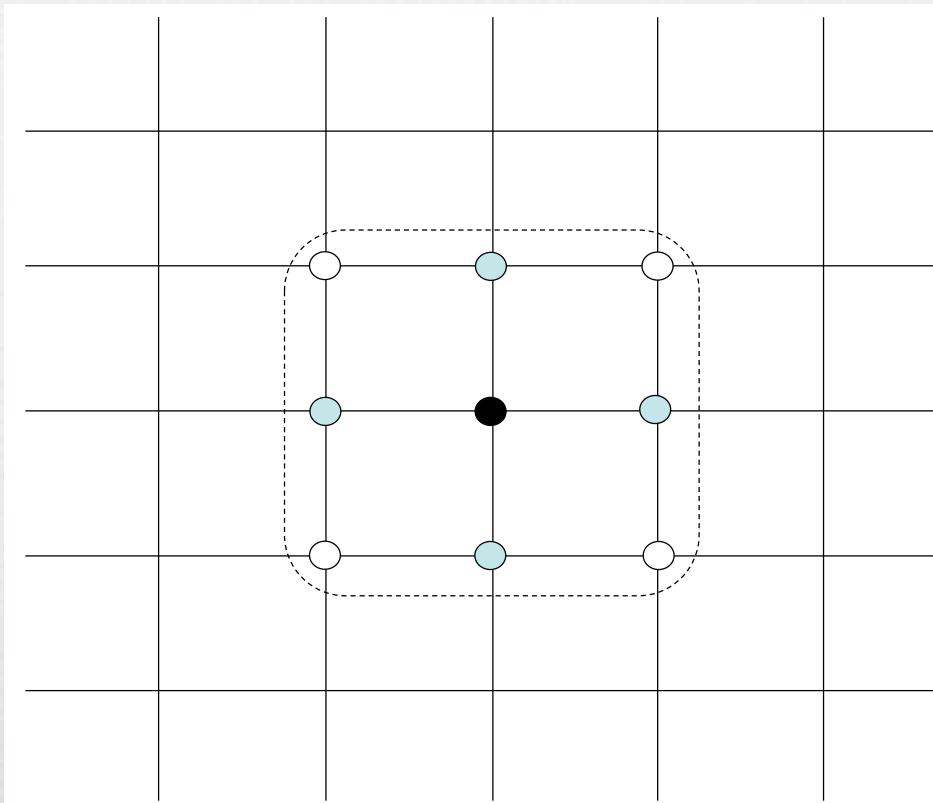


RELAXATION METHOD: TWO FULLY WORKED OUT EXAMPLES

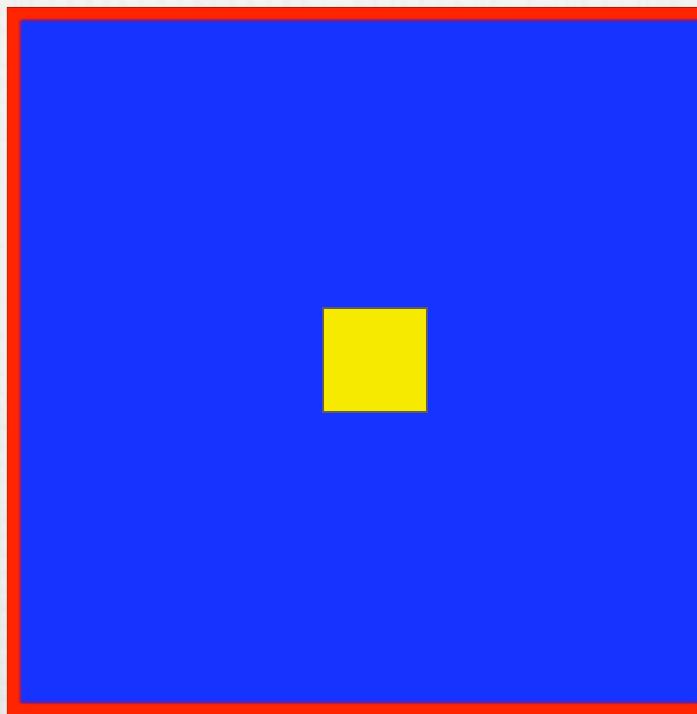
(GAUSS-SEIDEL METHOD)

BACKGROUND

$$U_{i,j} = \frac{1}{4} [U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j-1}] + \pi\rho(i\Delta, j\Delta)\Delta^2$$



EXAMPLE 1



V=0 VOLT

V=4 VOLT

INITIAL CONFIGURATION

0	0	0	0	0
0	0.00	0.00	0.00	0
0	0.00	4.00	0.00	0
0	0.00	0.00	0.00	0
0	0	0	0	0

Red are boundary conditions

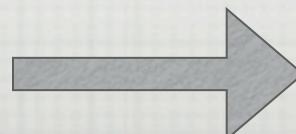
Blue defines the problem

SLOW LOOP

FAST LOOP

0	0	0	0	0
0	0.00	0.00	0.00	0
0	0.00	4.00	0.00	0
0	0.00	0.00	0.00	0
0	0	0	0	0

NEXT STEP



0	0	0	0	0
0	0.00	1.00	0.25	0
0	1.00	4.00	1.06	0
0	0.25	1.06	0.53	0
0	0	0	0	0

FIRST STEP

0	0	0	0	0
0	0.00	1.00	0.25	0
0	1.00	4.00	1.06	0
0	0.25	1.06	0.53	0
0	0	0	0	0

SECOND STEP

0	0	0	0	0
0	0.50	1.19	0.56	0
0	1.19	4.00	1.27	0
0	0.56	1.27	0.64	0
0	0	0	0	0

THIRD STEP

0	0	0	0	0
0	0.59	1.29	0.64	0
0	1.29	4.00	1.32	0
0	0.64	1.32	0.66	0
0	0	0	0	0

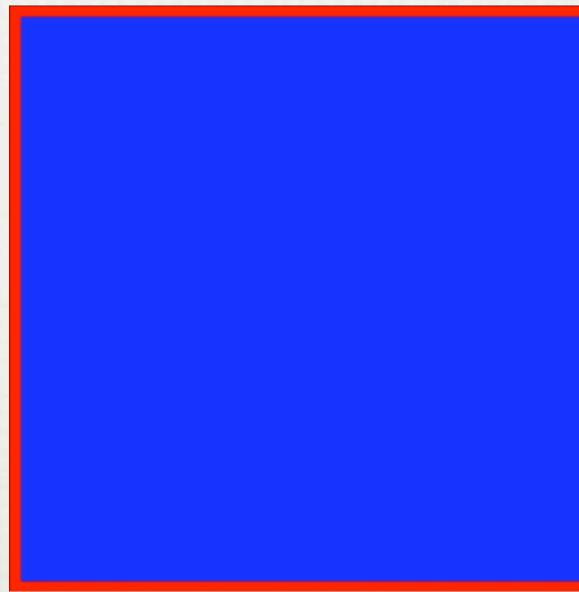
FOURTH STEP

0	0	0	0	0
0	0.64	1.32	0.66	0
0	1.32	4.00	1.33	0
0	0.66	1.33	0.66	0
0	0	0	0	0

FIFTH STEP

0	0	0	0	0
0	0.67	1.33	0.67	0
0	1.33	4.00	1.33	0
0	0.67	1.33	0.67	0
0	0	0	0	0

EXAMPLE 2



V=1 VOLT

V=0 VOLT

1.00	1.00	1.00	1.00	1.00
1.00	0.00	0.00	0.00	1.00
1.00	0.00	0.00	0.00	1.00
1.00	0.00	0.00	0.00	1.00
1.00	1.00	1.00	1.00	1.00

(0)

1.00	1.00	1.00	1.00	1.00
1.00	0.50	0.38	0.59	1.00
1.00	0.38	0.19	0.45	1.00
1.00	0.59	0.45	0.72	1.00
1.00	1.00	1.00	1.00	1.00

(1)

1.00	1.00	1.00	1.00	1.00
1.00	0.69	0.62	0.77	1.00
1.00	0.62	0.53	0.75	1.00
1.00	0.77	0.75	0.88	1.00
1.00	1.00	1.00	1.00	1.00

(2)

1.00	1.00	1.00	1.00	1.00
1.00	0.81	0.78	0.88	1.00
1.00	0.78	0.77	0.88	1.00
1.00	0.88	0.88	0.94	1.00
1.00	1.00	1.00	1.00	1.00

(3)

1.00	1.00	1.00	1.00	1.00
1.00	0.89	0.88	0.94	1.00
1.00	0.88	0.88	0.94	1.00
1.00	0.94	0.94	0.97	1.00
1.00	1.00	1.00	1.00	1.00

(4)

1.00	1.00	1.00	1.00	1.00
1.00	0.94	0.94	0.97	1.00
1.00	0.94	0.94	0.97	1.00
1.00	0.97	0.97	0.99	1.00
1.00	1.00	1.00	1.00	1.00

(5)

1.00	1.00	1.00	1.00	1.00
1.00	0.97	0.97	0.99	1.00
1.00	0.97	0.97	0.99	1.00
1.00	0.99	0.99	0.99	1.00
1.00	1.00	1.00	1.00	1.00

(6)

1.00	1.00	1.00	1.00	1.00
1.00	0.99	0.99	0.99	1.00
1.00	0.99	0.99	0.99	1.00
1.00	0.99	0.99	1.00	1.00
1.00	1.00	1.00	1.00	1.00

(7)

1.00	1.00	1.00	1.00	1.00
1.00	0.99	0.99	1.00	1.00
1.00	0.99	0.99	1.00	1.00
1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00

(8)

1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00
1.00	1.00	1.00	1.00	1.00

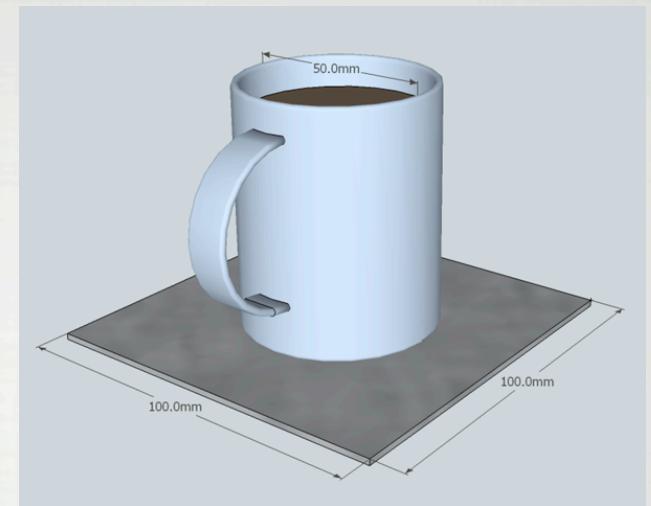
(FINAL)

PHY-4810

COMPUTATIONAL PHYSICS

LECTURE 11: SOLVING PDES; THE HEAT EQUATION

$$\frac{\partial u}{\partial t} - \alpha \nabla^2 u = 0$$

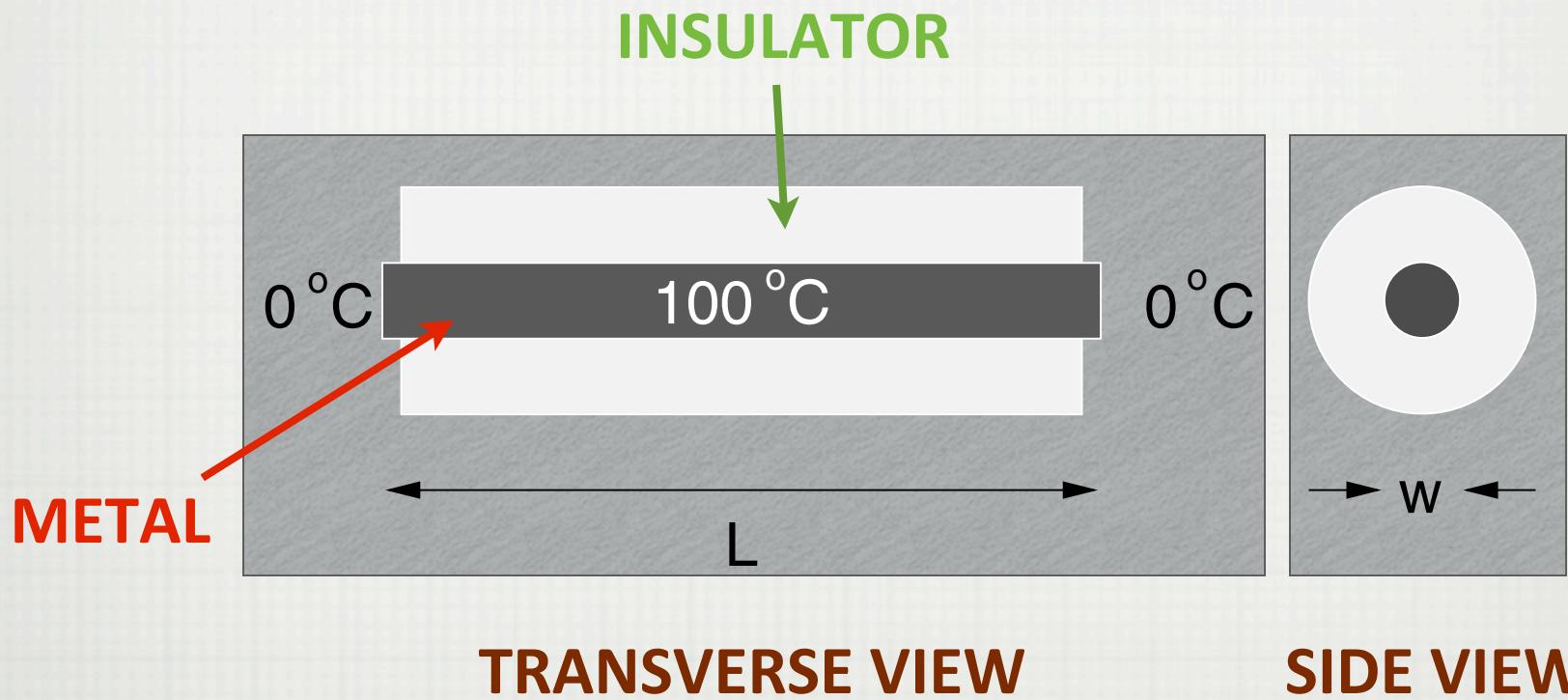


DEFINITION

Partial differential equations (PDEs) are a type of differential equation involving an unknown function (or functions) of several independent variables and their partial derivatives with respect to those variables.

$$F(x_1, \dots, x_n, u, \frac{\partial}{\partial x_1} u, \dots, \frac{\partial}{\partial x_n} u, \frac{\partial^2}{\partial x_1 \partial x_1} u, \dots, \frac{\partial^2}{\partial x_1 \partial x_2} u, \dots) = 0$$

PROBLEM WE WILL SOLVE TODAY



HOW DOES THE TEMPERATURE CHANGE WITH
TIME ALONG THE ROD?

PHYSICS: HEAT EQUATION

Translation: Heat flows from high to low temperature (“-” sign) at a rate that depends on the materials (“K”: thermal conductivity)

$$\mathbf{H} = -K \nabla T(\mathbf{x}, t)$$

Total amount of thermal energy: depends on the heat capacity (“C”: specific heat) and density (rho).

$$Q(t) = \int d\mathbf{x} C \rho(\mathbf{x}) T(\mathbf{x}, t)$$

A (TINY) BIT OF MATH

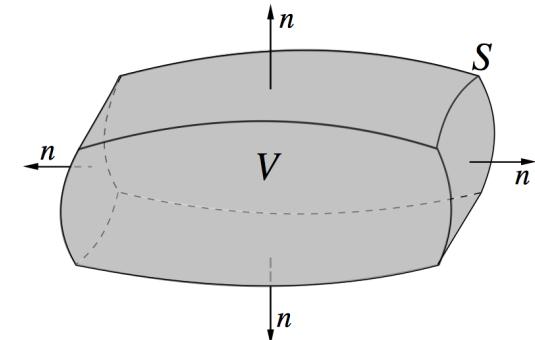
Total change in heat is given by the amount of heat going through the surface.

$$\frac{d}{dt}Q + \int_S ds \mathbf{H} \cdot \mathbf{n} = 0$$

D.T. $\rightarrow \frac{d}{dt}Q = - \int_V d\mathbf{x} (\nabla \cdot \mathbf{H})$

P.S. $\Rightarrow \frac{\partial}{\partial t} T = \frac{K}{C\rho(\mathbf{x})} \nabla^2 T$

DIVERGENCE THEOREM



$$\iiint_V (\nabla \cdot \mathbf{F}) dV = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS.$$

PREVIOUS SLIDE

$$Q(t) = \int d\mathbf{x} C \rho(\mathbf{x}) T(\mathbf{x}, t)$$
$$\mathbf{H} = -K \nabla T(\mathbf{x}, t)$$

HEAT EQUATION (PARABOLIC)

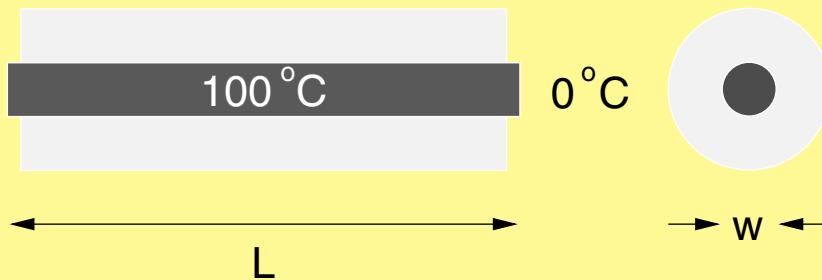
$$\frac{\partial}{\partial t} T = \frac{K}{C\rho(\mathbf{x})} \nabla^2 T$$

Boundary condition	Elliptic (Poisson equation)	Hyperbolic (Wave equation)	Parabolic (Heat equation)
Dirichlet open surface	Underspecified	Underspecified	<i>Unique and stable (1D)</i>
Dirichlet closed surface	<i>Unique and stable</i>	Overspecified	Overspecified
Neumann open surface	Underspecified	Underspecified	<i>Unique and stable (1D)</i>
Neumann closed surface	<i>Unique and stable</i>	Overspecified	Overspecified
Cauchy open surface	Unphysical	<i>Unique and stable</i>	Overspecified
Cauchy closed surface	Overspecified	Overspecified	Overspecified

EXAMPLE OF BOUNDARY CONDITIONS:

$$T(x,t=0)=100$$

$$T(x=0,t)=T(x=L,t)=0$$



NUMERICAL SOLUTION

- *Same philosophy as that adopted for Laplace/Poisson: convert the differential equation into a discretized form.*
- *Time derivative (forward difference):*

$$\frac{\partial T(x, t)}{\partial t} \simeq \frac{T(x, t + \Delta t) - T(x, t)}{\Delta t}$$

- *Laplacian [1D] (central difference):*

$$\frac{\partial^2 T(x, t)}{\partial x^2} \simeq \frac{T(x + \Delta x, t) + T(x - \Delta x, t) - 2T(x, t)}{(\Delta x)^2}$$

- *Finally, with i running over space variable and j over time:*

$$T_{i,j+1} = T_{i,j} + \eta [T_{i+1,j} + T_{i-1,j} - 2T_{i,j}] \quad \eta = \frac{K\Delta t}{C\rho\Delta x^2}$$

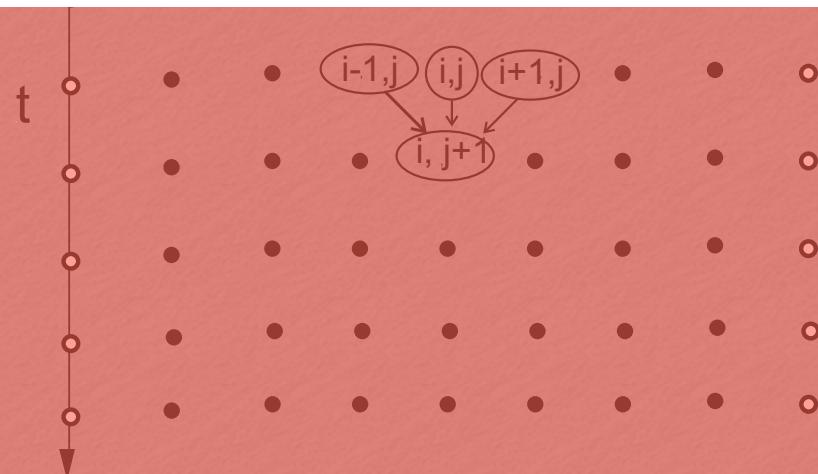
TO SUM UP

- This algorithm is called “explicit” since it provides a solution explicitly in terms of other solutions

$$T_{i,j+1} = T_{i,j} + \eta [T_{i+1,j} + T_{i-1,j} - 2T_{i,j}]$$



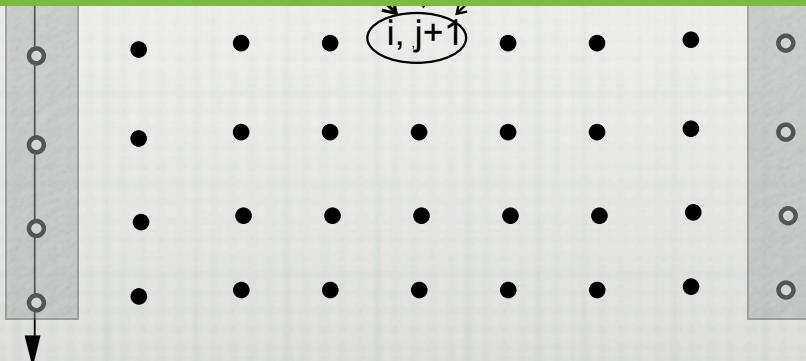
What would be an “implicit” algorithm?



LEAPFROG

- We only know the answers on three sides of the grid: we can't use the relaxation method used for electrostatics problems
- Note that this is a simple consequence of the fact we do not know the future in advance!

LEAPFROG ALGORITHM



VALUES

WE NEED TO
JUMP FROM
SOLUTION TO
SOLUTION

RESULT OF VON NEUMANN ANALYSIS

- *How can we choose parameters for discretization?*

$$T_{i,j+1} = T_{i,j} + \eta [T_{i+1,j} + T_{i-1,j} - 2T_{i,j}]$$

- *Analysis shows that:*

$$\eta = \frac{K\Delta t}{C\rho\Delta x^2} < \frac{1}{4}$$

DISCRETIZATION
IN X CANNOT BE
AS SMALL AS WE
CAN FOR A FIXED
DISCRETIZATION
IN TIME!

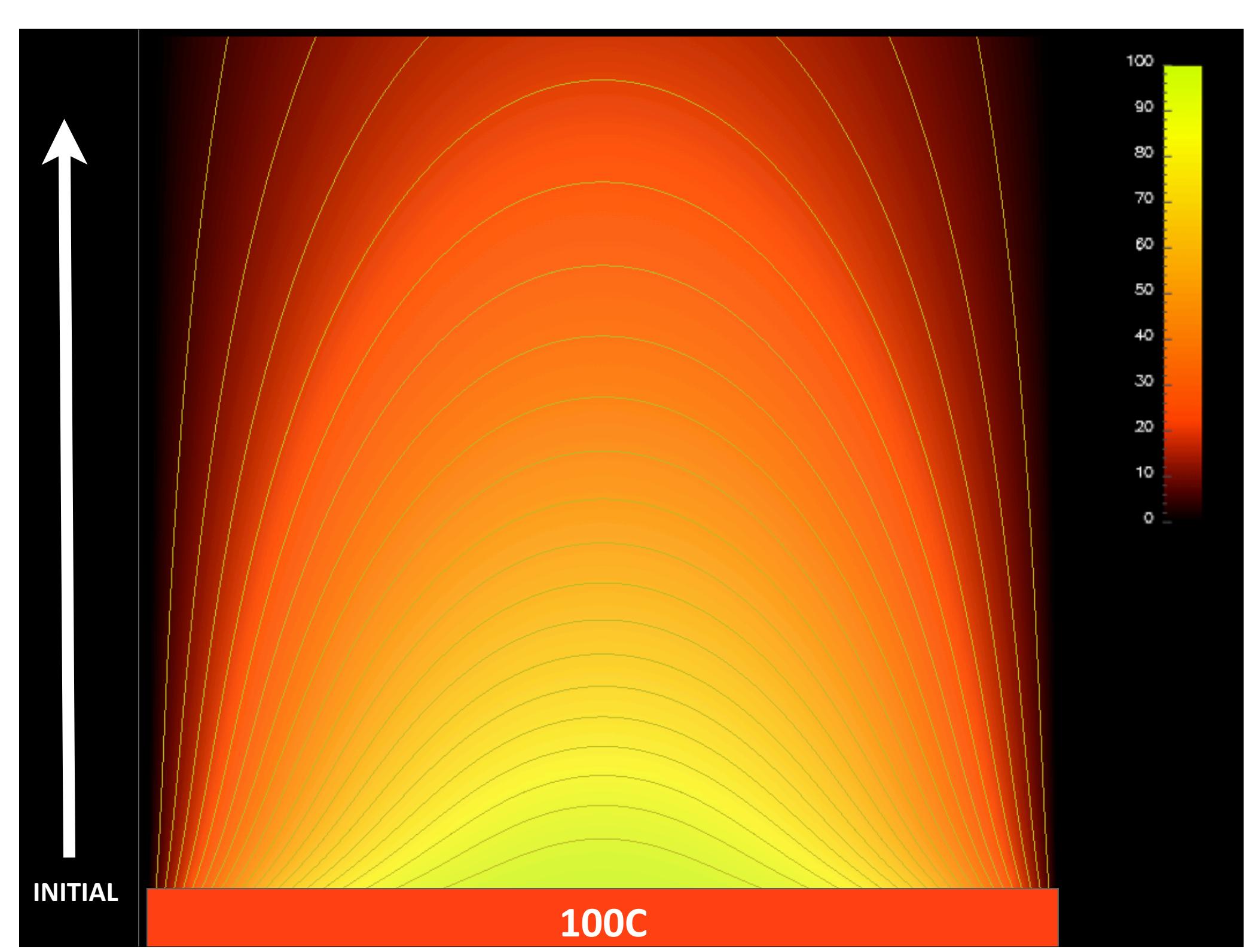
APPLICATION #1: HOT ROD PLACED BETWEEN TWO COLD RESERVOIRS

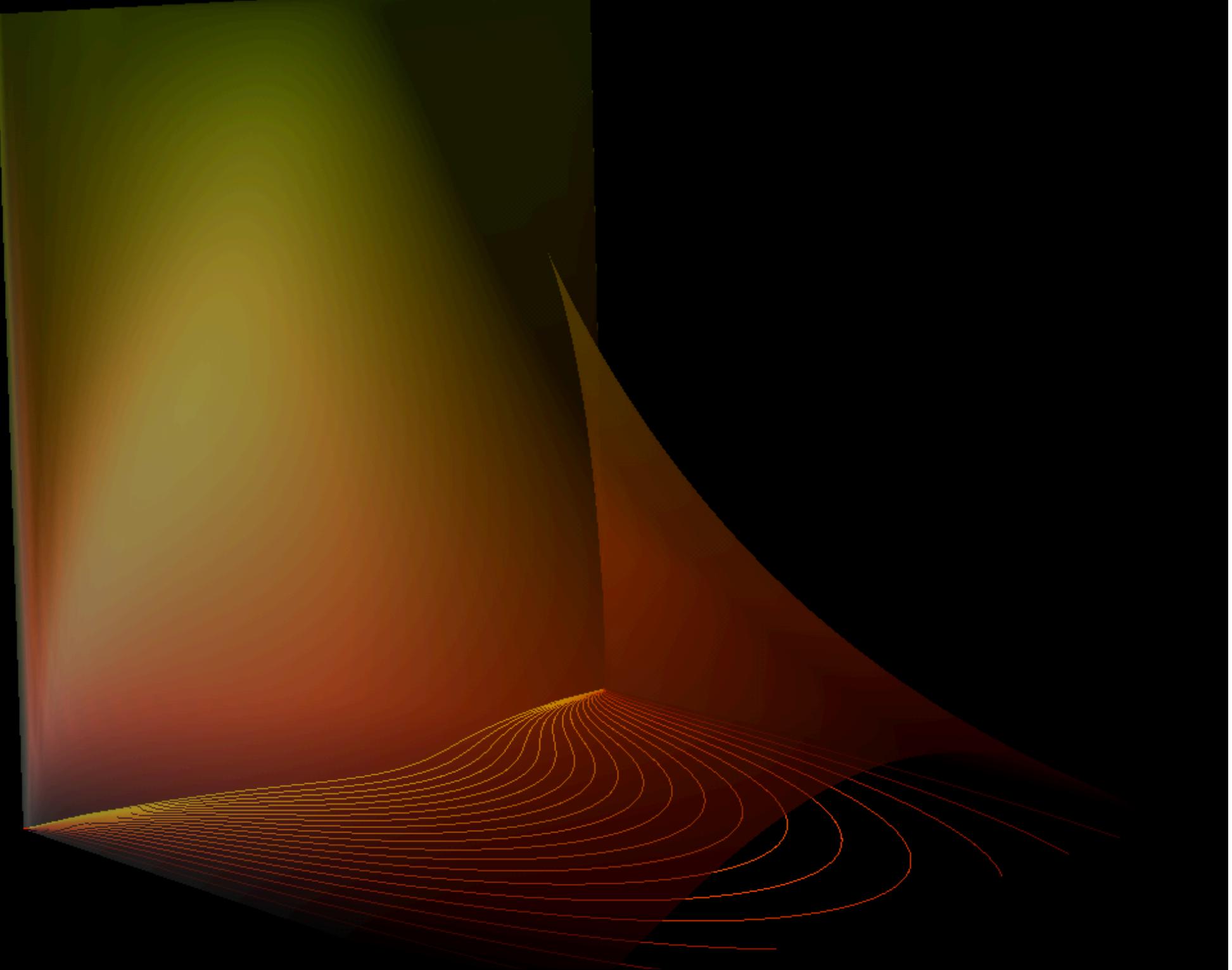
100C

0C

0C

HOW DOES TEMPERATURE CHANGE WITH TIME?

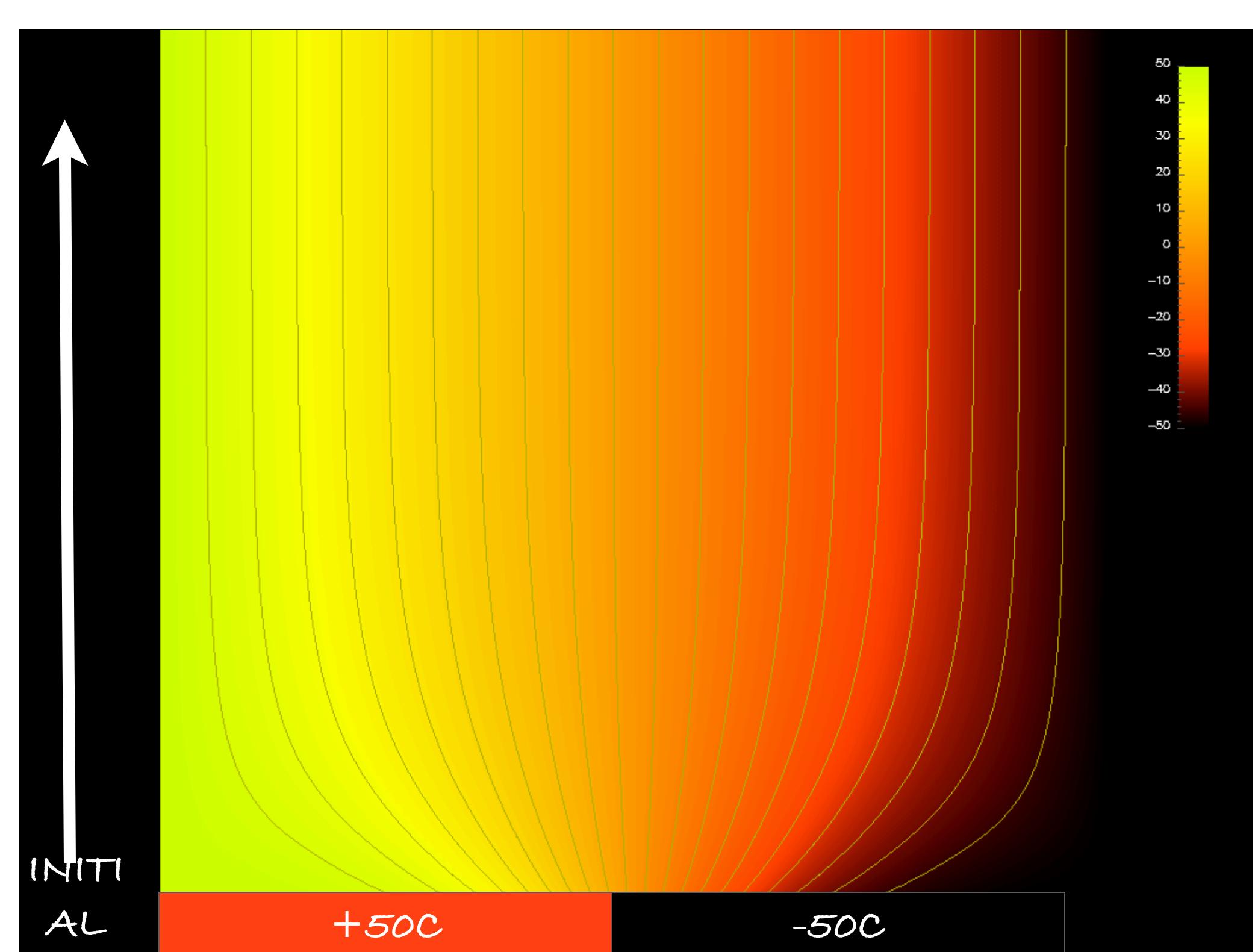




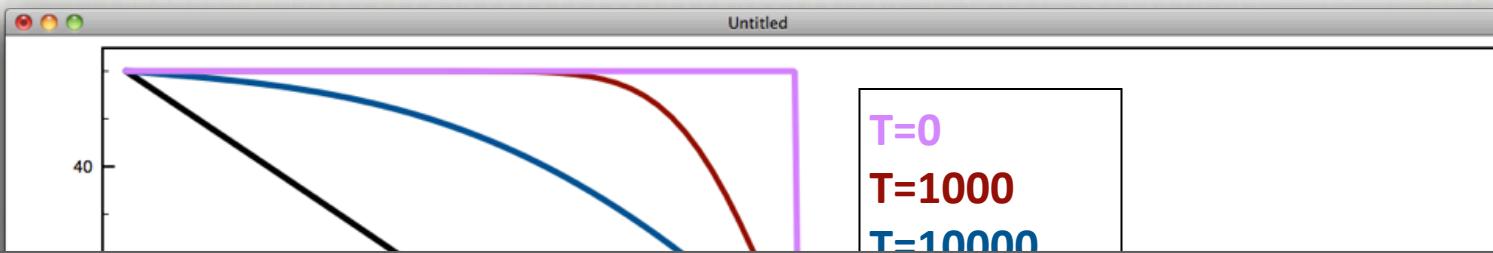
APPLICATION #2: TWO RODS BROUGHT TOGETHER

-50C

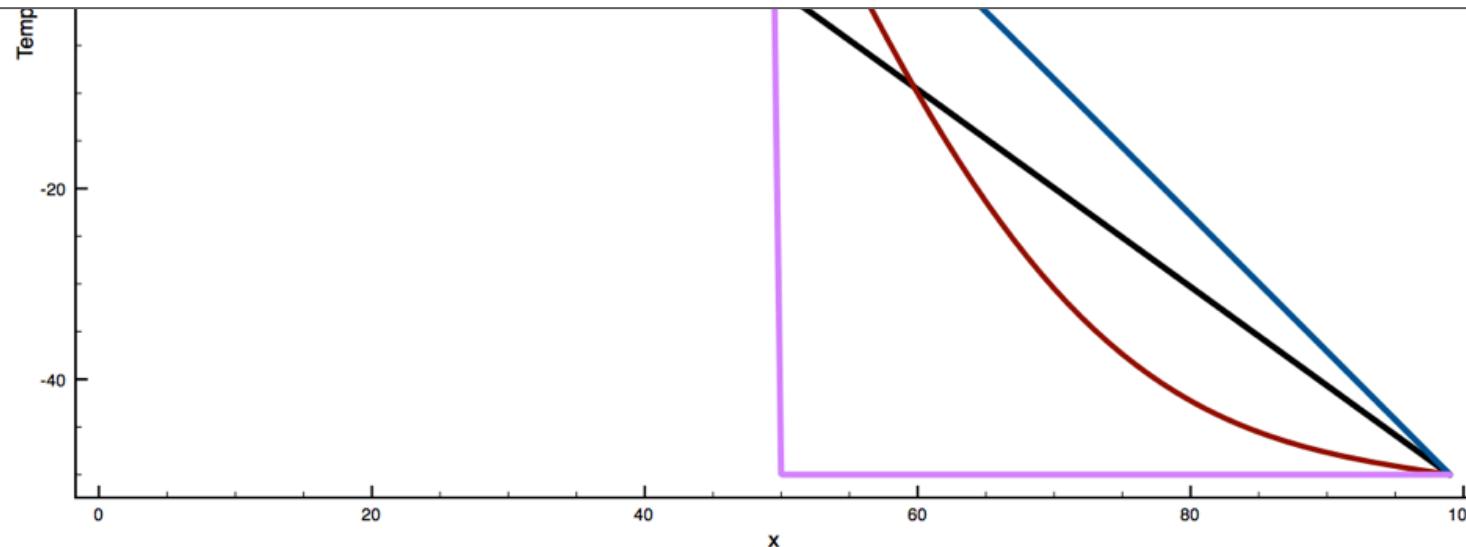
+50C



TWO DIFFERENT MATERIALS



$$T_{i,j+1} = T_{i,j} + \eta [T_{i+1,j} + T_{i-1,j} - 2T_{i,j}] \quad \eta = \frac{K\Delta t}{C\rho\Delta x^2}$$



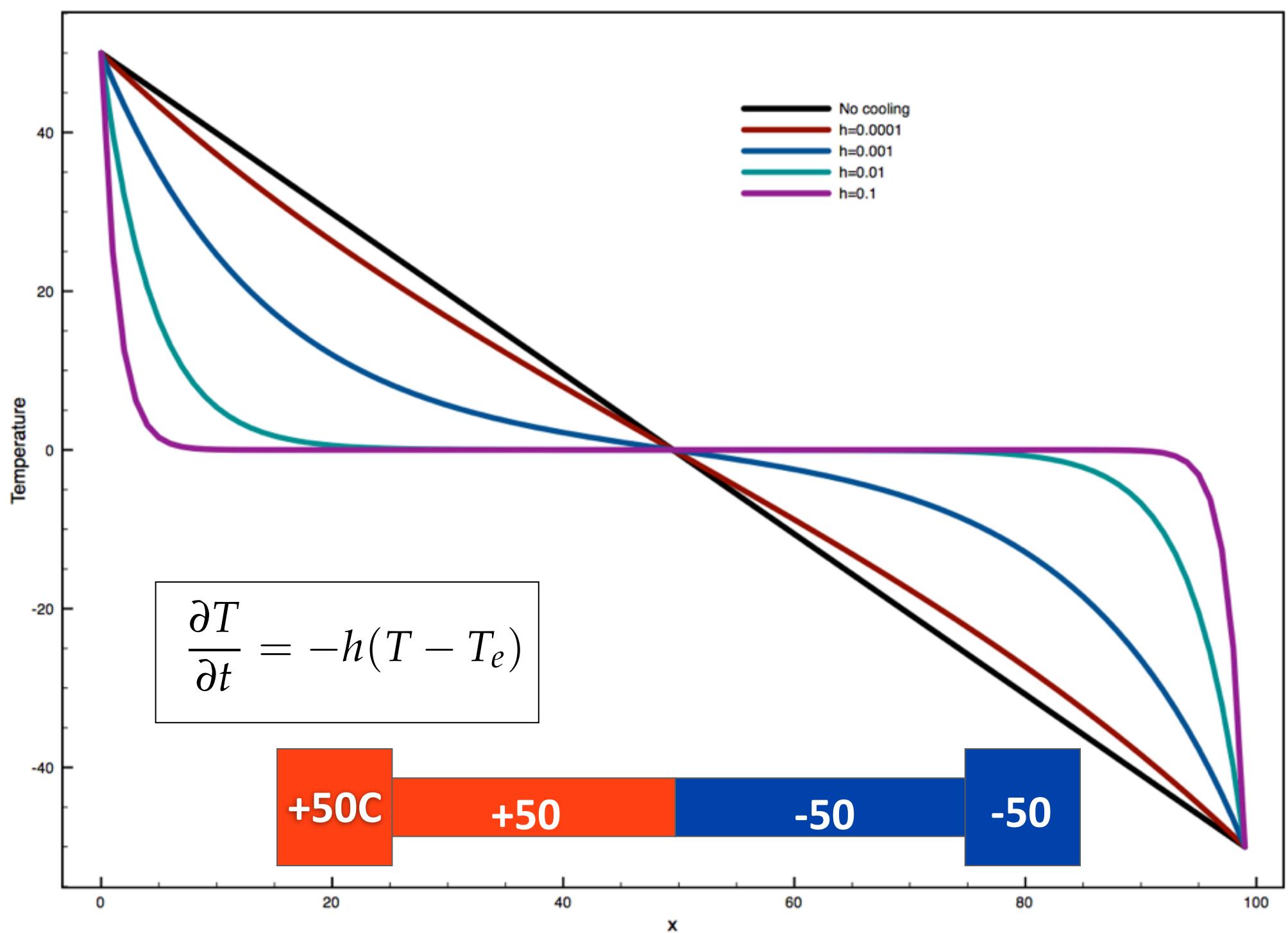
RADIATING BAR: NEWTON-COOLING

- A bar in contact with an environment at temperature T_e radiates energy out.
- Newton's law of cooling (radiation) says that the rate of temperature change due to radiation is

$$\frac{\partial T}{\partial t} = -h(T - T_e)$$

- Modified Heat Equation:

$$\frac{\partial T(x, t)}{\partial t} = \frac{K}{C\rho} \frac{\partial^2 T}{\partial^2 x} - hT(x, t)$$



SUMMARY

- *Leapfrog algorithm is used to solve parabolic (heat) equation*
- *We cannot use the relaxation method since we do not know all the information needed on the “four squares” of the multi-dimensional space*
- *Time boundary conditions are called “initial conditions”. We do not know the B.C. for $t \ggg 1$*
- *von Neumann principle indicates which value of dx and dt one should use for a given materials, to ensure convergence*