Homework 4: Partial Differential Equations (PDE's): Relaxation and Leapfrog Methods

A.L. Phillips II
Department of Physics, Astronomy, and Applied Physics,
Rensselaer Polytechnic Institute
`philla3@rpi.edu`

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Assignment 4

Elliptical PDE's

4.1 Jacobi, Gauss-Seidel Over/Under Relaxation Methods

1. Several methods are available to solve elliptical partial differential equations (PDEs). Amongst these are the Jacobian method, the Gauss-Seidel Method and the Over or Under Relaxation methods. The Jacobian and Gauss-Seidel methods are similar in that they use the same basic algorithm:

$$U_{i,j} = \frac{1}{4}[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j} - 1]$$

Note that the difference between Jacobi and Gauss-Seidel is that the Jacobian method does not update it's potential values until an entire loops over the grid is completed.

For the Gauss-Seidel method, a much different approach is taken. In this method, the following algorithm determines the new values of the potential ($styleU$) through the previous values of the potential plus a residual.

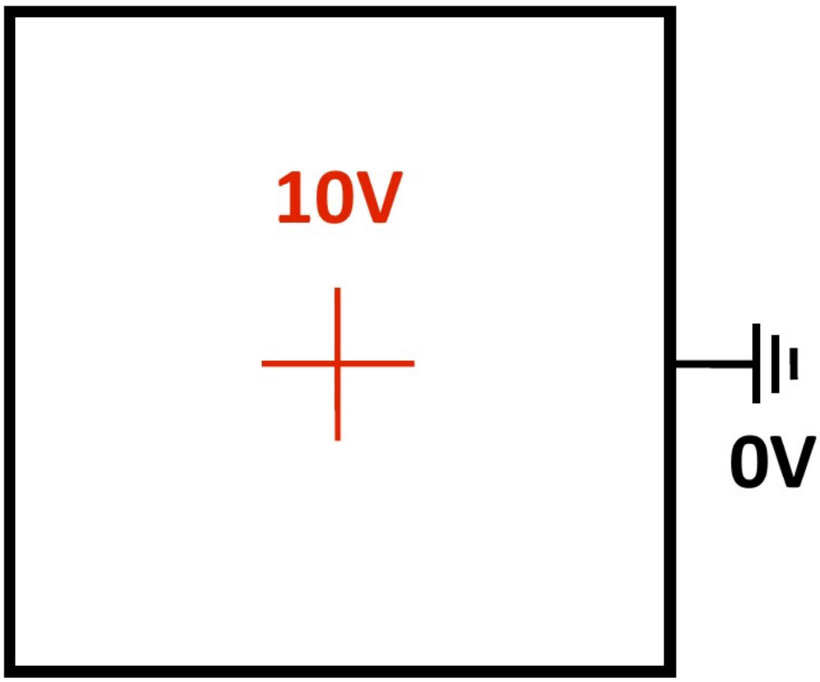
$$U_{i,j} = \frac{1}{4}[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j} - 1] + \omega(r)$$

$$r = U_{i,j} = \frac{1}{4}[U_{i+1,j} + U_{i-1,j} + U_{i,j+1} + U_{i,j} - 1] - U_{old})$$

$$\mathbf{A}\vec{x} = \vec{b}$$

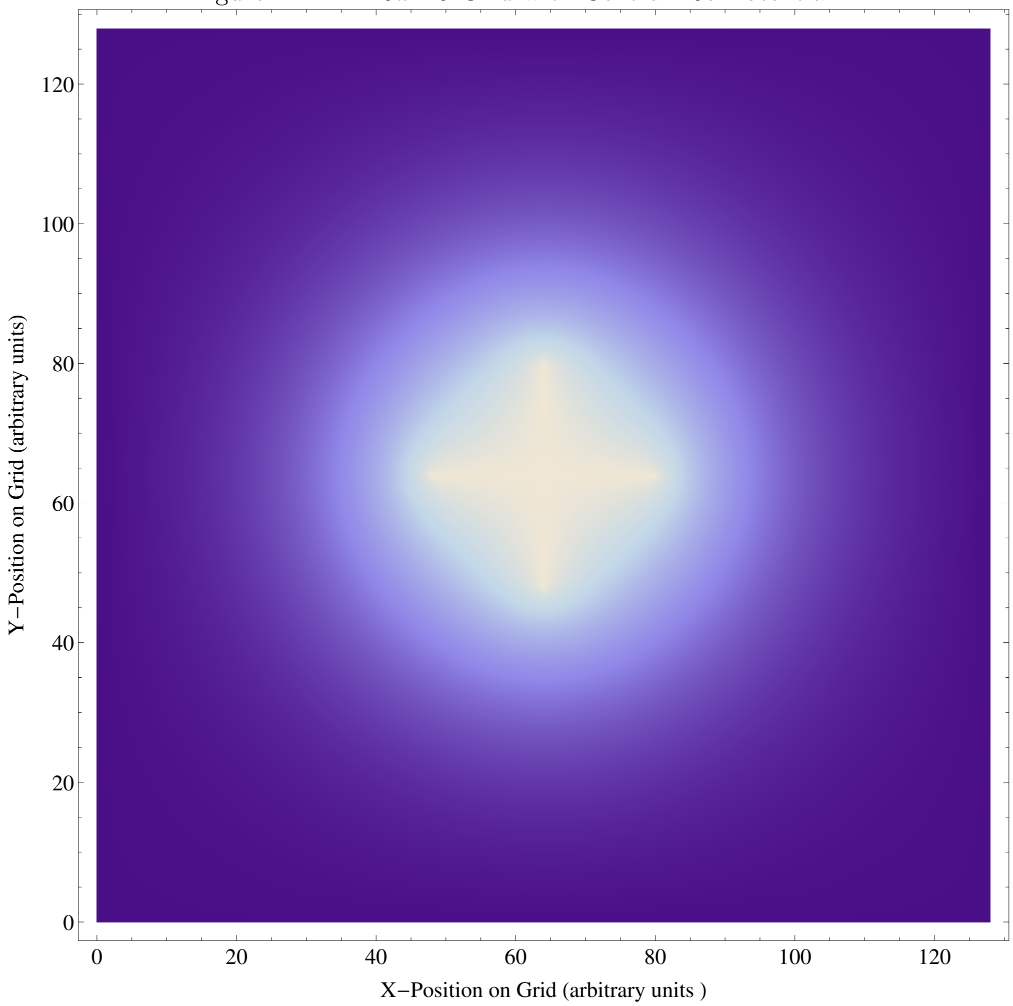
4.2 A Potential Plate

Figure 4.1: A 129x129 Grid with Central 10v Potential



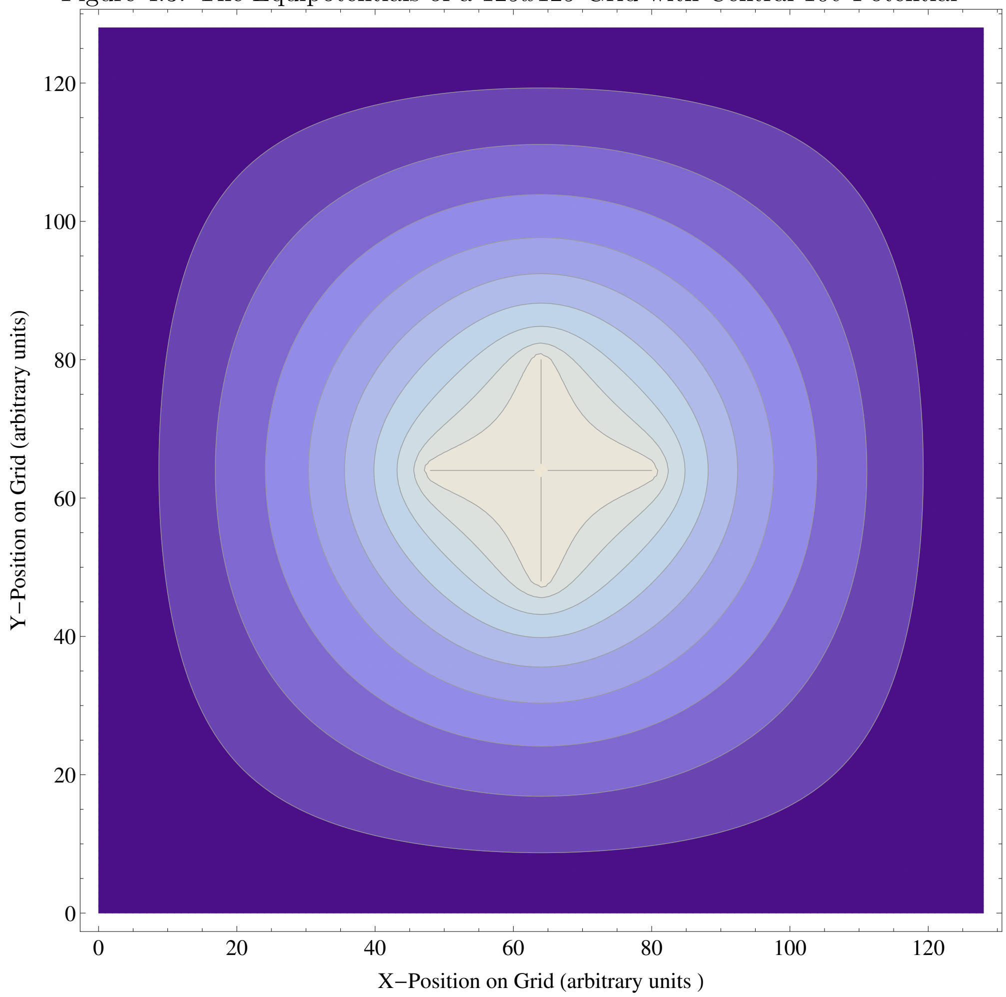
1. While the different methods have considerations which affect performance gains and loses, all output similar results when calculating the potential in the system shown above. The following plot depicts the spread of potential throughout the system after convergence:

Figure 4.2: A 129x129 Grid with Central 10v Potential



Notice that, as expected, the potential is greatest towards the central cross.

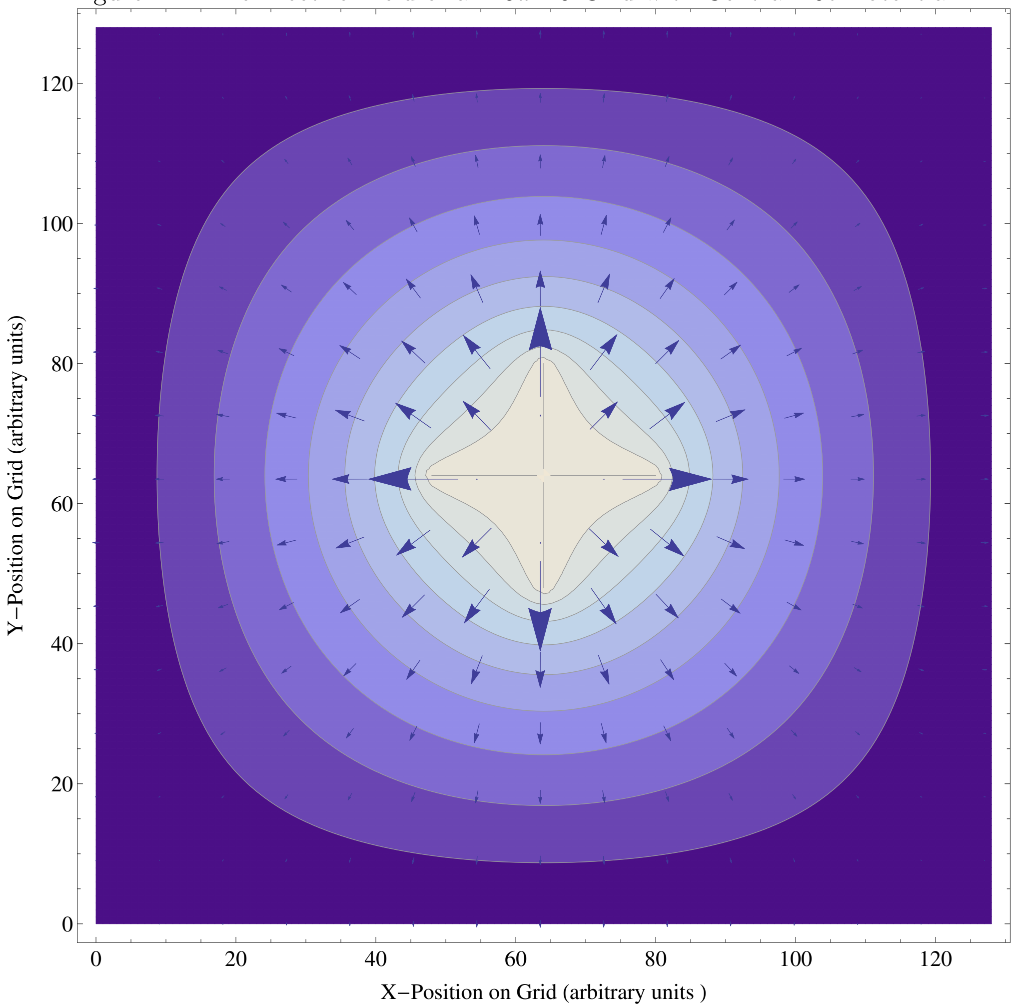
Figure 4.3: The Equipotentials of a $129x129$ Grid with Central $10v$ Potential



Notice that the steepest gradients are located directly between the cross and the grounded boundary.

Here the electric field may be computed using finite difference. This is done by evaluating the change in the horizontal and vertical component positions:

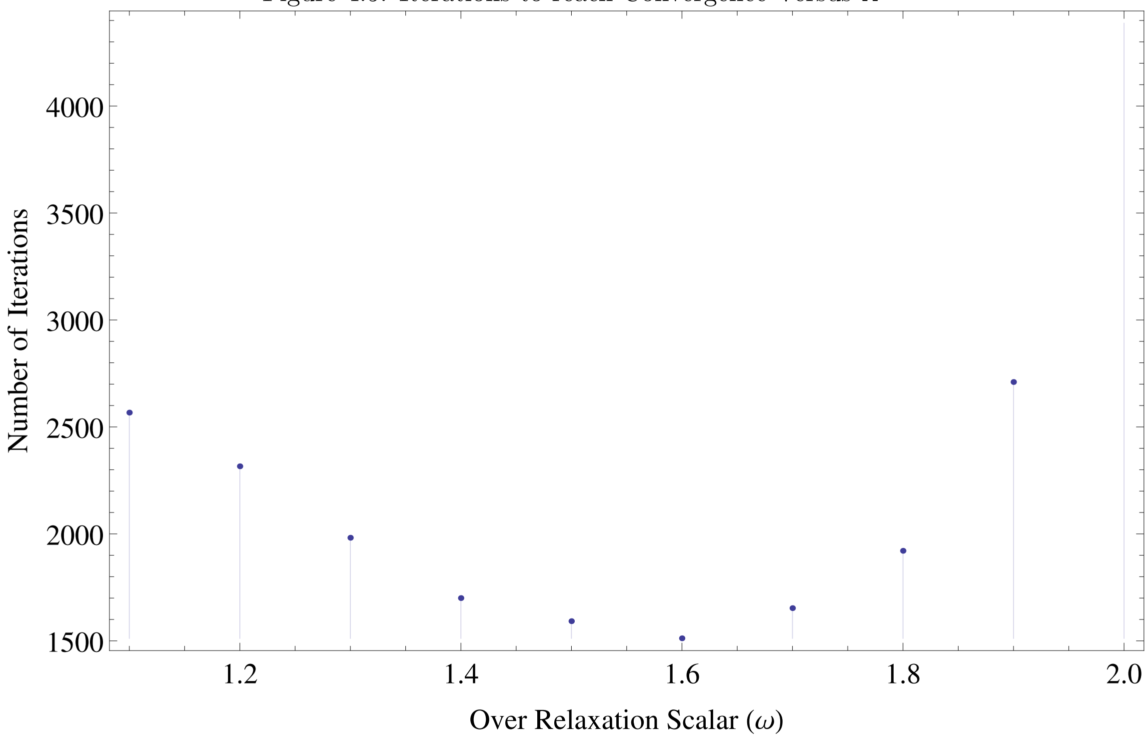
Figure 4.4: The Electric Field of a $129x129$ Grid with Central $10v$ Potential



Notice again that the steepest gradients are located directly between the cross and the grounded boundary.

2. The Over and Under Relaxation methods allow for accelerated convergence.

Figure 4.5: Iterations to reach Convergence Versus Ω



Note that the optimal $\Omega wasapproximately1.6andthatsolutionsbegandivergingatgreaterthanorequalto2.Hyperboll$