18.02 Problem Set 5

18.02 习题集 5

At MIT problem sets are referred to as ‘psets’. You will see this term used occasionally within the problems sets.

在麻省理工学院，习题集被称为“psets”。您会在习题集中偶尔看到这个术语。

The 18.02 psets are split into two parts ‘part I’ and ‘part II’. The part I are all taken from the supplementary problems. You will find a link to the supplementary problems and solutions on this website.

18.02 的习题分为两个部分“第一部分”和“第二部分”。第一部分全部来自补充习题。您将在此网站上找到补充习题和解决方案的链接。

The intention is that these help the student develop some fluency with concepts and techniques.

目的是帮助学生在概念和技巧上发展一定的流利度。

Students have access to the solutions while they do the problems, so they can check their work or get a little help as they do the problems. After you finish the problems go back and redo the ones for which you needed help from the solutions.

学生在做题时可以访问解答，因此他们可以检查自己的工作或在做题时获得一些帮助。完成题目后，回去重新做那些需要从解答中获得帮助的题目。

The part II problems are more involved. At MIT the students do not have access to the solutions while they work on the problems. They are encouraged to work together, but they have to write their solutions independently.

第二部分的问题更复杂。在麻省理工学院，学生在解决问题时无法访问答案。他们被鼓励一起合作，但必须独立写出他们的解决方案。

Part I (15 points)

第一部分（15 分）

At MIT the underlined problems must be done and turned in for grading. The ‘Others’ are some suggested choices for more practice.

在麻省理工学院，划线的问题必须完成并提交以供评分。“其他”是一些建议的选择以便进行更多练习。

A listing like ’ § $1\,\mathrm{B}: \underline{2}, 5\,\underline{\mathrm{b}}, \underline{10}$ ’ means do the indicated problems from supplementary problems section 1B.

像 ' § $1\,\mathrm{B}: \underline{2}, 5\,\underline{\mathrm{b}}, \underline{10}$ ' 这样的列表意味着做补充问题部分 1B 中指示的问题。

1 Differentials. Chain rule

1 微分。链式法则

§2C: 1 $\underline{\mathbf{ad}}, \underline{\mathbf{2}}, \underline{\mathbf{3}}, \mathbf{5}\underline{\mathbf{ab}}$ Others: 1bc

§2C: 1 $\underline{\mathbf{ad}}, \underline{\mathbf{2}}, \underline{\mathbf{3}}, \mathbf{5}\underline{\mathbf{ab}}$ 其他: 1bc

§2E: í⊥, 2be, 8a; Others: 1ab, 2d, 4, 5, 7

§2E: í⊥, 2be, 8a; 其他: 1ab, 2d, 4, 5, 7

2 Gradient and directional derivatives

2 梯度和方向导数

§ $\mathbf{2D}: \mathbf{1}\underline{\mathbf{ae}}, \mathbf{2}\,\underline{\mathbf{b}}, \mathbf{3}\underline{\mathbf{a}}, \underline{\mathbf{8}}, \underline{\mathbf{9}}$; Others: $\mathbf{1bc}, \mathbf{2a}, \mathbf{3\ b}, \mathbf{4}, \mathbf{5}$ § $\mathbf{2D}: \mathbf{1}\underline{\mathbf{ae}}, \mathbf{2}\,\underline{\mathbf{b}}, \mathbf{3}\underline{\mathbf{a}}, \underline{\mathbf{8}}, \underline{\mathbf{9}}$; 其他: $\mathbf{1bc}, \mathbf{2a}, \mathbf{3\ b}, \mathbf{4}, \mathbf{5}$

§2E: 7

Part II (17 points)

第二部分（17 分）

Problem 1 (4: 1,2,1) 问题 1 (4: 1,2,1)
In laminar flow in a cylinder (for example, blood flow in a vein or artery), the resistance R to the flow is related to the length w and radius r of the cylinder by the law of Poiseuille: $R = k\frac{w}{r^4}$ for some constant k.

在圆柱体中的层流（例如，血液在静脉或动脉中的流动），流动的阻力 R 与圆柱体的长度 w 和半径 r 通过泊肃叶定律相关：
$R = k\frac{w}{r^4}$ ，其中 k 为某个常数。

a) Compute the linear approximation $\mathrm{d}R$ to the change in R, in terms of the changes in w and r.

a) 计算 R 的线性近似 $\mathrm{d}R$ ，以 w 和 r 的变化为基础。

b) Compute the linear approximation $\frac{\mathrm{d}R}{R}$ to the relative change in R in terms of $\frac{\mathrm{d}w}{w}$ = the relative change in w and $\frac{\mathrm{d}r}{r}$ = the relative change in r.

b) 计算相对变化 R 的线性近似 $\frac{\mathrm{d}R}{R}$ ，以相对变化 w 和 r 的相对变化 $\frac{\mathrm{d}w}{w}$ = 为基础。

c) For relative changes in w and r of about the same sizes, which variable contributes more to the relative change in R? Also, in order to produce the greatest relative change in R, should the changes in w and r both have the same sign or opposite signs (and why)?

c) 对于 w 和 r 的相对变化大致相同的情况，哪个变量对 R 的相对变化贡献更大？此外，为了在 R 中产生最大的相对变化， w 和 r 的变化应该是同号还是异号（为什么）？

Problem 2 (3) 问题 2 (3)

Let $f(x,y,z,t)$ be a smooth function, and let $\nabla f = \langle f_x, f_y, f_z \rangle$ be the gradient in the space variables only. Let $\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ be a smooth curve, and $\mathbf{v} = \mathbf{r}'(t)$; and suppose we use the notation $\frac{Df}{Dt} = \frac{d}{dt} f(\mathbf{r}(t), t)$.

设 $f(x,y,z,t)$ 为一个光滑函数，设 $\nabla f = \langle f_x, f_y, f_z \rangle$ 为仅在空间变量中的梯度。设 $\mathbf{r} = \mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ 为一条光滑曲线，设 $\mathbf{v} = \mathbf{r}'(t)$ ；并假设我们使用符号 $\frac{Df}{Dt} = \frac{d}{dt} f(\mathbf{r}(t), t)$ 。

Use the Chain Rule to show that $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f$.

使用链式法则证明 $\frac{Df}{Dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f$ 。

Background: The notation $\frac{D}{Dt}$ comes from the physics of fluid motion, where it is called the convective derivative (or material or substantial derivative, and by several other names), and means the rate of change along a moving path of some physical quantity (scalar or vector) which is being transported by fluid currents.

背景：符号 $\frac{D}{Dt}$ 来自流体运动的物理学，在其中被称为对流导数（或物质导数或实质导数，以及其他几个名称），意味着沿着某个物理量（标量或向量）被流体流动所运输的移动路径的变化率。

In this macroscopic model, the fluid is pictured as a continuum of point masses rather than as individual molecules. At a location (x,y,z) in space and a time t, the point mass has a density $\rho = \rho(x,y,z,t)$, and a velocity $\mathbf{v} = \mathbf{v}(x,y,z,t)$. This means that the vector $\mathbf{v}(x,y,z,t)$ points in direction tangent to the path of a particle at (x,y,z,t) in the flow, and has magnitude equal to the instantaneous speed of the particle located at that point and which is moving in the flow.

在这个宏观模型中，流体被视为点质量的连续体，而不是单个分子。在空间中的位置 (x,y,z) 和时间 t ，点质量具有密度 $\rho = \rho(x,y,z,t)$ 和速度 $\mathbf{v} = \mathbf{v}(x,y,z,t)$ 。这意味着向量 $\mathbf{v}(x,y,z,t)$ 指向流动中粒子在 (x,y,z,t) 处路径的切线方向，并且其大小等于位于该点并在流动中移动的粒子的瞬时速度。

Now suppose that the curve $\mathbf{r} = \mathbf{r}(t)$ is a path of a point mass in the flow, so that (by definition) $\mathbf{r}'(t) = \mathbf{v}(\mathbf{r}(t), t)$. The convective derivative $\frac{Df}{Dt}$ of f along this path is the time rate of change of f using only the values of $f(x,y,z,t)$ for which the space variables (x,y,z) are restricted to the path $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ of a particle in the flow. For this reason you will see the convective derivative described as the rate of change of the quantity f "moving along the flow" or "moving with an element of the fluid" (and other similar language).

现在假设曲线 $\mathbf{r} = \mathbf{r}(t)$ 是流动中点质量的路径，因此（根据定义） $\mathbf{r}'(t) = \mathbf{v}(\mathbf{r}(t), t)$ 。沿着这条路径的对流导数 $\frac{Df}{Dt}$ 是 f 的时间变化率，仅使用 $f(x,y,z,t)$ 的值，其中空间变量 (x,y,z) 被限制在流动中粒子的路径 $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$ 。因此，您将看到对流导数被描述为"沿着流动移动"或"与流体元素一起移动"的量 f 的变化率（以及其他类似的表述）。

Problem 3 (5: 1,2,2) (continuation)

问题 3 (5: 1,2,2) (续)

Now take the case $f = \rho$, the density of the fluid. A fluid flow is called incompressible if $\frac{D\rho}{Dt} = 0$.

现在考虑情况 $f = \rho$ ，流体的密度。如果 $\frac{D\rho}{Dt} = 0$ ，则流体流动被称为不可压缩。

As discussed above, this means that the mass density is constant along the paths of the flow. Any substance (like water, at moderate pressures) which has the property that its density is constant in all variables (x,y,z,t) will of course be incompressible, which is the usual way one pictures something which cannot be compressed.

如上所述，这意味着质量密度在流动路径上是恒定的。任何具有在所有变量 (x,y,z,t) 中密度恒定特性的物质（如水，在适度压力下）当然是不可压缩的，这通常是人们想象无法被压缩的事物的方式。

However, incompressibility is in general a property of the flow rather than just the fluid itself, since it says only that the rate of change of the density moving along the flow is zero. The following examples illustrate this.

然而，不可压缩性通常是流动的特性，而不仅仅是流体本身，因为它仅表示沿流动移动的密度变化率为零。以下示例说明了这一点。

a) Suppose that the density function depends only on time t but is constant in the space variables (x,y,z), that is, $\rho = \rho(t)$. Then show that the flow is incompressible if and only if the density $\rho(t)$ is constant in all the variables (x,y,z,t) (that is, the constant-density case discussed above).

a) 假设密度函数仅依赖于时间 t ，但在空间变量 (x,y,z) 中是常数，即 $\rho = \rho(t)$ 。然后证明流动是不可压缩的当且仅当密度 $\rho(t)$ 在所有变量 (x,y,z,t) 中是常数（即上述讨论的常密度情况）。

b) Next suppose instead that the density depends only on the space variables (x,y,z) but not (explicity) on t, so that $\rho = \rho(x,y,z)$. An incompressible flow in this case is called stratified.

b) 接下来假设密度仅依赖于空间变量（ x,y,z ），而不（明确地）依赖于 t ，因此 $\rho = \rho(x,y,z)$ 。在这种情况下，称为分层的不可压流动。

Use the result of problem 2 to give the condition on ρ and $\mathbf{v}$ for stratified flow.

使用问题 2 的结果给出分层流动的 ρ 和 $\mathbf{v}$ 的条件。

A flow is called steady if the density ρ and the velocity field $\mathbf{v}$ of the flow do not depend explicitly on the time t, i.e. $\rho = \rho(x,y,z)$ and $\mathbf{v} = \mathbf{v}(x,y,z)$. In this case, the term streamlines is used for the paths of the particles in the flow, since they keep their same shapes over time.

如果流动的密度 ρ 和流动的速度场 $\mathbf{v}$ 不显式依赖于时间 t ，则称该流动为稳态流动，即 $\rho = \rho(x,y,z)$ 和 $\mathbf{v} = \mathbf{v}(x,y,z)$ 。在这种情况下，粒子在流动中的路径被称为流线，因为它们随时间保持相同的形状。

c) Suppose one has a 2D stratified steady flow, so that $\rho = \rho(x,y)$ and $\mathbf{v} = \mathbf{v}(x,y)$, and suppose also that the density varies only with the height y. Draw a picture of the streamlines for such as flow. Then explain why they must follow this pattern, and why the term "stratified" fits in this case.

c) 假设有一个二维分层稳态流动，因此 $\rho = \rho(x,y)$ 和 $\mathbf{v} = \mathbf{v}(x,y)$ ，并且假设密度仅随高度 y 变化。画出这种流动的流线图。然后解释为什么它们必须遵循这种模式，以及为什么"分层"这个术语适用于这种情况。

(This could be, for example, a cross-section of a very regular ocean current, if it is an incompressible steady flow whose density varies only with the depth.)

（这可以是一个非常规则的海洋洋流的横截面，例如，如果它是一个不可压缩的稳定流动，其密度仅随深度变化。）

Problem 4 (5: 1, 1, 1,1,1)

问题 4 (5: 1, 1, 1, 1, 1)

For the linear function $f(x, y) = 4 - x - 4y$:

对于线性函数 $f(x, y) = 4 - x - 4y$:

a) Sketch the portion of the graph in the first octant

a) 在第一象限中绘制图形的部分

b) Compute the gradient of f.

b) 计算 f 的梯度。

c) Find the point on the level curve $f(x, y) = 0$ such that the line in the gradient direction passes through the origin, and then sketch in the gradient at that point.

c) 找到在水平曲线 $f(x, y) = 0$ 上的点，使得沿梯度方向的直线通过原点，然后在该点上绘制梯度。

d) Compute the directional derivative of f in the direction $\mathbf{w} = -2\mathbf{i} - \mathbf{j}$

d) 计算在方向 $\mathbf{w} = -2\mathbf{i} - \mathbf{j}$ 上 f 的方向导数

e) Sketch in the slope triangles for the rates of change of f in the gradient direction and in the direction of $\mathbf{w}$.

e) 在梯度方向和 $\mathbf{w}$ 方向上绘制 f 的变化率的斜率三角形。