Implementation Exercise:

Return Predictability and Dynamic Asset Allocation: How Often Should Investors Rebalance?

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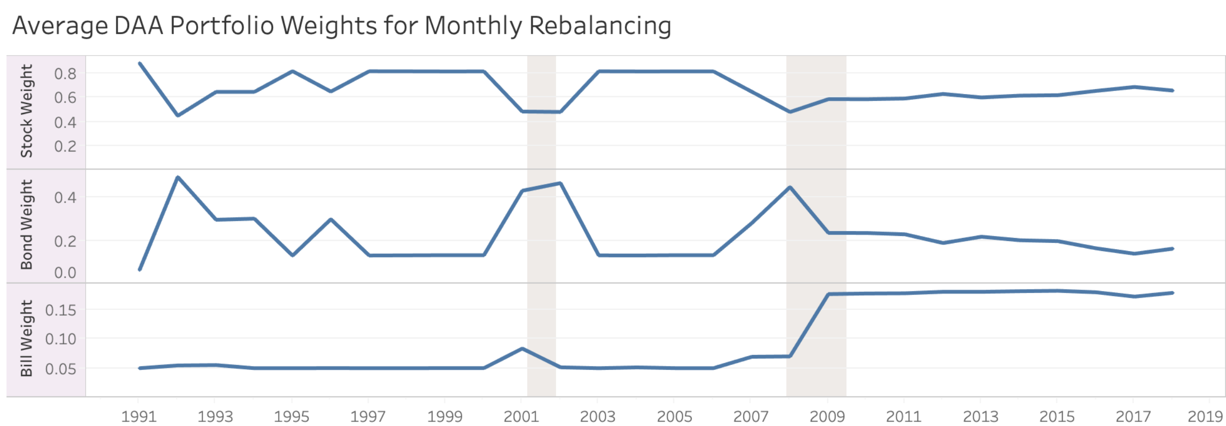
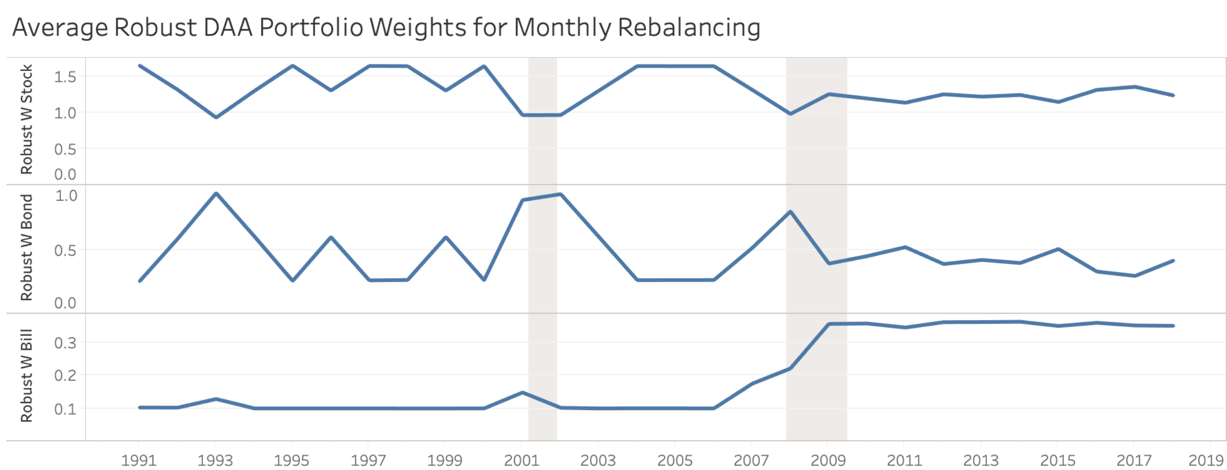
Results Overview

1. The optimal portfolio wegiths for both original optimization problem and simple robust allocation problem reduced the proportion of investment in stock during recession and market crashes





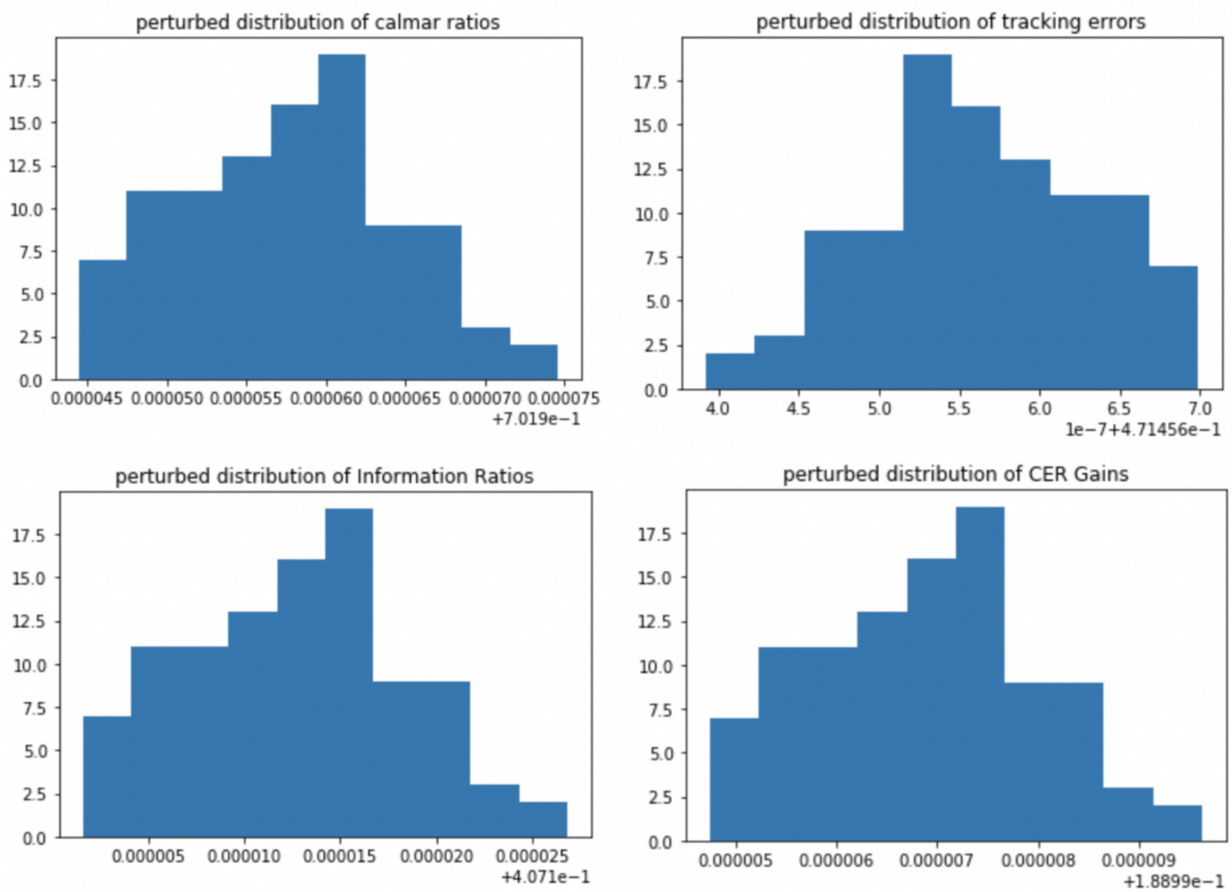
2. Based on various performance measures, the Robust Allocation was the most optimal strategy and the original formulation was more optimal than the benchmark scenario; that is, to invest 65% in stock, 30% in bond, and 5% in bill.

	Annualized Return	Standard Deviation	Maximum Drawdown	Calmar Ratio	Average Excess Return	Tracking Error	Information Ratio	CER Gain
Benchmark	8.25%	0.07723	0.41463	0.19896	-	-	-	-
Original Formulation	12.86%	0.09470	0.52184	0.24643	4.61%	0.31001	0.14872	1.64%
Robust Allocation	16.78%	0.09299	0.47305	0.35482	8.54%	0.31080	0.27463	4.34%
Robust vs Original	-	-	-	-	3.92%	0.06326	0.62042	2.71%

`Robust vs Original` values are obtained by treating `original formulation` as the benchmark portfolio

3. The robust allocation was not sensitive to small perturbations to Σ_{BL}.

The distriution of perturbed performance measures from `N=100` simulation was as following:



1. Processing data from Bloomberg Terminal

In [1]:
```
import pandas as pd
import numpy as np
import matplotlib.pyplot as plt
import multiprocessing
from gurobipy import *
```

In [2]:
```
def process_xlsx(filename,filetype = 'Bloomberg'):
    """
    process .xlsx files downloaded from Bloomberg and Factset
    """
    file = './data/' + filename
    if filetype == 'Bloomberg':
        df = pd.read_excel(file,header=5)
        df.rename(columns = {'Unnamed: 0':'dates'},inplace=True)
        df.loc[:,'Dates']=pd.to_datetime(df['Dates'])
    if filetype == 'Bonds':
        df = pd.read_excel(file,header=3,parser='Date')
    if filetype == 'Bills':
        df = pd.read_excel(file,header=20,parser='Date')
    return df.reset_index(drop=True)
```

In [3]:
```
other_df = process_xlsx('others.xlsx')
other_df.head()
```

Out[3]:

	Dates	SPX Index	SPXDIV Index	CPI INDX Index	CT10 Govt	CB3 Govt	MOODCBAA Index	MOODCAAA Index	IP Index
0	NaT	329.08	NaN	NaN	NaN	NaN	NaN	NaN	NaN
1	1990-02-28	331.89	NaN	128.0	NaN	NaN	10.14	9.22	64.0446
2	1990-03-30	339.94	0.31	128.6	NaN	NaN	10.21	9.37	64.3580
3	1990-04-30	330.80	0.98	128.9	NaN	NaN	10.30	9.46	64.2602
4	1990-05-31	361.23	2.74	129.1	NaN	NaN	10.41	9.47	64.3973

In [4]:
```
df_bonds = process_xlsx('./facset/US10YY-TU1.xlsx','Bonds')
df_bonds.head()
```

Out[4]:

	Date	Close	Volume	Change	% Change	Total Return (Gross)	Cumulative Return %	Open	High	Low
0	1990-01-31	8.43	NaN	NaN	NaN	8.43	0.000000	NaN	NaN	NaN
1	1990-02-28	8.51	NaN	0.080000	0.948991	8.51	0.948992	NaN	NaN	NaN
2	1990-03-30	8.65	NaN	0.139999	1.645116	8.65	2.609727	NaN	NaN	NaN
3	1990-04-30	9.04	NaN	0.390000	4.508675	9.04	7.236062	NaN	NaN	NaN
4	1990-05-31	8.60	NaN	-0.440000	-4.867252	8.60	2.016607	NaN	NaN	NaN

In [5]:
```
df_bill = process_xlsx('./facset/TRYUS3M-FDS.xlsx','Bills')
df_bill.head()
```

Out[5]:

	Date	Close	Volume	Change	% Change	Total Return (Gross)	Cumulative Return %	Open	High	Low
0	1990-01-31	7.74	NaN	NaN	NaN	7.74	0.000000	NaN	NaN	NaN
1	1990-02-28	7.77	NaN	0.03	0.387600	7.77	0.387597	NaN	NaN	NaN
2	1990-03-30	7.80	NaN	0.03	0.386103	7.80	0.775194	NaN	NaN	NaN
3	1990-04-30	7.79	NaN	-0.01	-0.128208	7.79	0.645995	NaN	NaN	NaN
4	1990-05-31	7.75	NaN	-0.04	-0.513478	7.75	0.129199	NaN	NaN	NaN

2. Processing Explanatory Variables

2.1 log(D/P)

More specifically, I am computing the following:

$$log(\sum_{s=1}^{12} D_{t-(s-1)}) - log(P_t)$$

where `D` stands for dividens paid by `SP500` constituents at time t. And `P` stands for `SP500` at time t.

As a sanity check, I wanted to check there are only two `NaN` values for `SPXDIV Index`

In [6]:
```
other_df['SPXDIV Index'].isna().sum()
```

Out[6]: 2

In [7]:
```
other_df.head()
```

Out[7]:

	Dates	SPX Index	SPXDIV Index	CPI INDX Index	CT10 Govt	CB3 Govt	MOODCBAA Index	MOODCAAA Index	IP Index
0	NaT	329.08	NaN	NaN	NaN	NaN	NaN	NaN	NaN
1	1990-02-28	331.89	NaN	128.0	NaN	NaN	10.14	9.22	64.0446
2	1990-03-30	339.94	0.31	128.6	NaN	NaN	10.21	9.37	64.3580
3	1990-04-30	330.80	0.98	128.9	NaN	NaN	10.30	9.46	64.2602
4	1990-05-31	361.23	2.74	129.1	NaN	NaN	10.41	9.47	64.3973

In [8]:
```
df = pd.DataFrame([])
df['Dates'] = other_df['Dates']
df['log_DP'] = (other_df['SPXDIV Index'].rolling(12).sum().apply('log')
                - other_df['SPX Index'].apply('log'))
```

In [9]:
```
df[['Dates','log_DP']][13:].head()
```

Out[9]:

	Dates	log_DP
13	1991-02-28	NaN
14	1991-03-29	-3.184739
15	1991-04-30	-3.187664
16	1991-05-31	-3.221650
17	1991-06-28	-3.175793

Compare 14th row with following for sanity check:

In [10]:
```
np.log(sum(other_df['SPXDIV Index'][2:14])) \
- np.log(other_df['SPX Index'][14])
```

Out[10]: `-3.1847388834548687`

2.2 Inflation

Inflation is calculated by CPI Index. The research paper writes following:
We account for the delay in the release of monthly CPI data when computing the forecast
My susequent search suggested that CPI Index is released with lag of 1 month.
Thus, I adjust for the issue by shifting CPI Index level by 1.

$$inflation_t = \frac{CPI_t}{CPI_{t-1}}$$

However, at time t, the most recent available inflation rate is that of time t-1.

In [11]:
```
df['inflation'] = other_df['CPI INDX Index'].pct_change().shift()
```

In [12]:
```
df[['Dates','inflation']].head()
```

Out[12]:

	Dates	inflation
0	NaT	NaN
1	1990-02-28	NaN
2	1990-03-30	NaN
3	1990-04-30	0.004687
4	1990-05-31	0.002333

Following is a sanity check. inflation rate for 2nd row is consistent with third row of inflation columns. The rationale for this is explained at the Markdown Cell above

In [13]:
```
(other_df['CPI INDX Index'][2]/other_df['CPI INDX Index'][1] -1)
```

Out[13]:
```
0.004687499999999956
```

2.3 Term Spread

Term spread refers to the difference between 10-year Treasury bond yield and the three-month Treasury bill yield.

In [14]:
```
df['Term_Spread'] = df_bonds['Close'] - df_bill['Close']
```

In [15]:
```
df[['Dates','Term_Spread']].head()
```

Out[15]:

	Dates	Term_Spread
0	NaT	0.690001
1	1990-02-28	0.740000
2	1990-03-30	0.849999
3	1990-04-30	1.250000
4	1990-05-31	0.850000

2.4 Default Spread

The difference between Moody's BAA and AAA rated corporate Bond yields

In [16]:
```
df['Default_Spread']=other_df['MOODCBAA Index'] \
                                        - other_df['MOODCAAA Index']
```

In [17]:
```
df[['Dates','Default_Spread']].head()
```

Out[17]:

	Dates	Default_Spread
0	NaT	NaN
1	1990-02-28	0.92
2	1990-03-30	0.84
3	1990-04-30	0.84
4	1990-05-31	0.94

As with `term spread` , a few observation would be sufficient for a sanity check as it only requires a subtraction

In [18]:
```
other_df[['Dates','MOODCBAA Index','MOODCAAA Index']].head()
```

Out[18]:

	Dates	MOODCBAA Index	MOODCAAA Index
0	NaT	NaN	NaN
1	1990-02-28	10.14	9.22
2	1990-03-30	10.21	9.37
3	1990-04-30	10.30	9.46
4	1990-05-31	10.41	9.47

2.5 Moving Averages

Average of `SPX Index` for the past 12-months.

$$MA_t = \frac{\sum_{i=0}^{11} SPX_{t-i}}{12}$$

In [19]:
```
Moving_Average_12 = other_df['SPX Index'].rolling(12).mean()
```

In [20]:
```
Moving_Average_12[10:].head()
```

Out[20]:
```
10           NaN
11    332.680000
12    333.917500
13    336.849167
14    339.789167
Name: SPX Index, dtype: float64
```

As a sanity check: the moving average of first 12 numbers are as following:

In [21]:
```
np.mean(other_df['SPX Index'][:12])
```

Out[21]:
```
332.68000000000006
```

In [22]:
```
Moving_Average_2 = other_df['SPX Index'].rolling(2).mean()
```

In [23]:
```
Moving_Average_2.head()
```

Out[23]:
```
0        NaN
1    330.485
2    335.915
3    335.370
4    346.015
Name: SPX Index, dtype: float64
```

As a sanity check: the moving average of first 12 numbers are as following:

In [24]:
```
np.mean(other_df['SPX Index'][:2])
```

Out[24]:
```
330.485
```

In [25]:
```
Bond_Moving_Average_12 = df_bonds['Close'].rolling(12).mean()
```

In [26]:
```
Bond_Moving_Average_12[10:].head()
```

Out[26]:
```
10         NaN
11    8.557500
12    8.524167
13    8.483333
14    8.433333
Name: Close, dtype: float64
```

As a sanity check: the moving average of first 12 numbers are as following:

In [27]:
```
df_bonds['Close'][:12].mean()
```

Out[27]:
```
8.557499965031942
```

In [28]:
```
Bond_Moving_Average_6 = df_bonds['Close'].rolling(6).mean()
```

In [29]:
```
Bond_Moving_Average_6[5:].head()
```

Out[29]:
```
5    8.610000
6    8.598333
7    8.656667
8    8.685000
9    8.620000
Name: Close, dtype: float64
```

In [30]:
```
np.mean(df_bonds['Close'][:6])
```

Out[30]:
```
8.610000133514404
```

2.5.1 MA(1,12)

It is a dummy variable based on `SPX Index` level and `Moving_Average_12`
If `SPX Index` value is greater than its 12 months moving average, then assign 1. Otherwise assign 0.

In [31]:
```
MA_nan = ((other_df['SPX Index']-Moving_Average_12)
          .apply(lambda x: np.nan if np.isnan(x) else 0))
```

In [32]:
```
df['MA_1_12']=((other_df['SPX Index']-Moving_Average_12)
               .apply(lambda x: 1 if x>0 else 0) + MA_nan)
```

In [33]:
```
df[['Dates','MA_1_12']][9:].head()
```

Out[33]:

	Dates	MA_1_12
9	1990-10-31	NaN
10	1990-11-30	NaN
11	1990-12-31	0.0
12	1991-01-31	1.0
13	1991-02-28	1.0

As a sanity check, since Moving_Average_12 is already checked, following shows that lambda function performs as expected.

In [34]:
```
(other_df['SPX Index']-Moving_Average_12)[9:].head()
```

Out[34]:
```
9            NaN
10           NaN
11     -2.460000
12     10.012500
13     30.220833
Name: SPX Index, dtype: float64
```

2.5.2 MA(2,12)

It is a dummy variable based on `Moving_Average_2` and `Moving_Average_12`
If `Moving_Average_2` value is greater than `Moving_Average_12` value, then assign 1. Otherwise assign 0.

In [35]:
```
df['MA_2_12']=((Moving_Average_2-Moving_Average_12)
               .apply(lambda x: 1 if x>0 else 0) + MA_nan)
```

In [36]:
```
df[['Dates','MA_2_12']][9:].head()
```

Out[36]:

	Dates	MA_2_12
9	1990-10-31	NaN
10	1990-11-30	NaN
11	1990-12-31	0.0
12	1991-01-31	1.0
13	1991-02-28	1.0

2.5.3 MOMBY(6)

It is a dummy variable based on `Bond_Moving_Average_6` and the current bond yield.
If the bond yield is greater than `Bond_Moving_Average_6` by more than 5 basis points, assign -1.
Else if the bond yield is less than `Bond_Moving_Average_6` by more than 5 basis points, assign 1.
Otherwie, assign 0.

In [37]:
```
MOMBY_6_nan = (df_bonds['Close']-Bond_Moving_Average_6
              ).apply(lambda x: np.nan if np.isnan(x) else 0)
```

In [38]:
```
df['MOMBY_6']=((df_bonds['Close']-Bond_Moving_Average_6
              ).apply(lambda x: -1 if x>0.05 else (1 if x<0.05 else 0))
               + MOMBY_6_nan)
```

In [39]:
```
df[['Dates','MOMBY_6']][5:].head()
```

Out[39]:

	Dates	MOMBY_6
5	1990-06-29	1.0
6	1990-07-31	1.0
7	1990-08-31	-1.0
8	1990-09-28	-1.0
9	1990-10-31	1.0

2.5.3 MOMBY(12)

In [40]:
```
MOMBY_12_nan = (df_bonds['Close']-Bond_Moving_Average_12
              ).apply(lambda x: np.nan if np.isnan(x) else 0)
```

In [41]:
```
df['MOMBY_12']=((df_bonds['Close']-Bond_Moving_Average_12)
                .apply(lambda x: -1 if x>0.05 else (1 if x<0.05 else 0))
                + MOMBY_12_nan)
```

In [42]:
```
df[['Dates','MOMBY_12']][11:].head()
```

Out[42]:

	Dates	MOMBY_12
11	1990-12-31	1.0
12	1991-01-31	1.0
13	1991-02-28	1.0
14	1991-03-29	1.0
15	1991-04-30	1.0

2.6 MOM

It is a dummy variable that depends on `SPX Index` and its lagged values.

2.6.1 MOM(9)

If the difference between `SPX Index` and its 9 months lagged value is positive then assign 1. Otherwise assign 0.

In [43]:
```
MOM_9_nan = (other_df['SPX Index']-other_df['SPX Index'].shift(9)
            ).apply(lambda x: np.nan if np.isnan(x) else 0)
df['MOM_9']=(other_df['SPX Index']-other_df['SPX Index'].shift(9)
            ).apply(lambda x: 1 if x>0 else 0) + MOM_9_nan
```

In [44]:
```
df[['Dates','MOM_9']][8:].head()
```

Out[44]:

	Dates	MOM_9
8	1990-09-28	NaN
9	1990-10-31	0.0
10	1990-11-30	0.0
11	1990-12-31	0.0
12	1991-01-31	1.0

Following checks the result

In [45]:
```
(other_df['SPX Index']-other_df['SPX Index'].shift(9))[8:].head()
```

Out[45]:
```
8        NaN
9     -25.08
10     -9.67
11     -9.72
12     13.13
Name: SPX Index, dtype: float64
```

2.6.2 MOM(12)

If the difference between `SPX Index` and its 12 months lagged value is positive then assign 1. Otherwise assign 0.

In [46]:
```
MOM_12_nan = (other_df['SPX Index']-other_df['SPX Index'].shift(12)
              ).apply(lambda x: np.nan if np.isnan(x) else 0)
df['MOM_12']=(other_df['SPX Index']-other_df['SPX Index'].shift(12)
              ).apply(lambda x: 1 if x>0 else 0) + MOM_12_nan
```

In [47]:
```
df[['Dates','MOM_12']][11:].head()
```

Out[47]:

	Dates	MOM_12
11	1990-12-31	NaN
12	1991-01-31	1.0
13	1991-02-28	1.0
14	1991-03-29	1.0
15	1991-04-30	1.0

2.7 Output Gap

The deviation of the log of industrial production from a quadratic trend.
I believe this data is available from Bloomberg Terminal. However, it is simple to compute.
Therefore, I decided to simply compute it.
A quadratic trend is of the following form:

$$y_t = \beta_0 + \beta_0 x_t + \beta_0 x_t^2 + \epsilon_t$$

$\widehat{\beta}$ is estimated by

$$\widehat{\beta} = (X^T X)^{-1} X^T y$$

following provides an illustration of this estimation procedure.

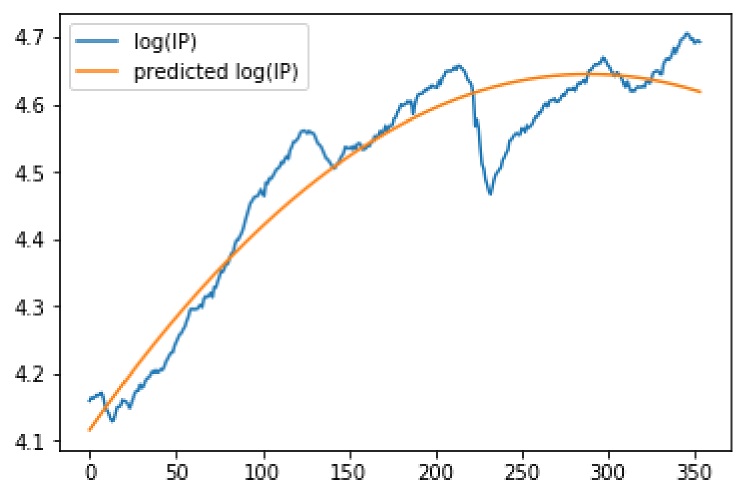
In [48]:

```
def beta_est(obs):
    """
    estimates beta of ols to minimize l2 norm
    """
    y = np.log(other_df['IP Index'].dropna())[:obs]
    X = np.concatenate((np.ones(obs).reshape(-1,1),
                        np.arange(obs).reshape(-1,1)),
                       axis=1)
    X = np.concatenate((X,(np.arange(obs)**2)
                        .reshape(-1,1)),
                       axis=1)
    beta_hat = np.dot(
                  np.matmul(
                      np.linalg.inv(
                          1e-6 *np.eye(3)
                          + np.matmul(X.transpose(),X)),
                            X.transpose()),
                  y.values.reshape(-1,1))
    return beta_hat
```

In [49]:

```
n = len(np.log(other_df['IP Index'].dropna()))
X = np.concatenate((np.ones(n).reshape(-1,1)
                    ,np.arange(n).reshape(-1,1)),
                   axis=1)
X = np.concatenate((X,(np.arange(n)**2)
                    .reshape(-1,1)),
                   axis=1)
beta_hat = beta_est(n)
y_pred = np.matmul(X,beta_hat)
y = np.log(other_df['IP Index'].dropna())
plot_df = np.concatenate((np.arange(n)
                          .reshape(-1,1),
                          y.values.reshape(-1,1),
                          y_pred),
                         axis=1)
plot_df = pd.DataFrame(plot_df)
plot_df.columns = ['obs','log(IP)','predicted log(IP)']
plt.plot('obs','log(IP)',data=plot_df)
plt.plot('obs','predicted log(IP)',data=plot_df)
plt.legend()
```

Out[49]: <matplotlib.legend.Legend at 0x126a44978>



However, the research estimates $\widehat{\beta}$ from data available up to each point in time. Thus, I will repeat the calculation above to every time step.

In [50]:

```
def output_gap_computer(obs):
    """
    computes the deviation of the log of industrial
    production from a quadratic trend.
    """
    return y[obs]-np.dot(X[:obs],beta_est(obs))[-1]
```

In [51]:

```
# Next Cell parallelize for loop. It is equivalent to

# output_gap2 = []
# for i in range(1,n+1):
#     output_gap2 += [output_gap_computer(i)]
```

In [52]:
```
pool = multiprocessing.Pool(4)
output_gap =[*pool.map(output_gap_computer, range(1, n+1))]
```

In [53]:
```
df['output_gap']=np.concatenate(([np.nan],
                                np.array(output_gap)
                                .reshape(-1)))
```

In [54]:
```
df[['Dates','output_gap']].head()
```

Out[54]:

	Dates	output_gap
0	NaT	NaN
1	1990-02-28	4.159576e-06
2	1990-03-30	2.422823e-09
3	1990-04-30	-5.646037e-09
4	1990-05-31	5.029228e-04

2.7 SPX Index return

In [55]:
```
df['r_SPX'] = other_df['SPX Index'].pct_change().shift(-1)
df[['Dates','r_SPX']].head()
```

Out[55]:

	Dates	r_SPX
0	NaT	0.008539
1	1990-02-28	0.024255
2	1990-03-30	-0.026887
3	1990-04-30	0.091989
4	1990-05-31	-0.008886

2.8 Bond return and yield

In [56]:
```
def compound_return(cum_return):
    sol = (cum_return.values/100 + 1)
    for i in range(2,len(sol)):
       sol[i] = sol[i]/(cum_return.values/100 + 1)[i-1]
    return sol - 1

```

In [57]:
```
df['y_bond'] = df_bonds['Close']/100
df[['Dates','y_bond']].head()
```

Out[57]:

	Dates	y_bond
0	NaT	0.0843
1	1990-02-28	0.0851
2	1990-03-30	0.0865
3	1990-04-30	0.0904
4	1990-05-31	0.0860

In [58]:
```
df['r_bond'] = (compound_return(
                    df_bonds['Cumulative Return %'].shift(-1)))
df[['Dates','r_bond']].head()
```

Out[58]:

	Dates	r_bond
0	NaT	0.009490
1	1990-02-28	0.026097
2	1990-03-30	0.045087
3	1990-04-30	-0.048673
4	1990-05-31	-0.019767

To check the compound_return function indeed takes cumulative return as input and computes the compounding rate of return: I will manually compute `r_bond` on `1990-03-30`. The numbers correspond to cumulative returns as can be seen from the cell below:

$$(1.026097) = (1.009490)(1 + r) \Rightarrow r = 0.016451$$
$$(1.072361) = (1.009490)(1.016451)(1 + r) \Rightarrow r = 0.045087$$

In [59]:
```
(df_bonds['Cumulative Return %']/100 + 1).head()
```

Out[59]:
```
0    1.000000
1    1.009490
2    1.026097
3    1.072361
4    1.020166
Name: Cumulative Return %, dtype: float64
```

2. Bill return and yield

In [60]:

```
df['y_bill'] = df_bill['Close']/100
df[['Dates','y_bill']].head()
```

Out[60]:

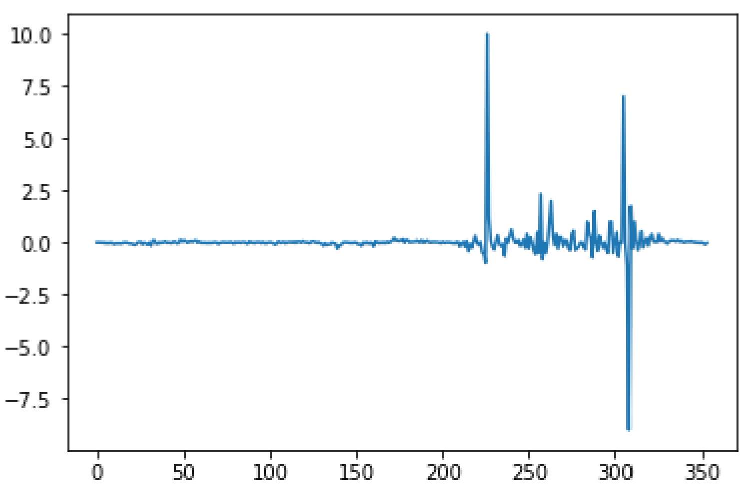
	Dates	y_bill
0	NaT	0.0774
1	1990-02-28	0.0777
2	1990-03-30	0.0780
3	1990-04-30	0.0779
4	1990-05-31	0.0775

In [61]:

```
df['r_bill'] = compound_return(\
                         df_bill['Cumulative Return %'].shift(-1))
plt.plot(df['r_bill'])
df[['Dates','r_bill']].head()
```

Out[61]:

	Dates	r_bill
0	NaT	0.003876
1	1990-02-28	0.007752
2	1990-03-30	-0.001282
3	1990-04-30	-0.005135
4	1990-05-31	-0.001290



To check the compound_return function indeed takes cumulative return as input and computes the compounding rate of return: I will manually compute `r_bill` on `1990-03-30` . The numbers correspond to cumulative returns as can be seen from the cell below:

$$(1 + 0.007752) = (1 + 0.003876)(1 + r) \Rightarrow r = 0.003861$$
$$(1 + 0.006460) = (1 + 0.003876)(1 + 0.003861)(1 + r) \Rightarrow r = -0.001282$$

In [62]:
```
(df_bill['Cumulative Return %']/100 + 1).head()
```

Out[62]:
```
0    1.000000
1    1.003876
2    1.007752
3    1.006460
4    1.001292
Name: Cumulative Return %, dtype: float64
```

In [63]:
```
1.006460/((1+0.003861)*(1+0.003876)) -1
```

Out[63]:
```
-0.001282027007208475
```

3. Return Forecasts

After preprocessing available data is from 1991-03-29 to 2019-07-31

In [64]:
```
processed_df = df.loc[14:353].reset_index(drop=True)
processed_df.tail()
```

Out[64]:

	Dates	log_DP	inflation	Term_Spread	Default_Spread	MA_1_12	MA_2_12	MOMBY_6	MO
335	2019-02-28	-3.780938	-0.000198	0.3123	1.16	1.0	1.0	1.0	
336	2019-03-29	-3.785870	0.001741	0.0657	1.07	1.0	1.0	1.0	
337	2019-04-30	-3.818848	0.004089	0.1212	1.01	1.0	1.0	1.0	
338	2019-05-31	-3.740195	0.003187	-0.1581	0.96	0.0	1.0	1.0	
339	2019-06-28	-3.792573	0.000773	-0.0816	1.04	1.0	1.0	1.0	

3.1 Stock Returns

First step is to find truncated PCA for different number of eigenvalues.
PCA is implemented on the following matrix. Each variable corresponds to a set of observations and hence is a column vector.
[$log(\frac{D}{P})$, Inflation, Term Spread, Default Spread, Output Gap, MA(1,12), MA(2,12), MOM(9), MOM(12)]

Second step is to come up with a decision rule on how to truncate eigenvalues. The research paper utilizes (1) out-of-sample R^2 denoted as R^2_{OS} and (2) Clark and West statistic.
where

$$R^2_{OS} = 1 - \frac{\sum_{t=1}^{T}(r_t - \hat{r}_t)}{\sum_{t=1}^{T}(r_t - \bar{r}_{t-1})}$$

and

$\widehat{r}_t$ is the fitted value using data up to t-1
$\overline{r}_{t-1}$ is the historal average using data upto t-1

Apart from R^2_{OS}, the paper utilizes Clark and West (2007) test.
Refer: Approximately Normal Tests for Equal Predictive Accuracy in Nested Models.
Clark and West claims that test of mean squared prediction error (MSPE) typically exhibits a stylised pattern. That is, `the MSPE under Null (parsimonious model) is relatively smaller than it is expected to be because of the efficiency of parsimonious model and noises from estimating more parameters`. Therefore, authors propose an alternative hypothesis test as following:
For the hypothesis testing
H0: Parsimonious model (constant) MSPE is equal to or better than that of the larger model, H1: Larger model is better.

$$\hat{f}_{t+1} = (y_{t+1} - \hat{y}_{\text{pars}:t,t+1})^2 - [(y_{t+1} - \hat{y}_{\text{large}:t,t+1})^2 - (\hat{y}_{\text{pars}:t,t+1} - \hat{y}_{\text{large}:t,t+1})^2]$$

$$\overline{f} = \frac{1}{T}\sum_{t=1}^{T} \hat{f}_{t+1}$$

$$s^2_{\hat{f}-\overline{f}} = \frac{1}{T-1}\sum_{t=1}^{T}(\hat{f}_{t+1} - \overline{f})^2$$

Test statistics is:

$$CW = \frac{\overline{f}}{s_{\hat{f}-\overline{f}}/\sqrt{T}}$$

the mean of $\hat{f}_{t+\tau}$ denoted as $\overline{\hat{f}_{t+\tau}}$. With 10% significance level, reject null if $\overline{\hat{f}_{t+\tau}}$ > 1.282. With 5% significance level, reject null if $\overline{\hat{f}_{t+\tau}}$ > 1.645. For one step forecast errors, the usual least squares standard errors can be used. For autocorrelated forecast errors, an autocorrelation consistent standard error should be used.

In [65]:
```
def truncated_PC(X,dim):
    eig, V = np.linalg.eig(np.matmul(X.transpose(),X))
    approx_X =  np.matmul(X,V[:,:dim])
    return approx_X
```

In [66]:
```
def PC_approx_error(X,dim):
    """
    PC approximation errors in terms of frobenius norms
    """
    eig, V = np.linalg.eig(np.matmul(X.transpose(),X))
    approx_A =  np.matmul(np.matmul(V[:,:dim],
                            np.diag(eig[:dim])),
                            V[:,:dim].transpose())
    error = np.linalg.norm(approx_A - np.matmul(\
                                X.transpose(),X), ord='fro')
    return error
```

In [67]:
```
def PC_fit(X,r,dim):
    """
    estimates SPX Index return in a way that minimizes l2 norm
    """
    X = np.concatenate((np.ones(X.shape[0])
                        .reshape(-1,1),
                        X),
                       axis=1)
    beta = \
    np.matmul(
        np.linalg.inv(\
            1e-6*np.eye(dim+1) + np.matmul(X.transpose(),X)),
        np.matmul(X.transpose(),r))
    return beta
```

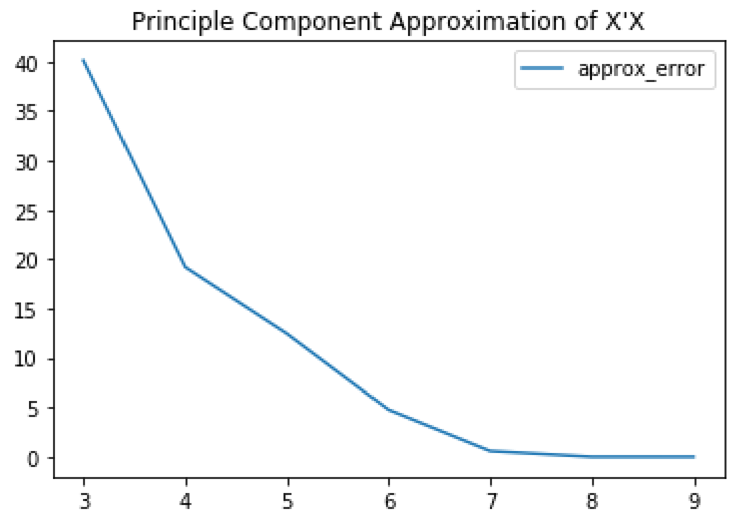
3.1.1 Principle Component Regression

In [68]:
```
stock_explanatory_variables = \
            ['log_DP','inflation','Term_Spread','Default_Spread',\
            'output_gap','MA_1_12','MA_2_12','MOM_9','MOM_12']
X = processed_df[stock_explanatory_variables].values
```

The plot of PC approximation errors in terms of frobenius norm is as following:

In [69]:
```
PC_plot_df = pd.DataFrame([])
PC_plot_df['dim'] = range(3,X.shape[1]+1)
PC_plot_df['approx_error'] = [PC_approx_error(X,dim) for dim \
                                in range(3,X.shape[1]+1)]
plt.plot('dim','approx_error',data=PC_plot_df)
plt.title("Principle Component Approximation of X'X")
plt.legend()
```

Out[69]: <matplotlib.legend.Legend at 0x10d21eb00>

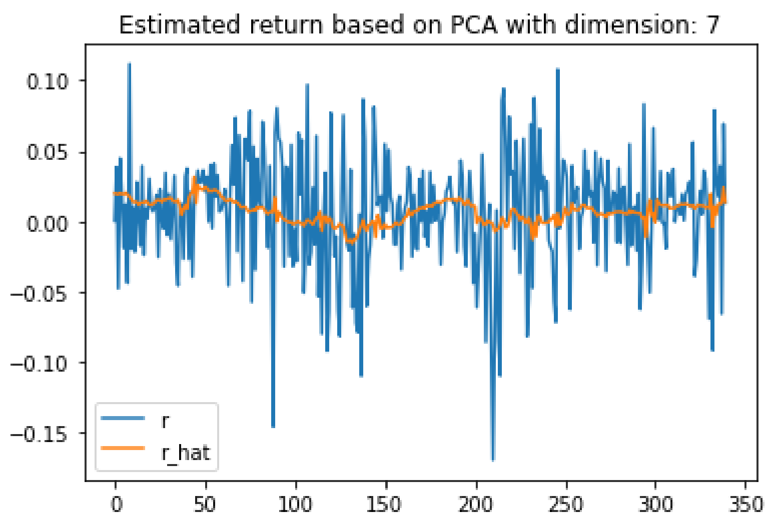


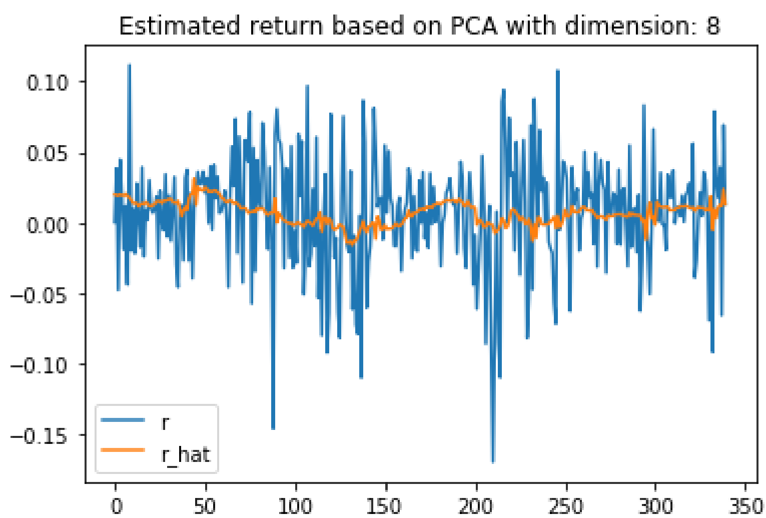
Above plot confirms that the principle component approximation works as expected

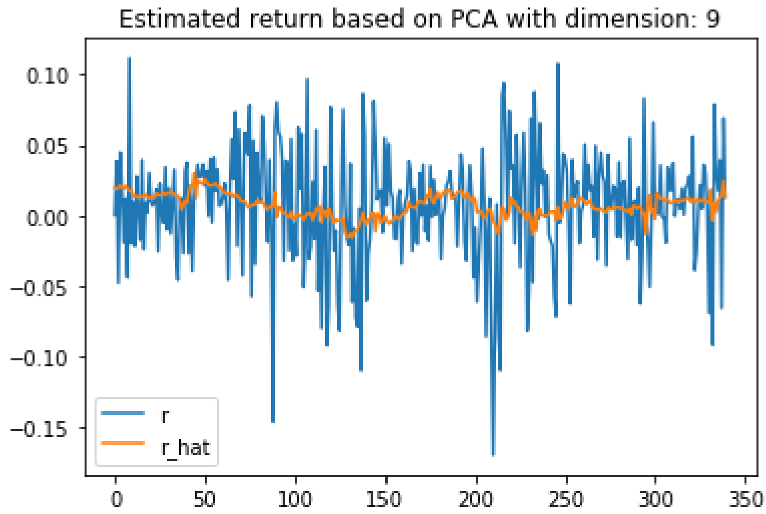
Following figures compare actual return with estimated return based on PC regression.

In [70]:

```
for i in range(7,10):
    dim = i
    r = processed_df['r_SPX'].values
    X_PCA = truncated_PC(X,dim)
    w = PC_fit(X_PCA,r,dim)
    X_PCA_intercept = np.concatenate(
                        (np.ones(X_PCA.shape[0])
                         .reshape(-1,1),
                         X_PCA),axis=1)
    r_hat = np.matmul(X_PCA_intercept,w)
    plt.figure(i)
    PC_plot_df2 = pd.DataFrame([])
    PC_plot_df2['r_hat'] = r_hat
    PC_plot_df2['r'] = r
    plt.plot('r',data=PC_plot_df2)
    plt.plot('r_hat',data=PC_plot_df2)
    plt.title(f'Estimated return based on PCA with dimension: {dim}')
    plt.legend()

```







3.1.2 R^2_{OS} Computation

Following the logic of the original paper, I will compute R^2_{OS} for monthly (h=1), quartherly (h=3), semi-annual (h=6), and annual (h=12). And out-of-sample forecasts are estimated by recursive estimation windows. For example, for monthly estimation, initial 200 samples are used exclusively for fitting the model. The 201st sample is forecasted by the model fitted by 200 samples. The 202nd sample is estimated by the model fitted using 201 samples. And so on.

Below illustrations show that monthly R^2_{OS} gives the highest value range. This is consistent with the original paper.

In [71]:
```
X[:200].shape
```

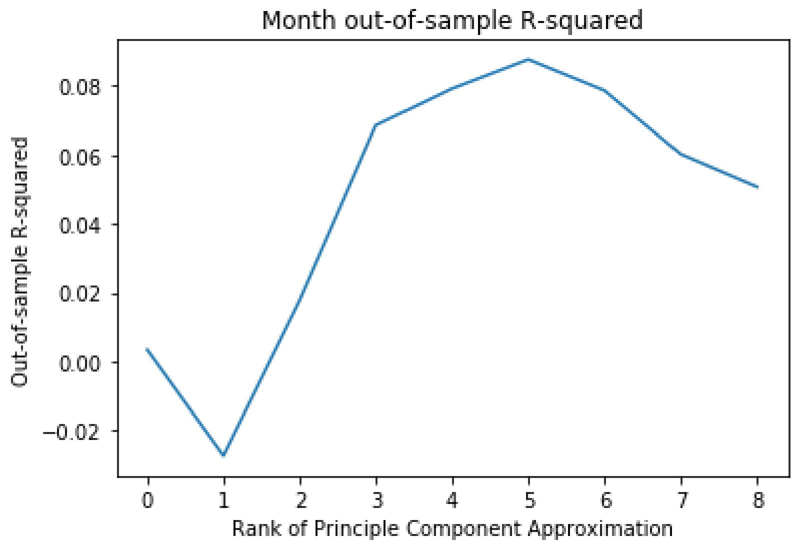
Out[71]:
```
(200, 9)
```

In [72]:
```python
def Rsquared_OS(X,r,dim,OS_start):
    """
    computes out-of-sample rsquared.
    First computes PCA only using explanatory variables
    without augmenting the data with a constant = 1.
    For regression result, added a constant to capture
    the y-intercept.
    PCA_fit by default adds the constant column.
    Therefore, I only add the constant column to obtain
    one step ahead forecast using the weights obtained
    by the PCA_fit
    """
    numerator = 0
    denominator = 0
    for i in range(OS_start,len(r)):
        X_PCA = truncated_PC(X[:i],dim)
        w = PC_fit(X_PCA,r[:i],dim)
        X_PCA_OS = truncated_PC(X[:i+1],dim)
        X_PCA_OS_intercept = np.concatenate(
            (np.ones(X_PCA_OS.shape[0])
             .reshape(-1,1),
             X_PCA_OS),
            axis=1)
        r_hat = np.matmul(X_PCA_OS_intercept[-1],w)
        numerator += (r[i] - r_hat)**2
        denominator += (r[i]-r[:i].mean())**2
    R_squared_OS = 1 - numerator/(denominator + 1e-6)
    return R_squared_OS
```

3.1.2.1 Month R^2_{OS}

In [73]:
```
plt.plot([Rsquared_OS(X,r,i,200) for i in range(1,X.shape[1]+1)])
plt.xlabel('Rank of Principle Component Approximation')
plt.ylabel('Out-of-sample R-squared')
plt.title('Month out-of-sample R-squared')
```

Out[73]:
```
Text(0.5, 1.0, 'Month out-of-sample R-squared')
```



3.1.2.2 Quarter R^2_{OS}

For consistency, I utilized 66 quarterly observations exclusively for fitting the model. Following dataframe shows that 66^{th} quarterly observation corresponds to 200^{th} monthly observation.

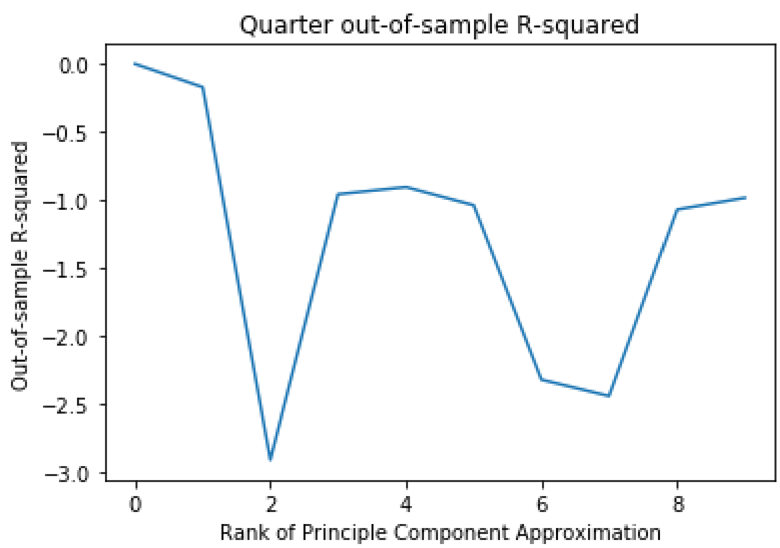
In [74]:
```
(processed_df[2:].reset_index()
                  .set_index('Dates')
                  .resample('3M')
                  .agg('last')
                  .reset_index()[64:].head())
```

Out[74]:

	Dates	index	log_DP	inflation	Term_Spread	Default_Spread	MA_1_12	MA_2_12	MOMBY_
64	2007-05-31	194	-3.843610	0.003001	0.2994	0.92	1.0	1.0	-1.
65	2007-08-31	197	-3.779418	0.001781	0.6221	0.86	1.0	1.0	1.
66	2007-11-30	200	-3.788723	0.003083	0.8589	0.96	1.0	1.0	1.
67	2008-02-29	203	-3.710938	0.003448	1.7002	1.29	0.0	0.0	1.
68	2008-05-31	206	-3.774398	0.002314	2.2153	1.36	0.0	0.0	-1.

In [75]:
```
X_quarter = (processed_df[2:]
                 .set_index('Dates')
                 .resample('3M')
                 .agg('last')[stock_explanatory_variables].values)
r_quarter = (processed_df[2:]
                 .set_index('Dates')
                 .resample('3M')
                 .agg('last')['r_SPX'].values)
plt.plot([Rsquared_OS(X_quarter,r_quarter,i,66)
             for i in range(X.shape[1]+1)])
plt.xlabel('Rank of Principle Component Approximation')
plt.ylabel('Out-of-sample R-squared')
plt.title('Quarter out-of-sample R-squared')
```

Out[75]: `Text(0.5, 1.0, 'Quarter out-of-sample R-squared')`



3.1.2.3 Semi-annual R^2_{OS}

For consistency, I utilized 33 semi-annually observations exclusively for fitting the model. Following dataframe shows that 33rd quarterly observation corresponds to 200th monthly observation.

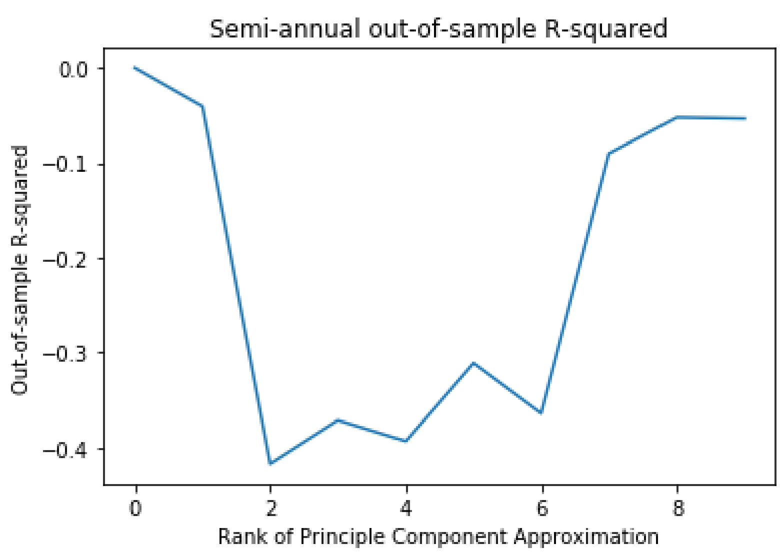
In [76]:
```
(processed_df[2:].reset_index()
                 .set_index('Dates')
                 .resample('6M')
                 .agg('last')
                 .reset_index()[32:].head())
```

Out[76]:

	Dates	index	log_DP	inflation	Term_Spread	Default_Spread	MA_1_12	MA_2_12	MOMBY_
32	2007-05-31	194	-3.843610	0.003001	0.2994	0.92	1.0	1.0	-1
33	2007-11-30	200	-3.788723	0.003083	0.8589	0.96	1.0	1.0	1
34	2008-05-31	206	-3.774398	0.002314	2.2153	1.36	0.0	0.0	-1
35	2008-11-30	212	-3.339371	-0.008598	2.9100	3.09	0.0	0.0	1
36	2009-05-31	218	-3.346824	0.001007	3.3204	2.52	0.0	0.0	-1

In [77]:
```
X_semi = (processed_df[2:]
          .set_index('Dates')
          .resample('6M')
          .agg('last')[stock_explanatory_variables].values)
r_semi = (processed_df[2:]
          .set_index('Dates')
          .resample('6M')
          .agg('last')['r_SPX'].values)
plt.plot([Rsquared_OS(X_semi,r_semi,i,33)
          for i in range(X.shape[1]+1)])
plt.xlabel('Rank of Principle Component Approximation')
plt.ylabel('Out-of-sample R-squared')
plt.title('Semi-annual out-of-sample R-squared')
```

Out[77]: Text(0.5, 1.0, 'Semi-annual out-of-sample R-squared')



3.1.2.4 Annual R^2_{OS}

For consistency, I utilized 33 semi-annually observations exclusively for fitting the model. Following dataframe shows that 33rd quarterly observation corresponds to 200th monthly observation.

In [78]:
```
(processed_df.reset_index()
 .set_index('Dates')
 .resample('Y')
 .agg(lambda x: x[-2])
 .reset_index()[15:].head())
```

Out[78]:

	Dates	index	log_DP	inflation	Term_Spread	Default_Spread	MA_1_12	MA_2_12	MOMBY_
15	2006-12-31	188	-3.814893	-0.004438	-0.4390	0.87	1.0	1.0	1
16	2007-12-31	200	-3.788723	0.003083	0.8589	0.96	1.0	1.0	1
17	2008-12-31	212	-3.339371	-0.008598	2.9100	3.09	0.0	0.0	1
18	2009-12-31	224	-3.617410	0.003002	3.1402	1.13	1.0	1.0	1
19	2010-12-31	236	-3.755558	0.003482	2.6254	1.05	1.0	1.0	-1

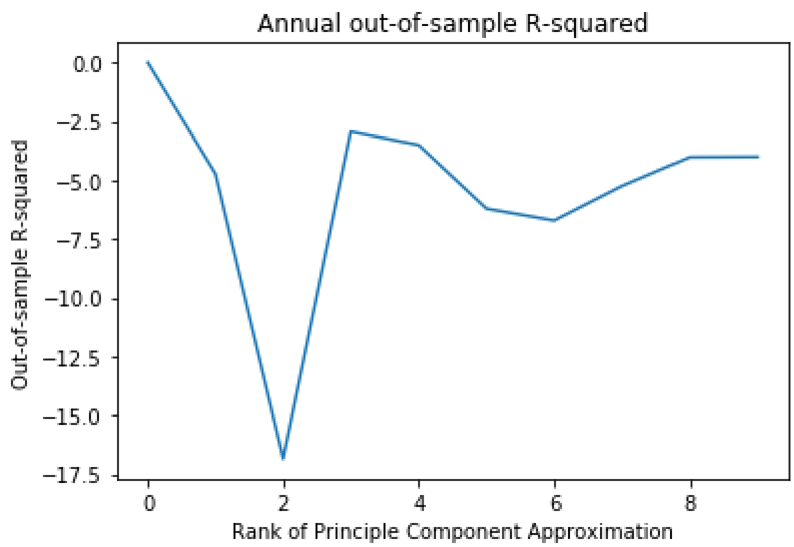
In [79]:

```python
X_annual = (processed_df.set_index('Dates')
            .resample('Y')
            .agg(lambda x: x[-2])[stock_explanatory_variables]
            .values)
r_annual = (processed_df.set_index('Dates')
            .resample('Y')
            .agg(lambda x: x[-2])['r_SPX'].values)
plt.plot([Rsquared_OS(X_annual,r_annual,i,16)
          for i in range(X.shape[1]+1)])
plt.xlabel('Rank of Principle Component Approximation')
plt.ylabel('Out-of-sample R-squared')
plt.title('Annual out-of-sample R-squared')
```

```
/Users/gimdong-geon/python3_cooking/lib/python3.7/site-packages/numpy/cor
e/numeric.py:538: ComplexWarning: Casting complex values to real discards
the imaginary part
  return array(a, dtype, copy=False, order=order)
```

Out[79]:

```
Text(0.5, 1.0, 'Annual out-of-sample R-squared')
```

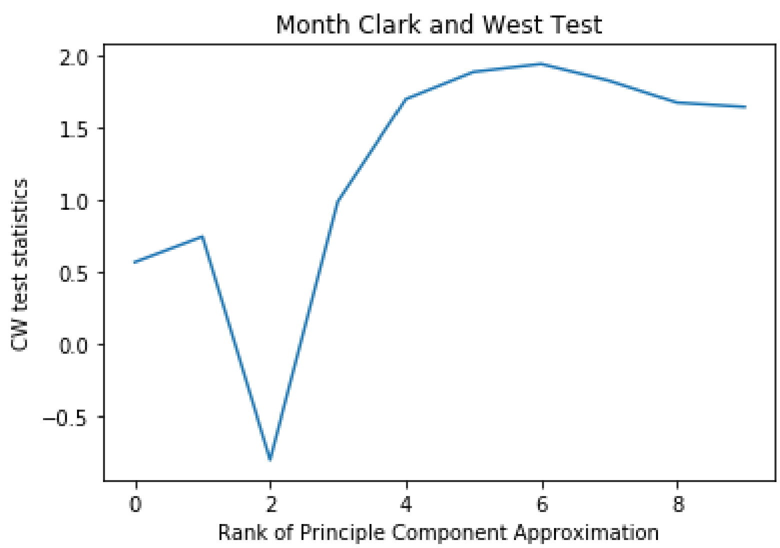


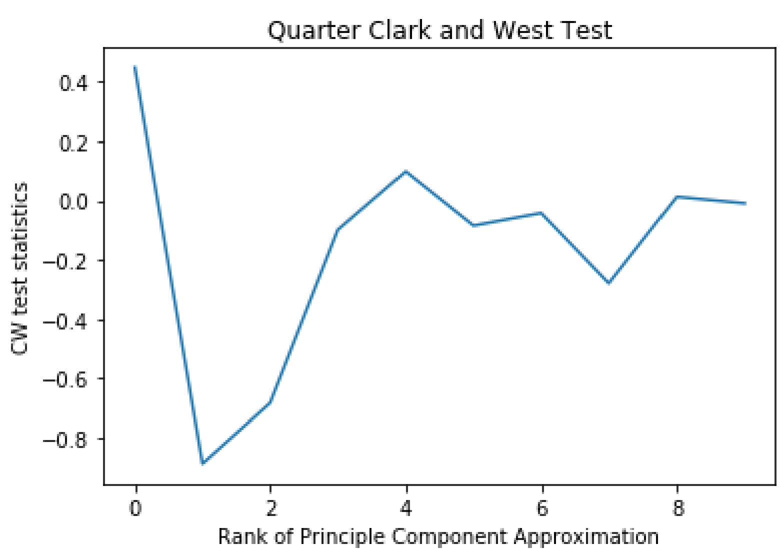
3.1.3 Clark and West(2007) Test Statistics Computation

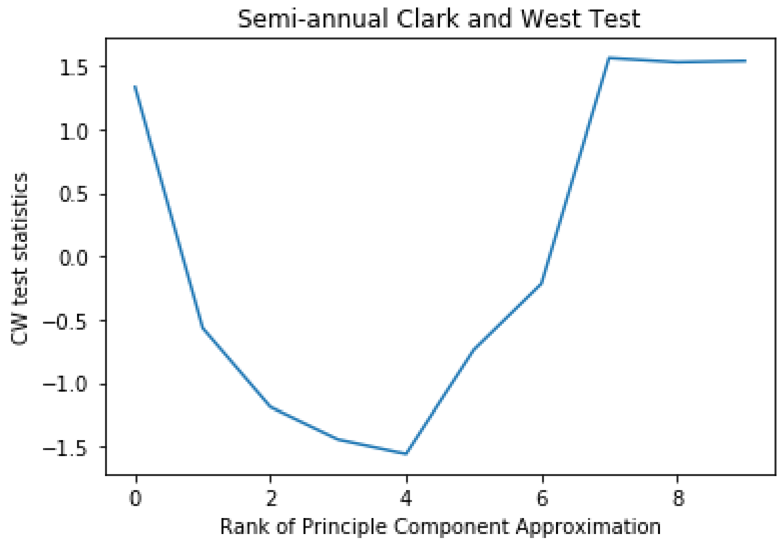
In [80]:

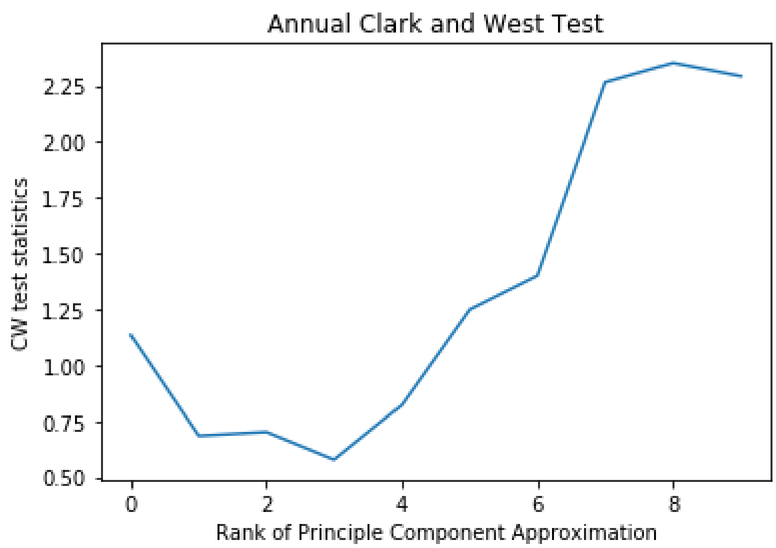
```
def CW_test(X,r,dim,OS_start):
    """
    computes Clark and West test statistics.
    First computes PCA only using explanatory
    variables without augmenting the data with
    a constant = 1.
    For regression result, added a constant to
    capture the y-intercept.
    PCA_fit by default adds the constant column.
    Therefore, I only add the constant column to obtain
    one step ahead forecast using the weights obtained
    by the PCA_fit

    """
    denom = len(r) - OS_start
    num = []
    for i in range(OS_start,len(r)):
        X_PCA = truncated_PC(X[:i],dim)
        w = PC_fit(X_PCA,r[:i],dim)
        X_PCA_OS = truncated_PC(X[:i+1],dim)
        X_PCA_OS_intercept = np.concatenate(
            (np.ones(X_PCA_OS.shape[0])
             .reshape(-1,1),X_PCA_OS),axis=1)
        r_hat = np.matmul(X_PCA_OS_intercept[-1],w)
        num += [(r[i]-r[:i].mean())**2
                - (r[i] - r_hat)**2
                + (r[:i].mean() - r_hat)**2]
    f_bar = np.array(num).mean()
    CW = np.sqrt(denom) * f_bar / \
         np.std(np.array(num) - f_bar,ddof=1)
    return CW
```

In [81]:
```
my_dict = {'month':200,'quarter':66,'semi':33,'annual':16}

i=0
for item in my_dict.keys():
    plt.figure(i)
    if item == 'month':
        plt.plot([CW_test(X,r,i,my_dict[item]) for i in range(X.shape[1
        plt.xlabel('Rank of Principle Component Approximation')
        plt.ylabel('CW test statistics')
        plt.title('Month Clark and West Test')
    else:
        eval(f'plt.plot([CW_test(X_{item},r_{item},i,my_dict[item]) for
        plt.xlabel('Rank of Principle Component Approximation')
        plt.ylabel('CW test statistics')
        if item == 'semi':
            plt.title(f'{item[0].upper()}{item[1:]}' + '-annual Clark a
        else:
            plt.title(f'{item[0].upper()}{item[1:]} Clark and West Test
    i+=1
```









3.2 Bond Returns

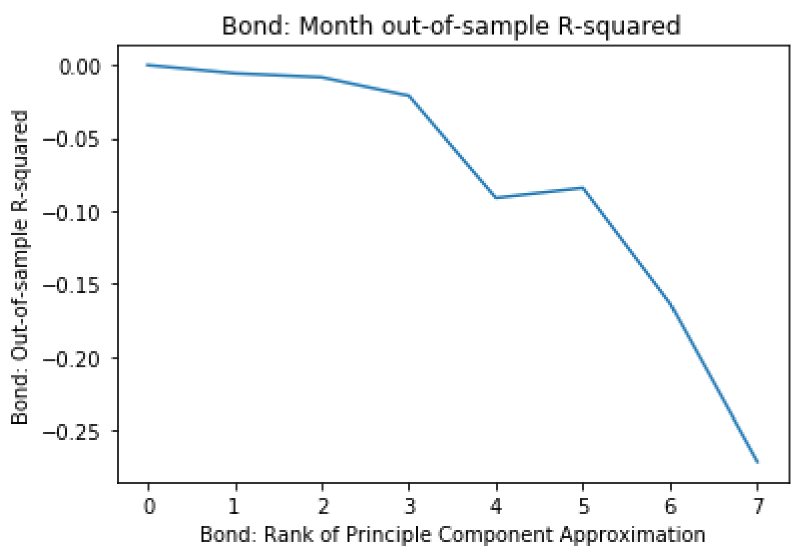
The logic of this section is equivalent to the stock returns. First section deals with principal component, the second section the R^2_{OS} and the third section the Clark West test statistics.

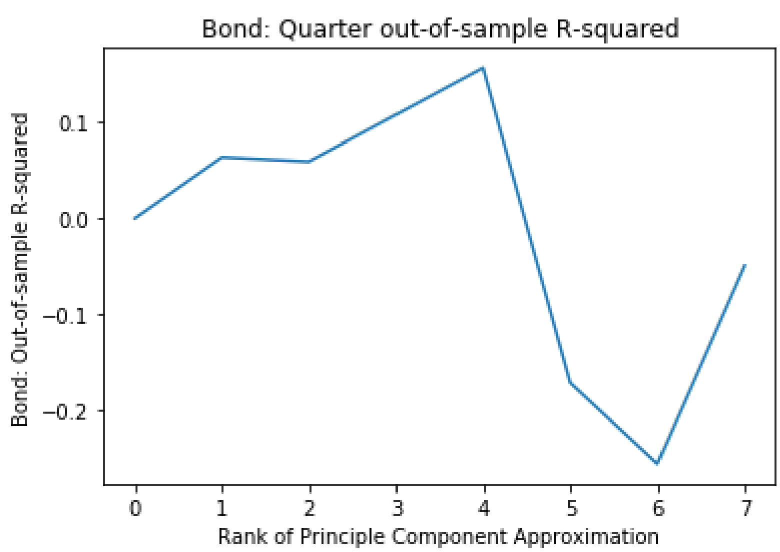
In [82]:

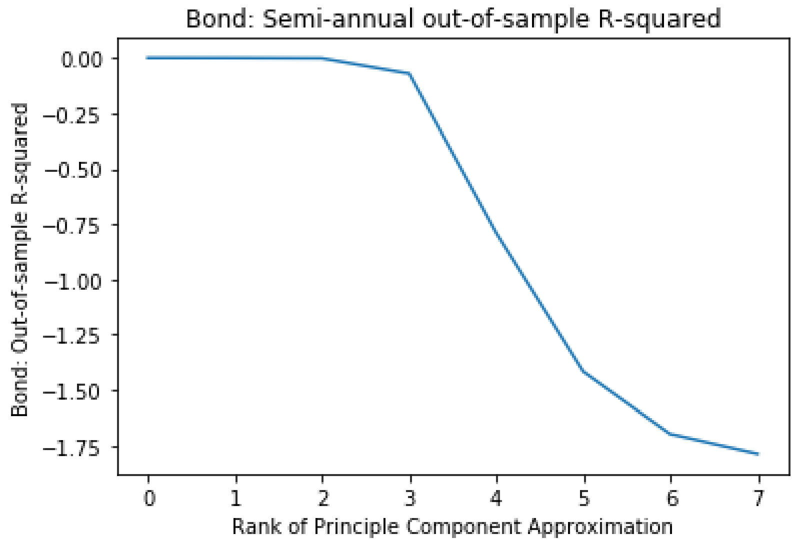
```
bonds_explanatory_variables = ['y_bond','inflation','Term_Spread','Defa
                               'output_gap','MOMBY_6','MOMBY_12']
X = processed_df[bonds_explanatory_variables].values
r = processed_df['r_bond'].values

X_quarter = (processed_df[2:].set_index('Dates').resample('3M')
             .agg('last')[bonds_explanatory_variables].values)
y_quarter = (processed_df[2:].set_index('Dates')
             .resample('3M').agg('last')['r_bond'].values)
X_semi = (processed_df[2:].set_index('Dates').resample('6M')
          .agg('last')[bonds_explanatory_variables].values)
y_semi = (processed_df[2:].set_index('Dates')
          .resample('6M').agg('last')['r_bond'].values)
X_annual = (processed_df.set_index('Dates').resample('Y')
        .agg(lambda x: x[-2])[bonds_explanatory_variables].values)
y_annual = (processed_df.set_index('Dates').resample('Y')
        .agg(lambda x: x[-2])['r_bond'].values)

```

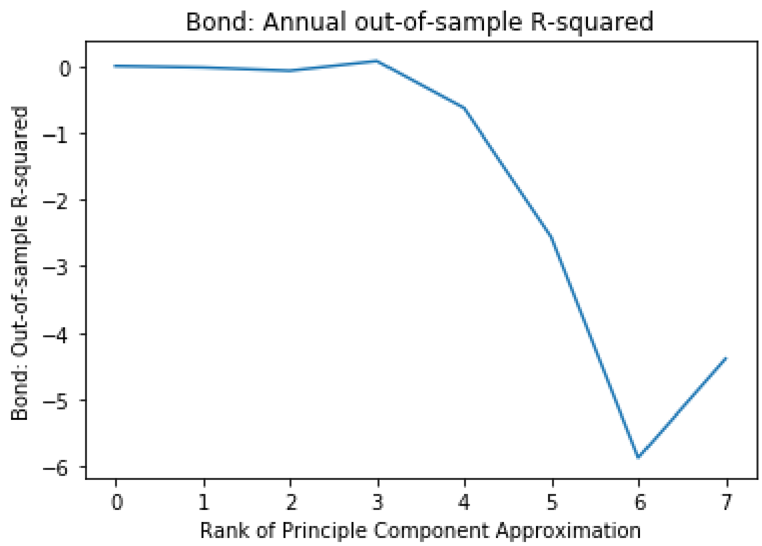
In [83]:

```
my_dict = {'month':200,'quarter':66,'semi':33,'annual':16}

i=0
for item in my_dict.keys():
    plt.figure(i)
    if item == 'month':
        plt.plot([Rsquared_OS(X,r,i,my_dict[item]) for i in range(X.sha
        plt.xlabel('Bond: Rank of Principle Component Approximation')
        plt.ylabel('Bond: Out-of-sample R-squared')
        plt.title('Bond: Month out-of-sample R-squared')
    else:
        eval(f'plt.plot([Rsquared_OS(X_{item},r_{item},i,my_dict[item])
        plt.xlabel('Rank of Principle Component Approximation')
        plt.ylabel('Bond: Out-of-sample R-squared')
        if item == 'semi':
            plt.title('Bond: '+f'{item[0].upper()}{item[1:]}' + '-annua
        else:
            plt.title('Bond: '+ f'{item[0].upper()}{item[1:]} out-of-sa
    i+=1


```



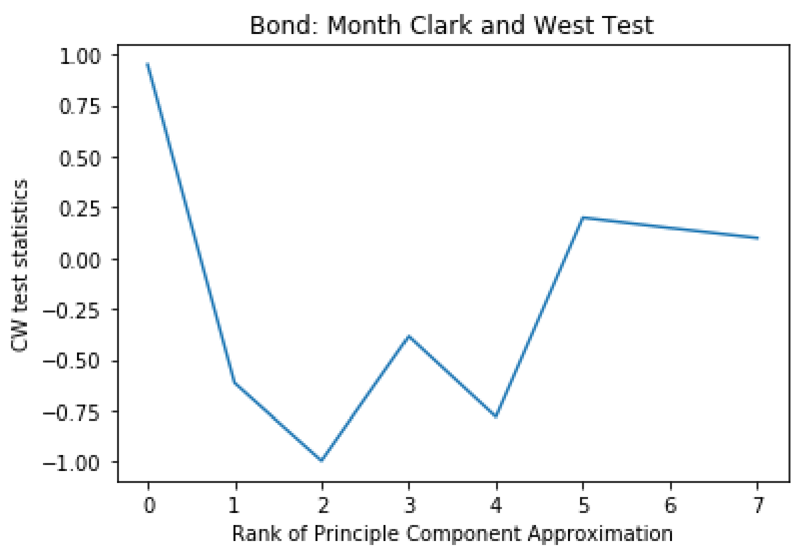


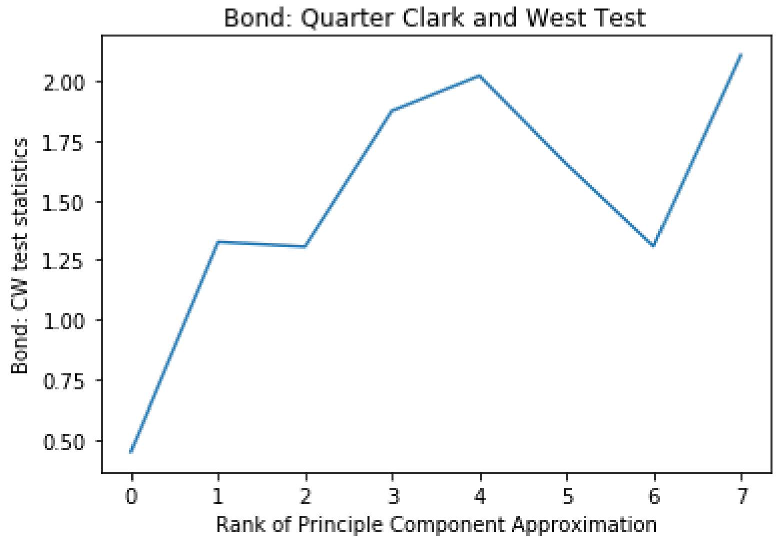

Bond: Semi-annual out-of-sample R-squared
Bond: Out-of-sample R-squared
0.00
−0.25
−0.50
−0.75
−1.00
−1.25
−1.50
−1.75
0 1 2 3 4 5 6 7
Rank of Principle Component Approximation

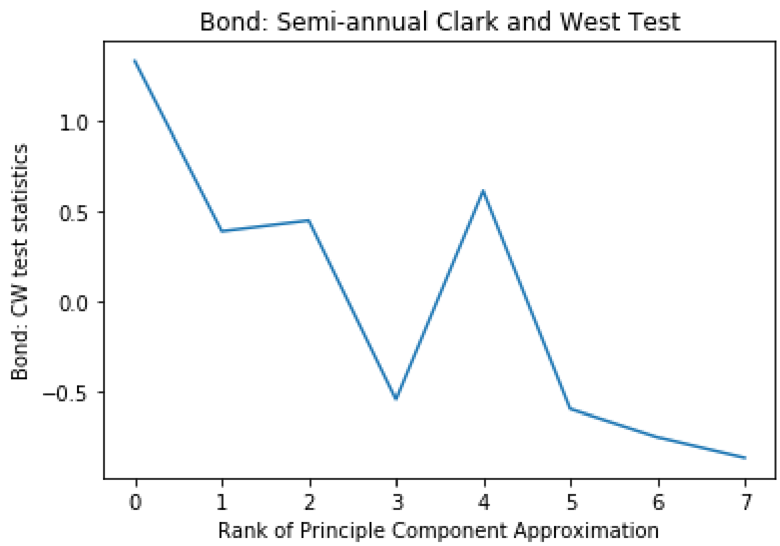

Bond: Annual out-of-sample R-squared
Bond: Out-of-sample R-squared
0
−1
−2
−3
−4
−5
−6
0 1 2 3 4 5 6 7
Rank of Principle Component Approximation

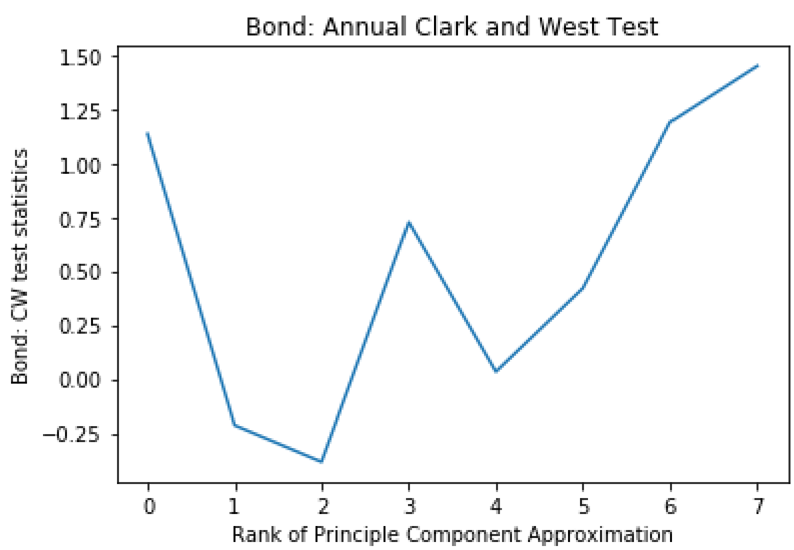
In [84]:

```
my_dict = {'month':200,'quarter':66,'semi':33,'annual':16}

i=0
for item in my_dict.keys():
    plt.figure(i)
    if item == 'month':
        plt.plot([CW_test(X,r,i,my_dict[item])
                  for i in range(X.shape[1]+1)])
        plt.xlabel('Rank of Principle Component Approximation')
        plt.ylabel('CW test statistics')
        plt.title('Bond: Month Clark and West Test')
    else:
        ev_str = f'plt.plot([CW_test(X_{item},r_{item},i,my_dict[item])
        ev_str += 'for i in range(X.shape[1]+1)])'
        eval(ev_str)
        plt.xlabel('Rank of Principle Component Approximation')
        plt.ylabel('Bond: CW test statistics')
        if item == 'semi':
            plt.title('Bond: '+f'{item[0].upper()}{item[1:]}'
                      + '-annual Clark and West Test')
        else:
            plt.title('Bond: '+
                      f'{item[0].upper()}{item[1:]} Clark and West Test
    i+=1
```







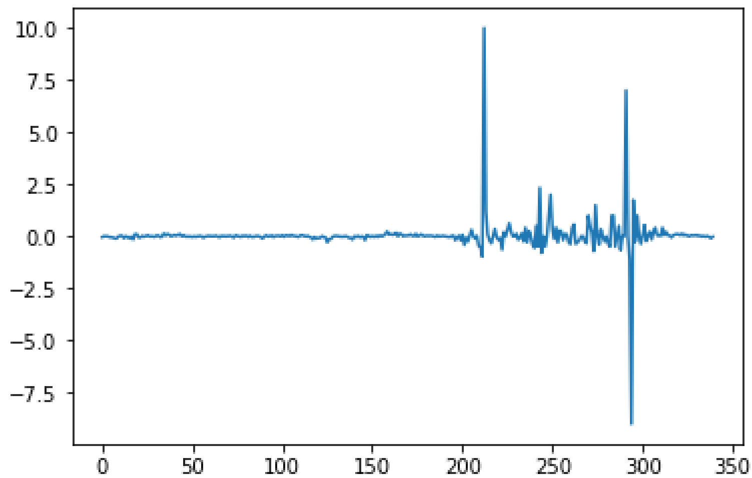


3.3 Bill Returns

The result is not as expected for bill returns. I suspect that the compound return data provided by the Factset is not accurate. For visualization of the data, Please refer to the next cell. The data shows the 1-month return. However, the maximum return was 10.0 and minimum value was -9. Clearly, return of -9 does not make sense. As I have checked the function I used to compute the return, I should find another source to obtain the return data. However, I do not currently know where I can retrieve the data.

In [85]:
```
plt.plot( processed_df['r_bill'].values)
```

Out[85]:
```
[<matplotlib.lines.Line2D at 0x126e30be0>]
```



In [86]:
```
processed_df['r_bill'].values.min()
```

Out[86]:
```
-8.999999999997765
```

In [87]:
```
X = processed_df['y_bill'].values.reshape(-1,1)
r = processed_df['r_bill'].values

X_quarter = (processed_df[2:].set_index('Dates').resample('3M')
             .agg('last')[['y_bill']].values)
y_quarter = (processed_df[2:].set_index('Dates')
             .resample('3M').agg('last')['r_bill'].values)
X_semi = (processed_df[2:].set_index('Dates').resample('6M')
          .agg('last')[['y_bill']].values)
y_semi = (processed_df[2:].set_index('Dates')
          .resample('6M').agg('last')['r_bill'].values)
X_annual = (processed_df.set_index('Dates').resample('Y')
        .agg(lambda x: x[-2])[['y_bill']].values)
y_annual = (processed_df.set_index('Dates').resample('Y')
        .agg(lambda x: x[-2])['r_bill'].values)
```

In [88]:
```
my_dict = {'month':200,'quarter':66,'semi':33,'annual':16}

i=0
for item in my_dict.keys():
    if item=='month':
        print('Month Rsquared_OS: ', Rsquared_OS(X,r,1,my_dict[item]))
    else:
        eval( f'print(item," Rsquared_OS: ",Rsquared_OS(X_{item},r_{ite

```

```
Month Rsquared_OS:  -0.009179932272639357
quarter  Rsquared_OS:  -0.10034942055336504
semi  Rsquared_OS:  -0.036215348588125584
annual  Rsquared_OS:  -0.025368552452170956
```

In [89]:
```
my_dict = {'month':200,'quarter':66,'semi':33,'annual':16}

i=0
for item in my_dict.keys():
    if item=='month':
        print('Month Rsquared_OS: ', CW_test(X,r,1,my_dict[item]))
    else:
        eval( f'print(item," Rsquared_OS: ",CW_test(X_{item},r_{item},1

```

```
Month Rsquared_OS:  -0.15740600529154933
quarter  Rsquared_OS:  -0.5358886092568653
semi  Rsquared_OS:  0.44965609947281726
annual  Rsquared_OS:  0.0908765054611409
```

4. Portfolio Performance Evaluation

Choice of the number of Principal Components for Month: Stock: 7, Bond: 2

4.1 Return Estimate

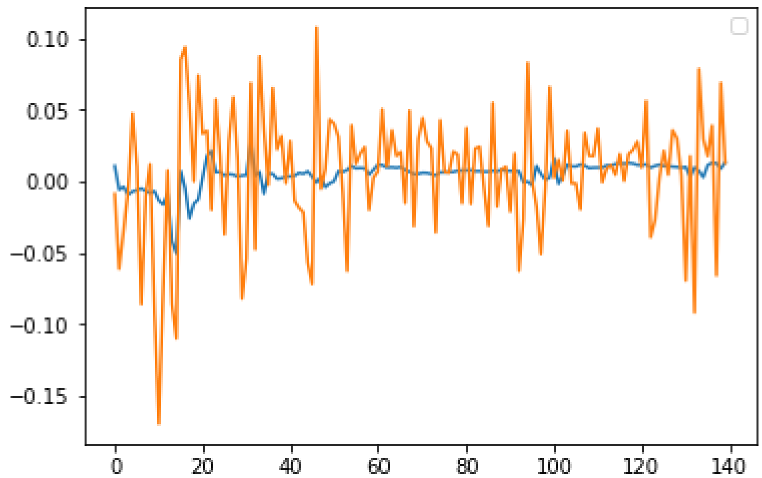
In [90]:
```
r_actual = processed_df[['r_SPX','r_bond','r_bill']].values
```

In [91]:
```
r_hat_SPX = []
for i in range(r_actual.shape[0]):
    X_stock_PCA = truncated_PC(
            processed_df[stock_explanatory_variables][:i].values,7)
    w_stock = PC_fit(X_stock_PCA,r_actual[:i,0],7)
    x_stock_PCA_new = \
        np.concatenate(
            (np.ones(1),
             truncated_PC(
                 processed_df
                 [stock_explanatory_variables][:i+1]
                 .values,7)[-1,:]))
    r_hat_SPX+=[np.dot(x_stock_PCA_new,w_stock)]
```

In [92]:
```
plt.plot(r_hat_SPX[200:])
plt.plot(r_actual[200:,0])
plt.legend()
```

```
No handles with labels found to put in legend.
```

Out[92]: <matplotlib.legend.Legend at 0x127a0ef60>

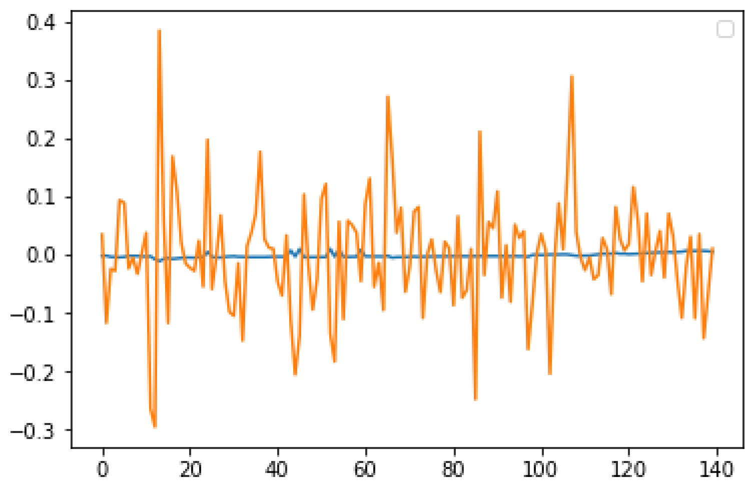


In [93]:
```
r_hat_bond = []
for i in range(r_actual.shape[0]):
    X_bond_PCA = truncated_PC(
        processed_df
        [bonds_explanatory_variables][:i].values,2)
    w_bond = PC_fit(X_bond_PCA,r_actual[:i,1],2)
    x_bond_PCA_new = \
      np.concatenate(
        (np.ones(1),
         truncated_PC(
             processed_df
             [bonds_explanatory_variables][:i+1].values,2)[-1,:]))
    r_hat_bond+=[np.dot(x_bond_PCA_new,w_bond)]
```

In [94]:
```
plt.plot(r_hat_bond[200:])
plt.plot(r_actual[200:,1])
plt.legend()
```

```
No handles with labels found to put in legend.
```

Out[94]: <matplotlib.legend.Legend at 0x126d82b70>

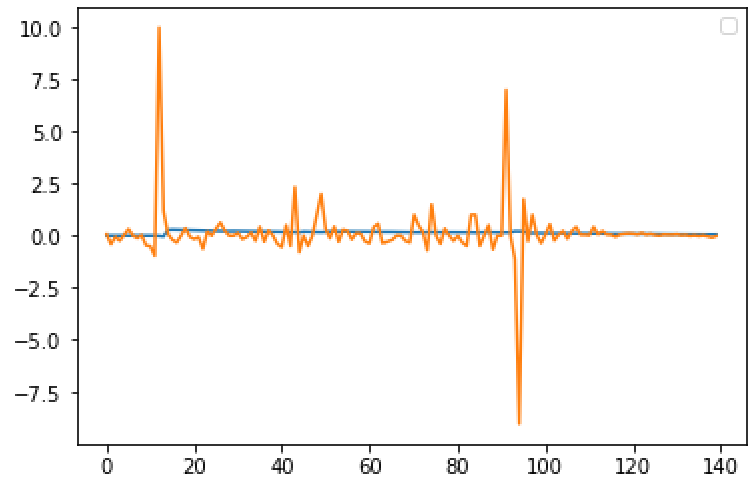


In [95]:
```
r_hat_bill = []
X_bill = np.concatenate((np.ones(r_actual.shape[0]).reshape(-1,1),
                         processed_df['y_bill'].values.reshape(-1,1)),a
for i in range(r_actual.shape[0]):
    w_bill = np.matmul(
                np.matmul(
                    np.linalg.inv(1e-6*np.eye(2) + np.matmul(
                        X_bill[:i-1,:].transpose(),X_bill[:i-1,:]))
                        X_bill[:i-1,:].transpose()),
                    r_actual[:i-1,2])
    r_hat_bill+=[np.dot(X_bill[i-1,:],w_bill)]



```

In [96]:
```
plt.plot(r_hat_bill[200:])
plt.plot(r_actual[200:,2])
plt.legend()
```

```
No handles with labels found to put in legend.
```

Out[96]: ```<matplotlib.legend.Legend at 0x127acdd30>```



In [97]:
```
r_hat = pd.DataFrame(r_hat_SPX,columns = ['r_hat_SPX'])
r_hat['r_hat_bond'] = pd.Series(r_hat_bond)
r_hat['r_hat_bill'] = pd.Series(r_hat_bill)
r_hat = r_hat.astype(float)
```

```
/Users/gimdong-geon/python3_cooking/lib/python3.7/site-packages/pandas/co
re/dtypes/cast.py:702: ComplexWarning: Casting complex values to real dis
cards the imaginary part
  return arr.astype(dtype, copy=True)
```

4.2 EWMA Covariance Estimate

I referred the formula of Exponentially Weighted Moving Average from Table 5.1 of Riskmetrics - technical document. It is as following:

$$Cov(r^i, r^j) = (1-\lambda)\sum_{t=0}^{T-1} \lambda^t (r_t^i - \bar{r}_t^i)(r_t^j - \bar{r}_t^j) \qquad i, j \in \{stock, bond, bill\}$$

The way I compute this amount is:

$$X = \begin{pmatrix} r_0^{stock} - \bar{r}^{stock} & r_0^{bond} - \bar{r}^{bond} & r_0^{bill} - \bar{r}^{bill} \\ r_1^{stock} - \bar{r}^{stock} & r_1^{bond} - \bar{r}^{bond} & r_1^{bill} - \bar{r}^{bill} \\ r_3^{stock} - \bar{r}^{stock} & r_2^{bond} - \bar{r}^{bond} & r_2^{bill} - \bar{r}^{bill} \end{pmatrix}$$

$$\Rightarrow \tilde{X} = \sqrt{1-\lambda}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \lambda^{0.5} & 0 \\ 0 & 0 & \lambda \end{pmatrix} X = (\tilde{r}^{stock}, \tilde{r}^{bond}, \tilde{r}^{bill})$$

$$\Rightarrow \tilde{X}^T \tilde{X} = \begin{pmatrix} (\tilde{r}^{stock})^T \tilde{r}^{stock} & (\tilde{r}^{stock})^T \tilde{r}^{bond} & (\tilde{r}^{stock})^T \tilde{r}^{bill} \\ (\tilde{r}^{bond})^T \tilde{r}^{stock} & (\tilde{r}^{bond})^T \tilde{r}^{bond} & (\tilde{r}^{bond})^T \tilde{r}^{bill} \\ (\tilde{r}^{bill})^T \tilde{r}^{stock} & (\tilde{r}^{bill})^T \tilde{r}^{bond} & (\tilde{r}^{bill})^T \tilde{r}^{bill} \end{pmatrix}$$

which is the desired matrix

In [98]:
```
def EWMA(t):
    decay = 0.94
    A = np.matmul(np.diag([np.sqrt((1-decay)*decay**i)
                           for i in range(t+1)]),r_actual[:t+1])
    return np.matmul(A.T,A)
```

In [99]:
```
EWMA(200)
```

Out[99]:
```
array([[ 0.00119016, -0.00066904, -0.00047128],
       [-0.00066904,  0.00164213,  0.00131968],
       [-0.00047128,  0.00131968,  0.00323162]])
```

4.3 Black Litterman μ and Σ

4.3.1 Covariance Matrix of Active Views

$$\omega_j = \left(\frac{1-c_j}{c_j}\right) p_j \hat{\Sigma}_{t:t+h} p_j^T \quad \text{for } j = \text{stock, bond, or bill}$$

where $p_{stock}, p_{bond}, p_{bill}$ are first, second and third rows of $P = I_3$. And, $c_{stock} = 0.75, c_{bond} = 0.50, c_{bill} = 0.25$. Note that larger the c is, less relevant is the investor view and when c equals 1 then investor view is considered majestic. The rationale for the choice of parameter is that if regression result is credit-worthy (characterized by high R_{OS}^2) then do not utilize information about views.

I slightly modify the weights `c` because of poor quality of bond/bill returns data I obtained.

In [100]:
```
omega_stock = [(1/0.75 -1)*EWMA(t)[0,0]
               for t in range(r_actual.shape[0])]
omega_bond  = [(1/0.5 -1)*EWMA(t)[1,1]
               for t in range(r_actual.shape[0])]
omega_bill  = [(1/0.25 -1)*EWMA(t)[2,2]
               for t in range(r_actual.shape[0])]
```

In [101]:
```
Omega = []
for i in range(len(omega_stock)):
    Omega += [np.diag([omega_stock[i],omega_bond[i],omega_bill[i]])]
```

4.3.2 μ_{BL} And Σ_{BL}

$$\mu_{BL} = \mu + \hat{\Sigma}P^T(P\hat{\Sigma}P^T + \Omega)^{-1}(V - P\mu)$$
$$\Sigma_{BL} = \hat{\Sigma} - \hat{\Sigma}P^T(P\hat{\Sigma}P^T + \Omega)^{-1}P\hat{\Sigma}$$

Da Silva(2009) claims that Black-Litterman was derived under the mean-variance portfolio optimization rather than optimizing the common active management performance measure, the information ratio. And, this resulted in a bias that could lead to unintentional trades.

The authors' remedy for this issue pertains to the practice of obtaining implied equilibrium excess returns through $\mu = \gamma \Sigma \omega_B$ where γ is a risk-aversion coefficient. The author asserts to set $\mu = 0$. And, $\hat{r}$ is considered to be the active views V and P is assumed to be I_3. In summary,

$$\begin{aligned} \mu_{BL} &= \hat{\Sigma}(\hat{\Sigma} + \Omega)^{-1}\hat{r} \\ \Sigma_{BL} &= \hat{\Sigma} - \hat{\Sigma}(\hat{\Sigma} + \Omega)^{-1}\hat{\Sigma} \end{aligned}$$

In [102]:

```
mu_BL = []
Sigma_BL = []
for i in range(len(r_actual)):
    mu_BL+=[np.dot(
                np.matmul(EWMA(i),
                    np.linalg.inv(
                        1e-6 + EWMA(i) + Omega[i])),
                r_hat.values[i] )]
    Sigma_BL+=[EWMA(i)
              -np.matmul(
                  np.matmul(
                      EWMA(i),
                      np.linalg.inv(
                          1e-6 +EWMA(i)+Omega[i]))
                  ,EWMA(i))]
```

4.3.3 Black-Litterman Return Expectation and Variance

$$Return|view \sim N(\mu_{BL}, \Sigma_{BL})$$

$$\Rightarrow E[P_t^i] = P_0^i exp\left(\mu_{BL,i} + \frac{1}{2}\Sigma_{BL,(i,i)}\right) \quad \text{where} \quad i \in \{stock, bond, bill\}$$

$$Cov[P_t^i, P_t^j] = P_0^i P_0^i e^{\mu_{BL,i}+\mu_{BL,j}} e^{\frac{1}{2}\left(\Sigma_{BL,(ii)}+\Sigma_{BL,(jj)}\right)} \odot \left(e^{\Sigma_{BL,(ij)}-1}\right)$$

In [103]:

```
m=[]
S=[]
for i in range(len(r_actual)):
    m += [np.exp(mu_BL[i]
          +0.5*np.diag(Sigma_BL[i]))
                 .reshape(-1,1)
          - 1]
    S += [np.multiply(
            np.matmul(
                np.exp(
                    mu_BL[i]
                    +0.5*np.diag(Sigma_BL[i]))
                     .reshape(-1,1),
                np.exp(
                    mu_BL[i]
                    +0.5*np.diag(Sigma_BL[i]))
                     .reshape(-1,1).T)
            ,np.exp(Sigma_BL[i]) - 1)]
```

4.3.4 DAA Portfolio Optimization

Initial attempt:

Optimization problem is:

$$\begin{aligned} \max_{w} \quad & (w - w_{bench})^T m \\ \text{s.t.} \quad & \|R(w - w_{bench})\|_2^2 \le (h/12)TE^2 \\ & w^T 1_3 = 1 \\ & w \ge w_{LB} \end{aligned}$$

where R is the Cholesky Decomposition of S

In [104]:
```
def cholesky(A):
    """
    computes left cholesky matrix. Advantage of this
    matrix over np.linalg.cholesky
    is that first few observation of S matrix is not
    positive definite which creates
    an error message.
    """
    L = np.eye(3)
    L[1:,0]=-A[1:,0]/A[0,0]
    tmp = np.matmul(L,A)
    L2 = np.eye(3)
    L2[2,1] = -tmp[2,1]/tmp[1,1]
    diag = np.sqrt(np.matmul(
                    np.matmul(np.matmul(L2,tmp),L.T),L2.T))
    Linv=np.eye(3)
    Linv[1:,0] = -L[1:,0]/L[0,0]
    L2inv=np.eye(3)
    L2inv[2,1] = -L2[2,1]/L2[1,1]
    Left = np.matmul(np.matmul(Linv,L2inv),diag)
    return Left
```

Sanity Check:

In [105]:
```
R = cholesky(S[20]).T
R.T - np.linalg.cholesky(S[20])
```

Out[105]:
```
array([[0., 0., 0.],
       [0., 0., 0.],
       [0., 0., 0.]])
```

In [106]:
```
for i in range(len(r_actual))[:5]:
    print(cholesky(S[i]))
```

```
[[ 5.32797217e-05  0.00000000e+00  0.00000000e+00]
 [-5.73580103e-04  2.58327850e-07  0.00000000e+00]
 [-6.21356947e-03  2.69271475e-05  4.56954959e-06]]
[[       nan 0.               nan]
 [       nan 0.0007352        nan]
 [       nan 0.0078131        nan]]
[[ 0.00785924  0.          0.        ]
 [-0.000789    0.00282752  0.        ]
 [ 0.00037972  0.00165482  0.00759579]]
[[ 0.00665097  0.          0.        ]
 [-0.00130537  0.00472013  0.        ]
 [ 0.00032736  0.00192706  0.00942859]]
[[ 1.00566426e-02  0.00000000e+00  0.00000000e+00]
 [-1.12037556e-03  5.42509194e-03  0.00000000e+00]
 [ 2.50722230e-05  4.10683401e-03  7.22966998e-03]]

/Users/gimdong-geon/python3_cooking/lib/python3.7/site-packages/ipykernel
_launcher.py:15: RuntimeWarning: invalid value encountered in sqrt
  from ipykernel import kernelapp as app
```

S is not psd for many observations. So I avoided cholesky

$$
\begin{aligned}
\max_{w} \quad & w^T m \\
\text{s.t.} \quad & (w - w_{bench})^T S (w - w_{bench}) \ \leq (h/12) T E^2 \\
& w^T 1_3 = 1 \\
& w \geq w_{LB}
\end{aligned}
$$

To get constraints, I use fstring with list comprehension as below:

In [107]:
```
cone = '+'.join([f'((w{i}-w_bench[{i}])*S[i][{i},{j}]*(w{j}-w_bench[{j}
                 for i in range(3) for j in range(3)])
cone
```

Out[107]:
```
'((w0-w_bench[0])*S[i][0,0]*(w0-w_bench[0]))+((w0-w_bench[0])*S[i][0,1]*
(w1-w_bench[1]))+((w0-w_bench[0])*S[i][0,2]*(w2-w_bench[2]))+((w1-w_bench
[1])*S[i][1,0]*(w0-w_bench[0]))+((w1-w_bench[1])*S[i][1,1]*(w1-w_bench
[1]))+((w1-w_bench[1])*S[i][1,2]*(w2-w_bench[2]))+((w2-w_bench[2])*S[i]
[2,0]*(w0-w_bench[0]))+((w2-w_bench[2])*S[i][2,1]*(w1-w_bench[1]))+((w2-w
_bench[2])*S[i][2,2]*(w2-w_bench[2]))'
```

In [114]:
```
w_LB = 0.05*np.ones(3)
w_bench = np.array([0.65,0.3,0.05])
TE=0.02
w = np.array([np.nan,np.nan,np.nan]).reshape(-1,3)
for i in range(len(r_actual)):
    # Create a new model

        model = Model("qcp")
        model.setParam('OutputFlag', 0)
        # Create variables
        w0 = model.addVar(name="w0")
        w1 = model.addVar(name="w1")
        w2 = model.addVar(name="w2")

        # Set objective: x
        obj = m[i][0][0]*w0 + m[i][1][0]*w1 + m[i][2][0]*w2
        model.setObjective(obj, GRB.MAXIMIZE)

        model.addConstr(w0 + w1 + w2 == 1, "c0")
        model.addConstr(w0  >= w_LB[0], "c1")
        model.addConstr(w1  >= w_LB[1], "c2")
        model.addConstr(w2  >= w_LB[2], "c3")

        # Add second-order cone:
        eval('model.addConstr(' + cone + '<= (TE**2 *(1/12)), "qc0")')
        model.optimize()

        if i%100==0:
            print(f'{i}th observation: ')
            for v in model.getVars():
                print('%s %g' % (v.varName, v.x))
            print('Obj: %g' % obj.getValue())
        w = np.concatenate((w,np.array([model.getVars()[i].x
                                        for i in range(3)]).reshape(-1,3)))
w = w[1:,:]
```

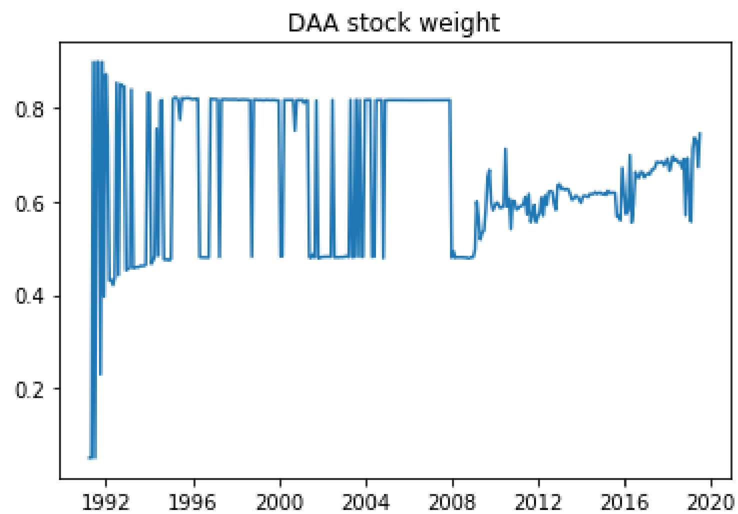
```
Academic license - for non-commercial use only
0th observation:
w0 0.0500041
w1 0.0500182
w2 0.899978
Obj: 0.00750459
100th observation:
w0 0.818077
w1 0.131605
w2 0.0503178
Obj: 0.00521456
200th observation:
w0 0.818
w1 0.131969
w2 0.0500305
Obj: 0.00632291
300th observation:
w0 0.700523
w1 0.131539
```

```
w2 0.167938
Obj: 0.0127552
```

In [115]:
```
weights_df = pd.DataFrame(w)
weights_df['Dates'] = processed_df['Dates']
weights_df.columns = ['w_stock','w_bond','w_bill','Dates']
```

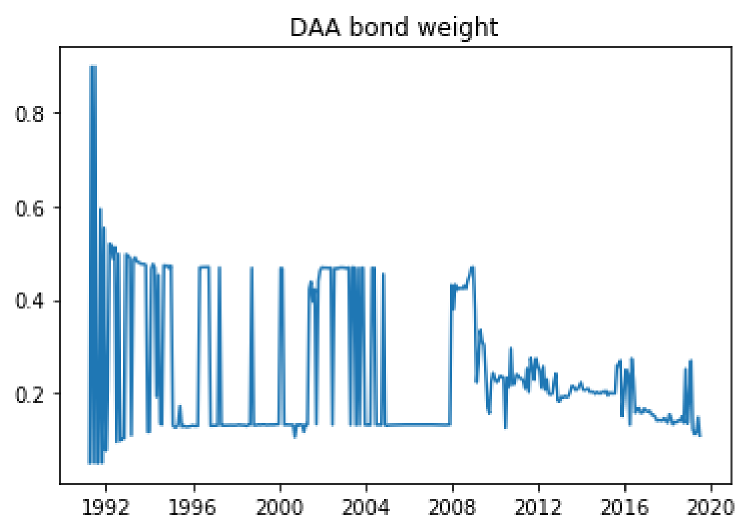
In [116]:
```
plt.plot('Dates','w_stock',data=weights_df)
plt.title("DAA stock weight")
```

Out[116]: `Text(0.5, 1.0, 'DAA stock weight')`



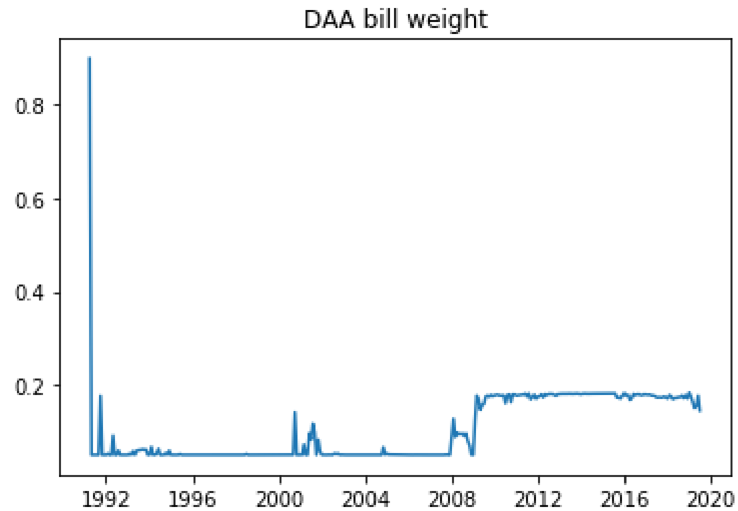
In [117]:
```
plt.plot('Dates','w_bond',data=weights_df)
plt.title("DAA bond weight")
```

Out[117]: `Text(0.5, 1.0, 'DAA bond weight')`



In [118]:
```
plt.plot('Dates','w_bill',data=weights_df)
plt.title("DAA bill weight")
```

Out[118]: Text(0.5, 1.0, 'DAA bill weight')



4.3.5 Portfolio Performance Statistics

4.3.5.1 Annualized Return

Below cell tries to adjusts for erratic data. Bill return is not expected to change by 500% over a month.

In [119]:
```
np.place(w[:,2], np.abs(r_actual[:,2])>5, 0)
w[np.abs(r_actual[:,2])>5]
```

Out[119]:
```
array([[0.48131974, 0.46832747, 0.        ],
       [0.61944123, 0.19848686, 0.        ],
       [0.56876547, 0.25783333, 0.        ]])
```

In [120]:
```
r_portfolio_actual = np.multiply(r_actual,w).sum(axis=1)
r_portfolio_bench = np.multiply(r_actual,w_bench).sum(axis=1)
```

In [121]:
```
DAA_geo_return = ((((r_portfolio_actual[200:]+1)
                   .prod()**(1/r_portfolio_actual[200:].shape[0]))
                  -1) * 12)
DAA_geo_return
```

Out[121]: 0.12859850020765418

In [122]:
```
Bench_geo_return =(((r_portfolio_bench[200:]+1).prod()**(1/r_portfolio_
Bench_geo_return
```

Out[122]: 0.0824949523121612

Annualized geometric return of DAA portfolio from November 2007 (observation index 200) to July is 13.97% and that of benchmark portfolio is 7.49%

4.3.5.2 Standard Deviation

In [123]:
```
DAA_std = r_portfolio_actual[200:].std(ddof=1)
DAA_std
```

Out[123]: 0.09470120187816364

In [124]:
```
Bench_std = r_portfolio_bench[200:].std(ddof=1)
Bench_std
```

Out[124]: 0.0772344189520475

4.3.5.3 Maximum Drawdown

In [125]:
```
DAA_mdd = 1-(r_portfolio_actual[200:] + 1).cumprod().min()
DAA_mdd
```

Out[125]: 0.5218382528879826

In [126]:
```
Bench_mdd = 1-(r_portfolio_bench[200:] + 1).cumprod().min()
Bench_mdd
```

Out[126]: 0.4146251668854053

4.3.5.4 Calmar Ratio

In [127]:
```
DAA_Calmar = DAA_geo_return/DAA_mdd
DAA_Calmar
```

Out[127]: 0.2464336401096664

In [128]:
```
Bench_Calmar = Bench_geo_return/Bench_mdd
Bench_Calmar
```

Out[128]: 0.198962723203344

4.3.5.4 Average Excess Return

In [129]:
```
Avg_Excess_Return = DAA_geo_return - Bench_geo_return
Avg_Excess_Return
```

Out[129]: 0.04610354789549298

4.3.5.5 Tracking Error

In [130]:
```
Tracking_Error = ((r_portfolio_actual[200:]
                  - r_portfolio_bench[200:])
                 .std(ddof=1) *np.sqrt(12))
Tracking_Error
```

Out[130]: 0.3100078946188472

4.3.5.6 Information Ratio

In [131]:
```
IR = Avg_Excess_Return/Tracking_Error
IR
```

Out[131]: 0.14871733493167008

4.3.5.7 CER Gain

Certainty Equivalent Return is the return an investor would want to be guaranteed for his investment. The investor is assumed to have power utility with risk aversion coefficients of two. This means the investor is risk-averse because utility function is concave. Power Utility fuction is as following:

$$U(x) = \frac{x^{1-RAA}}{1-RAA}$$

To compute the CER, for each return observation, compute $U(1 + return)$. Denote the average of the return as $\overline{U}$ Then, we recover CER from the following equation.

$$\overline{U} = \frac{(1 + CER)^{1-RAA}}{1-RAA}$$

For the ease of writing code, this is equivalent to:

$$CER = [(1 - RRA)\,\overline{U}\,]^{\frac{1}{1-RAA}} - 1$$

As the final step, the return is annualized.

In [132]:
```
U_bar_actual = (-(1+r_portfolio_actual)**(-1)).mean()
U_bar_bench = (-(1+r_portfolio_bench)**(-1)).mean()
```

In [133]:
```
CER_actual = ((-U_bar_actual)**(-1) -1)*12
CER_bench = ((-U_bar_bench)**(-1) -1)*12
```

In [134]:
```
CER_gain = CER_actual - CER_bench
CER_gain
```

Out[134]: 0.016350129132626456

4.4 Transaction Cost and Performance Evaluation

4.4.1 Transaction Cost

In [135]:
```
transaction_cost = np.arange(0,0.0455,0.0005) # 50 basis points increme
transaction_cost
```

Out[135]:
```
array([0.    , 0.0005, 0.001 , 0.0015, 0.002 , 0.0025, 0.003 , 0.0035,
       0.004 , 0.0045, 0.005 , 0.0055, 0.006 , 0.0065, 0.007 , 0.0075,
       0.008 , 0.0085, 0.009 , 0.0095, 0.01  , 0.0105, 0.011 , 0.0115,
       0.012 , 0.0125, 0.013 , 0.0135, 0.014 , 0.0145, 0.015 , 0.0155,
       0.016 , 0.0165, 0.017 , 0.0175, 0.018 , 0.0185, 0.019 , 0.0195,
       0.02  , 0.0205, 0.021 , 0.0215, 0.022 , 0.0225, 0.023 , 0.0235,
       0.024 , 0.0245, 0.025 , 0.0255, 0.026 , 0.0265, 0.027 , 0.0275,
       0.028 , 0.0285, 0.029 , 0.0295, 0.03  , 0.0305, 0.031 , 0.0315,
       0.032 , 0.0325, 0.033 , 0.0335, 0.034 , 0.0345, 0.035 , 0.0355,
       0.036 , 0.0365, 0.037 , 0.0375, 0.038 , 0.0385, 0.039 , 0.0395,
       0.04  , 0.0405, 0.041 , 0.0415, 0.042 , 0.0425, 0.043 , 0.0435,
       0.044 , 0.0445, 0.045 ])
```

In [136]:
```
r_sign = np.sign(r_portfolio_actual)
```

In [137]:
```
transaction_df = processed_df[['Dates']]
transaction_df['tc_0_bp'] = r_portfolio_actual
for i in range(1,len(transaction_cost)):
    string = f'transaction_df["tc_{i*50}_bp"] = np.multiply(r_portfolic
    string += '(1-r_sign*transaction_cost[i]))'
    exec(string)
transaction_df.set_index('Dates', inplace=True)
```

```
/Users/gimdong-geon/python3_cooking/lib/python3.7/site-packages/ipykernel
_launcher.py:2: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the caveats in the documentation: http://pandas.pydata.org/pandas-doc
s/stable/indexing.html#indexing-view-versus-copy (http://pandas.pydata.or
g/pandas-docs/stable/indexing.html#indexing-view-versus-copy)

/Users/gimdong-geon/python3_cooking/lib/python3.7/site-packages/ipykernel
_launcher.py:1: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the caveats in the documentation: http://pandas.pydata.org/pandas-doc
s/stable/indexing.html#indexing-view-versus-copy (http://pandas.pydata.or
g/pandas-docs/stable/indexing.html#indexing-view-versus-copy)
  """Entry point for launching an IPython kernel.
```

In [138]:

```
transaction_df.head()
```

Out[138]:

Dates	tc_0_bp	tc_50_bp	tc_100_bp	tc_150_bp	tc_200_bp	tc_250_bp	tc_300_bp	tc_350_bp	tc_
1991-03-29	-0.036231	-0.036249	-0.036267	-0.036285	-0.036303	-0.036321	-0.036340	-0.036358	-0.
1991-04-30	0.006605	0.006602	0.006598	0.006595	0.006592	0.006589	0.006585	0.006582	0.
1991-05-31	-0.041896	-0.041917	-0.041938	-0.041959	-0.041980	-0.042001	-0.042022	-0.042043	-0.
1991-06-28	-0.002216	-0.002217	-0.002218	-0.002220	-0.002221	-0.002222	-0.002223	-0.002224	-0.
1991-07-31	0.013559	0.013552	0.013545	0.013538	0.013531	0.013525	0.013518	0.013511	0.

5 rows × 91 columns

In [139]:

```
r_sign_bench = np.sign(r_portfolio_bench)

transaction_benchmark_df = processed_df[['Dates']]
transaction_benchmark_df['tc_0_bp'] = r_portfolio_bench
for i in range(1,len(transaction_cost)):
    string = f'transaction_benchmark_df["tc_{i*50}_bp"]'
    string += '=np.multiply(r_portfolio_bench,'
    string += '(1-r_sign_bench*transaction_cost[i]))'
    exec(string)
transaction_benchmark_df.set_index('Dates', inplace=True)
```

```
/Users/gimdong-geon/python3_cooking/lib/python3.7/site-packages/ipykernel
_launcher.py:4: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the caveats in the documentation: http://pandas.pydata.org/pandas-doc
s/stable/indexing.html#indexing-view-versus-copy (http://pandas.pydata.or
g/pandas-docs/stable/indexing.html#indexing-view-versus-copy)
  after removing the cwd from sys.path.
/Users/gimdong-geon/python3_cooking/lib/python3.7/site-packages/ipykernel
_launcher.py:1: SettingWithCopyWarning:
A value is trying to be set on a copy of a slice from a DataFrame.
Try using .loc[row_indexer,col_indexer] = value instead

See the caveats in the documentation: http://pandas.pydata.org/pandas-doc
s/stable/indexing.html#indexing-view-versus-copy (http://pandas.pydata.or
g/pandas-docs/stable/indexing.html#indexing-view-versus-copy)
  """Entry point for launching an IPython kernel.
```

In [140]:
```
transaction_benchmark_df.head()
```

Out[140]:

Dates	tc_0_bp	tc_50_bp	tc_100_bp	tc_150_bp	tc_200_bp	tc_250_bp	tc_300_bp	tc_350_bp	tc_
1991-03-29	-0.002896	-0.002898	-0.002899	-0.002901	-0.002902	-0.002904	-0.002905	-0.002906	-0.
1991-04-30	0.026753	0.026740	0.026726	0.026713	0.026700	0.026686	0.026673	0.026659	0.
1991-05-31	-0.024340	-0.024352	-0.024364	-0.024377	-0.024389	-0.024401	-0.024413	-0.024425	-0.
1991-06-28	0.027612	0.027598	0.027584	0.027571	0.027557	0.027543	0.027529	0.027515	0.
1991-07-31	-0.002939	-0.002941	-0.002942	-0.002943	-0.002945	-0.002946	-0.002948	-0.002949	-0.

5 rows × 91 columns

4.4.2 Performance Evaluation

4.4.2.1 Annualized Return

In [141]:
```
tc_perf_df = pd.DataFrame([f'{i*50} bp'
                           for i in range(0,len(transaction_cost))])
tc_perf_df.columns=['transaction_cost']
```

In [142]:
```
tc_return = []
tc_bench_return = []
for i in range(0,len(transaction_cost)):
    string = f"tc_return += [(((transaction_df"
    string += f"['tc_{i*50}_bp'].values[200:]+1)"
    string += f'.prod()**(1/transaction_df["tc_'
    string += f'{i*50}_bp"].values[200:].shape[0]))-1) * 12]'
    exec(string)
    string2 = f"tc_bench_return += [(((transaction_"
    string2 += f"benchmark_df['tc_{i*50}_bp'].values[200:]+1)"
    string2 += f'.prod()**(1/transaction_benchmark_df'
    string2 += f'["tc_{i*50}_bp"].values[200:].shape[0]))-1) * 12]'
    exec(string2)
```

In [143]:
```
tc_perf_df['annual_return'] = tc_return
tc_perf_df['annual_return_bench'] = tc_bench_return
```

In [144]:
```
tc_perf_df.head()
```

Out[144]:

	transaction_cost	annual_return	annual_return_bench
0	0 bp	0.128599	0.082495
1	50 bp	0.128205	0.082196
2	100 bp	0.127812	0.081897
3	150 bp	0.127419	0.081598
4	200 bp	0.127025	0.081300

4.4.2.2 Standard Deviation

In [145]:
```
aug_str = '.values[200:].std(ddof=1)'
tc_perf_df['std'] = [eval(f'transaction_df["tc_{i*50}_bp"]'+aug_str)
                              for i in range(0,len(transaction_cost))]
```

In [146]:
```
tc_perf_df['std_bench'] = \
[eval(f'transaction_benchmark_df["tc_{i*50}_bp"]'+aug_str)
      for i in range(0,len(transaction_cost))]
```

In [147]:
```
tc_perf_df[['transaction_cost','std_bench']].head()
```

Out[147]:

	transaction_cost	std_bench
0	0 bp	0.077234
1	50 bp	0.077230
2	100 bp	0.077226
3	150 bp	0.077221
4	200 bp	0.077217

4.4.2.3 Maximum Drawdown

In [148]:
```
aug_str = '.values[200:]+1).cumprod().min()'
tc_perf_df['max_drawdown'] = \
[eval(f'1-(transaction_df["tc_{i*50}_bp"]'+aug_str)
      for i in range(0,len(transaction_cost))]
```

In [149]:
```
tc_perf_df['max_drawdown_bench'] = \
[eval(f'1-(transaction_benchmark_df["tc_{i*50}_bp"]'+aug_str)
      for i in range(0,len(transaction_cost))]
```

In [150]:
```
tc_perf_df[['transaction_cost','max_drawdown',\
            'max_drawdown_bench']].head()
```

Out[150]:

	transaction_cost	max_drawdown	max_drawdown_bench
0	0 bp	0.521838	0.414625
1	50 bp	0.522094	0.414861
2	100 bp	0.522350	0.415096
3	150 bp	0.522606	0.415332
4	200 bp	0.522861	0.415567

4.4.2.4 Calmar Ratio

In [151]:
```
tc_perf_df['calmar'] =\
    tc_perf_df['annual_return'].values/ \
    tc_perf_df['max_drawdown'].values
```

In [152]:
```
tc_perf_df['calmar_bench'] = \
    tc_perf_df['annual_return_bench'].values/ \
    tc_perf_df['max_drawdown_bench'].values
```

In [153]:
```
tc_perf_df[['transaction_cost','calmar','calmar_bench']].head()
```

Out[153]:

	transaction_cost	calmar	calmar_bench
0	0 bp	0.246434	0.198963
1	50 bp	0.245560	0.198129
2	100 bp	0.244686	0.197297
3	150 bp	0.243814	0.196466
4	200 bp	0.242943	0.195635

4.4.2.5 Average Excess Return

In [154]:
```
tc_perf_df['avg_excess_return'] =\
tc_perf_df['annual_return'] - tc_perf_df['annual_return_bench']
```

In [155]:
```
tc_perf_df[['transaction_cost','avg_excess_return']].head()
```

Out[155]:

	transaction_cost	avg_excess_return
0	0 bp	0.046104
1	50 bp	0.046009
2	100 bp	0.045915
3	150 bp	0.045820
4	200 bp	0.045726

4.4.2.6 Tracking Error

In [156]:
```
string = f'(transaction_df["tc_{i*50}_bp"].values[200:] '
string += f'- transaction_benchmark_df["tc_{i*50}_bp"]'
string += '.values[200:]).std(ddof=1)*np.sqrt(12)'
tc_perf_df['tracking_error'] = \
[eval(string) for i in range(0,len(transaction_cost))]
```

In [157]:
```
tc_perf_df[['transaction_cost','tracking_error']].head()
```

Out[157]:

	transaction_cost	tracking_error
0	0 bp	0.30494
1	50 bp	0.30494
2	100 bp	0.30494
3	150 bp	0.30494
4	200 bp	0.30494

4.4.2.7 Information Ratio

In [158]:
```
tc_perf_df['IR'] = \
    tc_perf_df['avg_excess_return'].values/ \
    tc_perf_df['tracking_error'].values
```

In [159]:
```
tc_perf_df[['transaction_cost','IR']].head()
```

Out[159]:

	transaction_cost	IR
0	0 bp	0.151189
1	50 bp	0.150879
2	100 bp	0.150569
3	150 bp	0.150260
4	200 bp	0.149950

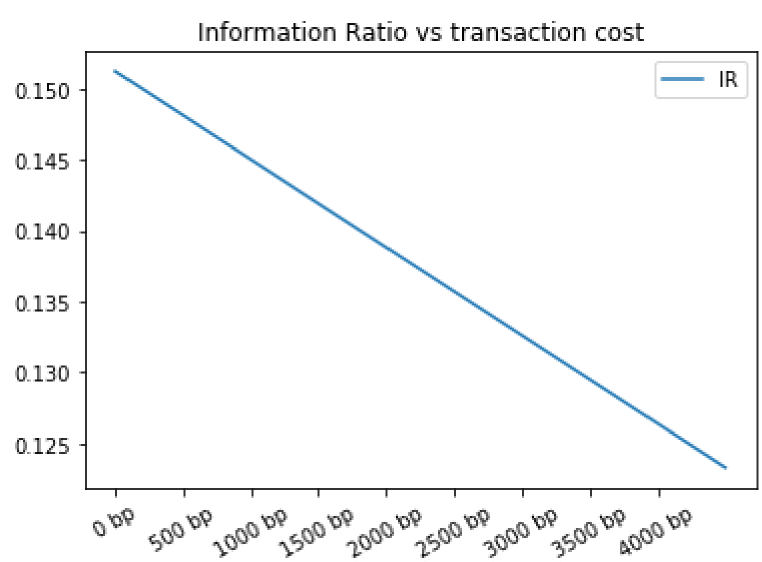
4.4.2.7 CER Gain

In [160]:
```
cer_gain=[]
for i in range(transaction_df.shape[1]):
    string = f'transaction_df["tc_{i*50}_bp"].values[200:]'
    string2=f'transaction_benchmark_df["tc_{i*50}_bp"].values[200:]'
    exec(f'U_bar_tc = (-(1+{string})**(-1)).mean()')
    exec(f'U_bar_bench_tc = (-(1+ {string2})**(-1)).mean()')
    CER_actual_tc = ((-U_bar_tc)**(-1) -1)*12
    CER_bench_tc = ((-U_bar_bench_tc)**(-1) -1)*12
    cer_gain += [CER_actual_tc-CER_bench_tc]
tc_perf_df['CER_gain'] = cer_gain
```

4.4.3 Plot

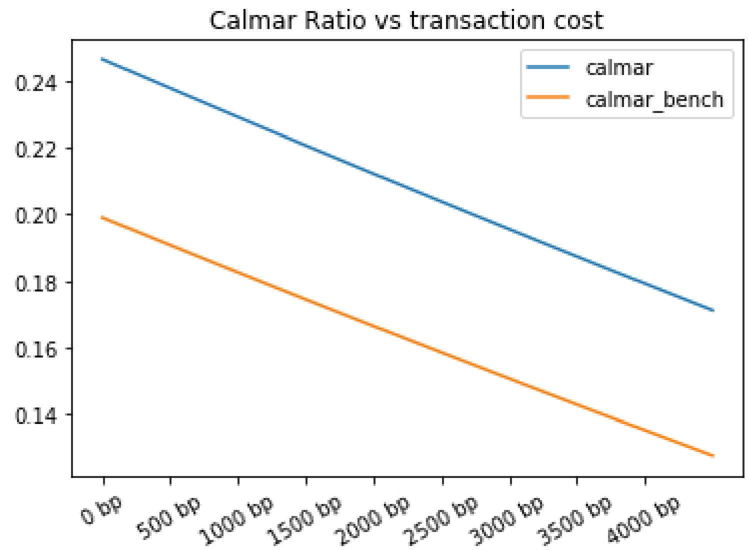
In [161]:
```
plt.plot('transaction_cost','IR',data=tc_perf_df)
plt.title('Information Ratio vs transaction cost')
plt.xticks(np.arange(0,90,10),rotation=30)
plt.legend()
```

Out[161]: <matplotlib.legend.Legend at 0x127879ef0>



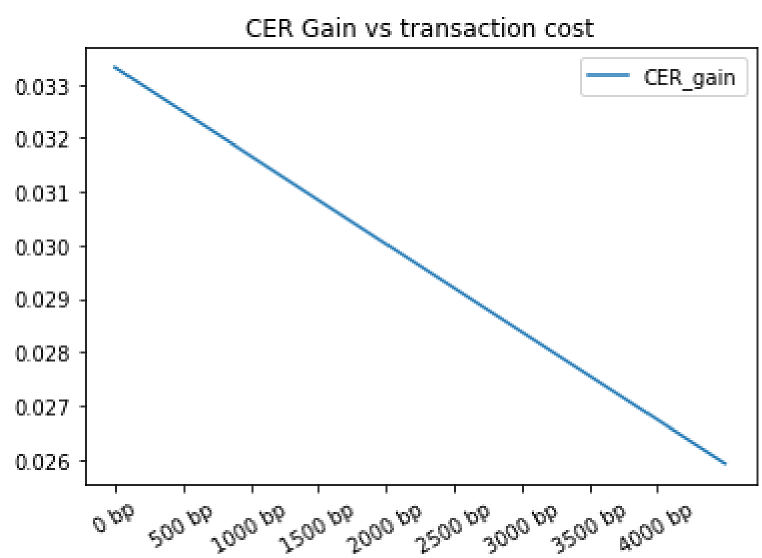
In [162]:
```python
plt.plot('transaction_cost','calmar',data=tc_perf_df)
plt.plot('transaction_cost','calmar_bench',data=tc_perf_df)
plt.title('Calmar Ratio vs transaction cost')
plt.xticks(np.arange(0,90,10),rotation=30)
plt.legend()
```

Out[162]:
```
<matplotlib.legend.Legend at 0x126cc0978>
```



In [163]:
```python
plt.plot('transaction_cost','CER_gain',data=tc_perf_df)
plt.title('CER Gain vs transaction cost')
plt.xticks(np.arange(0,90,10),rotation=30)
plt.legend()
```

Out[163]:
```
<matplotlib.legend.Legend at 0x1278deb00>
```



5. Data Export

Exporting data to csv

In [164]:
```
weights_df.to_csv('weight.csv')
```

6. Simple Robust Allocation

6.1 Optimization

$$
\begin{aligned}
\max_{w} \quad & \min_{m \in \hat{\theta}_m} (w - w_{bench})^T m \\
\text{s.t.} \quad & (w - w_{bench})^T S (w - w_{bench}) \le (h/12) T E^2 \\
& w^T 1_3 = 1 \\
& w \ge w_{LB}
\end{aligned}
$$

where

$$\hat{\theta}_m \equiv \{ m : (m - \mu_{BL})^T \Sigma_{BL}^{-1} (m - \mu_{BL}) \le q^2 \}$$

q is the critical value of Chi-squared distribution for 95% confidence level and degree of Freedom equal to the rank of Σ_{BL}

Then, suppose SVD of Σ_{BL} is given by $\Sigma_{BL} = E \Lambda E^T$ This implies

$$
\begin{aligned}
u &\equiv \frac{1}{q} \Lambda^{-\frac{1}{2}} E^T (m - \mu_{BL}) \\
\implies & \quad m = \mu_{BL} + q E \Lambda^{\frac{1}{2}} u
\end{aligned}
$$

Then,

$$\theta_m = \{ \mu_{BL} + q E \Lambda^{\frac{1}{2}} u : u^T u \le 1 \}$$

This translates to

$$
\begin{aligned}
\min_{m \in \hat{\theta}_m} (w - w_{bench})^T m = \min_{m \in \hat{\theta}_m} w^T m & \\
&= \min_{u^T u \le 1} \{ w^T \left(\mu_{BL} + q E \Lambda^{\frac{1}{2}} u \right) \} \\
&= w^T \mu_{BL} + q \min_{u^T u \le 1} \left\langle \Lambda^{\frac{1}{2}} E^T w, u \right\rangle \\
&= w^T \mu_{BL} - q \left\| \Lambda^{\frac{1}{2}} E^T w \right\| \\
&= w^T \mu_{BL} - q \sqrt{w^T \Sigma_{BL} w}
\end{aligned}
$$

where $\langle \cdot \rangle$ refers to dot product. The last equality follows from the fact minimum of the dot product happens where the angle θ between two vectors yields $\cos(\theta) = -1$. This leads to the following optimization problem:

$$\begin{aligned} \max_{w} \quad & w^T m - z \\ \text{s.t.} \quad & (w - w_{bench})^T S \, (w - w_{bench}) \; \leq (h/12) T E^2 \\ & q\sqrt{w^T \Sigma_{BL} w} \leq z \\ & w^T 1_3 = 1 \\ & w \geq w_{LB} \end{aligned}$$

In [165]:
```
q=np.sqrt(7.815)
q
```

Out[165]: 2.7955321496988725

In [166]:
```
cone1 = '+'.join([f'((w{i}-w_bench[{i}])*Sigma_BL[i][{i},{j}]*(w{j}-w_b
                   for i in range(3) for j in range(3)])
cone1
```

Out[166]:
```
'((w0-w_bench[0])*Sigma_BL[i][0,0]*(w0-w_bench[0]))+((w0-w_bench[0])*Sigm
a_BL[i][0,1]*(w1-w_bench[1]))+((w0-w_bench[0])*Sigma_BL[i][0,2]*(w2-w_ben
ch[2]))+((w1-w_bench[1])*Sigma_BL[i][1,0]*(w0-w_bench[0]))+((w1-w_bench
[1])*Sigma_BL[i][1,1]*(w1-w_bench[1]))+((w1-w_bench[1])*Sigma_BL[i][1,2]*
(w2-w_bench[2]))+((w2-w_bench[2])*Sigma_BL[i][2,0]*(w0-w_bench[0]))+((w2-
w_bench[2])*Sigma_BL[i][2,1]*(w1-w_bench[1]))+((w2-w_bench[2])*Sigma_BL
[i][2,2]*(w2-w_bench[2]))'
```

In [167]:

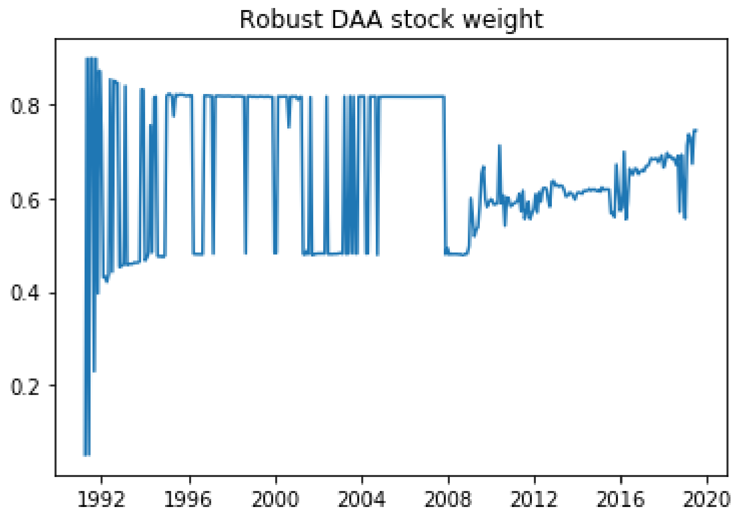
```
w_LB = 0.05*np.ones(3)
w_bench = np.array([0.65,0.3,0.05])
TE=0.02
w_r = np.array([np.nan,np.nan,np.nan]).reshape(-1,3)
for i in range(20,len(r_actual)):
    # Create a new model
        model = Model("qcp")
        model.setParam('OutputFlag', 0)

        # Create variables
        w0 = model.addVar(name="w0")
        w1 = model.addVar(name="w1")
        w2 = model.addVar(name="w2")
        z  = model.addVar(name="z")
        # Set objective: x
        obj = m[i][0][0]*w0 + m[i][1][0]*w1 + m[i][2][0]*w2 - z
        model.setObjective(obj, GRB.MAXIMIZE)

        model.addConstr(w0 + w1 + w2 == 1, "c0")
        model.addConstr(w0  >= w_LB[0], "c1")
        model.addConstr(w1  >= w_LB[1], "c2")
        model.addConstr(w2  >= w_LB[2], "c3")

        # Add second-order cone:
        eval('model.addConstr(' + cone + '<= (TE**2 *(1/12)), "qc0")')
        eval('model.addConstr(' + cone1 + '<= z/q**2, "qc1")')
        model.optimize()
        if i%100==0:
            print(f'{i}th observation: ')
            for v in model.getVars():
                print('%s %g' % (v.varName, v.x))
            print('Obj: %g' % obj.getValue())

        w_r = np.concatenate((w,np.array([model.getVars()[i].x
                                for i in range(3)]).reshape(-1,3)))

w_r = w_r[1:,:]
```

```
100th observation:
w0 0.818121
w1 0.13171
w2 0.0501692
z 0.000259754
Obj: 0.00495497
200th observation:
w0 0.81799
w1 0.131932
w2 0.0500783
z 0.000260148
Obj: 0.00606276
300th observation:
w0 0.700172
w1 0.131759
w2 0.168069
z 0.00025143
Obj: 0.0125038
```

In [168]:
```
r_weights_df = pd.DataFrame(w_r)
r_weights_df['Dates'] = processed_df['Dates']
r_weights_df.columns = ['robust_w_stock','robust_w_bond',
                        'robust_w_bill','Dates']
r_weights_df.to_csv('r_weight.csv')
```

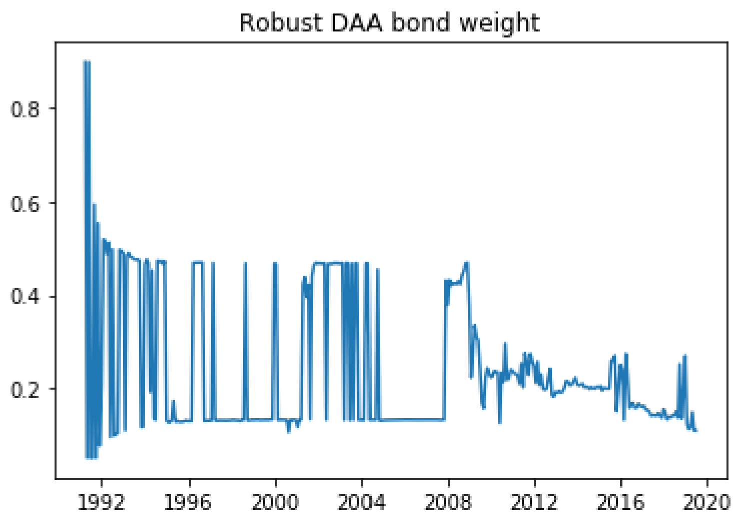
In [169]:
```
plt.plot('Dates','robust_w_stock',data=r_weights_df)
plt.title("Robust DAA stock weight")
```

Out[169]: Text(0.5, 1.0, 'Robust DAA stock weight')



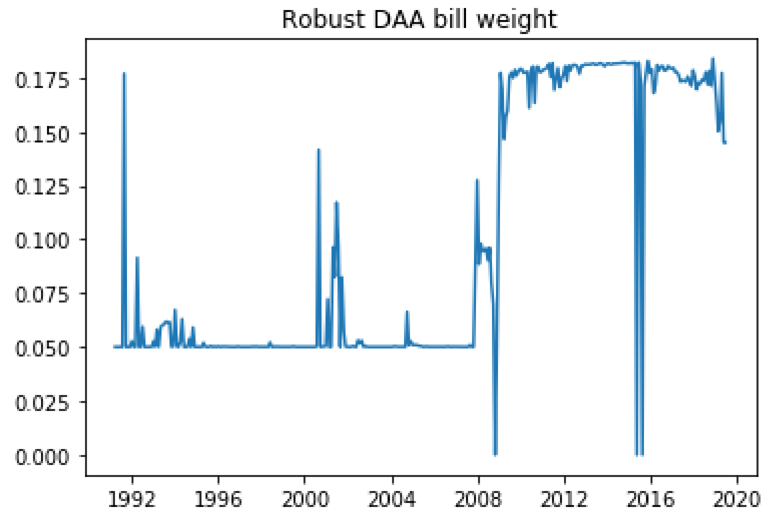
In [170]:
```
plt.plot('Dates','robust_w_bond',data=r_weights_df)
plt.title("Robust DAA bond weight")
```

Out[170]: Text(0.5, 1.0, 'Robust DAA bond weight')



In [171]:
```
plt.plot('Dates','robust_w_bill',data=r_weights_df)
plt.title("Robust DAA bill weight")
```

Out[171]: Text(0.5, 1.0, 'Robust DAA bill weight')



6.2 Performance Evaluation

Below cell attempts to ensure that the proportion of investment in bill is reduced to 0 for outliers in the bill returns.

In [172]:
```
np.place(w_r[:,2], np.abs(r_actual[:,2])>5, 0)
w_r[np.abs(r_actual[:,2])>5]
```

Out[172]:
```
array([[0.48066532, 0.46929671, 0.        ],
       [0.61937458, 0.19851546, 0.        ],
       [0.56025247, 0.26876236, 0.        ]])
```

6.2.1 Geometric Return

In [173]:
```
r_portfolio_robust = np.multiply(r_actual,w_r).sum(axis=1)
r_portfolio_bench = np.multiply(r_actual,w_bench).sum(axis=1)
```

In [174]:
```
robust_DAA_geo_return = ((((r_portfolio_robust[200:]+1)
                         .prod()**(1/r_portfolio_robust[200:].shape[0]))
                        -1) * 12)
robust_DAA_geo_return
```

Out[174]: 0.16784655039664553

6.2.1 Standard Deviation of Return

In [175]:
```
robust_DAA_std = r_portfolio_robust[200:].std(ddof=1)
robust_DAA_std
```

Out[175]: 0.09299296754868039

6.2.2 Maximum Drawdown

In [176]:
```
robust_DAA_mdd = 1-(r_portfolio_robust[200:] + 1).cumprod().min()
robust_DAA_mdd
```

Out[176]: 0.47304784802558286

6.2.3 Calmar Ratio

In [177]:
```
robust_DAA_Calmar = robust_DAA_geo_return/robust_DAA_mdd
robust_DAA_Calmar
```

Out[177]: 0.3548193932119278

6.2.4 Average Excess Return

In [178]:
```
robust_Avg_Excess_Return = (
                        robust_DAA_geo_return - Bench_geo_return)
robust_Avg_Excess_Return
```

Out[178]: 0.08535159808448434

In [179]:
```
robust_Avg_Excess_Return_wrt_original = (
                          robust_DAA_geo_return - DAA_geo_return)
robust_Avg_Excess_Return_wrt_original
```

Out[179]: 0.03924805018899136

6.2.5 Tracking Error

In [180]:
```
robust_Tracking_Error = ((r_portfolio_robust[200:]
                        - r_portfolio_bench[200:])
                       .std(ddof=1) *np.sqrt(12))
robust_Tracking_Error
```

Out[180]: 0.3107965421699985

In [181]:
```
robust_Tracking_Error_wrt_original = ((
                          r_portfolio_robust[200:]
                        - r_portfolio_actual[200:])
                       .std(ddof=1) *np.sqrt(12))
robust_Tracking_Error_wrt_original
```

Out[181]: 0.06326300690528569

6.2.6 Information Ratio

In [182]:
```
robust_IR = robust_Avg_Excess_Return/robust_Tracking_Error
robust_IR
```

Out[182]: 0.2746220967857454

In [183]:
```
robust_IR_wrt_original = (
    robust_Avg_Excess_Return_wrt_original/
    robust_Tracking_Error_wrt_original)
robust_IR_wrt_original
```

Out[183]: 0.6203949528948511

6.2.7 CER Gain

In [184]:
```
robust_U_bar_actual = (-(1+r_portfolio_robust)**(-1)).mean()
robust_U_bar_bench = (-(1+r_portfolio_bench)**(-1)).mean()
```

In [185]:
```
robust_CER_actual = ((-robust_U_bar_actual)**(-1) -1)*12
CER_bench = ((-U_bar_bench)**(-1) -1)*12
```

In [186]:
```
CER_gain = robust_CER_actual - CER_bench
CER_gain
```

Out[186]: 0.04340854156391405

In [187]:
```
CER_gain_wrt_original = robust_CER_actual - CER_actual
CER_gain_wrt_original
```

Out[187]: 0.027058412431287593

6.3 Perturbations

The purpose of this section is to observe how sensitive the performance of the portfolio would be for different realization of Σ_{BL} by adding randomly generated matrices to Σ_{BL} .

In [188]:

```
N=100
w_collect = np.zeros((N,w.shape[0],w.shape[1]))
return_collect = []
std_collect = []
mdd_collect = []
calmar_collect = []
avg_excess_return_collect = []
tracking_error_collect = []
IR_collect = []
CER_gain_collect = []

perturb = np.random.random((N,340,3,3))*1e-4
for r in range(N):
    cone1 = '+'.join([f'((w{i}-w_bench[{i}])*(Sigma_BL[i]+perturb[r,i])
                      for i in range(3) for j in range(3)])
    w_LB = 0.05*np.ones(3)
    w_bench = np.array([0.65,0.3,0.05])
    TE=0.02
    w_r_perturb = np.array([np.nan,np.nan,np.nan]).reshape(-1,3)
    for i in range(20,len(r_actual)):
        # Create a new model
            model = Model("qcp")
            model.setParam('OutputFlag', 0)

            # Create variables
            w0 = model.addVar(name="w0")
            w1 = model.addVar(name="w1")
            w2 = model.addVar(name="w2")
            z  = model.addVar(name="z")
            # Set objective: x
            obj = m[i][0][0]*w0 + m[i][1][0]*w1 + m[i][2][0]*w2 - z
            model.setObjective(obj, GRB.MAXIMIZE)

            model.addConstr(w0 + w1 + w2 == 1, "c0")
            model.addConstr(w0  >= w_LB[0], "c1")
            model.addConstr(w1  >= w_LB[1], "c2")
            model.addConstr(w2  >= w_LB[2], "c3")

            # Add second-order cone:
            eval('model.addConstr(' + cone + '<= (TE**2 *(1/12)), "qc0"
            eval('model.addConstr(' + cone1 + '<= z/q**2, "qc1")')
            model.optimize()

            w_r_perturb = np.concatenate(
                (w,np.array([model.getVars()[i].x
                for i in range(3)]).reshape(-1,3)))

    w_collect[r] = w_r_perturb[1:,:]
    r_portfolio_perturb = np.multiply(
        r_actual,w_r_perturb[1:,:]).sum(axis=1)
    cum_return = np.array(
        [x if x>0 else 1 for x in (r_portfolio_perturb[200:]+1)])
    return_collect += [
        (((cum_return.prod()**(
            1/r_portfolio_perturb[200:].shape[0])) -1) * 12)]
    std_collect += [r_portfolio_perturb[200:].std(ddof=1)]
```

```
    mdd_collect += [1-cum_return.cumprod().min()]
    calmar_collect += [return_collect[-1]/mdd_collect[-1]]
    avg_excess_return_collect += [
        return_collect[-1] - Bench_geo_return]
    tracking_error_collect += [
        ((r_portfolio_perturb[200:]
            - r_portfolio_bench[200:])
            .std(ddof=1) *np.sqrt(12))]
    IR_collect += [
        avg_excess_return_collect[-1]/
        tracking_error_collect[-1]]
    perturb_U_bar_actual = (
        -(1+r_portfolio_perturb)**(-1)).mean()
    perturb_CER_actual = (
        (-perturb_U_bar_actual)**(-1) -1)*12
    CER_gain_collect += [perturb_CER_actual - CER_bench]
```

In [189]:

```
perturb_df = pd.DataFrame(std_collect)
i=1
for item in [mdd_collect,calmar_collect,
             avg_excess_return_collect,
             tracking_error_collect,
             IR_collect,CER_gain_collect]:
    exec(f"perturb_df['{i}'] = item")
    i+=1
columns = ['std','mdd','calmar',
           'avg_excess_return',
           'tracking_error','IR','CER_gain']
perturb_df.columns = columns
```

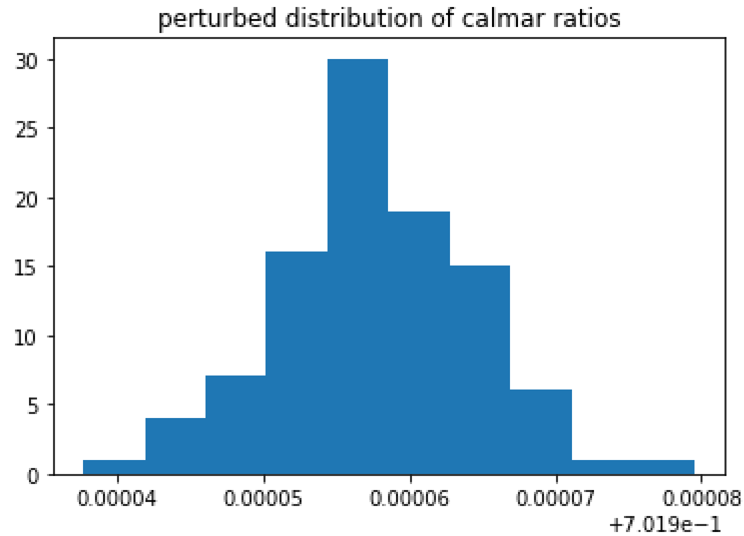
In [190]:

```
perturb_df.head()
```

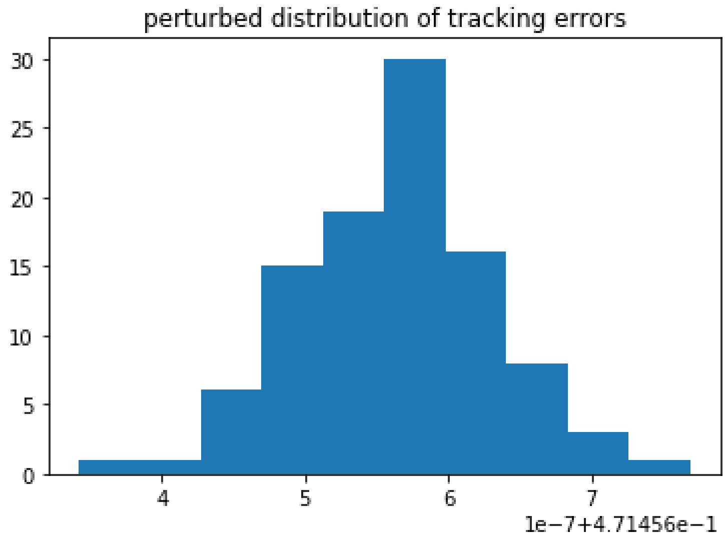
Out[190]:

	std	mdd	calmar	avg_excess_return	tracking_error	IR	CER_gain
0	0.191727	0.390951	0.701956	0.191935	0.471457	0.407111	0.188997
1	0.191727	0.390951	0.701959	0.191936	0.471457	0.407114	0.188997
2	0.191727	0.390951	0.701953	0.191934	0.471457	0.407109	0.188996
3	0.191727	0.390951	0.701948	0.191932	0.471457	0.407105	0.188995
4	0.191727	0.390951	0.701951	0.191933	0.471457	0.407107	0.188996

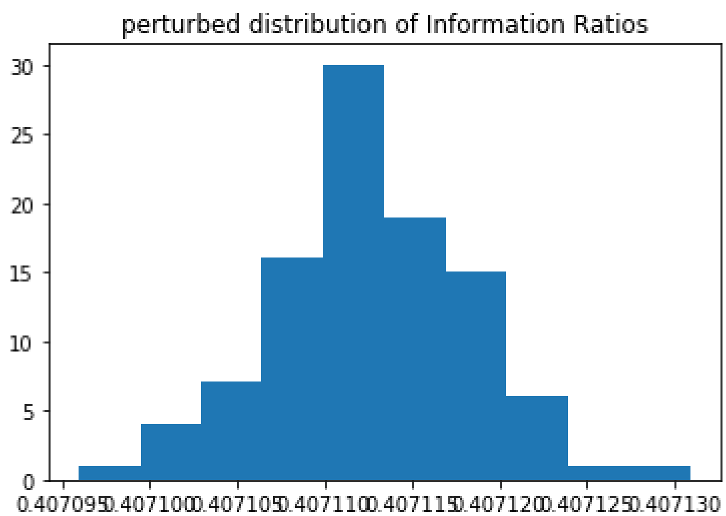
In [191]:
```
n_bins=10

plt.hist(perturb_df['calmar'])
plt.title('perturbed distribution of calmar ratios')
plt.show()
```



In [192]:
```
n_bins=10

plt.hist(perturb_df['tracking_error'])
plt.title('perturbed distribution of tracking errors')
plt.show()
```

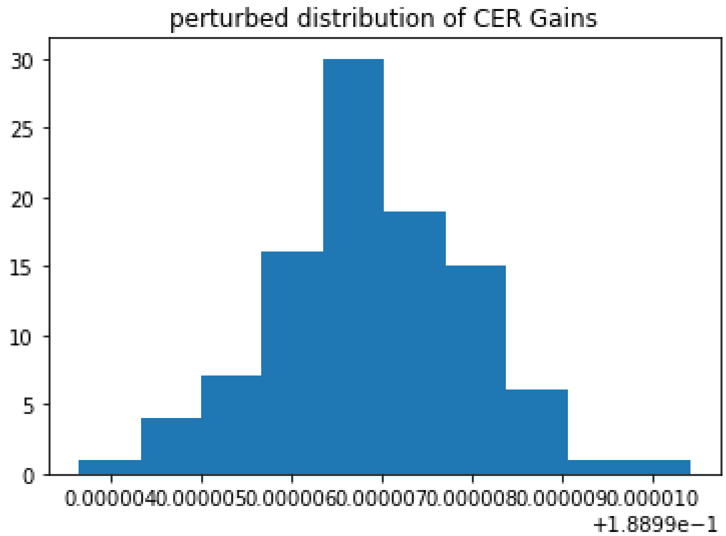


In [193]:
```
n_bins=10

plt.hist(perturb_df['IR'])
plt.title('perturbed distribution of Information Ratios')
plt.show()
```



In [194]:

```
n_bins=10

plt.hist(perturb_df['CER_gain'])
plt.title('perturbed distribution of CER Gains')
plt.show()
```



In [ ]:

```

```

In [ ]:

```

```