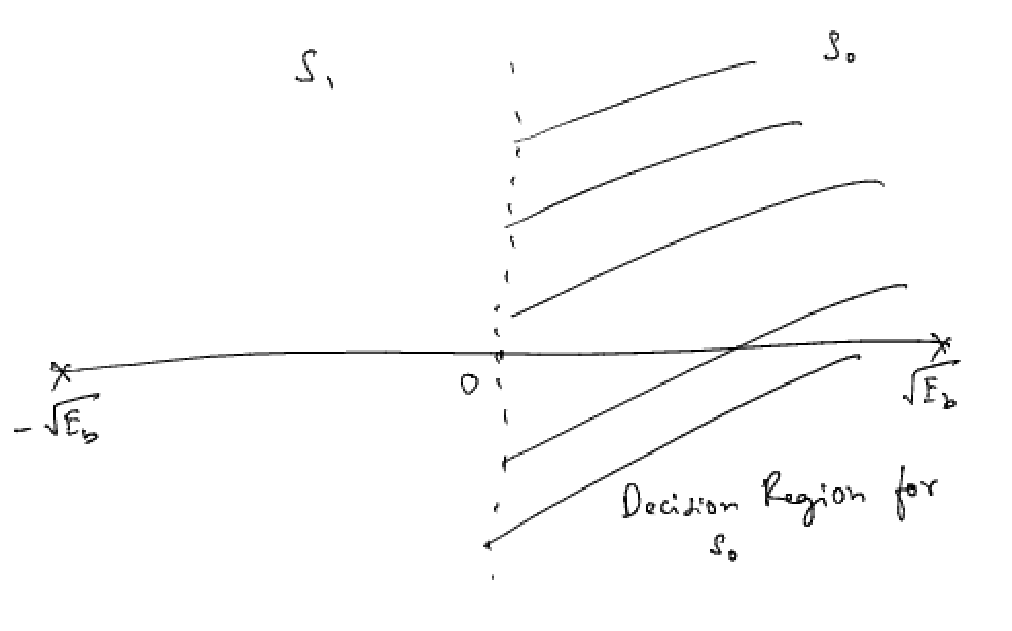


Figure 1.1:

4. Using the decision rule in Problem 2, obtain an expression for the probability of error for BPSK.

Solution: Since the symbols are equiprobable, it is sufficient if the error is calculated assuming that a 0 was sent. This results in

$$P_e = \Pr\left(y < 0 | s_0\right) = \Pr\left(\sqrt{E_b} + n < 0\right) \tag{4.1}$$

$$= \Pr\left(-n > \sqrt{E_b}\right) = \Pr\left(n > \sqrt{E_b}\right) \tag{4.2}$$

since n has a symmetric pdf. Let $w \sim \mathcal{N}(0,1)$. Then $n = \sqrt{\frac{N_0}{2}}w$. Substituting this in

(4.2),

$$P_e = \Pr\left(\sqrt{\frac{N_0}{2}}w > \sqrt{E_b}\right) = \Pr\left(w > \sqrt{\frac{2E_b}{N_0}}\right) \tag{4.3}$$

$$= Q\left(\sqrt{\frac{2E_b}{N_0}}\right) \tag{4.4}$$

where $Q\left(x\right) \triangleq \Pr\left(w > x\right), x \geq 0$.

5. The PDF of $w \sim \mathcal{N}\left(0, 1\right)$ is given by

$$p_w(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty \tag{5.1}$$

and the complementary error function is defined as

$$\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2}\, dt. \tag{5.2}$$

Show that

$$Q(x) = \frac{1}{2}\,\text{erfc}\left(\frac{x}{\sqrt{2}}\right) \tag{5.3}$$

6. Verify the bit error rate (BER) plots for BPSK through simulation and analysis for 0 to 10 dB.

Solution: The following code

```
codes/bpsk_ber.py
```

yields Fig. 6.1

7. Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}}\, d\theta \tag{7.1}$$

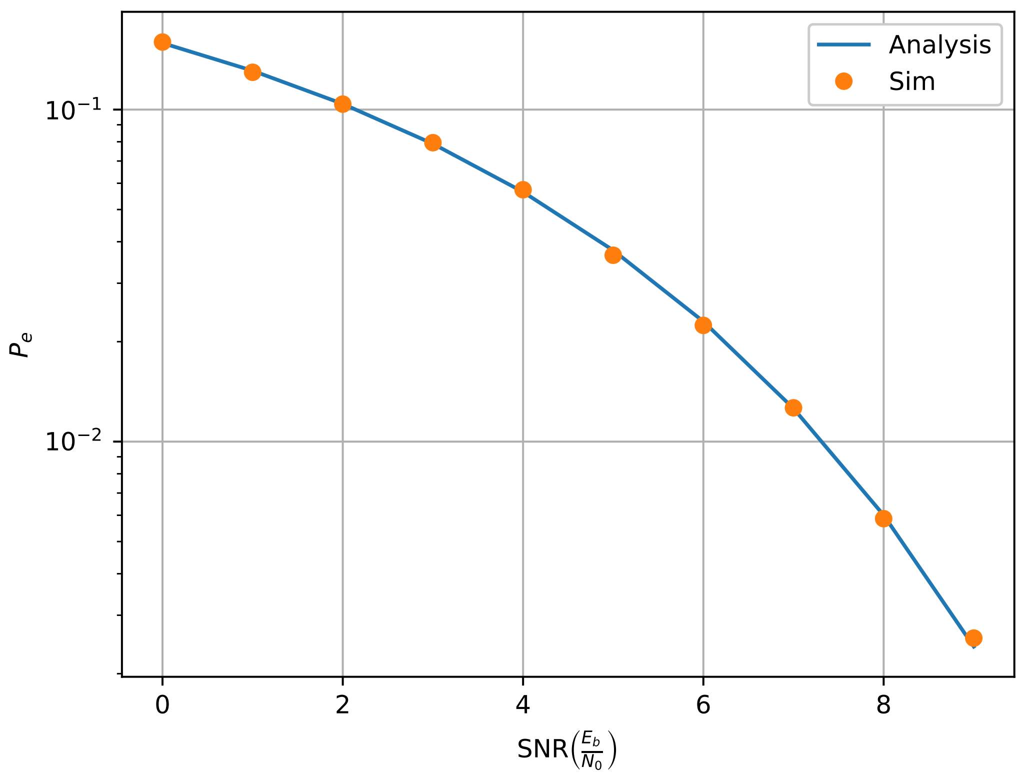


Figure 6.1:

9.2. Coherent BFSK

1. The signal constellation for binary frequency shift keying (BFSK) is given in Fig. 1.1. Obtain the equations for the received symbols.

   Solution: The received symbols are given by

$$\mathbf{y}|s_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{1.1}$$

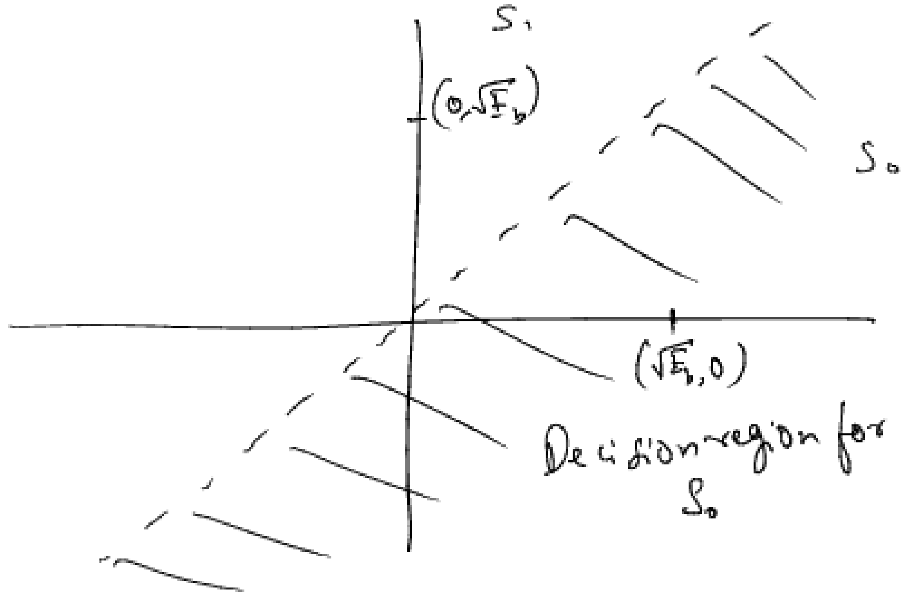


Figure 1.1:

and

$$\mathbf{y}|s_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix} + \begin{pmatrix} n_1 \\ n_2 \end{pmatrix}, \tag{1.2}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$. and $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$.

2. Obtain a decision rule for BFSK from Fig. 1.1.

   Solution: The decision rule is

$$y_1 \underset{s_1}{\overset{s_0}{\gtrless}} y_2 \tag{2.1}$$

3. Repeat the previous exercise using the MAP criterion.

4. Derive and plot the probability of error. Verify through simulation.

9.3. QPSK

1. Let

$$\mathbf{r} = \mathbf{s} + \mathbf{n} \tag{1.1}$$

where $\mathbf{s} \in \{s_0, s_1, s_2, s_3\}$ and

$$\mathbf{s}_0 = \begin{pmatrix} \sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_1 = \begin{pmatrix} 0 \\ \sqrt{E_b} \end{pmatrix}, \tag{1.2}$$

$$\mathbf{s}_2 = \begin{pmatrix} -\sqrt{E_b} \\ 0 \end{pmatrix}, \mathbf{s}_3 = \begin{pmatrix} 0 \\ -\sqrt{E_b} \end{pmatrix}, \tag{1.3}$$

$$E\left[\mathbf{n}\right] = \mathbf{0}, E\left[\mathbf{n}\mathbf{n}^T\right] = \sigma^2\mathbf{I} \tag{1.4}$$

(a) Show that the MAP decision for detecting $\mathbf{s}_0$ results in

$$|r|_2 < r_1 \tag{1.5}$$

(b) Express $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right)$ in terms of r_1, r_2. Let $X = n_2 - n_1, Y = -n_2 - n_1$, where $\mathbf{n} = (n_1, n_2)$. Their correlation coefficient is defined as

$$\rho = \frac{E\left[\left(X - \mu_x\right)\left(Y - \mu_y\right)\right]}{\sigma_x \sigma_y} \tag{1.6}$$

X and Y are said to be uncorrelated if $\rho = 0$

(c) Show that if X and Y are uncorrelated Verify this numerically.

(d) Show that X and Y are independent, i.e. $p_{XY}(x,y) = p_X(x)p_Y(y)$.

(e) Show that $X, Y \sim \mathcal{N}\left(0, 2\sigma^2\right)$.

(f) Show that $\Pr\left(\hat{\mathbf{s}} = \mathbf{s}_0 | \mathbf{s} = \mathbf{s}_0\right) = \Pr\left(X < A, Y < A\right)$.

(g) Find $\Pr\left(X < A, Y < A\right)$.

(h) Verify the above through simulation.

9.4. M-PSK

1. Consider a system where $\mathbf{s}_i = \begin{pmatrix} \cos\left(\frac{2\pi i}{M}\right) \\ \cos\left(\frac{2\pi i}{M}\right) \end{pmatrix}, i = 0, 1, \dots M-1$. Let

$$\mathbf{r}|s_0 = \begin{pmatrix} r_1 \\ r_2 \end{pmatrix} = \begin{pmatrix} \sqrt{E_s} + n_1 \\ n_2 \end{pmatrix} \tag{1.1}$$

where $n_1, n_2 \sim \mathcal{N}\left(0, \frac{N_0}{2}\right)$.

(a) Substituting

$$r_1 = R\cos\theta \tag{1.2}$$

$$r_2 = R\sin\theta \tag{1.3}$$

show that the joint pdf of R, θ is

$$p\left(R, \theta\right) = \frac{R}{\pi N_0} \exp\left(-\frac{R^2 - 2R\sqrt{E_s}\cos\theta + E_s}{N_0}\right) \tag{1.4}$$

(b) Show that

$$\lim_{\alpha\to\infty}\int_0^\infty (V-\alpha)\,e^{-(V-\alpha)^2}\,dV = 0 \tag{1.5}$$

$$\lim_{\alpha\to\infty}\int_0^\infty e^{-(V-\alpha)^2}\,dV = \sqrt{\pi} \tag{1.6}$$

(c) Using the above, evaluate

$$\int_0^\infty V\exp\left\{-\left(V^2-2V\sqrt{\gamma}\cos\theta+\gamma\right)\right\}\,dV \tag{1.7}$$

for large values of γ.

(d) Find a compact expression for

$$I = 1-\sqrt{\frac{\gamma}{\pi}}\int_{-\frac{\pi}{M}}^{\frac{\pi}{M}} e^{-\gamma\sin^2\theta}\cos\theta\,d\theta \tag{1.8}$$

(e) Find $P_{e|\mathbf{s}_0}$.

9.5. Noncoherent BFSK

9.5.1 Show that

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos\theta}\,d\theta \tag{9.5.1.1}$$

$$I_0(x) = \frac{1}{2\pi}\int_0^{2\pi} e^{x\cos(\theta-\phi)}\,d\theta \tag{9.5.1.2}$$

$$\frac{1}{2\pi}\int_0^{2\pi} e^{m_1\cos\theta+m_2\sin\theta}\,d\theta = I_0\left(\sqrt{m_1^2+m_2^2}\right) \tag{9.5.1.3}$$

where the modified Bessel function of order n (integer) is defined as

$$I_n(x) = \frac{1}{\pi} \int_0^{\pi} e^{x \cos\theta} \cos n\theta \, d\theta \tag{9.5.1.4}$$

9.5.2 Let

$$\mathbf{r}|0 = \sqrt{E_b} \begin{pmatrix} \cos\phi_0 \\ \sin\phi_0 \\ 0 \\ 0 \end{pmatrix} + \mathbf{n}_0, \mathbf{r}|1 = \sqrt{E_b} \begin{pmatrix} 0 \\ 0 \\ \cos\phi_1 \\ \sin\phi_1 \end{pmatrix} + \mathbf{n}_1 \tag{9.5.2.1}$$

where $\mathbf{n}_0, \mathbf{n}_1 \sim \mathcal{N}\left(\mathbf{0}, \frac{N_0}{2}\mathbf{I}\right)$.

(a) Taking $\mathbf{r} = (r_1, r_2, r_3, r_4)^T$,, find the pdf $p\left(\mathbf{r}|0, \phi_0\right)$ in terms of $r_1, r_2, r_3, r_4, \phi, E_b$ and N_0. Assume that all noise variables are independent.

(b) If ϕ_0 is uniformly distributed between 0 and 2π, find $p\left(\mathbf{r}|0\right)$. Note that this expression will no longer contain ϕ_0.

(c) Show that the ML detection criterion for this scheme is

$$I_0\left(k\sqrt{r_1^2 + r_2^2}\right) \underset{1}{\overset{0}{\gtrless}} I_0\left(k\sqrt{r_3^2 + r_4^2}\right) \tag{9.5.2.2}$$

where k is a constant.

(d) The above criterion reduces to something simpler. Can you guess what it is? Justify your answer.

(e) Show that

$$P_{e|0} = \Pr\left(r_1^2 + r_2^2 < r_3^2 + r_4^2 | 0\right) \tag{9.5.2.3}$$

(f) Show that the pdf of $Y = r_3^2 + r_4^2$ id

$$p_Y(y) = \frac{1}{N_0} e^{-\frac{y}{N_0}}, y > 0 \tag{9.5.2.4}$$

(g) Find

$$g(r_1, r_2) = \Pr\left(r_1^2 + r_2^2 < Y | 0, r_1, r_2\right). \tag{9.5.2.5}$$

(h) Show that $E\left[e^{-\frac{X^2}{2\sigma^2}}\right] = \frac{1}{\sqrt{2}} e^{-\frac{\mu^2}{4\sigma^2}}$ for $X \sim \mathcal{N}\left(\mu, \sigma^2\right)$.

(i) Now show that

$$E\left[g(r_1, r_2)\right] = \frac{1}{2} e^{-\frac{E_b}{2N_0}}. \tag{9.5.2.6}$$

9.5.3 Let $U, V \sim \mathcal{N}\left(0, \frac{k}{2}\right)$ be i.i.d. Assuming that

$$U = \sqrt{R}\cos\Theta \tag{9.5.3.1}$$

$$V = \sqrt{R}\sin\Theta \tag{9.5.3.2}$$

(a) Compute the jacobian for U, V with respect to X and Θ defined by

$$J = \det\begin{pmatrix} \frac{\partial U}{\partial R} & \frac{\partial U}{\partial \Theta} \\ \frac{\partial V}{\partial R} & \frac{\partial V}{\partial \Theta} \end{pmatrix} \tag{9.5.3.3}$$

(b) The joint pdf for R, Θ is given by,

$$p_{R,\Theta}(r, \theta) = p_{U,V}(u, v)\, J|_{u=\sqrt{r}\cos\theta, v=\sqrt{r}\sin\theta} \tag{9.5.3.4}$$

Show that

$$p_R(r) = \begin{cases} \frac{1}{k}e^{-\frac{r}{k}} & r > 0, \\ 0 & r < 0, \end{cases} \tag{9.5.3.5}$$

assuming that Θ is uniformly distributed between 0 to 2π.

(c) Show that the pdf of $Y = R_1 - R_2$, where R_1 and R_2 are i.i.d. and have the same distribution as R is

$$p_Y(y) = \frac{1}{2k}e^{-\frac{|y|}{k}} \tag{9.5.3.6}$$

(d) Find the pdf of

$$Z = p + \sqrt{p}\left[U\cos\phi + V\sin\phi\right] \tag{9.5.3.7}$$

where ϕ is a constant.

(e) Find $\Pr(Y > Z)$.

(f) If $U \sim \mathcal{N}\left(m_1, \frac{k}{2}\right), V \sim \mathcal{N}\left(m_2, \frac{k}{2}\right)$, where m_1, m_2, k are constants, show that the pdf of

$$R = \sqrt{U^2 + V^2} \tag{9.5.3.8}$$

is

$$p_R(r) = \frac{e^{-\frac{r+m}{k}}}{k} I_0\left(\frac{2\sqrt{mr}}{k}\right), \quad m = \sqrt{m_1^2 + m_2^2} \tag{9.5.3.9}$$

(g) Show that

$$I_0(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{4^n (n!)^2} \tag{9.5.3.10}$$

(h) If

$$p_Z(z) = \begin{cases} \frac{1}{k} e^{-\frac{z}{k}} & z \ge 0 \\ 0 & z < 0 \end{cases} \tag{9.5.3.11}$$

find $\Pr(R < Z)$.

9.6. Craig's Formula and MGF

9.6.1 The Moment Generating Function (MGF) of X is defined as

$$M_X(s) = E\left[e^{sX}\right] \tag{9.6.1.1}$$

where X is a random variable and $E[\cdot]$ is the expectation.

(a) Let $Y \sim \mathcal{N}(0, 1)$. Define

$$Q(x) = \Pr(Y > x), x > 0 \tag{9.6.1.2}$$

Show that

$$Q(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} e^{-\frac{x^2}{2\sin^2\theta}} \, d\theta \tag{9.6.1.3}$$

(b) Let $h \sim \mathcal{CN}\left(0, \frac{\Omega}{2}\right), n \sim \mathcal{CN}\left(0, \frac{N_0}{2}\right)$. Find the distribution of $|h|^2$.

(c) Let

$$P_e = \Pr\left(\Re\left\{h^* y\right\} < 0\right), \text{ where } y = \left(\sqrt{E_s} h + n\right), \tag{9.6.1.4}$$

Show that

$$P_e = \int_0^\infty Q\left(\sqrt{2x}\right) p_A(x)\, dx \tag{9.6.1.5}$$

where $A = \frac{E_s |h|^2}{N_0}$.

(d) Show that

$$P_e = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} M_A\left(-\frac{1}{\sin^2\theta}\right) d\theta \tag{9.6.1.6}$$

(e) compute $M_A(s)$.

(f) Find P_e.

(g) If $\gamma = \frac{\Omega E_s}{N_0}$, show that $P_e < \frac{1}{2\gamma}$.