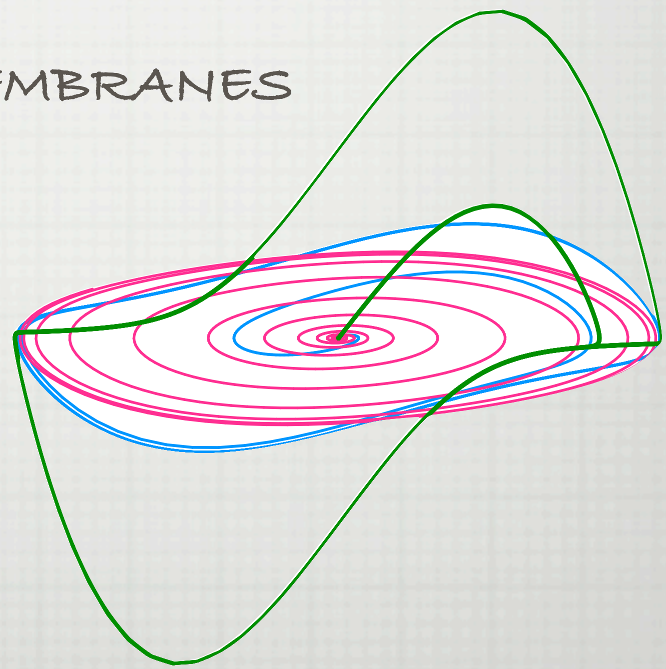
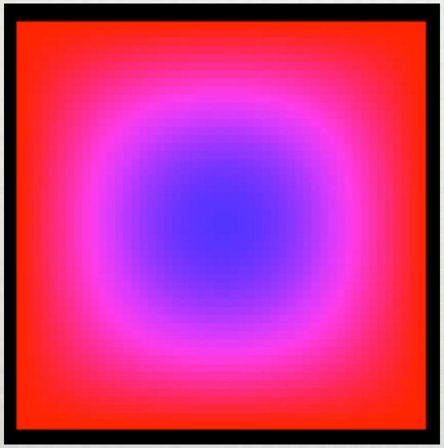
PHY-4810
COMPUTATIONAL PHYSICS

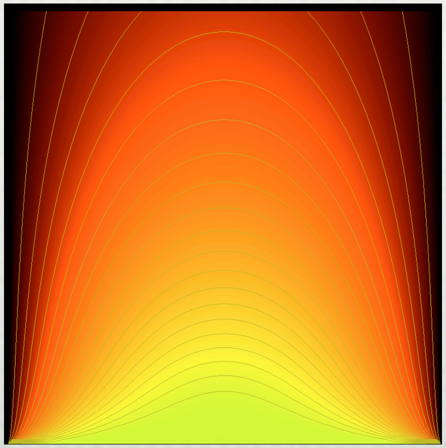
LECTURE 12: PDES; STRING, MEMBRANES



$$F\left(x_1, \ldots x_n, u, \frac{\partial}{\partial x_1}, \ldots \frac{\partial}{\partial x_n} u, \frac{\partial^2}{\partial x_1 \partial x_1} u, \ldots \frac{\partial^2}{\partial x_1 \partial x_2} u, \ldots\right) = 0$$

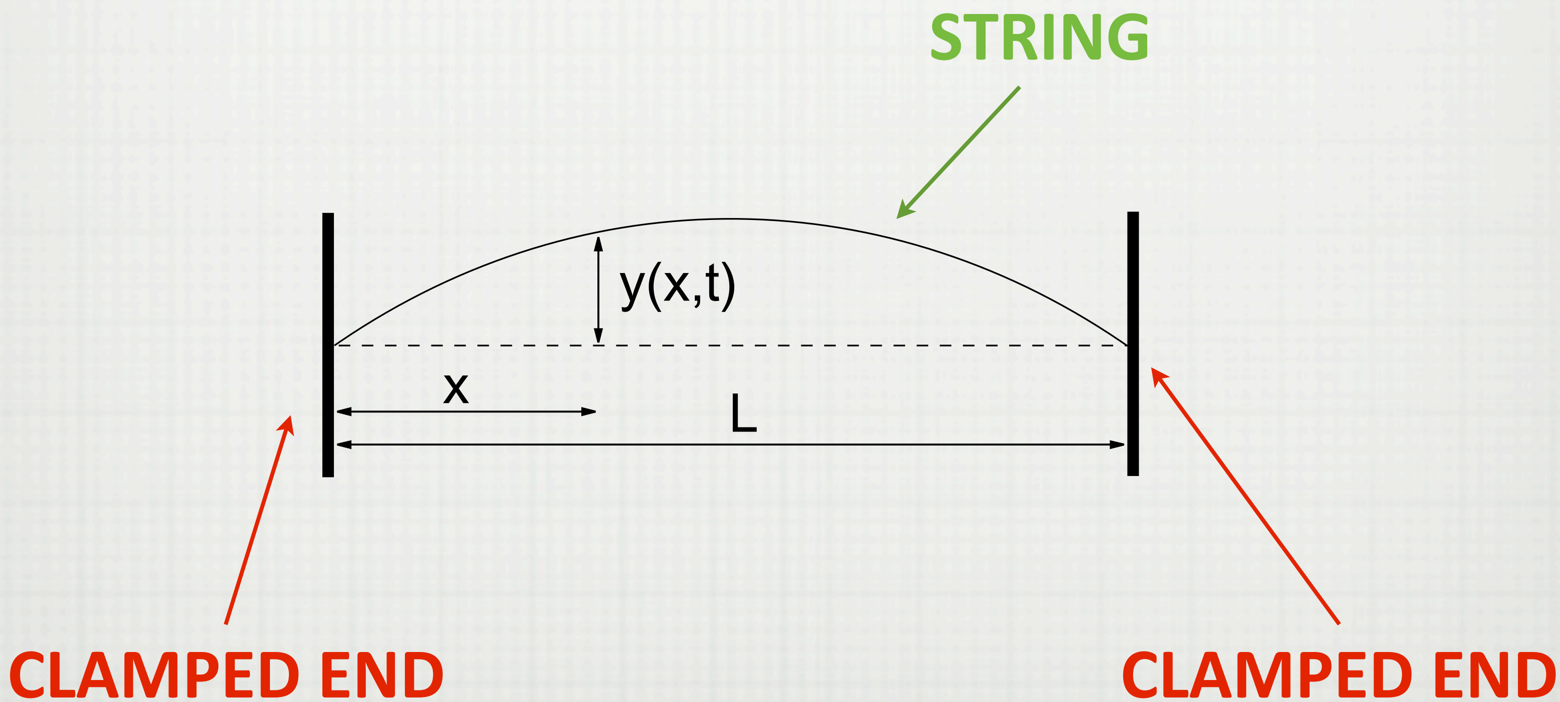
Elliptic	*Parabolic*	*Hyperbolic*
$d = AC - B^2 > 0$	$d = AC - B^2 = 0$	$d = AC - B^2 < 0$
$\nabla^2 U(x) = -4\pi\rho(x)$	$\nabla^2 U(\mathbf{x}, t) = a\partial U/\partial t$	$\nabla^2 U(\mathbf{x}, t) = c^{-2}\partial^2 U/\partial t^2$
Poisson's	Heat	Wave





TODAY

PROBLEM TO BE SOLVED

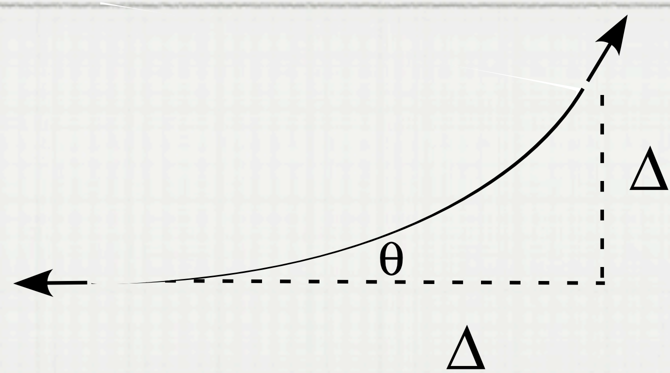


QUESTION: HOW DOES THE STRING VIBRATE?

PHYSICS: PROBLEM

- Consider a string of length L tied down at both ends.
- The string has a constant density ρ per unit length, a constant tension T, is subject to no frictional forces, and the tension is high enough that we may ignore the sagging of the string due to gravity.
- We assume that the displacement of the string y(x, t) from its rest position is in the vertical direction only, and that it is a function of the horizontal location along the string x and the time t.

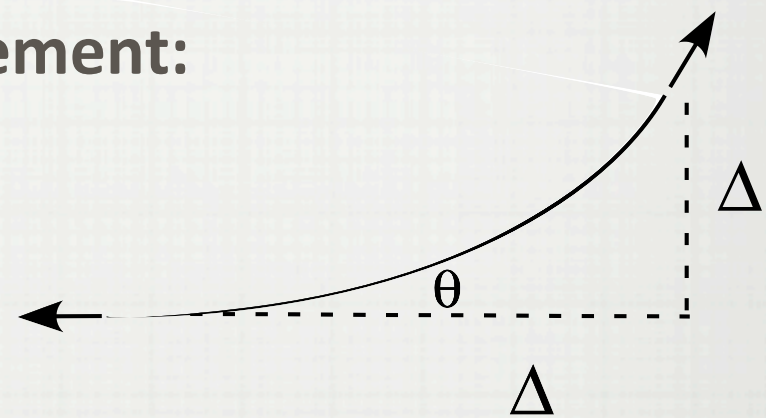
PHYSICS: SOLUTION



- To obtain a linear equation of motion, we assume that the string's relative displacement **y(x, t)/L** and slope **∂y/∂x** are small.
- We isolate an infinitesimal section **Δx** of the string.
- The difference in the vertical components of the tension on either end of the string produces the restoring force that accelerates this section of the string in the vertical direction.

NEWTON'S LAW

- Newton's law on the small element:



$$\sum F_y = \underbrace{\boxed{\rho\,\Delta x}}_{\text{MASS}}\ \underbrace{\boxed{\frac{\partial^2 y}{\partial t^2}}}_{\text{ACC.}}$$

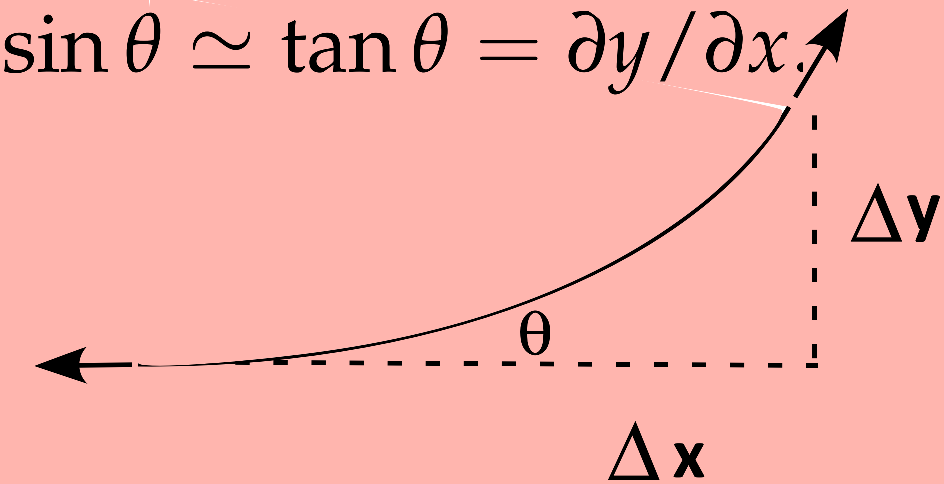
- Forces on the element:

$$\sum F_y = \boxed{T\sin[\theta(x+\Delta x)]} - \boxed{T\sin[\theta(x)]}$$

VERTICAL COMPONENTS OF TENSION ON DELTA

WAVE EQUATION

$$\sin\theta \simeq \tan\theta = \partial y/\partial x.$$



$$\sum F_y \;=\; T\sin[\theta(x+\Delta x)] - T\sin[\theta(x)]$$

$$=\; T\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - T\left.\frac{\partial y}{\partial x}\right|_{x} \;\simeq\; T\frac{\partial^2 y}{\partial x^2}$$

WAVE EQUATION

$$\sum F_y = T\sin[\theta(x+\Delta x)] - T\sin[\theta(x)]$$

$$= T\left.\frac{\partial y}{\partial x}\right|_{x+\Delta x} - T\left.\frac{\partial y}{\partial x}\right|_{x} \simeq T\frac{\partial^2 y}{\partial x^2}$$

$$\Rightarrow \quad \frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2} \qquad c = \sqrt{T/\rho}$$

C: SIGNAL VELOCITY

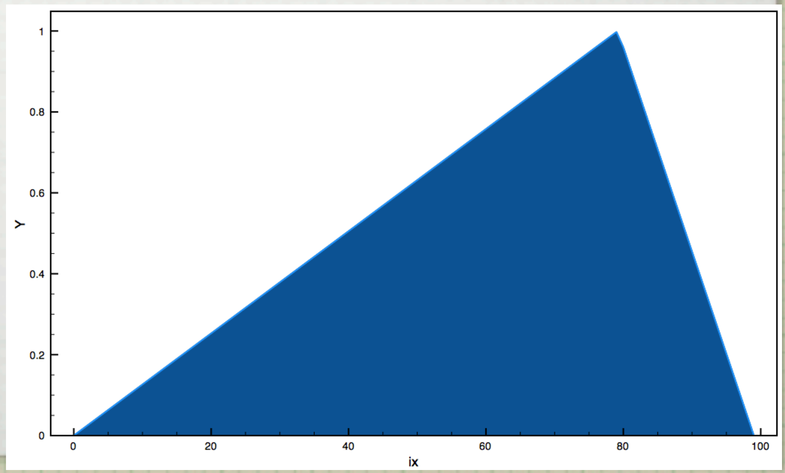
(LARGE T OR SMALL DENSITY -> HIGH VELOCITY)

HYPERBOLIC EQUATION

Boundary condition	Elliptic (Poisson equation)	Hyperbolic (Wave equation)	Parabolic (Heat equation)
Dirichlet open surface	Underspecified	Underspecified	*Unique and stable (1D)*
Dirichlet closed surface	*Unique and stable*	Overspecified	Overspecified
Neumann open surface	Underspecified	Underspecified	*Unique and stable (1D)*
Neumann closed surface	*Unique and stable*	Overspecified	Overspecified
Cauchy open surface	Unphysical	*Unique and stable*	Overspecified
Cauchy closed surface	Overspecified	Overspecified	Overspecified

INITIAL CONDITION: REPRESENTING HAND HITTING THE STRING

- **Gentle pluck**
- **We assume the pluck creates a triangular shape in the string, about 8/10 from the end of the string and of amplitude 1**



$$y(x,t=0) = \begin{cases} 1.25x/L & x \le 0.8L \\ (5-5x/L), & x > 0.8L \end{cases}$$

MORE INITIAL CONDITIONS

- **Because the wave-equation. is second-order in time, a second initial condition (beyond initial displacement) is needed to determine the solution.**
- **We interpret the "gentleness" of the pluck to mean the string is released from rest:**

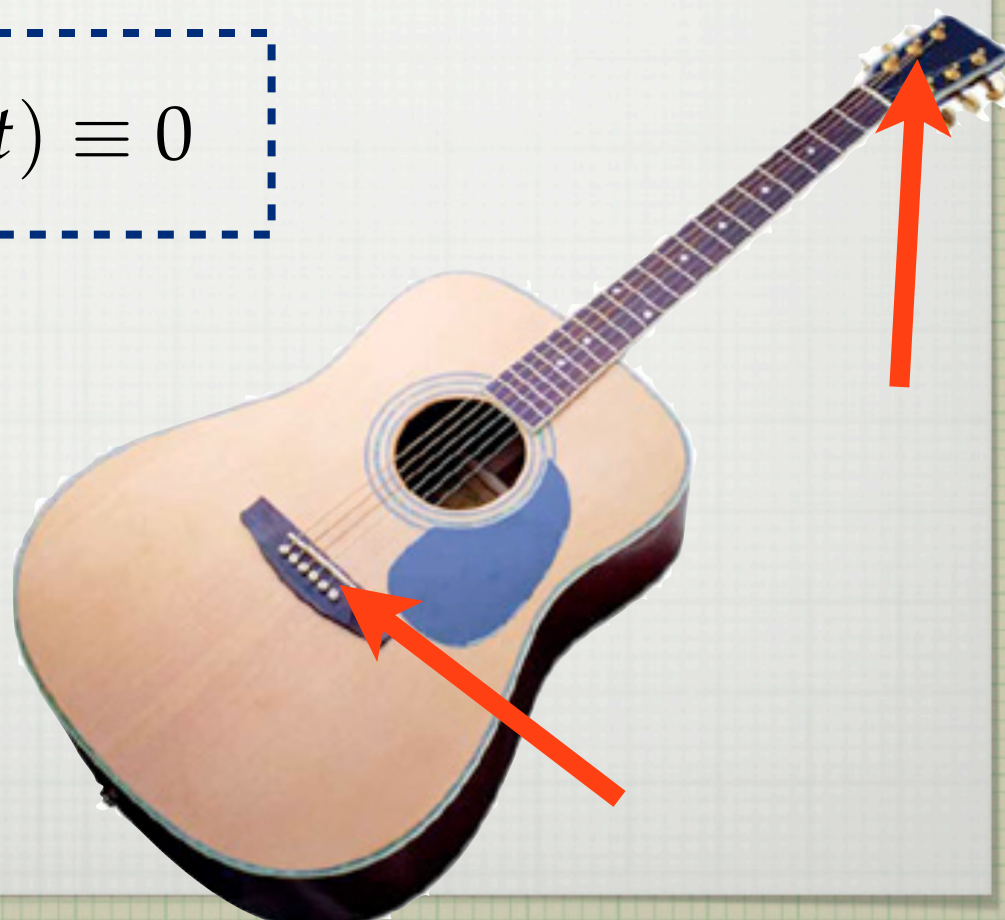
$$\frac{\partial y}{\partial t}(x, t = 0) = 0$$

- **After discretization:**

$$\boxed{\Rightarrow \qquad y_{i,0} = y_{i,1}}$$

BOUNDARY CONDITIONS

$$\boxed{y(0,t) \equiv 0 \qquad y(L,t) \equiv 0}$$



EXTENSION: WHAT ABOUT VIOLIN?

- **No Pluck in violins...**

ALL CONDITIONS: SUMMARY

- **Initial conditions (pluck + start from rest)**

$$y(x, t=0) = \begin{cases} 1.25x/L & x \leq 0.8L \\ (5 - 5x/L), & x > 0.8L \end{cases}$$

$$y_{i,0} = y_{i,1}$$

- **Boundary conditions (two ends are clamped)**

$$y(0, t) \equiv 0 \qquad y(L, t) \equiv 0$$

- **Need to solve the W.E. using these conditions**

$$\Rightarrow \qquad \frac{\partial^2 y(x,t)}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y(x,t)}{\partial t^2} \qquad c = \sqrt{T/\rho}$$

NUMERICAL SOLUTION: DISCRETIZATION

$$\frac{\partial^2 y}{\partial t^2} \simeq \frac{y_{i,j+1} + y_{i,j-1} - 2y_{i,j}}{(\Delta t)^2}$$

$$\frac{\partial^2 y}{\partial x^2} \simeq \frac{y_{i+1,j} + y_{i-1,j} - 2y_{i,j}}{(\Delta x)^2}$$

$$\frac{y_{i,j+1} + y_{i,j-1} - 2y_{i,j}}{c^2(\Delta t)^2} = \frac{y_{i+1,j} + y_{i-1,j} - 2y_{i,j}}{(\Delta x)^2}$$

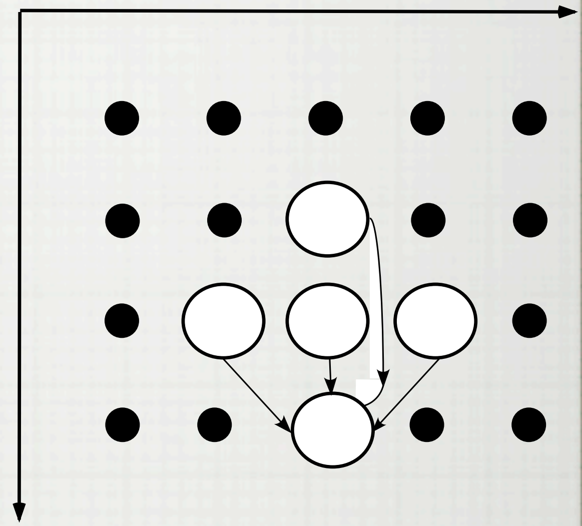
CENTRAL DIFFERENCE

$$y_{i,j+1} = 2y_{i,j} - y_{i,j-1} + \frac{c^2}{c'^2}\left[y_{i+1,j} + y_{i-1,j} - 2y_{i,j}\right]$$

$$c' = \Delta x / \Delta t$$

NUMERICAL SOLUTION: LEAPFROG

$$
\begin{aligned}
y_{i,j+1} &= 2y_{i,j} - y_{i,j-1} + \frac{c^2}{c'^2}\left[y_{i+1,j} + y_{i-1,j} - 2y_{i,j}\right] \\
c' &= \Delta x / \Delta t
\end{aligned}
$$



- We need values at two earlier times.
- We therefore need results at two different times to start the algorithm, provided by I.C.

$$
\begin{aligned}
\frac{\partial y}{\partial t}(x,0) &\simeq \frac{y(x,0) - y(x,-\Delta t)}{\Delta t} = 0 \\
\Rightarrow \qquad y_{i,0} &= y_{i,1}
\end{aligned}
$$

IN PRACTICE, CONVERGENCE CRITERIA

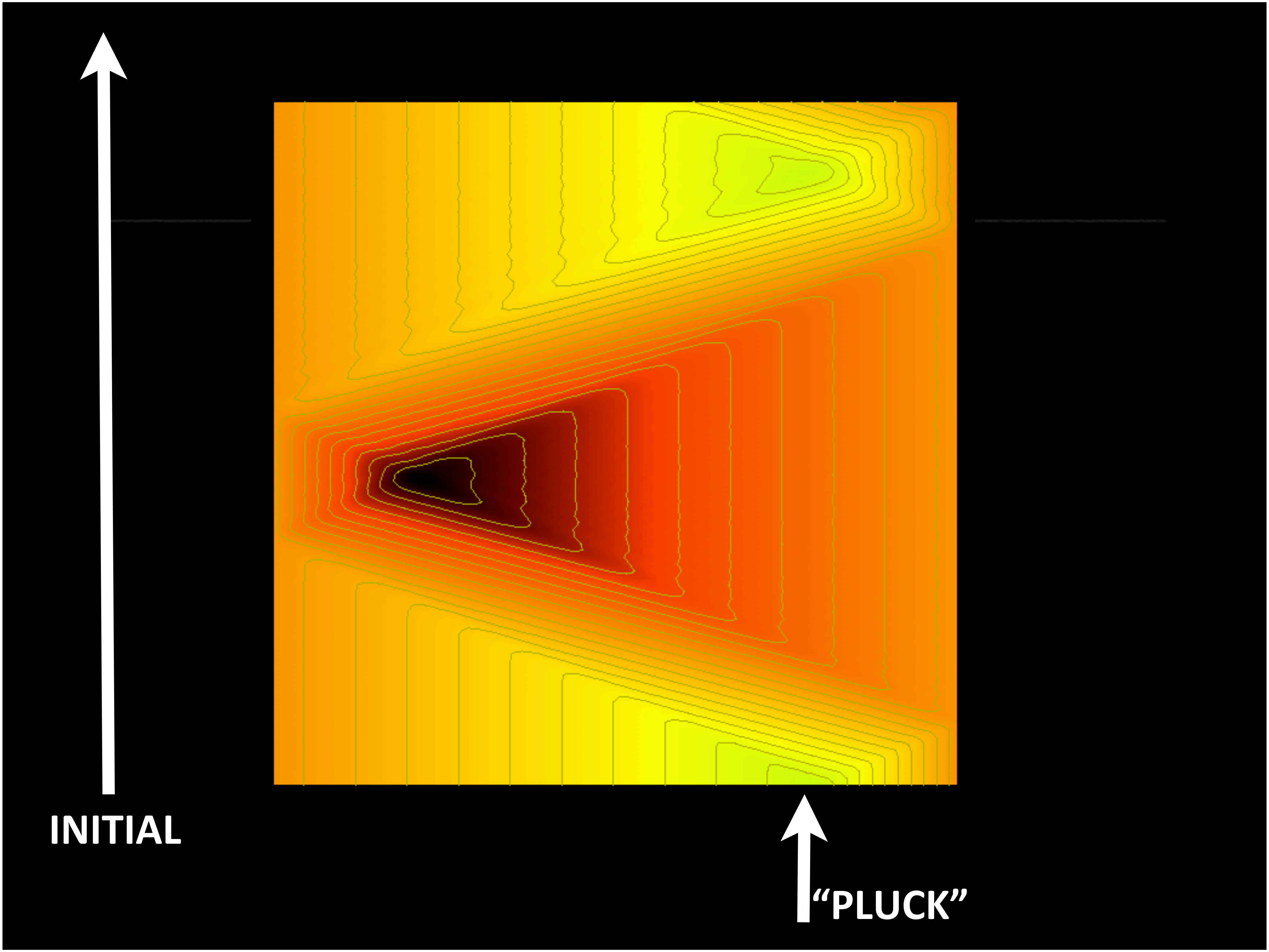
- Courant condition:

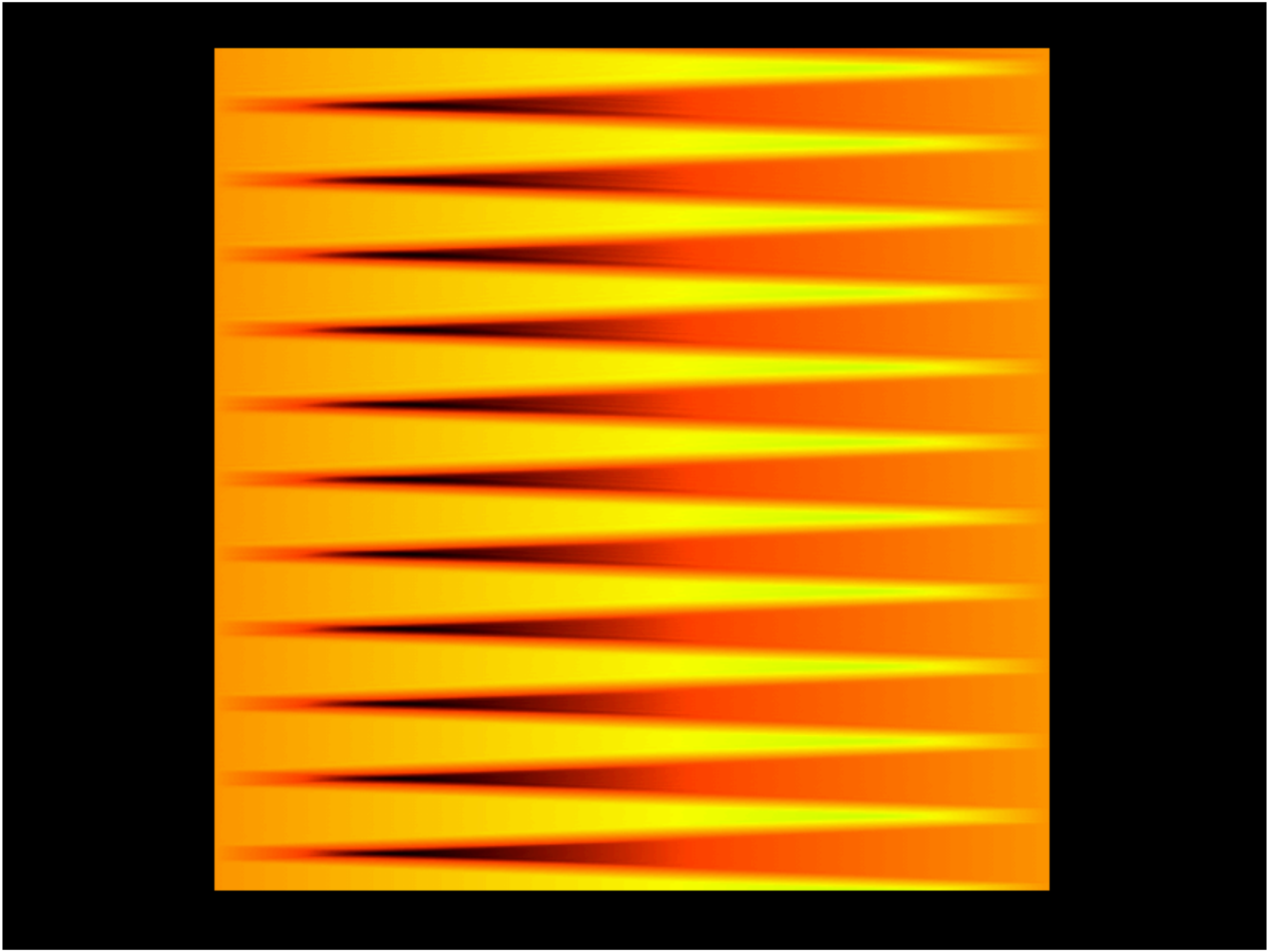
$$\boxed{c \leq c' = \Delta x / \Delta t}$$

- Better if time step is smaller, worse if space discretization is too small!

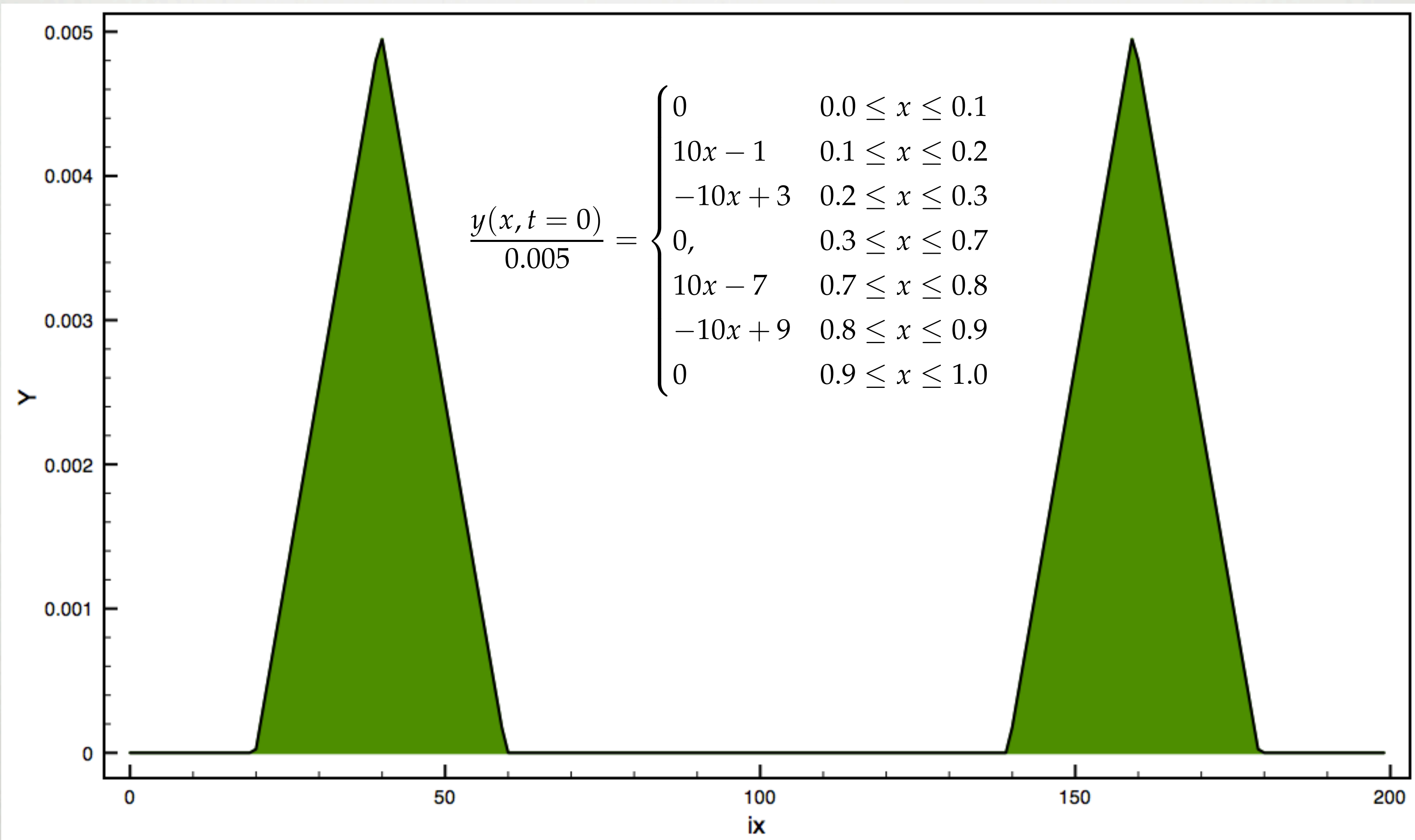
$$y_{i,j+1} = 2y_{i,j} - y_{i,j-1} + \frac{c^2}{c'^2}\left[y_{i+1,j} + y_{i-1,j} - 2y_{i,j}\right]$$

$$c' = \Delta x / \Delta t \qquad \boxed{c = \sqrt{T/\rho}}$$

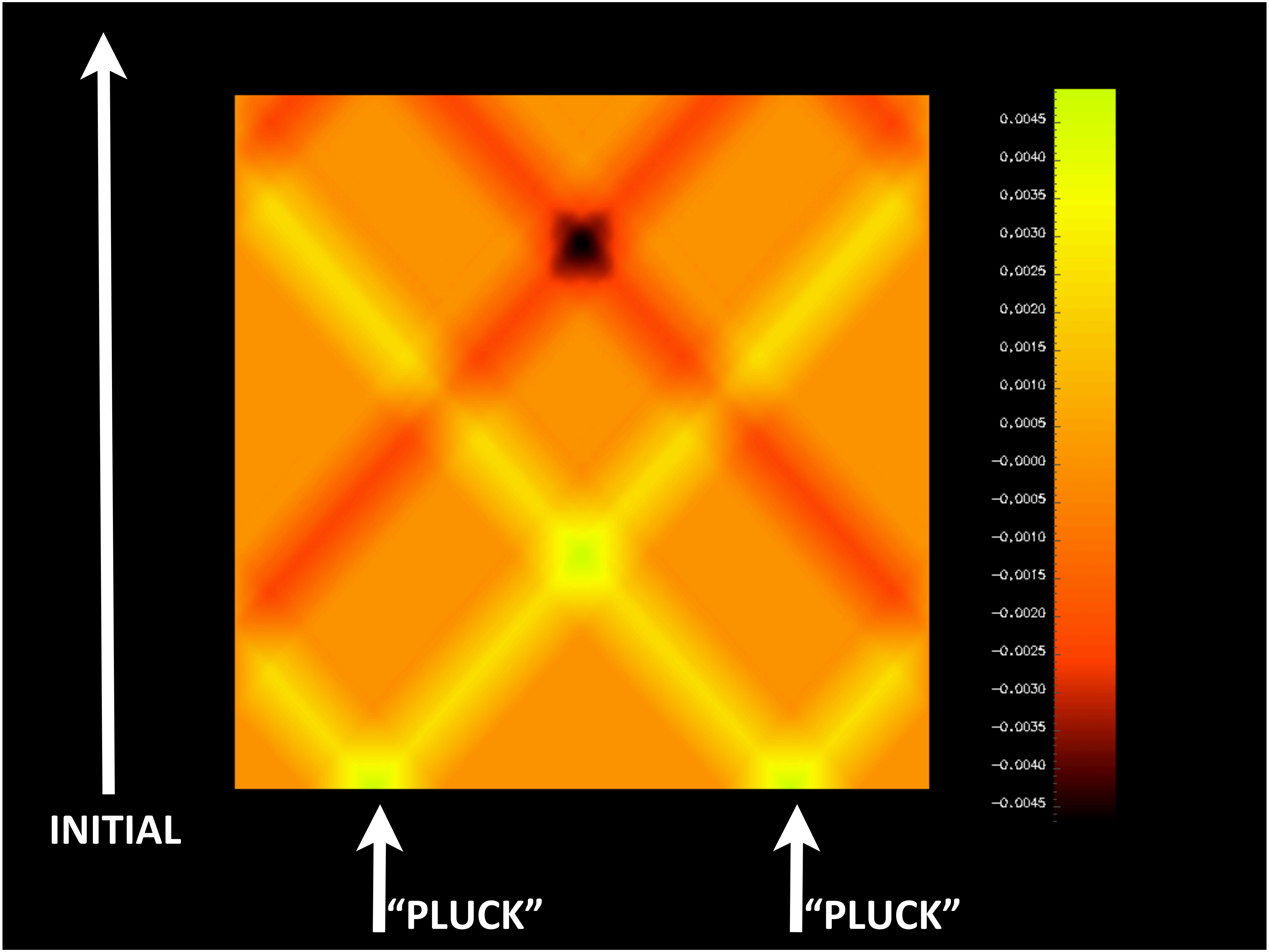

INITIAL
"PLUCK"



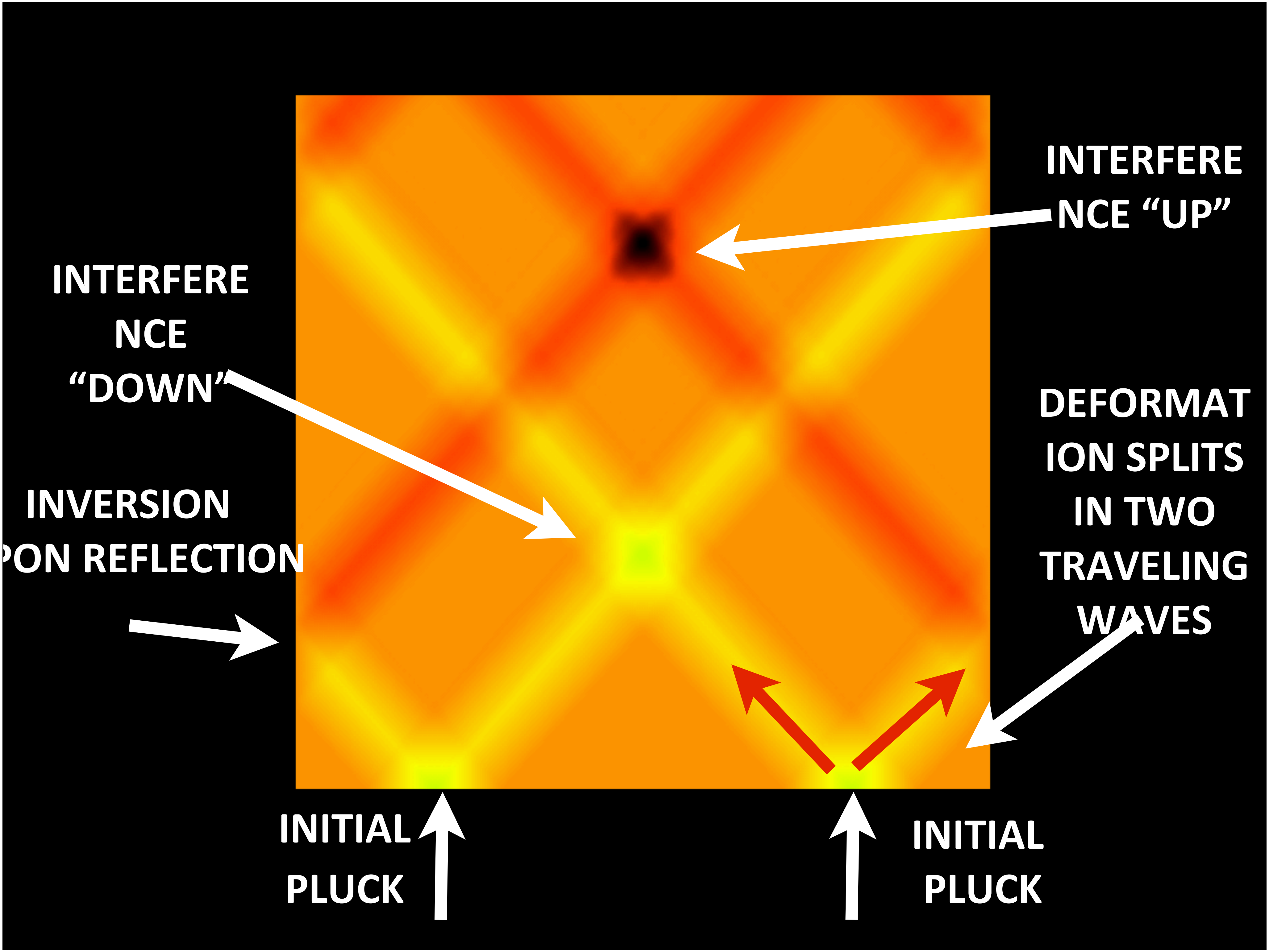
CHANGING INITIAL CONDITIONS: "DOUBLE-PLUCK"



$$\frac{y(x,t=0)}{0.005} = \begin{cases} 0 & 0.0 \le x \le 0.1 \\ 10x-1 & 0.1 \le x \le 0.2 \\ -10x+3 & 0.2 \le x \le 0.3 \\ 0, & 0.3 \le x \le 0.7 \\ 10x-7 & 0.7 \le x \le 0.8 \\ -10x+9 & 0.8 \le x \le 0.9 \\ 0 & 0.9 \le x \le 1.0 \end{cases}$$



0.0045
0.0040
0.0035
0.0030
0.0025
0.0020
0.0015
0.0010
0.0005
−0.0000
−0.0005
−0.0010
−0.0015
−0.0020
−0.0025
−0.0030
−0.0035
−0.0040
−0.0045
INITIAL
"PLUCK"
"PLUCK"


INTERFERE
NCE “UP”
INTERFERE
NCE
“DOWN”
INVERSION
PON REFLECTION
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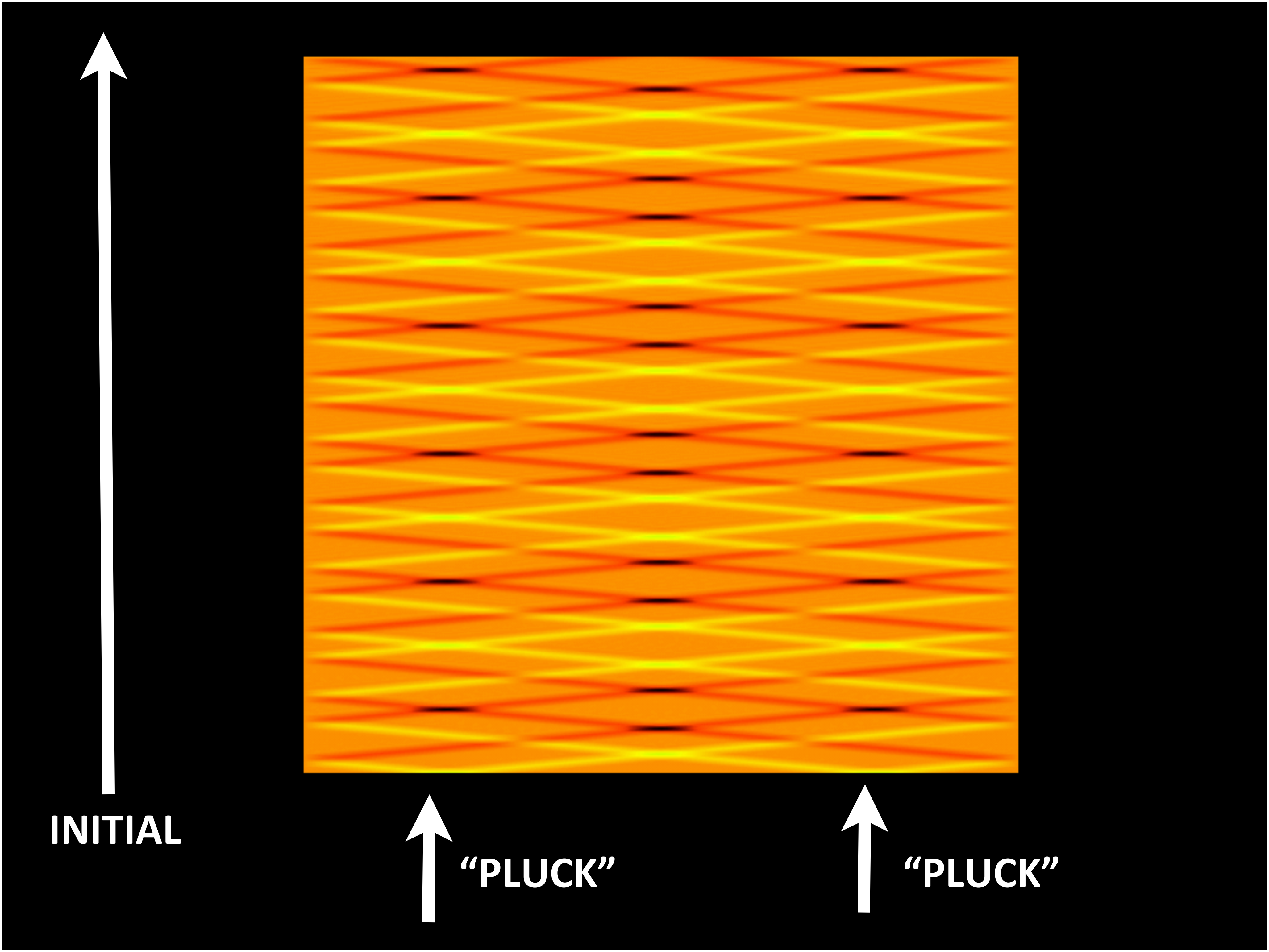
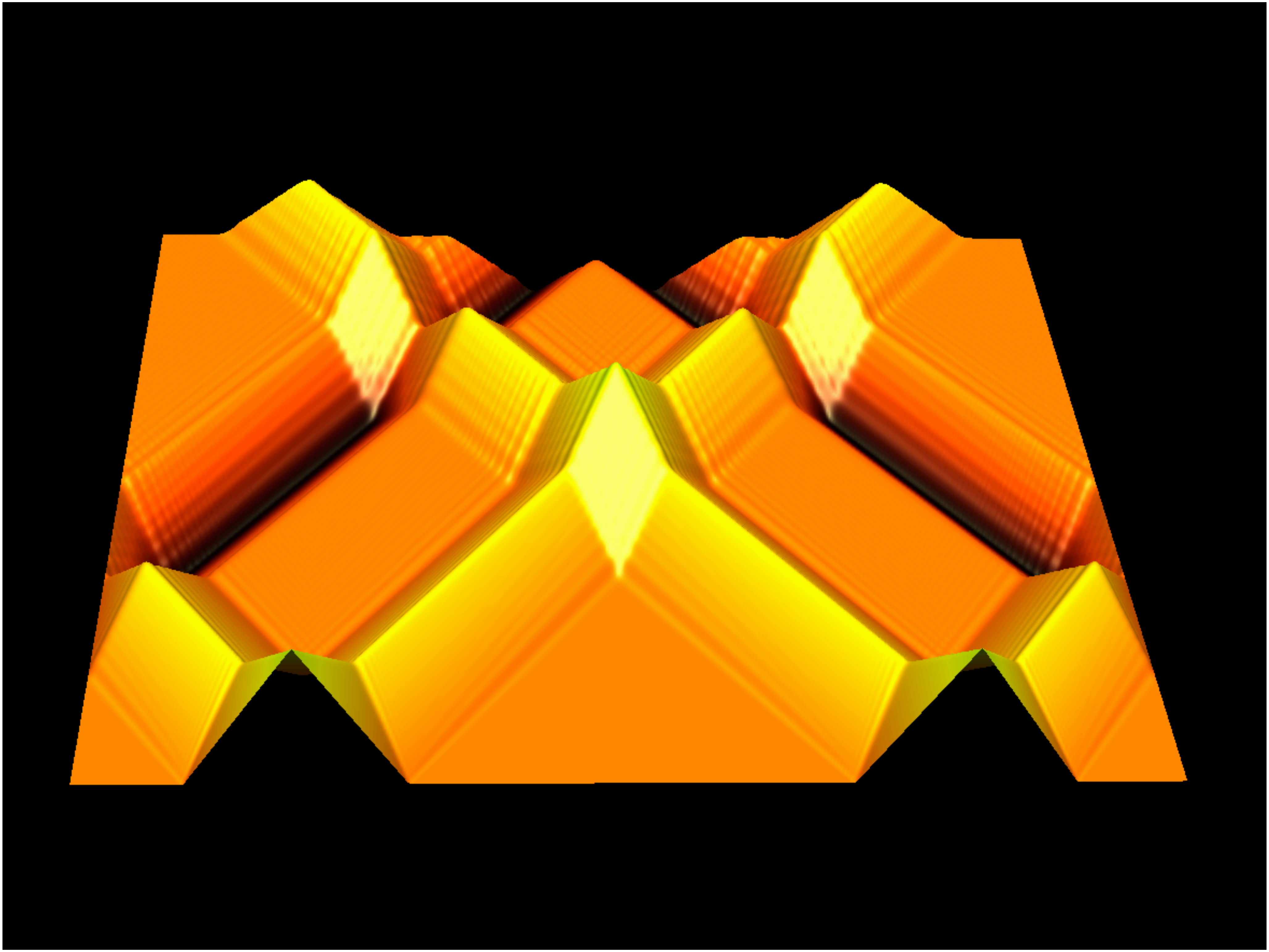
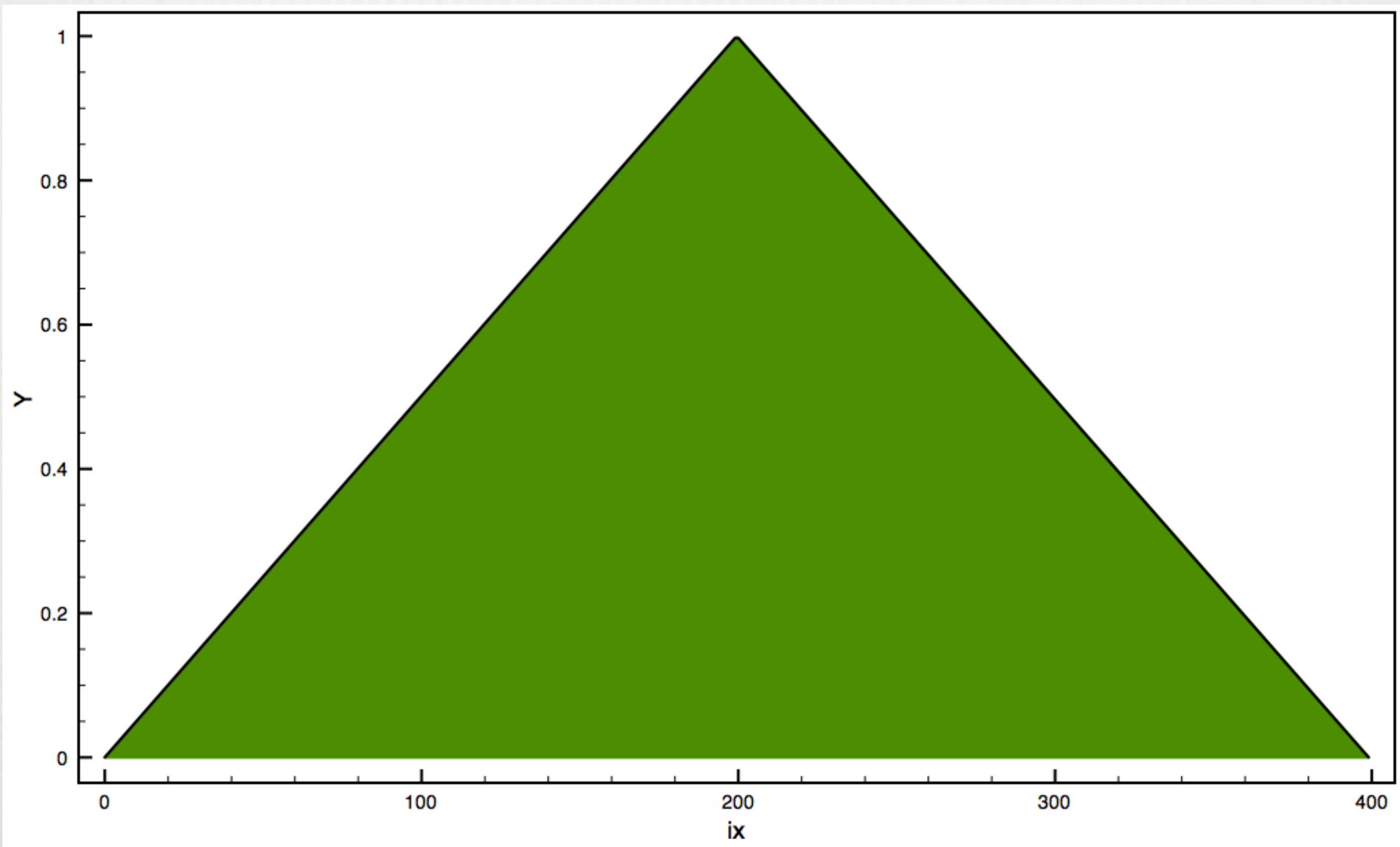
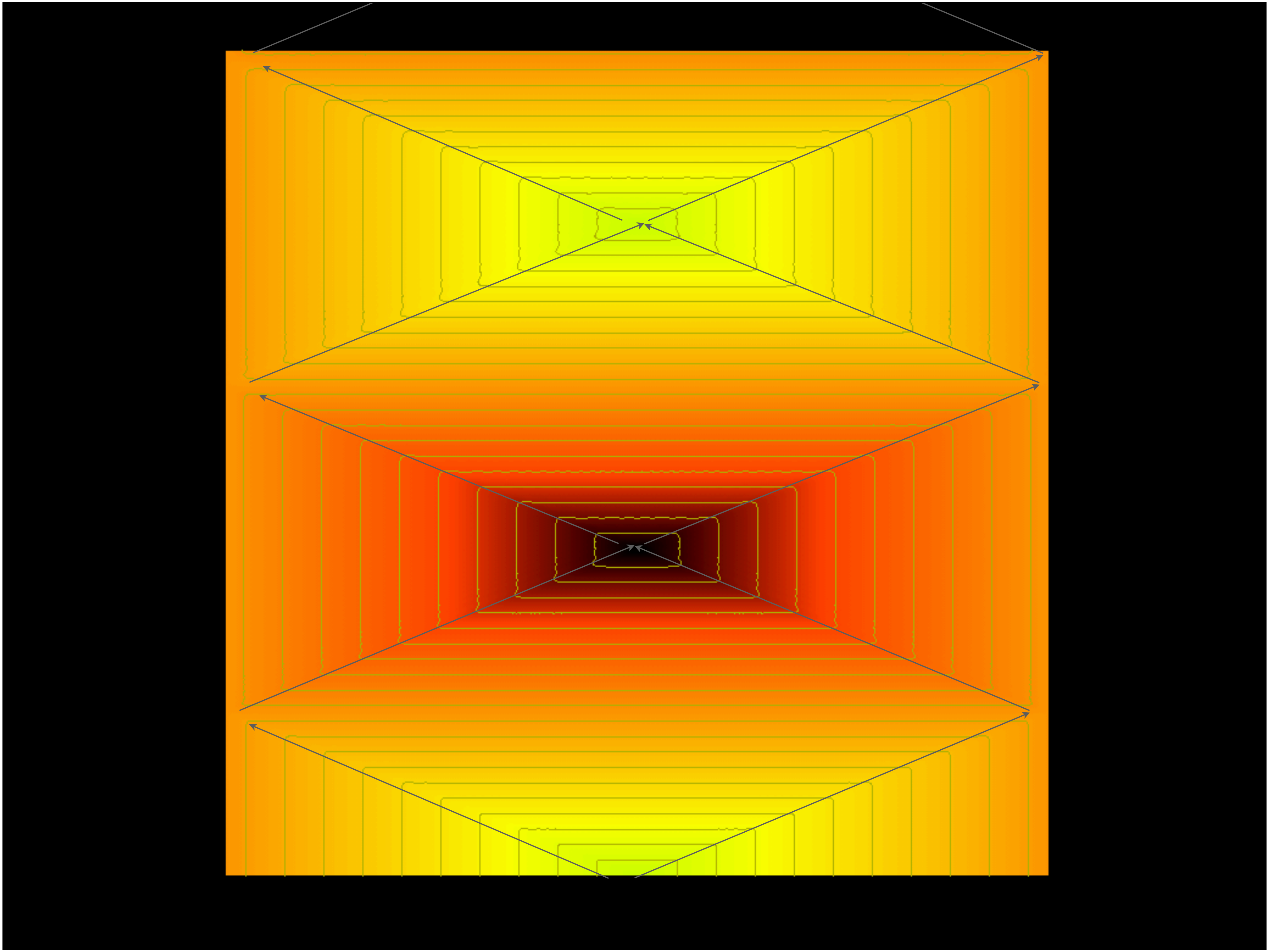
ION SPLITS
IN TWO
TRAVELING
WAVES
INITIAL
PLUCK
INITIAL
PLUCK


INITIAL
"PLUCK"
"PLUCK"



PLUCK IN THE MIDDLE





INCLUDING FRICTION

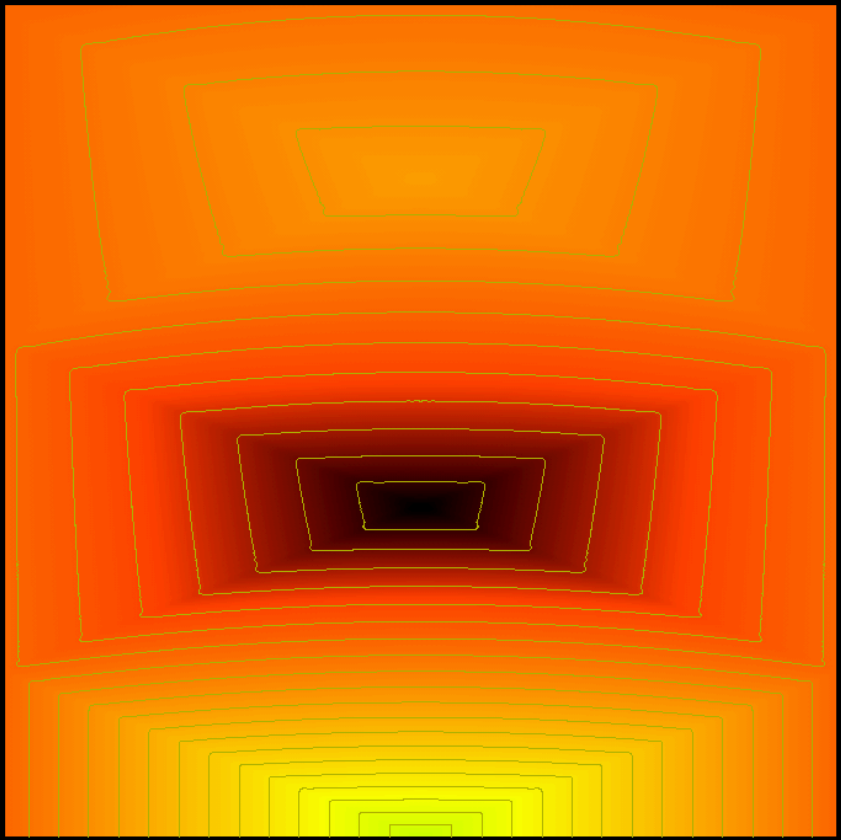
Real plucked strings do not vibrate forever because the real world contains friction.
The string elements is moving in a viscous fluid, such as air.

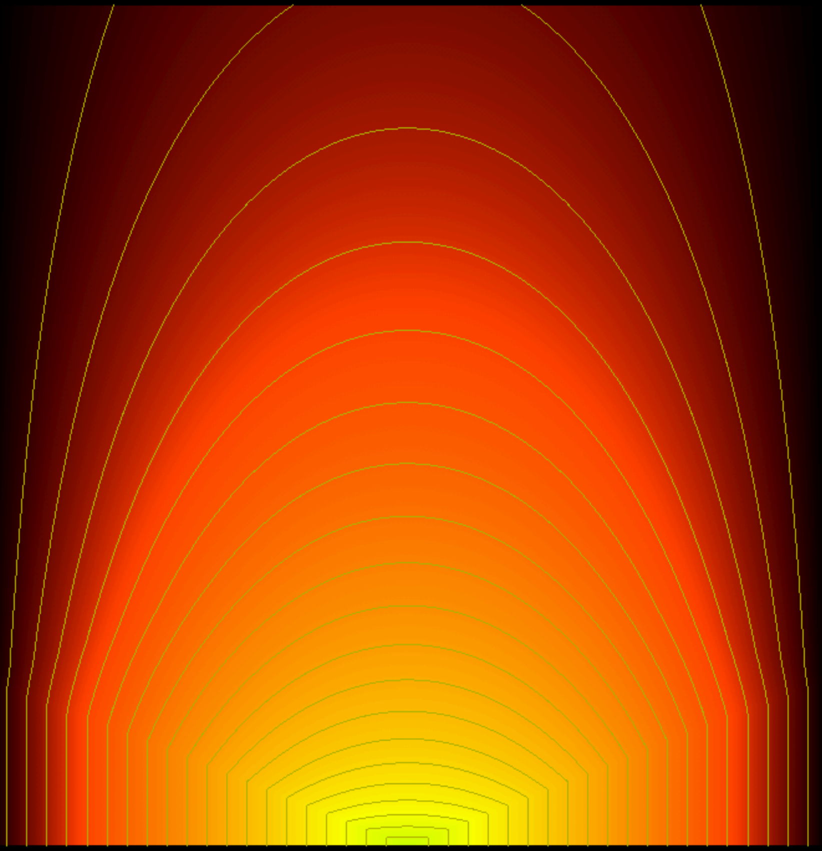
An approximate model for the frictional force is to have it point in a direction opposite to the (vertical) velocity of the string, proportional to that velocity, as well as proportional to the length of the element:

$$\boxed{F_f \simeq -2\kappa \Delta x \frac{\partial y}{\partial t}} \rightarrow \boxed{\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} - \frac{2\kappa}{\rho} \frac{\partial y}{\partial t}}$$

MODIFIED EQUATION

FRICTION (K=0.001) AND MORE FRICTION (K=0.01)





STRING OF VARYING TENSION AND DENSITY (FYI); PAGES 12-13 CHAP25

$$
\begin{aligned}
F &= ma \\
\Rightarrow \qquad \frac{\partial}{\partial x}\left[T(x)\,\frac{\partial y(x,t)}{\partial x}\right]\Delta x &= \rho(x)\Delta x\frac{\partial^2 u(x,t)}{\partial t^2}
\end{aligned}
$$

$$
\Rightarrow \qquad \frac{\partial T(x)}{\partial x}\frac{\partial y(x,t)}{\partial x} + T(x)\frac{\partial^2 y(x,t)}{\partial x^2} = \rho(x)\frac{\partial^2 y(x,t)}{\partial t^2}
$$

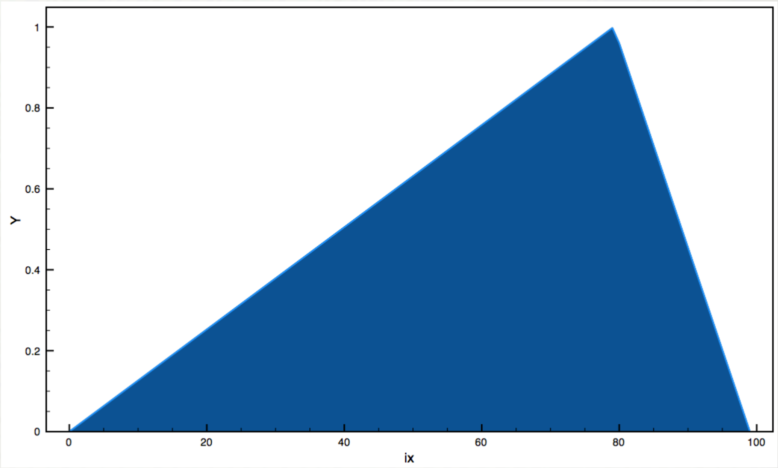
$$
\boxed{y_{i,j+1} = 2y_{i,j} - y_{i,j-1} + \frac{\alpha c^2(\Delta t)^2}{2\Delta x}[y_{i+1,j} - y_{i,j}] + \frac{c^2}{c'^2}[y_{i+1,j} + y_{i-1,j} - 2y_{i,j}]}
$$

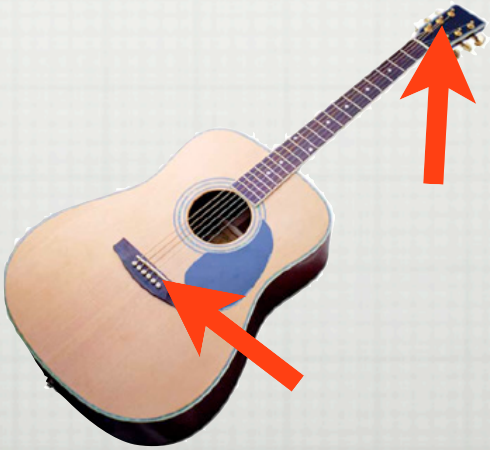
CODING (SEE "STRING.CPP"): I.C.

```
if(problem==1) {

//initial condition
    for (i=0;i<n;i++){
        donotcompute[i]=false;
        if(i<.8*n){
            T_1[i]=1.25*i/(n-1.);
        }
        else
        {
            T_1[i]=(5.0-5.0*i/(n-1.));
        }
        T0[i]=T_1[i];
    }
//boundary condition
    T_1[0]=0;
    T0[0]=0;
    T1[0]=T0[0];
    donotcompute[0]=true;

    T_1[n-1]=0;
    T0[n-1]=0;
    T1[n-1]=T0[n-1];
    donotcompute[n-1]=true;
}
```





CODING: SOLVING EQUATION

```
 // now we propagate the solution
   for (j=0;j<ntime;j++){
       if(j%2==0) {
           counter++;
           for (i=0;i<n;i++){
               profile << T0[i] << endl;
           }
       }
       for (i=0;i<n;i++){
           if(donotcompute[i]) continue;
           T1[i]=2*T0[i]-T_1[i]+c*(T0[min(i+1,n-1)]+T0[max(i-1,0)]-2*T0[i]);
           if(friction){
              T1[i]+=kappa*T_1[i];
              T1[i]/=(1.0+kappa);
           }
       }
//prepare for next iteration
       for (i=0;i<n;i++){
           T_1[i]=T0[i];
           T0[i]=T1[i];
       }
   }
```

$$\boxed{y_{i,j+1} = 2y_{i,j} - y_{i,j-1} + \frac{c^2}{c'^2}\left[y_{i+1,j} + y_{i-1,j} - 2y_{i,j}\right]}$$

$$\boxed{F_f \simeq -2\kappa\Delta x \frac{\partial y}{\partial t}}$$

SUMMARY

- ☐ **Leapfrog algorithm is used to solve hyperbolic (wave) equation**
- ☐ **We cannot use the relaxation method since we do not know all the information needed on the “four squares” of the multi-dimensional space**
- ☐ **Time boundary conditions are called “initial conditions”. We do not know the B.C. for t>>>1**
- ☐ **As opposed to the parabolic equation, we need to know solution at two previous times**

PDES SUMMARY