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Gravitational radiation in black-hole collisions at the speed of light. II. Reduction to two independent variables and calculation of the second-order news function

P. D. D'Eath
Department of Applied Mathematics and Theoretical Physics, University of Cambridge, Silver Street, Cambridge CB3 9EW, England

P. N. Payne
59, The Avenue, Nedlands 6009, Western Australia
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This paper describes analytical simplifications which make feasible the numerical calculation of the second-order news function, which gives partial information about the angular distribution of gravitational radiation emitted in the axisymmetric collision of two black holes at the speed of light. In the preceding paper, paper I, the curved radiative region of the space-time, produced after the collision of the two incoming plane-fronted shock waves, was treated using perturbation theory by making a large Lorentz boost to a frame in which a weak shock of energy λ scatters off a strong shock of energy $\nu \gg \lambda$. Calculations of gravitational radiation at first, second, ... order in (λ/ν) translate, when boosted back to the center-of-mass frame, into calculations of the coefficients $a_0(\hat{\tau}/\mu)$, $a_2(\hat{\tau}/\mu)$, ... in the convergent series expansion $c_0(\hat{\tau},\hat{\theta})=\sum_{n=0}^{\infty}a_{2n}(\hat{\tau}/\mu)\sin^{2n}\hat{\theta}$ expected for the news function c_0, where $\hat{\tau}$ is a retarded time coordinate, $\hat{\theta}$ is the angle from the symmetry axis, and μ is the energy of each incoming black hole in the center-of-mass frame. In paper I, $a_0(\hat{\tau}/\mu)$ was computed and $a_2(\hat{\tau}/\mu)$ was obtained as an integral expression too complicated to be tractable numerically. In the present paper a simpler expression for $a_2(\hat{\tau}/\mu)$ is derived, using the property that the perturbative field equations may all be reduced to equations in only two independent variables, because of a conformal symmetry at each order in perturbation theory. The Green's function for the perturbative field equations is found by reduction from the retarded flat-space Green's function in four dimensions, leading to expressions in terms of two variables for the second-order radiative metric components. From these, $a_2(\hat{\tau}/\mu)$ can be extracted after removing, by a gauge transformation, the $(\ln r)/r$ terms present in the second-order metric in the harmonic gauge used here (r being a radial coordinate). Numerical results are presented in the following paper, paper III, which discusses the implications for the energy emitted and the nature of the radiative space-time.

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I. INTRODUCTION

This is the second in a series of three papers concerned with gravitational radiation emitted in the axisymmetric collision of two black holes at the speed of light. The preceding paper, paper I [1], was concerned with describing the problem and with setting up an analytical treatment using a perturbation approach. The present paper describes analytical simplifications which make feasible the numerical calculation of the second-order news function, which gives partial information about the angular distribution of gravitational radiation. Results and conclusions concerning the radiation emitted and consequent mass loss are presented in the following paper, paper III [2].

In paper I the axisymmetric collision of two black holes traveling at the speed of light, each described in the center-of-mass frame before the collision by an impulsive plane-fronted shock wave with energy μ, was investigated in a new frame to which a large Lorentz boost had been applied. There the energy $\nu=\mu e^{\alpha}$ of the incoming shock 1, which initially lies on the hyperplane $z+t=0$ between two portions of Minkowski space, obeys $\nu \gg \lambda$, where $\lambda=\mu e^{-\alpha}$ is the energy of the incoming shock 2, which initially lies on the hypersurface $z-t=0$. In the boosted frame, to the future of the strong shock 1, the metric possesses the perturbation expansion

$$g_{ab} \sim \nu^2 \left[\eta_{ab} + \sum_{i=1}^{\infty} \left[\frac{\lambda}{\nu} \right]^i h_{ab}^{(i)} \right] , \tag{1.1}$$

(I3.18′)

with respect to suitable coordinates, where η_{ab} is the Minkowski metric. The problem of solving the Einstein field equations becomes a (singular) perturbation problem of finding $h_{ab}^{(1)} h_{ab}^{(2)}, \ldots$ by successively solving the linearized field equations at first, second, ... order in λ/ν, given characteristic initial data on the surface $u=0$ just to the future of the strong shock 1.

On boosting back to the center-of-mass frame, one finds that the perturbation series (1.1) gives an accurate description of the space-time geometry in the region in which gravitational radiation propagates at small angles away from the forward symmetry axis $\hat{\theta}=0$. By reflection symmetry, an analogous series also give a good description near the backward axis $\hat{\theta}=\pi$. The news function [3] $c_0(\hat{\tau},\hat{\theta})$, which describes the gravitational radiation arriving at future null infinity $\mathscr{I}^+$ in the center-of-mass frame, is expected to have the convergent series expansion

$$c_0(\hat{\tau},\hat{\theta})=\sum_{n=0}^{\infty} a_{2n}(\hat{\tau}/\mu)\sin^{2n}\hat{\theta}\ , \tag{1.2}$$

(I1.3′)

where $\hat{\tau}$ is a suitable retarded time coordinate. [The series of Eq. (I.13) has been modified here by making the replacement $a_{2n}(\hat{\tau})\rightarrow a_{2n}(\hat{\tau}/\mu)$, since $\hat{\tau}$ will always appear as an argument in the dimensionless combination $(\hat{\tau}/\mu)$ (see Sec. V of paper I).] The first-order perturbation calculation of $h_{ab}^{(1)}$ in Sec. IV of paper I, on boosting back to the center-of-mass frame, yielded $a_0(\hat{\tau}/\mu)$, in agreement with the expression found in Ref. [4] by studying the collision of two black holes at large but finite incoming Lorentz factor γ. This is such that, if the radiation were isotropic [i.e., if $a_{2n}(\hat{\tau}/\mu)$ were zero for $n\geq 1$], 25% of the initial energy 2μ would be emitted in gravitational waves. The second-order calculation of $h_{ab}^{(2)}$ in Sec. V of paper I, on boosting back to the center-of-mass frame, gave an integral expression for the next coefficient $a_2(\hat{\tau}/\mu)$ which unfortunately was so complicated as to be intractable numerically. In the present paper we shall show how the calculation of $a_2(\hat{\tau}/\mu)$ can be simplified analytically so as to enable us to compute this function numerically. As was shown in Sec. VI of paper I, if all the gravitational radiation in the space-time is (in a certain precise sense) accurately described by Eq. (1.2), then the mass of the (assumed) final static Schwarzschild black hole remaining after the collision can be determined from knowledge only of $a_0(\hat{\tau}/\mu)$ and $a_2(\hat{\tau}/\mu)$. Further, Penrose [5] has found an apparent horizon on the union of the two null planes on which the incoming shocks lie; if the cosmic censorship hypothesis [6] is correct, this gives a lower bound of $\sqrt{2}\mu$ for the energy of the final black hole (or holes). The computation of $a_2(\hat{\tau}/\mu)$ is thus linked to an interesting test of cosmic censorship.

In Sec. II of this paper we begin the process of finding a simpler form for $a_2(\hat{\tau}/\mu)$ by noting that, because of a conformal symmetry at each order in perturbation theory, the field equations obeyed by the metric perturbations $h_{ab}^{(1)},h_{ab}^{(2)},\ldots$ in Eq. (1.1) may all be reduced to equations in only two independent variables. The resulting reduced differential equations are studied in Sec. III; the equations are shown to be hyperbolic, and their characteristics are found. The retarded Green's function for the reduced differential operator is found in Sec. IV by reduction from the retarded flat-space Green's function in four dimensions. This allows the transverse components of the second-order metric perturbation $h_{ab}^{(2)}$ [from which $a_2(\hat{\tau}/\mu)$ can be found] to be expressed in two-dimensional form (Sec. V). The resulting integral expressions are considerably simpler than those found from a four-dimensional approach in Sec. V of paper I, thus making feasible the numerical computation of $a_2(\hat{\tau}/\mu)$, of which the results will be presented in paper III.

In order to extract $a_2(\hat{\tau}/\mu)$ from the metric perturbations, one has to deal with certain terms introduced in the metric as a result of the choice of the harmonic gauge, employed in the calculation of $h_{ab}^{(1)}$ and $h_{ab}^{(2)}$. As is well known [7], this gauge leads to the appearance of $(\ln r)/r$ terms in the metric tensor at second and higher orders in perturbation theory, where r is a radial coordinate. In Sec. VI of this paper we calculate the $(\ln r)/r$ term in the transverse part of $h_{ab}^{(2)}$, and show how to eliminate this term by finding an explicit coordinate transformation to a Bondi coordinate system [3] at first order in perturbation theory. In Sec. VII we show that, while the construction of this coordinate transformation can be carried on to second order, knowledge of the full second-order gauge transformation is not needed in order to calculate the second-order news function, which describes the gravitational radiation at this order. Section VIII discusses the ambiguity in the second-order news function caused by the freedom to make supertranslations [3]; use of this freedom is in fact essential in order to put the news function in a form which is square integrable at each order in perturbation theory. (The complete news function must be square-integrable, in order that the mass loss in gravitational waves be finite.) Some comments are included in Sec. IX.

II. REDUCTION TO TWO DIMENSIONS

We shall now show that the (four-dimensional) field equations satisfied by the metric perturbations $h_{ab}^{(1)},h_{ab}^{(2)},\ldots$ in Eq. (1.1) may all be reduced to two-dimensional form.

Consider the C^0 form of the infinitely boosted black hole metric:

$$\begin{aligned}ds^2=&d\hat{u}\,d\hat{v}+[1+4\mu\hat{u}\,\theta(\hat{u})\hat{\rho}^{-2}]^2d\hat{\rho}^2\\&+\hat{\rho}^2[1-4\mu\hat{u}\,\theta(\hat{u})\hat{\rho}^{-2}]^2d\phi^2\ ,\end{aligned} \tag{2.1}$$

(I2.4′)

where $\theta(\hat{u})$ is the Heaviside step function. On using the discontinuous coordinate transformation

$$\begin{aligned}x&=\hat{x}-4\mu\hat{u}\,\theta(\hat{u})\hat{x}\hat{\rho}^{-2}\ ,\\y&=\hat{y}-4\mu\hat{u}\,\theta(\hat{u})\hat{y}\hat{\rho}^{-2}\ ,\\u&=\hat{u}\ ,\\v&=\hat{v}+8\mu\theta(\hat{u})\ln\hat{\rho}-16\mu^2\hat{u}\,\theta(\hat{u})\hat{\rho}^{-2}\ ,\end{aligned} \tag{2.2}$$

(I2.3′)

where $x=\rho\cos\phi$, $y=\rho\sin\phi$, $\hat{x}=\hat{\rho}\cos\phi$, $\hat{y}=\hat{\rho}\sin\phi$, this may be put in the form

$$ds^2=du\,dv+dx^2+dy^2-4\mu\ln(x^2+y^2)\delta(u)du^2\ , \tag{2.3}$$

describing an impulsive plane-fronted wave between two portions of Minkowski space-time.

Let L denote the Lorentz transformation

$$(\hat{u},\hat{v},\hat{\rho},\phi)\xrightarrow{L}(\hat{u}',\hat{v}',\hat{\rho}',\phi')=(e^{-\beta}\hat{u},e^{\beta}\hat{v},\hat{\rho},\phi) \tag{2.4}$$

[using Eq. (2.2) it can be shown that L is a Lorentz transformation, even though the careted coordinates are not Minkowskian in $\hat{u}>0$], and let C denote the conformal transformation

$$(\hat{u},\hat{v},\hat{\rho},\phi)\xrightarrow{C}(\hat{u}',\hat{v}',\hat{\rho}',\phi')=(e^{-\beta}\hat{u},e^{-\beta}\hat{v},e^{-\beta}\hat{\rho},\phi)\ . \tag{2.5}$$

Then under CL,

$$(\hat{u},\hat{v},\hat{\rho},\phi)\xrightarrow{CL}(\hat{u}',\hat{v}',\hat{\rho}',\phi')=(e^{-2\beta}\hat{u},\hat{v},e^{-\beta}\hat{\rho},\phi) \tag{2.6}$$

and

$$d\hat{u}\,d\hat{v}+[1+4\mu\hat{u}\,\theta(\hat{u})\hat{\rho}^{-2}]^2d\hat{\rho}^2+\hat{\rho}^2[1-4\mu\hat{u}\,\theta(\hat{u})\hat{\rho}^{-2}]d\phi^2$$
$$\xrightarrow{CL}e^{2\beta}\{d\hat{u}\,'d\hat{v}\,'+[1+4\mu\hat{u}\,'\theta(\hat{u}\,')\hat{\rho}\,'^{-2}]^2d\hat{\rho}\,'^2+\hat{\rho}\,'^2[1-4\mu\hat{u}\,'\theta(\hat{u}\,')\hat{\rho}\,'^{-2}]^2d\phi^2\}\ . \tag{2.7}$$

Thus the transformation CL is a conformal symmetry of (2.1). (This is easy to understand physically: the Lorentz transformation L increases the apparent energy of the wave from μ to μe^{β}; but this energy provides the only length scale present in the metric. If, therefore, using C, we scale down all lengths by a factor e^{β}, then the apparent energy of the wave is reduced to μ again.) For a wave traveling in the opposite direction the effect of CL is

$$d\hat{u}\,d\hat{v}+[1-4\mu\hat{v}\,\theta(-\hat{v})\hat{\rho}^{-2}]^2d\hat{\rho}^2+\hat{\rho}^2[1+4\mu\hat{v}\,\theta(-\hat{v})\hat{\rho}^{-2}]^2d\phi^2$$
$$\xrightarrow{CL}e^{2\beta}\{d\hat{u}'\,d\hat{v}'+[1-4\mu e^{-2\beta}\hat{v}'\theta(-\hat{v}')\hat{\rho}'^{-2}]^2d\hat{\rho}'^2+\hat{\rho}'^2[1+4\mu e^{-2\beta}\hat{v}'\theta(-\hat{v}')\hat{\rho}'^{-2}]^2d\phi^2\}\ . \tag{2.8}$$

Now consider the axisymmetric collision of two such waves, viewed in the "boosted frame" in which the waves have energies $\nu=\mu e^{\alpha}$ and $\lambda=\mu e^{-\alpha}$, and in which the precollision metric is given by

$$\begin{aligned}ds^2=&d\hat{u}\,d\hat{v}+[1+4\nu\hat{u}\,\theta(\hat{u})\hat{\rho}^{-2}]^2d\hat{\rho}^2\\&+[-8\lambda\hat{v}\theta(-\hat{v})\hat{\rho}^{-2}+16\lambda^2\hat{v}^{\,2}\theta(-\hat{v})\hat{\rho}^{-4}]d\hat{\rho}^2\\&+\hat{\rho}^2[1-4\nu\hat{u}\,\theta(\hat{u})\hat{\rho}^{-2}]^2d\phi^2\\&+\hat{\rho}^2[8\lambda\hat{v}\theta(-\hat{v})\hat{\rho}^{-2}+16\lambda^2\hat{v}^{\,2}\theta(-\hat{v})\hat{\rho}^{-4}]d\phi^2\ .\end{aligned} \tag{2.9}$$

(I3.5′)

Let us denote by $g_{ab}(\nu,\lambda,\hat{X})$ this explicit form (2.9) of the precollision metric in the boosted frame $[\hat{X}\equiv(\hat{u},\hat{v},\hat{\rho},\phi)]$. From Eqs. (2.7) and (2.8), we see that $g_{ab}(\nu,\lambda,\hat{X})$ (and hence also the initial data on $\hat{u}=0^+$) transforms as

$$g_{ab}(\nu,\lambda,\hat{X})\xrightarrow{CL}e^{2\beta}g_{ab}(\nu,\lambda e^{-2\beta},\hat{X}\,') \tag{2.10}$$

under CL. The map CL has a natural continuation into the region in $\hat{u}>0$ where the weak shock appears a small perturbation to the flat background of the strong shock: namely, (2.6) with the coordinates being those of the strong shock background. In the uncareted coordinate system, which is related to the careted system through Eq. (2.2) with μ replaced by ν, the metric possesses the perturbation expansion (1.1):

$$g_{ab}(X)\sim\nu^2\left[\eta_{ab}+\sum_{i=1}^{\infty}\left[\frac{\lambda}{\nu}\right]^i h_{ab}^{(i)}(X)\right]\ . \tag{2.11}$$

(The coordinates have been rendered dimensionless, using $X\to X/\nu$ [Eq. (I3.11)], to obtain Eq. (2.11).) Since the metric to the future of the strong shock is determined solely by the initial data on $\hat{u}=0^+$, and since this initial data transforms as (2.10), the effect of CL on (2.11) is

$$\nu^2\left[\eta_{ab}+\sum_{i=1}^{\infty}\left[\frac{\lambda}{\nu}\right]^i h_{ab}^{(i)}(X)\right]\xrightarrow{CL}e^{2\beta}\nu^2\left[\eta_{ab}+\sum_{i=1}^{\infty}\left[\frac{\lambda e^{-2\beta}}{\nu}\right]^i h_{ab}^{(i)}(X')\right] \tag{2.12}$$

(where the explicit forms of the $h_{ab}^{(i)}$ are identical in the two expansions). Hence the transformation CL does not effect the intrinsic nature of the perturbation problem: it merely alters the value of the perturbation parameter. In other words, the space-time possesses a conformal symmetry at each order in perturbation theory:

$$h_{ab}^{(i)}(X)\xrightarrow{CL}e^{2(i-1)\beta}h_{ab}^{(i)}(X') \tag{2.13}$$

(where of course, $X\xrightarrow{CL}X'$).

We can use Eq. (2.13) to determine something of the behavior of the $h_{ab}^{(i)}(X)$. From Eq. (2.6) we deduce

$$\begin{aligned}&g_{\hat{u}'\hat{u}'}=e^{4\beta}g_{\hat{u}\hat{u}},\quad g_{\hat{u}'\hat{\rho}'}=e^{3\beta}g_{\hat{u}\hat{\rho}}\ ,\\&g_{\{\hat{\rho}'\hat{\rho}'\}\{\hat{u}'\hat{v}'\}}=e^{2\beta}g_{\{\hat{\rho}\hat{\rho}\}\{\hat{u}\hat{v}\}}\ ,\\&g_{\hat{v}'\hat{\rho}'}=e^{\beta}g_{\hat{v}'\hat{\rho}}\ ,\quad g_{\{\hat{v}'\hat{v}'\}\{\phi'\phi'\}}=g_{\{\hat{v}\hat{v}\}\{\phi\phi\}}\ .\end{aligned} \tag{2.14}$$

Using the coordinate transformation of Eq. (2.2) with μ replaced by ν, we can show that identical relationships hold between the uncareted coordinate systems:

$$\begin{aligned}&g_{u'u'}=e^{4\beta}g_{uu}\ ,\quad g_{u'\rho'}=e^{3\beta}g_{u\rho}\ ,\\&g_{\{\rho'\rho'\}\{u'v'\}}=e^{2\beta}g_{\{\rho\rho\}\{uv\}}\ ,\\&g_{v'\rho'}=e^{\beta}g_{v\rho},\quad g_{\{v'v'\}\{\phi'\phi'\}}=g_{\{vv\}\{\phi\phi\}}\ .\end{aligned} \tag{2.15}$$

Combining Eq. (2.15) with Eq. (2.13), we deduce

$$\begin{aligned} h_{uu}^{(i)}(X') &= e^{2(1+i)\beta} h_{uu}^{(i)}(X) , \\ h_{u\rho}^{(i)}(X') &= e^{(2i+1)\beta} h_{u\rho}^{(i)}(X) , \\ h_{\{\substack{\rho\rho \\ uv}\}}^{(i)}(X') &= e^{2\beta i} h_{\{\substack{\rho\rho \\ uv}\}}^{(i)}(X) , \\ h_{v\rho}^{(i)}(X') &= e^{(2i-1)\beta} h_{v\rho}^{(i)}(X) , \\ h_{\{\substack{vv \\ \phi\phi}\}}^{(i)}(X') &= e^{(2i-2)\beta} h_{\{\substack{vv \\ \phi\phi}\}}^{(i)}(X) . \end{aligned} \tag{2.16}$$

From Eq. (2.6) we note that the values of $\hat{v}, \hat{u}\hat{\rho}^{-2}$ and ϕ are left unchanged by the map CL. Using Eq. (2.2) with μ replaced by ν we can show that the corresponding combinations of uncareted coordinates that are left invariant by CL are

$$r \equiv 8\ln(\nu\rho) - \sqrt{2}v , \quad q \equiv u\rho^{-2} , \phi \tag{2.17}$$

[where we have removed a factor of ν from the coordinates, as in Eq. (I3.11), and redefined u and v by $u = (1/\sqrt{2})(z+t)$ and $v = (1/\sqrt{2})(z-t)$]. The lines on which q, r, and ϕ are constant may be interpreted geometrically as the orbits of the conformal symmetry CL.

If we express each $h_{ab}^{(i)}$ as $h_{ab}^{(i)}(q,r,\rho)$ (ϕ is ignorable) then the only coordinate that changes in value when CL is applied is $\rho (\rho \xrightarrow{CL} \rho' = e^{-\beta}\rho)$. Used in conjunction with Eq. (2.16) this tells us that

$$\begin{aligned} h_{uu}^{(i)} &= fn(q,r)\rho^{-(2i+2)} , \\ h_{u\rho}^{(i)} &= fn(q,r)\rho^{-(2i+1)} , \\ h_{\{\substack{\rho\rho \\ uv}\}} &= fn(q,r)\rho^{-2i} , \\ h_{v\rho}^{(i)} &= fn(q,r)\rho^{-(2i-1)} , \\ h_{\{\substack{vv \\ \phi\phi}\}}^{(i)} &= fn(q,r)\rho^{-(2i-2)} . \end{aligned} \tag{2.18}$$

Thus each metric perturbation has a very simple dependence on ρ.

In an appropriate gauge the field equations for the $h_{ab}^{(i)}$ are all of the form $\Box h_{ab}^{(i)} = S_{ab}^{(i)}$, where $S_{ab}^{(i)}$ is a function of $h_{ab}^{(i-1)}, \dots, h_{ab}^{(1)}$ and their derivatives ($S_{ab}^{(1)} = 0$). Since each $h_{ab}^{(i)}$ is of the form $fn(q,r)\rho^{-k}$, its corresponding $S_{ab}^{(i)}$ must be of the form $fn(q,r)\rho^{-(k+2)}$. This indicates that we can eliminate ρ from the field equations by separation of variables, thereby reducing them to two-dimensional differential equations.

Let us now perform the reduction to two dimensions explicitly, starting with the first-order perturbations $h_{ab}^{(1)}$. Consider the flat-space wave equation

$$\Box\psi \equiv 2\frac{\partial^2\psi}{\partial u\,\partial v} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{2\psi}{\partial\rho}\right] + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} = 0 , \tag{2.19}$$

where the boundary condition is

$$\psi|_{u=0} = e^{im\phi}\rho^{-n} f[8\ln(v\rho) - \sqrt{2}v] ,$$

$$f(x) = 0, \quad \forall x < 0 \tag{2.20}$$

[here m and n are integers and apart from the above restriction $f(x)$ is arbitrary]. The field equations for $h_{ab}^{(1)}$ are special cases of the general system (2.19) and (2.20). We know from our preceding arguments that ψ must be of the form $e^{im\phi}\rho^{-n}\chi(q,r)$ in $u \geq 0$ [where q and r are defined in Eq. (2.17)]. From Eq. (2.17) we find

$$\begin{aligned} \left[\frac{\partial}{\partial u}\right]_{v,\rho,\phi} &= \frac{1}{\rho^2}\left[\frac{\partial}{\partial q}\right]_{r,\rho,\phi} , \\ \left[\frac{\partial}{\partial v}\right]_{u,\rho,\phi} &= -\sqrt{2}\left[\frac{\partial}{\partial r}\right]_{q,\rho,\phi} , \\ \left[\frac{\partial}{\partial \rho}\right]_{u,v,\phi} &= \left[\frac{\partial}{\partial \rho}\right]_{q,r,\phi} - \frac{2q}{\rho}\left[\frac{\partial}{\partial q}\right]_{r,\rho,\phi} - \frac{8}{\rho}\left[\frac{\partial}{\partial r}\right]_{q,\rho,\phi} , \end{aligned} \tag{2.21}$$

and therefore

$$2\frac{\partial^2}{\partial u\,\partial v} + \frac{1}{\rho}\frac{\partial}{\partial\rho}\left[\rho\frac{\partial}{\partial\rho}\right] + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} = \frac{1}{\rho^2}\left[-2\sqrt{2}\frac{\partial^2}{\partial q\,\partial r} + \left[\rho\frac{\partial}{\partial\rho} - 2q\frac{\partial}{\partial q} + 8\frac{\partial}{\partial r}\right]\left[\rho\frac{\partial}{\partial\rho} - 2q\frac{\partial}{\partial q} + 8\frac{\partial}{\partial r}\right] + \frac{\partial^2}{\partial\phi^2}\right] . \tag{2.22}$$

Thus χ is the solution to

$$\mathcal{L}_{m,n}\chi \equiv \left[-2\sqrt{2}\frac{\partial^2}{\partial q\,\partial r} + \left[-n - 2q\frac{\partial}{\partial q} + 8\frac{\partial}{\partial r}\right]\left[-n - 2q\frac{\partial}{\partial q} + 8\frac{\partial}{\partial r}\right] - m^2\right]\chi = 0 , \tag{2.23}$$

where the boundary condition is $\chi|_{q=0} = f(r)$.

Of course, for the homogeneous wave equation (2.19), where the solution has the simple integral form given in Eq. (I4.5) [8,9], there is really no point in eliminating ρ and ϕ from the differential equation. However, consider the field equation for any one of the higher-order metric coefficients (i.e., $h_{ab}^{(i)}$, $i \geq 2$). It is an inhomogeneous flat-space wave equation

$$\Box\psi = S , \tag{2.24}$$

in which the source term is $S = e^{im\phi}\rho^{-(n+2)}H(q,r)$. [The boundary condition may be taken to be $\psi|_{u=0} = 0$, since any contribution to the solution from nonzero boundary conditions can be evaluated separately using Eq. (I4.5).] In contrast with the homogeneous case, the benefits to be gained by reducing Eq. (2.24) to

$$\mathcal{L}_{m,n}\chi=H , \tag{2.25}$$

(where, of course, $\psi=e^{im\phi}\rho^{-n}\chi$) are not insignificant. First, the geometrical configuration of the problem is now much easier to visualize. Previously, to calculate the solution at some space-time point P we would have had to integrate the source term S (suitably weighted) over the past null cone of P. Now we need simply to integrate the product of H and the Green's function for the differential operator $\mathcal{L}_{m,n}$ over some two-dimensional region in the (q,r) plane. This makes it much easier to estimate the various contributions to the solution from different parts of the integration region. Second, although we must now find and calculate the Green's function for $\mathcal{L}_{m,n}$, it turns out that there is a considerable computational gain which makes the numerical calculation of the solution practicable, whereas before it would have required a prohibitive amount of computer time.

III. THE REDUCED DIFFERENTIAL EQUATION

We shall now demonstrate that the differential operator $\mathcal{L}_{m,n}$ is hyperbolic and find its characteristics. Define new coordinates

$$\xi=\xi(q,r), \quad \eta=\eta(q,r) . \tag{3.1}$$

Now,

$$\mathcal{L}_{m,n}=-(2\sqrt{2}+32q)\frac{\partial^2}{\partial q\partial r}+4q^2\frac{\partial^2}{\partial q^2}+64\frac{\partial^2}{\partial r^2}+\cdots , \tag{3.2}$$

where the terms omitted are first and zeroth order in $\partial/\partial q$ and $\partial/\partial r$. We wish to choose ξ and η so that $\mathcal{L}_{m,n}$ is transformed to normal hyperbolic form [10], in which

$$L_{m,n}=f(\xi,\eta)\frac{\partial^2}{\partial\xi\partial\eta}+\cdots , \tag{3.3}$$

where the terms omitted are now of first and zeroth order in $\partial/\partial\xi$ and $\partial/\partial\eta$. Expressing $\mathcal{L}_{m,n}$ in terms of $\partial/\partial\xi$ and $\partial/\partial\eta$ we find that

$$\begin{aligned}\mathcal{L}_{m,n}=&\left[-(2\sqrt{2}+32q)\left[\frac{\partial\xi}{\partial q}\right]\left[\frac{\partial\xi}{\partial r}\right]+4q^2\left[\frac{\partial\xi}{\partial q}\right]^2+64\left[\frac{\partial\xi}{\partial r}\right]^2\right]\frac{\partial^2}{\partial\xi^2}\\&+\left[-(2\sqrt{2}+32q)\left[\frac{\partial\eta}{\partial q}\right]\left[\frac{\partial\eta}{\partial r}\right]+4q^2\left[\frac{\partial\eta}{\partial q}\right]^2+64\left[\frac{\partial\eta}{\partial r}\right]^2\right]\frac{\partial^2}{\partial\eta^2}\\&+\left\{-(2\sqrt{2}+32q)\left[\left[\frac{\partial\xi}{\partial q}\right]\left[\frac{\partial\eta}{\partial r}\right]+\left[\frac{\partial\xi}{\partial r}\right]\left[\frac{\partial\eta}{\partial q}\right]\right]+8q^2\left[\frac{\partial\xi}{\partial q}\right]\left[\frac{\partial\eta}{\partial q}\right]+128\left[\frac{\partial\xi}{\partial r}\right]\left[\frac{\partial\eta}{\partial r}\right]\right\}\frac{\partial^2}{\partial\eta\partial\xi}+\cdots .\end{aligned} \tag{3.4}$$

In order that Eq. (3.3) be satisfied, we must have

$$-(2\sqrt{2}+32q)\left[\frac{\partial\eta}{\partial q}\right]\left[\frac{\partial\eta}{\partial r}\right]+4q^2\left[\frac{\partial\eta}{\partial q}\right]^2+64\left[\frac{\partial\eta}{\partial r}\right]^2=0 ,$$

$$-(2\sqrt{2}+32q)\left[\frac{\partial\xi}{\partial q}\right]\left[\frac{\partial\xi}{\partial r}\right]+4q^2\left[\frac{\partial\xi}{\partial q}\right]^2+64\left[\frac{\partial\xi}{\partial r}\right]^2=0 . \tag{3.5}$$

In other words, $(\partial\xi/\partial q)/(\partial\xi/\partial r)$ and $(\partial\eta/\partial q)/(\partial\eta/\partial r)$ must be the two real roots of the quadratic equation

$$4q^2x^2-(2\sqrt{2}+32q)x+64=0 . \tag{3.6}$$

The discriminant of this quadratic is positive, so $\mathcal{L}_{m,n}$ is hyperbolic, and its characteristic coordinates ξ and η satisfy

$$\left[\frac{\partial\xi}{\partial q}\right]=\left[\frac{1+8\sqrt{2}q+\sqrt{(1+16\sqrt{2}q)}}{2\sqrt{2}q^2}\right]\left[\frac{\partial\xi}{\partial r}\right] \tag{3.7}$$

and

$$\left[\frac{\partial\eta}{\partial q}\right]=\left[\frac{1+8\sqrt{2}q-\sqrt{(1+16\sqrt{2}q)}}{2\sqrt{2}q^2}\right]\left[\frac{\partial\eta}{\partial r}\right], \tag{3.8}$$

where we have arbitrarily assigned the plus sign to ξ and the minus sign to η. For ease of calculation we now choose

$$\frac{\partial\xi}{\partial r}=1, \quad \frac{\partial\eta}{\partial r}=1 . \tag{3.9}$$

Solving Eqs. (3.7) and (3.8), subject to Eq. (3.9), we find

$$\xi=r+8\ln\left[\frac{\sqrt{(1+16\sqrt{2}q)}-1}{2}\right]-\frac{8}{[\sqrt{(1+16\sqrt{2}q)}-1]}-4 \tag{3.10}$$

and

$$\eta = r + 8\ln\left[\frac{\sqrt{(1+16\sqrt{2}q)}+1}{2}\right] + \frac{8}{[\sqrt{(1+16\sqrt{2}q)}+1]} - 4 , \tag{3.11}$$

where the constants of integration have been chosen for future convenience. The characteristics of $\mathcal{L}_{m,n}$ are the two families of lines

$$\xi = \text{const}, \quad \eta = \text{const} . \tag{3.12}$$

They play the usual role of limiting the speed of propagation of information, so that a point A cannot be influenced by another point B, if B lies outside the region bounded by the two past characteristics through A.

The explicit forms (3.10) and (3.11) for ξ and η have the following simple geometrical interpretations. In our dimensionless coordinates (u,v,ρ,ϕ), the null geodesic generators of the weak shock have the parametric representation (I3.20). That is,

$$\begin{aligned} u &= \sqrt{2}\Lambda , \\ v &= 4\sqrt{2}\ln(\nu\rho_0) - \frac{16\sqrt{2}\Lambda}{(\rho_0)^2} , \\ x &= x_0\left[1-\frac{8\Lambda}{(\rho_0)^2}\right] , \\ y &= y_0\left[1-\frac{8\Lambda}{(\rho_0)^2}\right] , \end{aligned} \tag{3.13}$$

(I3.20′)

where x and y are the usual Cartesian coordinates, so that $\rho=(x^2+y^2)^{1/2}$ and $\Lambda \ge 0$ is an affine parameter along each of the null geodesics. Let us now find the locus of intersection of these null geodesics with a surface S of constant ρ and ϕ, on which

$$\rho = k, \quad \phi = \phi_0 . \tag{3.14}$$

Consider a geodesic which comes through the collision surface at $\rho=\rho_0$, $\phi=\pi+\phi_0$. This geodesic will pass through the caustic at $\rho=0$ before hitting S (ϕ jumps from $\pi+\phi_0$ at $\rho=0^-$ to ϕ_0 at $\rho=0^+$). Hence, at its intersection with S,

$$\rho_0\left[\frac{8\Lambda}{(\rho_0)^2}-1\right] = k, \quad \phi=\phi_0 . \tag{3.15}$$

Solving for Λ we find

$$\Lambda = \left[1+\frac{k}{\rho_0}\right]\frac{(\rho_0)^2}{8} . \tag{3.16}$$

Substituting this into Eq. (3.13), we find that, at the point of intersection,

$$\begin{aligned} &\rho = k, \quad \phi=\phi_0 , \\ &u = \frac{(\rho_0)^2}{4\sqrt{2}}\left[1+\frac{k}{\rho_0}\right] , \\ &v = 4\sqrt{2}\ln(\nu\rho_0) - 2\sqrt{2}\left[1+\frac{k}{\rho_0}\right] . \end{aligned} \tag{3.17}$$

Expressing Eq. (3.17) in terms of r and q one finds

$$\begin{aligned} q &= \frac{1}{4\sqrt{2}}\left[\frac{k}{\rho_0}+1\right]\left[\frac{\rho_0}{k}\right]^2 , \\ r &= 8\ln\left[\frac{k}{\rho_0}\right] + 4\left[\frac{k}{\rho_0}+1\right] . \end{aligned} \tag{3.18}$$

Now by eliminating ρ_0/k from Eq. (3.18) it is easy to show that

$$r + 8\ln\left[\frac{\sqrt{(1+16\sqrt{2}q)}-1}{2}\right] - \frac{8}{[\sqrt{(1+16\sqrt{2}q)}-1]} - 4 = 0 \tag{3.19}$$

at the geodesic's intersection with S (here $q>0$).

Now consider a geodesic generator which originates at $\rho=\rho_0 \ge k$, $\phi=\phi_0$. A geodesic of this type will hit S *before* passing through the caustic. By following a similar argument to that of Eqs. (3.15)—(3.19) it is easy to show that, at the point of intersection,

$$r + 8\ln\left[\frac{\sqrt{(1+16\sqrt{2}q)}+1}{2}\right] + \frac{8}{[\sqrt{(1+16\sqrt{2}q)}+1]} - 4 = 0 , \tag{3.20}$$

where again $q>0$.

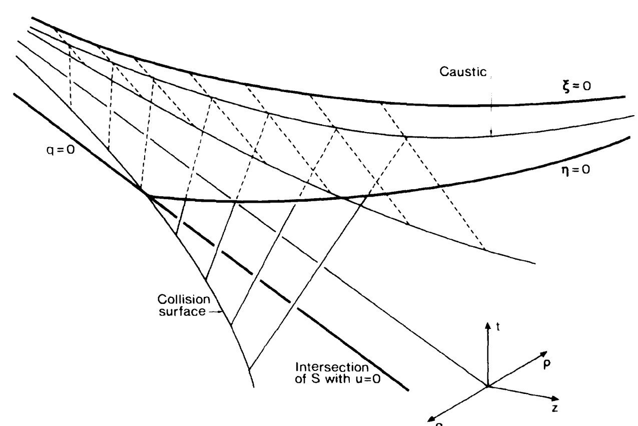

FIG. 1. The curved shock 2 is depicted, as viewed from the Minkowskian region III to its past, which lies above the incoming plane shock 1 ($u=0$). The heavy black lines all lie in the surface $S(\rho=k,\ \phi=\phi_0)$. The lines $\xi=0$ and $\eta=0$ mark the intersection of the null geodesic generators of shock 2 with S. The generators are drawn bold on the near side of S, and dashed on the far side.

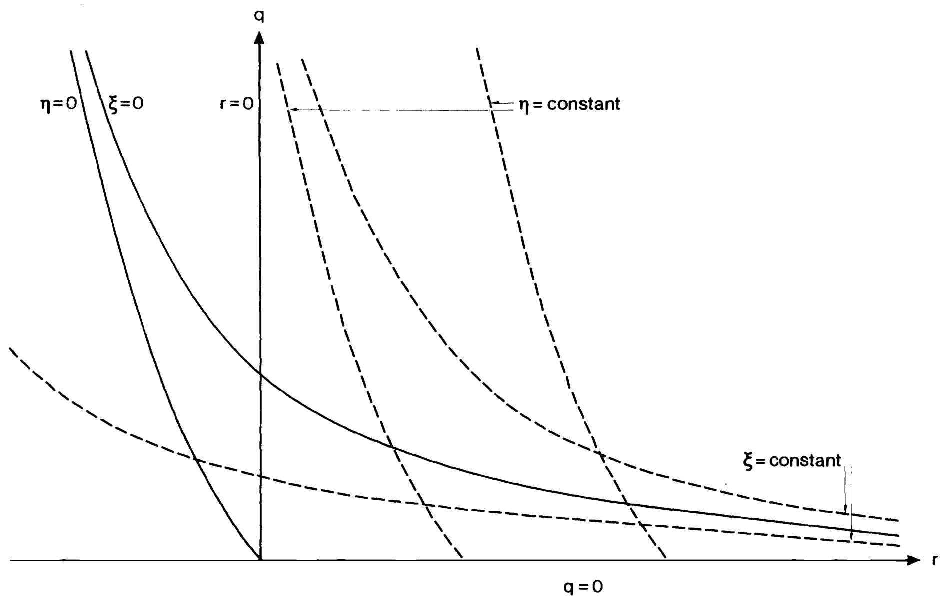

FIG. 2. Characteristics ξ=const and η=const are shown in the (q,r) plane. The lines $\xi=0$ and $\eta=0$ represent the curved surface of the weak shock 2. The caustic region has been mapped to their "intersection" at infinite q. In $\eta<0$, which corresponds to the flat space-time region III underneath the curved shock, all the metric perturbations vanish. Nonzero initial data are set on the characteristic surface $q=0$ for $r>0$. The gravitational radiation is found from the metric perturbations in the region immediately surrounding $\xi=0$, in the limit $r \to \infty$.

Thus in the near-field region, before it passes through the caustic, the weak shock intersects S at the line $\eta=0$ (see Fig. 1). This line also marks the boundary of the region between the weak shock and the collision surface, underneath the caustic, in which space-time is flat and all the metric perturbations are zero.

The line $\xi=0$, though, marks the intersection of the weak shock with the surface S after it has passed through the caustic and is propagating out towards null infinity (again see Fig. 1). However, we saw in Secs. III and IV of paper I that where the null geodesic generators of the weak shock intersect $\mathcal{J}^+$ the first-order news function has a logarithmic singularity, and that the news function is only significantly nonzero in the region immediately surrounding the weak shock. In our two-dimensional (q,r) plane this region is that in which $|\xi|$ is small and $r\to\infty$, $q\to 0$ (see Fig. 2). Thus out near null infinity the pulse of gravitational radiation is centered around $\xi=0$, and dies away asymptotically as we go far away on either side of this line. ξ is therefore a measure of retarded time.

The other characteristics (3.12) are obtained from $\xi=0$ and $\eta=0$ simply by translating these lines parallel to the q axis. The line $q=0$ is itself a characteristic, and so the boundary condition $\chi|_{q=0}=f(r)$ is sufficient to determine the solution uniquely in $q>0$.

IV. THE GREEN'S FUNCTION FOR THE REDUCED EQUATION

Let us now find the Green's function for the differential operator $\mathcal{L}_{m,n}$. It is defined by [10]

$$\mathcal{L}_{m,n}G_{m,n}(q,r;q_0,r_0)=\delta(q-q_0)\delta(r-r_0)\ , \qquad (4.1)$$

where $\mathcal{L}_{m,n}$ acts on the (q,r) part of $G_{m,n}$. Expressed in terms of the Green's function $G_{m,,n}$, the explicit solution to Eq. (2.25) at a point (q,r) is

$$\chi(q,r)=\int\int G_{m,n}(q,r;q_0,r_0)H(q_0,r_0)dq_0dr_0\ , \qquad (4.2)$$

subject to suitable boundary conditions.

To solve Eq. (4.1) we use a method of reduction. Because $\mathcal{L}_{m,n}$ is derived from the flat space d'Alembertian $\Box$ in the manner described in Sec. II, it is clear that $G_{m,n}$ will satisfy

$$\Box[e^{im\phi}\rho^{-n}G_{m,n}(q,r;q_0,r_0)]$$
$$=e^{im\phi}\rho^{-(n+2)}\delta(q-q_0)\delta(r-r_0)\ . \qquad (4.3)$$

We can solve Eq. (4.3) using the flat-space Green's is function $G(t,\mathbf{x};t_0,\mathbf{x}_0)$, whose explicit form is

$$G(t,\mathbf{x};t_0,\mathbf{x}_0)=\frac{-1}{4\pi}\frac{\delta(t-t_0-|\mathbf{x}-\mathbf{x}_0|)}{|\mathbf{x}-\mathbf{x}_0|}\ , \qquad (4.4)$$

where t and $\mathbf{x}$ have their usual meanings as time and position vector, respectively. [In choosing the retarded flat-space Green's function, we ensure that $G_{m,n}$ will be the retarded Green's function of $\mathcal{L}_{m,n}$]. Thus

$$e^{im\phi}\rho^{-n}G_{m,n}(q,r;q_0,r_0)=\frac{-1}{4\pi}\int\int_{u_1>0}\int_0^\infty\int_0^{2\pi}\frac{\delta[t-t_1-\sqrt{(z-z_1)^2+\rho^2-2\rho\rho_1\cos(\phi-\phi_1)+\rho_1^2}]}{\sqrt{(z-z_1)^2+\rho^2-2\rho\rho_1\cos(\phi-\phi_1)+\rho_1^2}}$$
$$\times e^{im\phi_1}\rho_1^{-(n+2)}\delta(q_1-q_0)\delta(r_1-r_0)\rho_1 dt_1 dz_1 d\rho_1 d\phi_1$$
$$=\frac{-e^{im\phi}}{4\pi}\int\int_{u_1>0}\int_0^\infty\int_0^{2\pi}\frac{\delta[t-t_1-\sqrt{(z-z_1)^2+\rho^2-2\rho\rho_1\cos\omega+\rho_1^2}]}{\sqrt{(z-z_1)^2+\rho^2-2\rho\rho_1\cos\omega+\rho_1^2}}$$
$$\times\cos(m\omega)\rho_1^{-(n+1)}\delta(q_1-q_0)\delta(r_1-r_0)dt_1 dz_1 d\rho_1 d\omega\ . \tag{4.5}$$

There is no $\sin(m\omega)$ term present in the second multiple integral of Eq. (4.5) because such a term clearly integrates to zero

If $f(\omega)$ is a function with n simple zeros $\omega_1,\ldots,\omega_n$, then

$$\int_{-\infty}^{\infty}\delta[f(\omega)]g(\omega)d\omega=\sum_{i=1}^{n}g(\omega_i)/|f'(\omega_i)|\ . \tag{4.6}$$

In Eq. (4.5) let us first integrate over ω. This involves evaluating an integral of the form (4.6). For the particular case (4.5),

$$f(\omega)=t-t_1-\sqrt{(z-z_1)^2+\rho^2-2\rho\rho_1\cos\omega+\rho_1^2} \tag{4.7}$$

and

$$g(\omega)=\frac{\cos(m\omega)}{\sqrt{(z-z_1)^2+\rho^2-2\rho\rho_1\cos\omega+\rho_1^2}}\ . \tag{4.8}$$

As a convenient shorthand, and for reasons which will become apparent, define $\cos\Omega_1$ by

$$\cos\Omega_1=\frac{(z-z_1)^2+\rho^2+\rho_1^2-(t-t_1)^2}{2\rho\rho_1}\ . \tag{4.9}$$

If the space-time coordinates (t,z,ρ) and (t_1,z_1,ρ_1) in Eq. (4.9) are such that $|\cos\Omega_1|>1$ then $f(\omega)$ will have no zero. On the other hand, if $|\cos\Omega_1|\le 1$ then $f(\omega)$ will have zeros at

$$\omega=\pm\arccos(\cos\Omega_1)\ . \tag{4.10}$$

We deduce that

$$e^{im\phi}\rho^{-n}G_{m,n}(q,r;q_0,r_0)=\frac{-e^{im\phi}}{2\pi}\int\int_{u_1>0}\int_0^\infty\frac{\cos(m\Omega_1)}{\rho\rho_1\sin\Omega_1}\theta(1-|\cos\Omega_1|)\rho_1^{-(n+1)}\delta(q_1-q_0)\delta(r_1-r_0)dt_1 dz_1 d\rho_1\ , \tag{4.11}$$

where $\cos(m\Omega_1)$ and $\sin\Omega_1$ are related to $\cos\Omega_1$ through the standard trigonometric formulae and $\theta(x)$ is the Heaviside function, defined by

$$\theta(x)=\begin{cases}1, & x\ge 0,\\ 0, & x<0\ .\end{cases} \tag{4.12}$$

Now reexpressing $\cos\Omega_1$, first in terms of (u,v,ρ) and then (q,r,ρ), we find that

$$\cos\Omega_1=\frac{2(u-u_1)(v-v_1)+\rho^2+\rho_1^2}{2\rho\rho_1}$$
$$=\frac{1}{\sqrt{2}}\left[\frac{\rho}{\rho_1}\right]\left[q-q_1\left[\frac{\rho_1}{\rho}\right]^2\right]$$
$$\times\left[8\ln\left[\frac{\rho}{\rho_1}\right]-(r-r_1)\right]+\frac{1}{2}\left[\frac{\rho}{\rho_1}+\frac{\rho_1}{\rho}\right]\ . \tag{4.13}$$

Also

$$dt_1 dz_1=\frac{\rho_1^2}{\sqrt{2}}dq_1 dr_1\ . \tag{4.14}$$

Integrating out the two remaining delta functions in Eq. (4.11), we find

$$e^{im\phi}\rho^{-n}G_{m,n}(q,r;q_0,r_0)$$
$$=\frac{-e^{im\phi}}{2\sqrt{2}\pi\rho}\int_0^\infty\frac{\cos(m\Omega_0)}{\sin\Omega_0}\theta(1-|\cos\Omega_0|)\rho_1{}^{-n}d\rho_1\ , \tag{4.15}$$

where $\cos\Omega_0$ is defined as in Eq. (4.9), except that (t_1,z_1,ρ_1) is replaced by (t_0,z_0,ρ_0). Making the substitution $\rho_1=y\rho$ in Eq. (4.15) we find that

$$G_{m,n}(q,r;q_0,r_0)=\frac{-1}{2\sqrt{2}\pi}\int_0^\infty\frac{\cos(m\epsilon)}{\sin\epsilon}\theta(1-|\cos\epsilon|)\frac{dy}{y^n}\ , \tag{4.16}$$

where now

$$\cos\epsilon=\frac{1}{2y}[1+y^2-\sqrt{2}(q-q_0y^2)(8\ln y+r-r_0)]\ . \tag{4.17}$$

One can show that $G_{m,n}$ does vanish outside the region bounded by the past-directed characteristics through (q,r), as it should.

V. THE SECOND-ORDER TRANSVERSE METRIC FUNCTIONS IN TWO-DIMENSIONAL FORM

Let

$$e=\rho^2E,\quad b=\rho^3B,\quad a=\rho^4A\ , \tag{5.1}$$

where E, B and A are the first-order metric functions defined by

$$h_{uu}^{(1)}=A,\quad h_{vv}^{(1)}=0\ ,$$
$$h_{xx}^{(1)}=-h_{yy}^{(1)}=(y^2-x^2)\rho^{-2}E\ ,$$
$$h_{uv}^{(1)}=0\ ,\quad h_{vx}^{(1)}=0\ , \tag{5.2}$$
$$h_{xy}^{(1)}=-2xy\rho^{-2}E\ ,\quad h_{ux}^{(1)}=x\rho^{-1}B\ ,$$
$$h_{vy}^{(1)}=0,\quad h_{uy}^{(1)}=y\rho^{-1}B\ . \tag{I3.13'}$$

When the first-order field equations are solved in harmonic (de Donder) gauge, subject to the characteristic initial data (I3.14) on $u=0^+$, one finds from Eq. (I4.18) and its obvious analogues for B and A that e, b, and a are functions only of (q,r), given by

$$\begin{aligned}
e&=\frac{-4\sqrt{2}}{\pi q}\int_0^\infty\int_0^{2\pi}\cos(2\phi')\theta(\text{Arg})\frac{dy}{y}d\phi'\ ,\\
b&=\frac{32}{\pi q}\int_0^\infty\int_0^{2\pi}\cos\phi'\theta(\text{Arg})\frac{dy}{y^2}d\phi'\ ,\\
a&=\frac{128\sqrt{2}}{\pi q}\int_0^\infty\int_0^{2\pi}\theta(\text{Arg})\frac{dy}{y^3}d\phi'\\
&\quad-\frac{32}{\pi q^2}\int_0^\infty\int_0^{2\pi}\text{Arg}\,\theta(\text{Arg})\frac{dy}{y^3}d\phi'\ ,
\end{aligned} \tag{5.3}$$

where $\theta(x)$ is Heaviside's function and

$$\text{Arg}=\sqrt{2}q(8\ln y+r)-(1+y^2)+2y\cos\phi'\ . \tag{5.4}$$

Also define $d^{(2)}$ and $e^{(2)}$ by

$$d^{(2)}=\rho^4D^{(2)},\quad e^{(2)}=\rho^4E^{(2)}\ , \tag{5.5}$$

where $D^{(2)}$ and $E^{(2)}$ are the second-order transverse metric functions introduced in Eq. (I5.16):

$$h_{xx}^{(2)}=D^{(2)}+(y^2-x^2)\rho^{-2}E^{(2)}\ ,$$
$$h_{yy}^{(2)}=D^{(2)}+(x^2-y^2)\rho^{-2}E^{(2)}\ , \tag{5.6}$$
$$h_{xy}^{(2)}=-2xy\rho^{-2}E^{(2)}\ . \tag{I5.16'}$$

Here $d^{(2)}$ and $e^{(2)}$ are functions of q and r only. We recall that the second-order news function in the boosted frame is defined in terms of $D^{(2)}$ and $E^{(2)}$ through

$$c_0^{(2)}=-\frac{1}{2}\left[\frac{\lambda}{\nu}\right]^2\lim_{|\mathbf{r}|\to\infty}\left[|\mathbf{r}|\frac{\partial}{\partial\tau}(D^{(2)}+E^{(2)})\right]\ , \tag{5.7}$$
$$\tag{I5.18'}$$

once the spurious gauge terms contributing to this equation are eliminated by transforming to Bondi coordinates (Secs. VI and VII). In harmonic gauge, the second-order metric functions $D^{(2)}$ and $E^{(2)}$ obey the inhomogeneous flat-space wave equations (I5.19) and (I5.20). On reduction, these imply that

$$\begin{aligned}
\mathcal{L}_{0,4}d^{(2)}&=4\sqrt{2}be_{,r}+2\sqrt{2}qbe_{,qr}-8\sqrt{2}qbe_{,rr}-(b_{,r})^2+2\sqrt{2}qb_{,q}e_{,r}-8\sqrt{2}e_{,r}b_{,r}-2\sqrt{2}e_{,q}e_{,r}\\
&\quad+4e^2+4qee_{,q}-16ee_{,r}-32qe_{,q}e_{,r}+4q^2(e_{,q})^2+64(e_{,r})^2\equiv S(q,r)
\end{aligned} \tag{5.8}$$

and

$$\begin{aligned}
\mathcal{L}_{2,4}e^{(2)}&=2ae_{,rr}-2\sqrt{2}be_{,r}+2\sqrt{2}qbe_{,qr}-8\sqrt{2}be_{,rr}-2\sqrt{2}qe_{,r}b_{,q}+8\sqrt{2}e_{,r}b_{,r}+(b_{,r})^2\\
&\quad-12qee_{,q}+32ee_{,r}-4q^2ee_{,qq}+32qee_{,rq}-64ee_{,rr}\equiv T(q,r)\ .
\end{aligned} \tag{5.9}$$

From Eq. (I5.17) we find that the boundary conditions on $d^{(2)}$ and $e^{(2)}$ are

$$d^{(2)}|_{q=0}=16r^2\theta(r),\quad e^{(2)}|_{q=0}=0\ . \tag{5.10}$$

It is not difficult to show that the contribution to $d^{(2)}$ from this surface term is

$$d_{\text{surf}}^{(2)}=\frac{16}{\pi q^2}\int_0^\infty\int_0^{2\pi}\text{Arg}\,\theta(\text{Arg})\frac{dy}{y^3}d\phi'\ , \tag{5.11}$$

where Arg is as in Eq. (5.4).

The Green function for $d^{(2)}$ is, from Eq. (4.16),

$$G_{\rm I}(q,r;q_0,r_0)=\frac{-1}{2\sqrt{2}\pi}\int_0^\infty \theta(1-|\cos\epsilon|)\frac{dy}{y^4\sin\epsilon} \qquad (5.12)$$

and that for $e^{(2)}$,

$$G_{\rm II}(q,r;q_e,r_0)=\frac{-1}{2\sqrt{2}\pi}\int_0^\infty \theta(1-|\cos\epsilon|)\frac{\cos(2\epsilon)dy}{y^4\sin\epsilon}\ , \qquad (5.13)$$

where, from Eq. (4.17),

$$\cos\epsilon=\frac{1}{2y}[1+y^2-\sqrt{2}(q-q_0y^2)(8\ln y+r-r_0)]\ . \qquad (5.14)$$

We saw in Sec. III that to calculate the news function we must take the field point (q,r) out to the region where r is very large and $|\xi|$ is of order 1 (and so q is small). From the definition (3.10) for ξ we find

$$\xi=r+8\ln(4\sqrt{2}q)-\frac{1}{\sqrt{2}q}-8+O(q)\ . \qquad (5.15)$$

Now let $x=y/4\sqrt{2}q$. Then when x, q_0 and r_0 are of order 1 and r is very large,

$$\cos\epsilon=\frac{1}{8\sqrt{2}qx}\left[1+32q^2x^2-\sqrt{2}(q-32x^4q^2q_0)\left[8\ln x+\frac{1}{\sqrt{2}q}+8+\xi-r_0+O(q)\right]\right]$$
$$=\frac{16\sqrt{2}q_0x^2-(8\ln x+\xi-r_0+8)}{8x}+O(q)\ , \qquad (5.16)$$

and thus

$$G_{\rm I}=\frac{-1}{2\sqrt{2}\pi}\frac{1}{(4\sqrt{2}q)^3}\int_0^\infty \theta(1-|\cos\epsilon|)\frac{dx}{x^4\sin\epsilon}+O(q^{-2}) \qquad (5.17)$$

and

$$G_{\rm II}=\frac{-1}{2\sqrt{2}\pi}\frac{1}{(4\sqrt{2}q)^3}\int_0^\infty \theta(1-|\cos|\epsilon|)\frac{\cos(2\epsilon)dx}{x^4\sin\epsilon}+O(q^{-2})\ , \qquad (5.18)$$

where $\cos\epsilon$ is equal to the first term on the right-hand side of Eq. (5.16). We can in fact ignore the $O(q^{-2})$ terms in Eqs. (5.17), (5.18) since they do not contribute to the news function.

In Sec. VI of paper I we derived a mass-loss formula [Eq. (I6.27)], which showed that if the gravitational radiation obeyed certain uniformity conditions, then the mass $m_{\rm final}$ of the final black hole (assumed to be a static Schwarzschild geometry) produced by the speed-of-light collision must be $m_{\rm final}=\frac{3}{2}\mu+4\int_{-\infty}^{\infty}a_2(\tau/\mu)d\tau$. Since it is the time integral of the news function which is required in this formula, and not the news function itself, the quantity that we shall compute directly (as described further in paper III) will be the combination $d^{(2)}+e^{(2)}$ of metric components, and not its time derivative. When we require the news function in paper III, we shall differentiate numerically.

The surface term (5.11) contributes

$$\frac{\sqrt{2}}{\pi q^3}\int_0^\infty\int_0^{2\pi}(8\ln x+\xi+8+8x\cos\phi')\times\theta(8\ln x+\xi+8+8x\cos\phi')\frac{dx\,d\phi'}{x^3} \qquad (5.19)$$

(plus irrelevant higher-order terms) to $d^{(2)}+e^{(2)}$ when $r\to\infty$ with ξ constant. There is, of course, also the source term

$$\iint_{\xi_0<\xi,\eta_0<\eta,q_0>0}[G_{\rm I}(q,r;q_0,r_0)S(q_0,r_0)+G_{\rm II}(q,r;q_0,r_0)T(q_0,r_0)]dq_0dr_0 \qquad (5.20)$$

to be added to this, where the source functions S and T are defined in Eqs. (5.8) and (5.9).

VI. ELIMINATING LOGARITHMIC TERMS FROM THE SECOND-ORDER TRANSVERSE METRIC COEFFICIENTS

It has been known for a long time (see Fock [7]) that harmonic gauges are complicated by the appearance of $(\ln|\mathbf{r}|)/|\mathbf{r}|$ terms in the metric tensor at second and higher orders in perturbation theory ($|\mathbf{r}|$ is the radial coordinate). Initially it was not clear what, if any, physical significance these terms had, nor if gravitational radiation theory could be properly defined in such coordinate systems. [Naive calculations, using $dE/d\Omega\,dt=r_{\rm ret}^2/32\pi\langle h_{jk}^{TT}h_{jk}^{TT}\rangle_{\rm av}$ (see Ref. [11]), of the power radiated per unit solid angle predict an infinite quantity of gravitational radiation.] However, it has been shown by Winicour and Isacacson [12] in the axisymmetric case, and by Madore [13] for the general case, that these $(\ln|\mathbf{r}|)/|\mathbf{r}|$ terms are coordinate artifacts which can be eliminated by transforming to a Bondi gauge, so that the news function is still well defined. In this section we calculate the $(\ln|\mathbf{r}|)/|\mathbf{r}|$ term in the transverse part of the second-order metric perturbation $h_{ab}^{(2)}$, and then show how to eliminate it by finding an explicit coordinate transformation to a Bondi gauge. The $(\ln|\mathbf{r}|)/|\mathbf{r}|$ terms in the metric are produced, in the source integral (5.20), by the region of (q_0,r_0) space corresponding to a source point near future null infinity $\mathcal{J}^+$, where (in particular) $q_0\ll 1$. It is thus necessary to estimate the magnitudes of the Green's functions $G_{\rm I},G_{\rm II}$ and of the source functions $S(q_0,r_0),T(q_0,r_0)$ when (q_0,r_0) lies in this region.

From Eqs. (5.17) and (5.18):

$$\mathcal{G}_{\{I,II\}}(q,r;q_0,r_0)=\frac{-1}{2\sqrt{2}\pi}\cdot\frac{1}{(4\sqrt{2}q)^3}\int_0^{\infty}\theta(1-|\cos\epsilon|)\begin{Bmatrix}1\\ \cos 2\epsilon\end{Bmatrix}\frac{dx}{x^4\sin\epsilon} \tag{6.1}$$

plus corrections negligible in the radiation zone, where

$$\cos\epsilon=\frac{16\sqrt{2}q_0x^2-(8\ln x+\xi-r_0+8)}{8x} . \tag{6.2}$$

If q_0 is small then

$$r=\xi_0-8\ln(4\sqrt{2}q_0)+\frac{1}{\sqrt{2}q_0}+8+O(q_0) . \tag{6.3}$$

Define y by $y=4\sqrt{2}q_0x$. Then

$$G_{\{I,II\}}=\frac{-1}{2\sqrt{2}\pi}\cdot\left[\frac{q_0}{q}\right]^3\int_0^{\infty}\theta(1-|\cos\epsilon|)\begin{Bmatrix}1\\ \cos 2\epsilon\end{Bmatrix}\frac{dy}{y^4\sin\epsilon}+\frac{O(q_0^4)}{q^3} \tag{6.4}$$

where now

$$\cos\epsilon=\frac{1+y^2-\sqrt{2}q_0(8\ln y+\xi-\xi_0)}{2y} . \tag{6.5}$$

Now assume that $(\xi-\xi_0)$ is of order 1 (so that we are restricting attention to the region immediately surrounding the weak shock). Let $y=1+[\sqrt{2}q_0(\xi-\xi_0)]^{1/2}z$. Then

$$\cos\epsilon=1+\frac{q_0}{\sqrt{2}}(\xi-\xi_0)(z^2-1)+q_0^{3/2}f(z)+O(q_0^2) , \tag{6.6}$$

where $f(z)=-f(-z)$ [the explicit form of $f(z)$ may be easily found, but it is not important here]. Using Eq. (6.6) it is not difficult to show that

$$G_I=G_{II}=\frac{-1}{2\sqrt{2}}\left[\frac{q_0}{q}\right]^3+\frac{O(q_0^4)}{q^3} \tag{6.7}$$

if $\xi_0\le\xi$, while if $\xi_0>\xi$ then they both vanish [there is no $q_0^{7/2}/q^3$ term in Eq. (6.7) because $f(z)$ is odd].

Let us now examine the behavior of the source functions $S(q_0,r_0)$ and $T(q_0,r_0)$ in this region [where ξ_0 is $O(1)$ and $q_0\ll 1$] of the (q_0,r_0) plane. From Eq. (5.4) we have

$$\mathrm{Arg}=\sqrt{2}q_0(8\ln y+r_0)-(1+y^2)+2y\cos\phi' . \tag{6.8}$$

Define $\overline{\mathrm{Arg}}=\mathrm{Arg}/2q_0$ and $x=y/4\sqrt{2}\,q_0$, then using Eq. (6.3) we find

$$\overline{\mathrm{Arg}}\simeq 8\ln x+\xi_0+8+8x\cos\phi'-16\sqrt{2}q_0(x^2-1) , \tag{6.9}$$

and, to lowest order in q_0

$$\begin{aligned} e&=\frac{-8\sqrt{2}}{\pi q_0}\int_0^{\infty}\int_0^{\pi}\cos(2\phi')\theta(\overline{\mathrm{Arg}})\frac{dx\,d\phi'}{x} ,\\ b&=\frac{8\sqrt{2}}{\pi q_0^2}\int_0^{\infty}\int_0^{\pi}\cos\phi'\theta(\overline{\mathrm{Arg}})\frac{dx\,d\phi'}{x^2} ,\\ a&=\frac{8\sqrt{2}}{\pi q_0^3}\int_0^{\infty}\int_0^{\pi}\left[1-\frac{\overline{\mathrm{Arg}}}{4}\right]\theta(\overline{\mathrm{Arg}})\frac{dx\,d\phi'}{x^3}\end{aligned} \tag{6.10}$$

(where in each case the terms that have been neglected are of order $q_0\ln q_0$ times the leading term). Each of the functions in Eq. (6.10) will possess a series expansion in q_0. That is

$$e=\frac{1}{q_0}[e_1(\xi_0)+q_0\ln(q_0)e_2(\xi_0)+\cdots] , \tag{6.11a}$$

$$b=\frac{1}{q_0^2}[b_1(\xi_0)+q_0\ln(q_0)b_2(\xi_0)+\cdots] , \tag{6.11b}$$

$$a=\frac{1}{q_0^3}[a_1(\xi_0)+q_0\ln(q_0)a_2(\xi_0)+\cdots] . \tag{6.11c}$$

[Among the functions $a_i(\xi_0)$, only $a_1(\xi_0)$ will be used in this section, and the notation of Eq. (6.11c) will not be used in any other section. There is thus no risk of confusion with the functions $a_0(\hat{r}/\mu), a_2(\hat{r}/\mu), \ldots$ appearing in the series (1.2).]

Now

$$\begin{aligned}\left[\frac{\partial}{\partial q}\right]_r&=\left[\frac{\partial}{\partial q}\right]_\xi+\left[\frac{\partial\xi}{\partial q}\right]_r\left[\frac{\partial}{\partial\xi}\right]_q ,\\ \left[\frac{\partial}{\partial r}\right]_q&=\left[\frac{\partial}{\partial\xi}\right]_q ,\end{aligned} \tag{6.12}$$

and when q is small $(\partial\xi/\partial q)_r=1/\sqrt{2}\,q^2+O(1/q)$. Substituting Eqs. (6.11,12) into Eqs. (5.8,9) we find that in the region of interest

$$S(q_0,r_0)=(q_0)^{-4}\{2b_1(\xi_0)e_1''(\xi_0)-[b_1'(\xi_0)]^2+2b_1'(\xi_0)e_1'(\xi_0)\}+O\left[\frac{\ln(q_0)}{(q_0)^3}\right] \tag{6.13}$$

and

$$T(q_0,r_0)=(q_0)^{-4}\{2a_1(\xi_0)e_1''(\xi_0)+2b_1(\xi_0)e_1''(\xi_0)-2b_1'(\xi_0)e_1'(\xi_0)+[b_1'(\xi_0)]^2-2e_1(\xi_0)e_1''(\xi_0)\}+O\left[\frac{\ln(q_0)}{(q_0)^3}\right] , \tag{6.14}$$

and therefore

$$S+T=(q_0)^{-4}\{2[a_1(\xi_0)+2b_1(\xi_0)-e_1(\xi_0)]e_1''(\xi_0)\}+O\left[\frac{\ln(q_0)}{(q_0)^3}\right]. \tag{6.15}$$

This expression may be simplified using the harmonic (de Donder) gauge conditions (I4.2) relating the first-order metric functions E,B,A defined in Eq. (5.2). These conditions give

$$A_{,v}+\frac{1}{\rho}B+B_{,\rho}=0,\quad B_{,v}-\frac{2}{\rho}E-E_{,\rho}=0\ . \tag{6.16}$$

Written in terms of e, b, and a, Eq. (6.16) becomes

$$\begin{aligned}&-\sqrt{2}a_{,r}-2b-2qb_{,q}+8b_{,r}=0\ ,\\&-\sqrt{2}b_{,r}+2qe_{,q}-8e_{,r}=0\ .\end{aligned} \tag{6.17}$$

Substituting Eqs. (6.11), (6.12) into the above we find

$$a_1'(\xi_0)+b_1'(\xi_0)=0,\quad b_1'(\xi_0)-e_1'(\xi_0)=0 \tag{6.18}$$

and hence

$$a_1(\xi_0)+2b_1(\xi_0)-e_1(\xi_0)=\lambda\ \ (\text{const})\ . \tag{6.19}$$

We can find λ most easily by examining the behavior of e, b, and a in Eq. (6.10) as $\xi_0\to-\infty$. Now if $\xi_0\ll-1$ then Arg will be zero unless x is large. This implies that $\lim_{\xi_0\to-\infty}b_1(\xi_0)=\lim_{\xi_0\to-\infty}a_1(\xi_0)=0$, since there are factors of x^{-2} and x^{-3} in the integrands of b and e, respectively. From Eq. (6.10),

$$e=\frac{-4\sqrt{2}}{\pi q_0}\int_{|\cos\epsilon|\le 1}\sin(2\epsilon)\frac{dx}{x}+o(q_0^{-1})\ , \tag{6.20}$$

where $\cos\epsilon=(1/8x)[16\sqrt{2}q_0(x^2-1)-(8\ln x+\xi_0+8)]$. We recall here that the standard notation $g(z)=o(f(z))$ as $z\to 0$ means that $g(z)/f(z)\to 0$ as $z\to 0$. In Eq. (6.20), the $o(q_0^{-1})$ term refers to the limit $q_0\to 0$. When $\xi_0\ll-1$ the lower bound on x is $x_L\simeq-(\xi_0+8)/8$ and the upper bound is $x_u\simeq 1/2\sqrt{2}q_0$. Hence

$$\begin{aligned}\lim_{\xi_0\to-\infty}e&=\frac{-8\sqrt{2}}{\pi q_0}\left\{\lim_{\xi_0\to-\infty}\int_{-(\xi_0+8)/8}^{\infty}\frac{-(\xi_0+8)}{8x}\left[1-\left[\frac{\xi_0+8}{8x}\right]^2\right]^{1/2}\frac{dx}{x}\right.\\&\qquad\left.+\int_0^{1/2\sqrt{2}q_0}2\sqrt{2}q_0x\sqrt{1-(2\sqrt{2}q_0x)^2}\frac{dx}{x}\right\}+o(q_0^{-1})\\&=\frac{-8\sqrt{2}}{\pi q_0}\left[\int_1^{\infty}y^{-3}\sqrt{(y^2-1)}dy+\int_0^1\sqrt{(1-z^2)}dz\right]+o(q_0^{-1})\\&=\frac{-4\sqrt{2}}{q_0}+o(q_0^{-1})\ .\end{aligned} \tag{6.21}$$

Thus $\lim_{\xi_0\to-\infty}e_1(\xi_0)=-4\sqrt{2}$, giving $\lambda=4\sqrt{2}$ in Eq. (6.19), and

$$S(q_0,r_0)+T(q_0,r_0)=8\sqrt{2}e_1''(\xi_0)q_0^{-4}+O[\ln(q_0)/(q_0)^3]\ . \tag{6.22}$$

The source term contribution to $d^{(2)}+e^{(2)}$ is

$$\iint_{q_0>0,\eta_0<\eta,\xi_0<\xi}(G_{\rm I}S+G_{\rm II}T)dq_0dr_0\ .$$

In this integral dq_0dr_0 may be replaced by $[\partial(q_0,r_0)/\partial(\xi_0,\eta_0)]d\xi_0d\eta_0$, where the characteristic coordinates ξ,η are defined in Eqs. (3.10), (3.11). When $q_0\ll 1$, $\partial(q_0,r_0)/\partial(\xi_0,\eta_0)\simeq\sqrt{2}q_0^2$. Substituting Eqs. (6.7), (6.22), into the source integral, we find the contribution to $d^{(2)}+e^{(2)}$ from the region where η_0 is large and ξ_0 is of order 1 to be

$$\frac{1}{q^3}\int_{\Lambda}^{\eta}\int_{-\infty}^{\xi}[-4\sqrt{2}q_0e_1''(\xi_0)+o(q_0)]d\xi_0d\eta_0\ . \tag{6.23}$$

Here ξ and η are the coordinates of the field point and Λ is a lower cut-off which is $O(1)$ (with respect to η) but is sufficiently large that the series expansions derived earlier are valid at $\eta_0=\Lambda$. Since $\sqrt{2}q_0=1/\eta_0+o(\eta_0^{-1})$ in the region of interest, (6.23) reduces to

$$-\frac{4\ln\eta}{q^3}e_1'(\xi)+\text{terms of }O(q^{-3})\text{ and less}\ . \tag{6.24}$$

The contribution to $D^{(2)}+E^{(2)}$ from the logarithmic term is therefore

$$-\frac{4\ln\eta}{q^3\rho^4}e_1'(\xi)\ . \tag{6.25}$$

Now

$$\begin{aligned}&\xi=r+8\ln(4\sqrt{2}q)-\frac{1}{\sqrt{2}q}-8+O(q)\ ,\\&\eta=r+O(\ln q)\ ,\\&r=8\ln(v\rho)-\sqrt{2}v,\quad q=u\rho^{-2}\ ,\\&\rho=|\mathbf{r}|\sin\theta\ ,\\&v=\frac{1}{\sqrt{2}}(-2|\mathbf{r}|\sin^2\frac{\theta}{2}-\tau)\ ,\\&u=\frac{1}{\sqrt{2}}(2|\mathbf{r}|\cos^2\frac{\theta}{2}+\tau)\ ,\end{aligned} \tag{6.26}$$

where $\tau = t - |\mathbf{r}|$ is a retarded time coordinate. Hence (6.25) equals

$$-4\sqrt{2}\left[\frac{\ln|\mathbf{r}|+\ln(2\sin^2\theta/2)}{|\mathbf{r}|}\right]\tan^2\frac{\theta}{2}\sec^2\frac{\theta}{2}e_1'(T) + o\left[\frac{1}{|\mathbf{r}|}\right], \quad (6.27)$$

where

$$T = \tau\sec^2\frac{\theta}{2} - 8\ln\left[\frac{2\tan\theta/2}{v}\right] + 8\ln 8 - 8 .$$

(Note that we are still using dimensionless coordinates here.)

It is easy to see that the contribution to $D^{(2)}+E^{(2)}$ from the source region in which q_0 and r_0 are $O(1)$ is $O(1/|\mathbf{r}|)$; while that from the region where η_0 is large and ξ_0 is $O(1/q_0)$ (i.e., between the weak-shock region and the initial surface) is $o(1/|\mathbf{r}|)$. The contribution from the surface term (5.19) is also $O(1/|\mathbf{r}|)$. Hence (6.27) contains the only $\ln r/r$ term in $D^{(2)}+E^{(2)}$ (from here on write r instead of $|\mathbf{r}|$ since there will be no danger of confusion with the two-dimensional (2D) coordinate of that name).

We shall now show that this $\ln r/r$ term is eliminated from $h^{(2)}_{\phi\phi}$ when we transform to a Bondi coordinate system. In such a coordinate system the metric has the form [3]

$$ds^2 = v^2\left\{-\left[1-\frac{2M}{\hat r}+O\left[\frac{1}{\hat r^2}\right]\right]d\hat\tau^2 - 2\left[1+O\left[\frac{1}{\hat r^2}\right]\right]d\hat\tau\, d\hat r - 2\left[\frac{(\partial c/\partial\hat\theta+2c\cot\hat\theta)}{\hat r}+O\left[\frac{1}{\hat r^2}\right]\right]\hat r\, d\hat\tau\, d\hat\theta + \hat r^2\left[1+\frac{2c}{\hat r}+O\left[\frac{1}{\hat r^2}\right]\right]d\hat\theta^2 + \hat r^2\sin^2\hat\theta\left[1-\frac{2c}{\hat r}+O\left[\frac{1}{\hat r^2}\right]\right]d\phi^2\right\} . \quad (6.28)$$

We shall endeavor to put our metric in this form, to first order in $e^{-2\alpha}$, by searching for an explicit coordinate transformation from our harmonic gauge to a Bondi gauge.

The first-order metric perturbation $h^{(1)}_{ab}$ is given by Eq. (5.2) in terms of the metric functions $E=\rho^{-2}e$, $B=\rho^{-3}b$ and $A=\rho^{-4}a$ [Eq. (5.1)]. In the asymptotic region of space-time "near" $\mathcal{I}^+$, $e=e_1(\xi)/q]+o(q^{-1})$, $b=\{[e_1(\xi)-e_0]/q^2\}+o(q^{-2})$ and $a=-\{[e_1(\xi)-e_0]/q^3\}+o(q^{-3})$, where $e_0=\lim_{\xi\to-\infty}e_1(\xi)=-4\sqrt{2}$. Hence [using Eq. (6.26)]

$$E = \frac{\sec^2\theta/2}{\sqrt{2}r}e_1(T)+o(r^{-1}) ,$$

$$B = \frac{\tan\theta/2\,\sec^2\theta/2}{r}[e_1(T)-e_0]+o(r^{-1}) , \quad (6.29)$$

$$A = -\frac{\sqrt{2}\tan^2\theta/2\,\sec^2\theta/2}{r}[e_1(T)-e_0]+o(r^{-1}) ,$$

where T was defined after Eq. (6.27). Now transforming Eq. (5.2) to coordinates (τ,r,θ,ϕ) in which the background metric is flat space-time in the Bondi form (6.28),

$$ds^2 = v^2[-d\tau^2 - 2d\tau\, dr + r^2(d\theta^2+\sin^2\theta\, d\phi^2)] , \quad (6.30)$$

we find

$$\begin{aligned}
h^{(1)}_{\tau\tau} &= \frac{-\tan^2\theta/2\sec^2\theta/2}{\sqrt{2}r}[e_1(T)-e_0]+o(r^{-1}) ,\\
h^{(1)}_{\tau r} &= o(r^{-1}) ,\\
h^{(1)}_{\tau\theta} &= r\left[\frac{\tan\theta/2\,\sec^2\theta/2}{\sqrt{2}r}[e_1(T)-e_0]+o(r^{-1})\right] ,\\
h^{(1)}_{\tau\phi} &= 0 ,\\
h^{(1)}_{rr} &= \frac{-2\sqrt{2}\sin^2\theta/2}{r}e_0+o(r^{-1}) ,\\
h^{(1)}_{r\theta} &= r\left[-\frac{\sqrt{2}\tan\theta/2\,\cos\theta}{r}e_0+o(r^{-1})\right] ,\\
h^{(1)}_{r\phi} &= 0 ,\\
h^{(1)}_{\theta\theta} &= r^2\left[\frac{-\sec^2\theta/2}{\sqrt{2}r}e_1(T)+\frac{2\sqrt{2}\sin^2\theta/2}{r}e_0 + o(r^{-1})\right] ,\\
h^{(1)}_{\theta\phi} &= 0 ,\\
h^{(1)}_{\phi\phi} &= r^2\sin^2\theta\left[\frac{\sec^2\theta/2}{\sqrt{2}r}e_1(T)+o(r^{-1})\right] .
\end{aligned} \quad (6.31)$$

If we make a gauge transformation

$$\tau=\hat\tau+e^{-2\alpha}\xi_{\hat\tau}, \quad r=\hat r+e^{-2\alpha}\xi_{\hat r}, \quad \theta=\hat\theta+e^{-2\alpha}\xi_{\hat\theta} \quad (6.32)$$

of the flat background metric (6.30), then (6.30) transforms to

$$ds^2=-d\hat{\tau}^2-2\,d\hat{\tau}\,d\hat{r}+\hat{r}^2(d\hat{\theta}^2+\sin^2\hat{\theta}d\phi^2)$$
$$+e^{-2\alpha}[-2(\xi_{\hat{\tau},\hat{\tau}}+\xi_{\hat{\tau},\hat{\tau}})d\hat{\tau}^2-2(\xi_{\hat{\tau},\hat{r}}+\xi_{\hat{r},\hat{r}}+\xi_{\hat{\tau},\hat{\tau}})d\hat{r}\,d\hat{\tau}-2\xi_{\hat{\tau},\hat{r}}d\hat{r}^2-2(\xi_{\hat{\tau},\hat{\theta}}+\xi_{\hat{r},\hat{\theta}}-r^2\xi_{\hat{\theta},\hat{\tau}})d\hat{\tau}\,d\hat{\theta}$$
$$-2(\xi_{\hat{\tau},\hat{\theta}}-r^2\xi_{\hat{\theta},\hat{r}})d\hat{r}\,d\hat{\theta}+2(\hat{r}\xi_{\hat{r}}+\hat{r}^2\xi_{\hat{\theta},\hat{\theta}})d\hat{\theta}^2+2(\hat{r}\xi_{\hat{r}}\sin^2\hat{\theta}+\hat{r}^2\sin\hat{\theta}\cos\hat{\theta}\xi_{\hat{\theta}})d\phi^2]+O(e^{-4\alpha})\;. \tag{6.33}$$

If $\xi_{\hat{a}}$ is to transform the metric into Bondi form then $\xi_{\hat{\tau}}$, $\xi_{\hat{r}}$, and $\xi_{\hat{\theta}}$ must all possess series expansions in $\hat{r}$. That is,

$$\begin{aligned}\xi_{\hat{\tau}}&=f_1(\hat{\tau},\hat{\theta})\ln\hat{r}+f_2(\hat{\tau},\hat{\theta})+o(1)\;,\\ \xi_{\hat{r}}&=g_1(\hat{\tau},\hat{\theta})\ln\hat{r}+g_2(\hat{\tau},\hat{\theta})+o(1)\;,\\ \xi_{\hat{\theta}}&=h_1(\hat{\tau},\hat{\theta})\frac{\ln\hat{r}}{\hat{r}}+\frac{1}{\hat{r}}h_2(\hat{\tau},\hat{\theta})+o(\hat{r}^{-1})\;.\end{aligned} \tag{6.34}$$

Let $h_{ab}^{(1)B}$ denote the Bondi metric perturbations and $h_{ab}^{(1)H}$ the harmonic ones. Clearly, $h_{rr}^{(1)H}-2\xi_{\hat{\tau},\hat{r}}=h_{\hat{r}\hat{r}}^{(1)B}$. More explicitly,

$$\begin{aligned}\frac{f_1(\hat{\tau},\hat{\theta})}{\hat{r}}&=\frac{-\sqrt{2}\sin^2(\theta/2)e_0}{r}+o(r^{-1})\\ &=\frac{-\sqrt{2}\sin^2(\hat{\theta}/2)e_0}{\hat{r}}+O(e^{-2\alpha})\;,\end{aligned} \tag{6.35}$$

and so $f_1(\hat{\tau},\hat{\theta})=-\sqrt{2}e_0\sin^2(\hat{\theta}/2)$. [The $O(e^{-2\alpha})$ term is irrelevant here since it affects only the second- and higher-order metric perturbations.] Also, $h_{\theta\theta}^{(1)H}+2(\hat{r}\xi_r+\hat{r}^2\xi_{\hat{\theta},\hat{\theta}})=h_{\hat{\theta}\hat{\theta}}^{(1)B}$, which when written out in full is

$$-\frac{\hat{r}}{\sqrt{2}}\sec^2\frac{\hat{\theta}}{2}e_1(T)+2\sqrt{2}\hat{r}\sin^2\frac{\hat{\theta}}{2}e_0+2\left[\left\{g_1(\hat{\tau},\hat{\theta})+\frac{\partial h_1(\hat{\tau},\hat{\theta})}{\partial\hat{\theta}}\right\}\hat{r}\ln\hat{r}+\left\{g_2(\hat{\tau},\hat{\theta})+\frac{\partial h_2(\hat{\tau},\hat{\theta})}{\partial\hat{\theta}}\right\}\hat{r}\right]=2\hat{r}e^{2\alpha}c^{(1)}\;. \tag{6.36}$$

The corresponding $\phi\phi$ equation is

$$\frac{\hat{r}}{\sqrt{2}}\sin^2\hat{\theta}\sec^2\frac{\hat{\theta}}{2}e_1(T)+2\sin^2\hat{\theta}\{[g_1(\hat{\tau},\hat{\theta})+\cot\hat{\theta}h_1(\hat{\tau},\hat{\theta})]\hat{r}\ln\hat{r}+[g_2(\hat{\tau},\hat{\theta})+\cot\hat{\theta}h_2(\hat{\tau},\hat{\theta})]\hat{r}\}=-2\hat{r}\sin^2\hat{\theta}e^{2\alpha}c^{(1)}\;. \tag{6.37}$$

The $\hat{r}\ln\hat{r}$ terms must vanish in both these equations, and so $\partial h_1(\hat{\tau},\hat{\theta})/\partial\hat{\theta}=\cot\hat{\theta}h_1(\hat{\tau},\hat{\theta})$, which when integrated yields $h_1(\hat{\tau},\hat{\theta})=k(\hat{\tau})\sin\hat{\theta}$ [whence $g_1(\hat{\tau},\hat{\theta})=-k(\hat{\tau})\cos\hat{\theta}$]. Multiplying Eq. (6.36) by $\sin^2\hat{\theta}$ and adding it to Eq. (6.37) leads to

$$\sqrt{2}\sin^2\frac{\hat{\theta}}{2}e_0+2g_2(\hat{\tau},\hat{\theta})+\frac{\partial h_2(\hat{\tau},\hat{\theta})}{\partial\hat{\theta}}+\cot\hat{\theta}h_2(\hat{\tau},\hat{\theta})=0\;, \tag{6.38}$$

while subtracting yields

$$-\sqrt{2}\sec^2\frac{\hat{\theta}}{2}e_1(T)+2\sqrt{2}\sin^2\frac{\hat{\theta}}{2}e_0+2\left[\frac{\partial h_2(\hat{\tau},\hat{\theta})}{\partial\hat{\theta}}-\cot\hat{\theta}h_2(\hat{\tau},\hat{\theta})\right]=4e^{2\alpha}c^{(1)}\;. \tag{6.39}$$

The most general form that $c(\hat{\tau},\hat{\theta})$ can take is

$$c=e^{-2\alpha}\left[\frac{-\sec^2\hat{\theta}/2}{2\sqrt{2}}e_1(T)+\frac{1}{\sqrt{2}}\sin^2\frac{\hat{\theta}}{2}e_0+\frac{1}{2}[\alpha'(\hat{\theta})\cot\hat{\theta}-\alpha''(\hat{\theta})]\right]+O(e^{-4\alpha})\;. \tag{6.40}$$

[The $\hat{r}\hat{\theta}$ equation, $\xi_{\hat{\tau},\hat{\theta}}+\xi_{\hat{r},\hat{\theta}}-\hat{r}^2\xi_{\hat{\theta},\hat{\tau}}=O(1)$, implies that the time-varying part of $\xi_{\hat{\theta}}$ is $o(\hat{r}^{-1})$. Then Eq. (6.38) ensures that the time-dependent part of $\xi_{\hat{r}}$ is $o(1)$. Hence

$$\lim_{\hat{r}\to\infty}\hat{r}^{-1}(\sin\hat{\theta})^{-2}\partial h_{\phi\phi}^{(1)B}/\partial\hat{\tau}=\lim_{r\to\infty}r^{-1}(\sin\theta)^{-2}\partial h_{\phi\phi}^{H}/\partial\tau\;,$$

which proves rigorously that the formula $c_0^{(1)}=-\frac{1}{2}e^{-2\alpha}\lim_{r\to\infty}r^{-1}(\sin\theta)^{-2}(\partial h_{\phi\phi}^{(1)H}/\partial\tau)$ used in Sec. IV of paper I to derive the first-order news function is correct, and leads directly to Eq. (6.40). In Eq. (6.40) the derivative of the first term is the news function found previously, the second term is included for convenience, and the third term incorporates the supertranslation freedom [3].]

Since the time-dependent parts of $\xi_{\hat{r}}$ and $\xi_{\hat{\theta}}$ are $o(1)$ and $o(\hat{r}^{-1})$ respectively, $g_i(\hat{\tau},\hat{\theta})=g_i(\hat{\theta})$ and $h_i(\hat{\tau},\hat{\theta})=h_i(\hat{\theta})$; whence $k(\hat{\tau})=K$. Now combining Eqs. (6.39) and (6.40) we find

$$\frac{\partial h_2(\hat{\theta})}{\partial\hat{\theta}}-\cot\hat{\theta}\,h_2(\hat{\theta})=\alpha'(\hat{\theta})\cot\hat{\theta}-\alpha''(\hat{\theta})\;. \tag{6.41}$$

Therefore $h_2(\hat{\theta})=L\sin\hat{\theta}-\alpha'(\hat{\theta})$, whence from Eq. (6.38),

$$g_2(\hat{\theta})=\frac{-1}{\sqrt{2}}\sin^2\left[\frac{\hat{\theta}}{2}\right]e_0-L\cos\hat{\theta}+\frac{1}{2}\alpha''(\hat{\theta})+\frac{1}{2}\cot\hat{\theta}\,\alpha'(\hat{\theta}).$$

The $\hat{r}\hat{r}$ equation, $h^{(1)B}{}_{\hat{r}\hat{r}}=h^{(1)H}{}_{\tau\tau}-2(\xi_{\hat{\tau},\hat{\tau}}+\xi_{\hat{r},\hat{\tau}})$, now implies that $\partial f_i(\hat{\tau},\hat{\theta})/\partial\hat{\tau}=0$, and so $f_2(\hat{\tau},\hat{\theta})=f(\hat{\theta})$. Making the appropriate substitutions, the $\hat{r}\hat{\theta}$ equation is

$$-\sqrt{2}\tan\frac{\hat\theta}{2}\cos\hat\theta\, e_0-2\left[\frac{-1}{\sqrt{2}}\sin\hat\theta e_0\ln\hat r+\frac{\partial f_2(\hat\theta)}{\partial\hat\theta}-K\sin\hat\theta(1-\ln\hat r)+L\sin\hat\theta-\alpha'(\hat\theta)\right]=0\;. \tag{6.42}$$

Therefore $K=e_0/\sqrt{2}$, and

$$\frac{\partial f_2(\hat\theta)}{\partial\hat\theta}=\frac{1}{\sqrt{2}}e_0\tan\frac{\hat\theta}{2}-L\sin\hat\theta+\alpha'(\hat\theta)\;. \tag{6.43}$$

Hence $f_2(\hat\theta)=-\sqrt{2}e_0\ln[\cos(\hat\theta/2)]+L\cos\hat\theta+\alpha(\hat\theta)$, where the constant of integration has been absorbed into $\alpha(\hat\theta)$. If we now examine the forms of $f_2(\hat\theta)$, $g_2(\hat\theta)$, and $h_2(\hat\theta)$, we find that all the terms in L may be eliminated by redefining $\alpha(\hat\theta)$. In sum,

$$\xi_{\hat\tau}=-\sqrt{2}\sin^2\frac{\hat\theta}{2}e_0\ln\hat r-\sqrt{2}e_0\ln\left[\cos\frac{\hat\theta}{2}\right]+\alpha(\hat\theta)+o(1)\;,$$

$$\xi_{\hat r}=\frac{-1}{\sqrt{2}}e_0\cos\hat\theta\ln\hat r-\frac{e_0}{\sqrt{2}}\sin^2\frac{\hat\theta}{2}+\frac{1}{2\sin\hat\theta}\frac{d}{d\hat\theta}[\sin\hat\theta\alpha'(\hat\theta)]+o(1)\;,$$

$$\xi_{\hat\theta}=\frac{1}{\sqrt{2}}e_0\sin\hat\theta\frac{\ln\hat r}{\hat r}-\frac{\alpha'(\hat\theta)}{\hat r}+o(\hat r^{-1})\;. \tag{6.44}$$

(It is obvious that the $\hat\tau\hat\theta$ and $\hat\tau\hat r$ equations are satisfied to the appropriate order in $\hat r$.)

In our harmonic gauge

$$g_{\phi\phi}=v^2r^2\sin^2\theta\left[1+e^{-2\alpha}\frac{\sec^2(\theta/2)e_1(T)}{\sqrt{2}r}+e^{-4\alpha}(D^{(2)}+E^{(2)})+O(e^{-6\alpha})+o(r^{-1})\right]\;. \tag{6.45}$$

If we now apply the transformation given by (6.44), then the new $g_{\phi\phi}$ is

$$g_{\phi\phi}=v^2\hat r^{\,2}\sin^2\hat\theta\left\{1+e^{-2\alpha}\left[\frac{(1/\sqrt{2})\sec^2(\theta/2)e_1(T)-\sqrt{2}\sin^2(\hat\theta/2)e_0-[\alpha'(\hat\theta)\cot\hat\theta-\alpha''(\hat\theta)]}{\hat r}\right]\right.$$
$$\left.+e^{-4\alpha}(D^{(2)}+E^{(2)})+O(e^{-6\alpha})+o(\hat r^{-1})\right\}\;. \tag{6.46}$$

Let

$$\hat T=\hat\tau\sec^2\left[\frac{\hat\theta}{2}\right]-8\ln\left[\frac{2\tan(\hat\theta/2)}{v}\right]+8\ln 8-8\;.$$

Then

$$T=\hat T+e^{-2\alpha}\sec^2\frac{\hat\theta}{2}\left[-\sqrt{2}\sin^2\frac{\hat\theta}{2}e_0\ln\hat r-\sqrt{2}e_0\ln\left[\cos\frac{\hat\theta}{2}\right]+\alpha(\hat\theta)\right]+o(1)+O(e^{-4\alpha})\;. \tag{6.47}$$

Hence

$$g_{\phi\phi}=v^2r^2\sin^2\hat\theta\left[1+e^{-2\alpha}\left[\frac{(1/\sqrt{2})\sec^2(\hat\theta/2)e_1(\hat T)-\sqrt{2}\sin^2(\hat\theta/2)e_0-[\alpha'(\hat\theta)\cot\hat\theta-\alpha''(\hat\theta)]}{\hat r}\right]\right.$$
$$+e^{-4\alpha}\left\{D^{(2)}+E^{(2)}+\frac{1}{\sqrt{2}}\frac{\sec^4(\hat\theta/2)}{\hat r}\left[-\sqrt{2}\sin^2\frac{\hat\theta}{2}e_0\ln\hat r-\sqrt{2}e_0\ln\left[\cos\frac{\hat\theta}{2}\right]+\alpha(\hat\theta)\right]e_1'(T)\right\}$$
$$\left.+O(e^{-6\alpha})+o(\hat r^{-1})\right]. \tag{6.48}$$

The logarithmic term in $D^{(2)}+E^{(2)}$ is [from (6.27)]

$$e_0\tan^2\frac{\hat\theta}{2}\sec^2\frac{\hat\theta}{2}e_1'(\hat T)\frac{\ln\hat r}{\hat r}\;, \tag{6.49}$$

which is clearly canceled by the other $(\ln\hat r)/\hat r$ term in Eq. (6.48). There is also a term of the form "$e_1'(\hat T)$ times an arbitrary function of $\hat\theta$" in the $e^{-4\alpha}$ term in Eq. (6.48). We postpone to Sec. VIII any discussion of this term, which at first sight seems to make the second-order news function ill-defined.

VII. TRANSFORMING TO A BONDI GAUGE AT SECOND ORDER

We shall now investigate the relationship between the harmonic and Bondi gauges at second order in $e^{-2\alpha}$. In other words, we shall consider gauge transformations of the form

$$\begin{aligned} \tau &= \hat{\tau} + e^{-2\alpha}\xi_{\hat{\tau}} + e^{-4\alpha}\xi^{(2)}_{\hat{\tau}} , \\ r &= \hat{r} + e^{-2\alpha}\xi_{\hat{r}} + e^{-4\alpha}\xi^{(2)}_{\hat{r}} , \\ \theta &= \hat{\theta} + e^{-2\alpha}\xi_{\hat{\theta}} + e^{-4\alpha}\xi^{(2)}_{\hat{\theta}} , \end{aligned} \tag{7.1}$$

and examine some of the properties that $\xi^{(2)}_{\hat{a}}$ must have if $(\hat{\tau},\hat{r},\hat{\theta},\phi)$ is to be a Bondi coordinate system. More specifically, we shall show that the time-dependent parts of $\xi^{(2)}_{\hat{r}}$ and $\xi^{(2)}_{\hat{\theta}}$ are $o(1)$ and $o(\hat{r}^{-1})$ respectively, so that the effect of $\xi^{(2)}_{\hat{a}}$ on the time-varying part of $h^{(2)}_{\phi\phi}$ is $o(\hat{r})$.

If we apply (7.1) to the flat-space metric (6.30), then the $e^{-4\alpha}$ term in the transformed metric in the hatted coordinate system will consist of two terms. The first will be identical to the $e^{-2\alpha}$ term in Eq. (6.33), except that $\xi_{\hat{a}}$ will be replaced by $\xi^{(2)}_{\hat{a}}$. The second term will have subterms that are each quadratic in $\xi_{\hat{a}}$ and its first derivatives. Using Eq. (6.44) one can show that it has the form

$$e^{-4\alpha}[o(\hat{r}^{-1})d\hat{\tau}^2 + o(\hat{r}^{-1})d\hat{\tau}d\hat{r} + o(1)d\hat{\tau}d\hat{\theta} + o(\hat{r}^{-1})d\hat{r}^2 + o(1)d\hat{r}\,d\hat{\theta} + o(r)d\hat{\theta}^2 + o(\hat{r})d\phi^2] . \tag{7.2}$$

The first-order metric $e^{-2\alpha}h^{(1)H}_{ab}(x^c)$ transforms to

$$e^{-2\alpha}[h^{(1)H}_{ab}(\hat{x}^d) + e^{-2\alpha}h^{(1)H}_{ab,e}(\hat{x}^d)\xi^{\hat{e}}](\delta^a_c + e^{-2\alpha}\xi^{\hat{a}}_{,\hat{c}})(\delta^b_f + e^{-2\alpha}\xi^{\hat{b}}_{,\hat{f}}) + O(e^{-6\alpha}) . \tag{7.3}$$

If we write out the $e^{-4\alpha}$ term in Eq. (7.3) in full, we find that it has the form

$$\begin{aligned} e^{-4\alpha}\{ &[fn(\hat{\tau},\hat{\theta})\frac{\ln\hat{r}}{\hat{r}} + fn(\hat{\tau},\hat{\theta})\frac{1}{\hat{r}} + o(\hat{r}^{-1})]d\hat{\tau}^2 + [fn(\hat{\tau},\hat{\theta})\ln\hat{r} + fn(\hat{\tau},\hat{\theta}) + o(1)]d\hat{\tau}d\hat{\theta} + o(\hat{r}^{-1})d\hat{\tau}d\hat{r} \\ &+ o(\hat{r}^{-1})d\hat{r}^2 + o(1)d\hat{r}\,d\hat{\theta} + [A(\hat{\tau},\hat{\theta})\hat{r}\ln\hat{r} + B(\hat{\tau},\hat{\theta})\hat{r} + o(\hat{r})]d\hat{\theta}^2 \\ &+ \sin^2\hat{\theta}[-A(\hat{\tau},\hat{\theta})\hat{r}\ln\hat{r} - B(\hat{\tau},\hat{\theta})\hat{r} + fn(\hat{\theta})\hat{r}\ln\hat{r} + fn(\hat{\theta})\hat{r} + o(\hat{r})]d\phi^2\} , \end{aligned} \tag{7.4}$$

where the explicit forms of each fn and of A and B can be calculated using Eqs. (6.31) and (6.44).

There is, of course, also a contribution to the second-order hatted metric from $h^{(2)H}_{ab}$ itself. Clearly $e^{-4\alpha}h^{(2)H}{}_{ab}(x^c)$ transforms to $e^{-4\alpha}h^{(2)H}{}_{ab}(\hat{x}^c) + O(e^{-6\alpha})$. The second-order metric perturbations $h^{(2)H}{}_{ab}$ may be written as

$$\begin{aligned} &h^{(2)H}_{tt} = A^{(2)}, \quad h^{(2)H}_{tx} = \rho^{-1}xB^{(2)} , \\ &h^{(2)H}_{ty} = \rho^{-1}yB^{(2)}, \quad h^{(2)H}_{tz} = C^{(2)} , \\ &h^{(2)H}_{zz} = G^{(2)}, \quad h^{(2)H}_{zx} = \rho^{-1}xF^{(2)} , \\ &h^{(2)H}_{zy} = \rho^{-1}yF^{(2)}, \quad h^{(2)H}_{xx} = D^{(2)} + (y^2 - x^2)\rho^{-2}E^{(2)} , \\ &h^{(2)H}_{xy} = -2xy\rho^{-2}E^{(2)}, \quad h^{(2)H}_{yy} = D^{(2)} + (x^2 - y^2)\rho^{-2}E^{(2)} . \end{aligned} \tag{7.5}$$

(I5.16′)

Each of the functions $A^{(2)}, \dots, G^{(2)}$ in Eq. (7.5) will possess a series expansion of the form $fn(\hat{\tau},\hat{\theta})[\ln\hat{r})/\hat{r}] + [fn(\hat{\tau},\hat{\theta})/\hat{r}] + o(\hat{r}^{-1})$. When expressed in terms of $A^{(2)}, \dots, G^{(2)}$, the second-order gauge conditions $\bar{h}^{(2)}{}_{ab},{}^{b} = 0$ are

$$\frac{\partial}{\partial\tau}(\tfrac{1}{2}A^{(2)} + D^{(2)} + \tfrac{1}{2}G^{(2)} + \sin\theta\, B^{(2)} + \cos\theta\, C^{(2)}) = o(r^{-1}) , \tag{7.6a}$$

$$\frac{\partial}{\partial\tau}(-B^{(2)} + \sin\theta\, E^{(2)} - \cos\theta\, F^{(2)} + \tfrac{1}{2}\sin\theta\, G^{(2)} - \tfrac{1}{2}\sin\theta\, A^{(2)}) = o(r^{-1}) , \tag{7.6b}$$

$$\frac{\partial}{\partial\tau}(-C^{(2)} - \sin\theta\, F^{(2)} - \tfrac{1}{2}\cos\theta\, G^{(2)} + \cos\theta\, D^{(2)} - \tfrac{1}{2}\cos\theta\, A^{(2)}) = o(r^{-1}) . \tag{7.6c}$$

Now

$$h^{(2)H}_{rr} = A^{(2)} + 2\cos\theta C^{(2)} + 2\sin\theta\, B^{(2)} + \cos^2\theta\, G^{(2)} + \sin(2\theta)F^{(2)} + \sin^2\theta(D^{(2)} - E^{(2)}) . \tag{7.7}$$

Multiplying Eq. (7.6b) by $\sin\theta$ and adding it to $\cos\theta$ times Eq. (7.6c), we find that

$$h^{(2)H}_{rr,\tau} = o(r^{-1}) . \tag{7.8}$$

Similarly one can show that

$$h^{(2)H}_{r\theta,\tau} = o(1), \quad h^{(2)H}_{\phi\phi,\tau} + \sin^2\theta h^{(2)H}_{\theta\theta,\tau} = o(r) . \tag{7.9}$$

$\xi^{(2)}_{\hat{a}}$ must be of the form

$$\begin{aligned} \xi^{(2)}_{\hat{\tau}} &= fn(\hat{\tau},\hat{\theta})(\ln\hat{r})^2 + fn(\hat{\tau},\hat{\theta})\ln\hat{r} + fn(\hat{\tau},\hat{\theta}) + o(1) , \\ \xi^{(2)}_{\hat{r}} &= fn(\hat{\tau},\hat{\theta})(\ln\hat{r})^2 + fn(\hat{\tau},\hat{\theta})\ln\hat{r} + fn(\hat{\tau},\hat{\theta}) + o(1) , \\ \xi^{(2)}_{\hat{\theta}} &= \frac{1}{\hat{r}}[fn(\hat{\tau},\hat{\theta})(\ln\hat{r})^2 + fn(\hat{\tau},\hat{\theta})\ln\hat{r} + fn(\hat{\tau},\hat{\theta}) + o(1)] . \end{aligned} \tag{7.10}$$

Collecting together all the contributions to the second-order hatted metric, we find that if the hatted coordinate system is to be Bondi then the $e^{-4\alpha}d\hat{r}d\hat{\theta}$ term implies that

$$\xi^{(2)}_{\hat{r},\hat{\theta}}+\xi^{(2)}_{\hat{r},\hat{\theta}}-\hat{r}^{\,2}\xi^{(2)}_{\hat{\theta},\hat{r}}=o(\hat{r})\ , \tag{7.11}$$

and so $\xi^{(2)}_{\hat{\theta},\hat{r}}=o(\hat{r}^{\,-1})$. In addition, the $e^{-4\alpha}d\phi^2$ term added to $\sin^2\hat{\theta}$ times the $e^{-4\alpha}d\hat{\theta}^{\,2}$ term implies that

$$2\xi^{(2)}_{\hat{r}}\sin\hat{\theta}+\hat{r}(\sin\hat{\theta}\xi^{(2)}_{\hat{\theta}})_{,\hat{\theta}}=fn(\hat{\theta})\ln\hat{r}+fn(\hat{\theta})+o(1)\ , \tag{7.12}$$

and thus $\xi^{(2)}_{r,\tau}=o(1)$. Therefore

$$\lim_{\hat{r}\to\infty}\hat{r}^{\,-1}(\sin\hat{\theta})^{-2}h^{(2)B}_{\phi\phi,\tau}=\lim_{r\to\infty}[r^{-1}(\sin\theta)^{-2}h^{(2)H}_{\phi\phi,\tau}+\text{ terms which can be calculated using only } h^{(1)H}_{\phi\phi} \text{ and } \xi_{\hat{a}}]\ . \tag{7.13}$$

Thus, when calculating the second-order news function, $\xi^{(2)}_{\hat{a}}$ need not be found explicitly, which is what we set out to prove.

VIII. THE AMBIGUITY IN THE SECOND-ORDER NEWS FUNCTION CAUSED BY THE SUPERTRANSITION FREEDOM

We saw earlier that in addition to the $(\ln\hat{r})/\hat{r}$ term (6.49), the $\ln\eta/q^3\rho^4$ term (6.25) contributes

$$-4\sqrt{2}\frac{\ln(2\sin^2(\theta/2))}{\hat{r}}\tan^2\frac{\hat{\theta}}{2}\sec^2\frac{\hat{\theta}}{2}e_1'(\hat{T})$$

to $h^{(2)B}_{\phi\phi}$ [see Eq. (6.27)]. Also, in addition to the $(\ln\hat{r})/\hat{r}$ term in Eq. (6.48), the gauge transformation (6.44) introduces a term

$$\frac{1}{\sqrt{2}}\frac{\sec^4(\hat{\theta}/2)}{\hat{r}}\left[8\ln\left[\cos\frac{\hat{\theta}}{2}\right]+\alpha(\hat{\theta})\right]e_1'(\hat{T})$$

into $h^{(2)B}_{\phi\phi}$. There is also a contribution to $h^{(2)B}_{\phi\phi}$ from the surface integral (5.19): it is of the form

$$\tan^2\frac{\hat{\theta}}{2}\sec^2\frac{\hat{\theta}}{2}\frac{fn(\hat{T})}{\hat{r}}\ .$$

The rest of the source integral $\int\int(G_{\rm I}S+G_{\rm II}T)dq_0dr_0$ of Eq. (5.20) also contributes a term of the form

$$\tan^2\frac{\hat{\theta}}{2}\sec^2\frac{\hat{\theta}}{2}\frac{fn(\hat{T})}{\hat{r}}\ .$$

In total, the $1/\hat{r}$ term in $h^{(2)B}_{\phi\phi}$ has the form

$$\hat{r}^{\,2}\sin^2\hat{\theta}\left\{\frac{\tan^2(\hat{\theta}/2)\sec^2(\hat{\theta}/2)}{\hat{r}}fn(\hat{T})-\left[\frac{\beta'(\hat{\theta})\cot\hat{\theta}-\beta''(\hat{\theta})}{\hat{r}}\right]\right.$$
$$\left.+\frac{\sec^4(\hat{\theta}/2)}{\hat{r}}\left[-4\sqrt{2}\sin^2\frac{\hat{\theta}}{2}\ln\left[2\sin^2\frac{\hat{\theta}}{2}\right]+4\sqrt{2}\ln\left[\cos\frac{\hat{\theta}}{2}\right]+\alpha(\hat{\theta})\right]e_1'(\hat{T})\right\}\ . \tag{8.1}$$

The angular dependence of the first term is expected from the analysis in Sec. V of paper I, where we found the form which the $\sin^2\hat{\theta}$ series (1.2) for the news function would take in the boosted frame. The second term, which is time independent, incorporates the standard supertranslation freedom. The additional terms, however, are somewhat unexpected. The $\ln(2\sin^2(\hat{\theta}/2))$ and $\ln(\cos(\hat{\theta}/2))$ terms, when transformed to the center-of-mass frame, do not have an angular dependence of $\sin^2\theta$. There is, as well, the term

$$\frac{\sec^4(\hat{\theta}/2)}{\hat{r}}\alpha(\hat{\theta})e_1'(\hat{T})\ ,$$

where $\alpha(\hat{\theta})$ is arbitrary, which seems to make the second-order news function ambiguous.

Moreover, there is an additional problem. We recall from Sec. IV of paper I that $e_1'(T)$ diverges as $\ln|\hat{T}|$ near $\hat{T}=0$. In addition, one can show that the function $fn(\hat{T})$ in the first term in (8.1) contains a certain (and calculable [14]) amount of $\ln|\hat{T}|$ singularity. Hence the second-order news function, which is related directly to the time derivative of (8.1), will contain $1/\hat{T}$ singular terms, which are not square integrable. And yet the news function must be square integrable, in order that the mass loss be finite.

It is, in fact, not difficult to resolve these puzzles. Consider any Bondi metric in which the various metric functions all have series expansions in powers of some perturbation parameter ϵ. In particular, the function c in Eq. (6.28) will have the form

$$c(\tau,\theta)=A_0(\tau,\theta)+\epsilon A_1(\tau,\theta)+\epsilon^2A_2(\tau,\theta)+\cdots\ . \tag{8.2}$$

Now suppose we make a supertranslation

$$\tau=\bar{\tau}+\epsilon f_1(\theta)+\epsilon^2 f_2(\theta)+\cdots \tag{8.3}$$

(note that $\bar{r} \sim r$, and that $\bar{\theta}=\theta$ on $\mathcal{I}^+$). Then c is unchanged, apart from the addition of a time-independent term [3]:

$$\bar{c}(\bar{\tau},\theta)=\sum_{j=0}^{\infty}\epsilon^j A_j\left[\bar{\tau}+\sum_{i=1}^{\infty}\epsilon^i f_i(\theta),\theta\right]+g(\theta)\ , \qquad (8.4)$$

where

$$g(\theta)=\frac{1}{2}\sum_{i=1}^{\infty}\epsilon^i[f_i'(\theta)\cot\theta-f_i''(\theta)]\ .$$

But if we now expand out each A_i we find

$$\begin{aligned}\bar{c}(\bar{\tau},\theta)=&A_0(\bar{\tau},\theta)+\epsilon[A_1(\bar{\tau},\theta)+f_1(\theta)A_0'(\bar{\tau},\theta)]\\&+\epsilon^2[A_2(\bar{\tau},\theta)+\tfrac{1}{2}(f,(\theta))^2A_0''(\bar{\tau},\theta)\\&+f_2(\theta)A_0'(\bar{\tau},\theta)+f_1(\theta)A_1'(\bar{\tau},\theta)]+\cdots+g(\theta)\ ,\end{aligned} \qquad (8.5)$$

where $'=\partial/\partial\tau$.

The barred and unbarred coordinate systems are physically indistinguishable, since it is impossible to tell what supertranslation state one is in. Owing to this complete freedom in the choice of origin of retarded time, only the leading term in the perturbation expansion for c (and the news function c_0) is unambiguously determined—all the higher-order terms being uncertain to the extent shown in Eq. (8.5). Of course the magnitude of the total news function, given by the sum of the series, remains unchanged; thus the amplitude of the gravitational radiation, which is the physically significant quantity, is well defined. We also note that $\int_{-\infty}^{\infty}(c_0)^2 d\tau$ remains invariant at each order in ϵ.

Such behavior is not limited just to perturbative news functions. A general news function $c_0(\tau,\theta)$ "supertranslates" to $c_0(\bar{\tau}+f(\theta),\theta)$. We can then (at least formally) expand out in powers of $f(\theta)$, to obtain

$$c_0(\bar{\tau},\theta)+f(\theta)c_0'(\bar{\tau},\theta)+\tfrac{1}{2}f(\theta)^2c_0''(\bar{\tau},\theta)+\cdots\ .$$

(In this way an isotropic distribution of radiation could be made to look nonisotropic.) However, as in the perturbative case, the magnitude of the news function at any given point on $\mathcal{I}^+$ remains unchanged, and none of the additional terms contribute to $\int_{-\infty}^{\infty}(c_0)^2 d\tau$.

We see from Eq. (8.5) that the second-order $g_{\phi\phi}$ may contain an arbitrary multiple of the time derivative of the first-order $g_{\phi\phi}$. This explains the origin of all the $e_1'(\hat{T})$ terms in Eq. (8.1). The $\ln(\hat{T})$ term in $fn(\hat{T})$ must also be due to an $e_1'(\hat{T})$ term that has been introduced by the "wrong" choice of supertranslation state. All these terms may therefore be eliminated by making an appropriate supertranslation. In fact, since it is the center-of-mass news function that we would like to be manifestly square integrable, we shall choose $\alpha(\hat{\theta})$ in Eq. (8.1) [and Eq. (6.44)] to ensure that, on matching back to the center-of-mass frame, the coefficient $a_2(\hat{\tau}/\mu)$ of $\sin^2\hat{\theta}$ in Eq. (1.2) contains no $1/\hat{T}$ term near $\hat{T}=0$.

In a way, it is fortunate that our news function is singular, for otherwise we would have no way of telling how much of a_0' is contained in a_2. The correct amount can be given in analytic form as an integral [14], but here we simply quote a numerical value. Let us describe the radiative part of the second-order gravitational field in terms of the quantity $(\hat{d}^{(2)}+\hat{e}^{(2)})=q^3(d^{(2)}+e^{(2)})$. We assume that the logarithmic (gauge) part of this quantity near null infinity has been subtracted off, and denote by $\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)$ the remaining $O(1)$ part near $\mathcal{I}^+$. This is related to the second-order asymptotic metric function $c^{(2)}$ by

$$c^{(2)}=\frac{-\nu}{\sqrt{2}}e^{-4\alpha}\tan^2\frac{\hat{\theta}}{2}\sec^2\frac{\hat{\theta}}{2}[\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)]\ , \qquad (8.6)$$

where

$$\xi=\hat{\tau}\sec^2\frac{\hat{\theta}}{2}-8\ln\left[\frac{2\tan(\hat{\theta}/2)}{\nu}\right]+8\ln 8-8\ .$$

The integral expression in Ref. [14] shows that the numerical coefficient of $\ln|\xi|$ in $\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)$ is 4.21867. Further, the coefficient of $\ln|\xi|$ in $e_1'(\xi)$ may be shown to be $\sqrt{2}/\pi$. In the numerical calculation we therefore subtract $(\pi/\sqrt{2})\times 4.21867\times e_1'(\xi)$ from $\hat{d}^{(2)}(\xi)+\hat{e}^{(2)}(\xi)$ before differentiating to find the news function.

IX. COMMENTS

In this paper we have seen how the second-order perturbation problem in the axisymmetric speed-of-light collision can be reduced to a problem in two independent variables, by exploiting the conformal symmetry (2.10) at each order of perturbation theory. The second-order metric coefficients can then be expressed in terms of two-dimensional integrals of a Green function multiplying a source function, plus surface contributions. Although the resulting metric is not in a Bondi gauge at null infinity, gauge transformations can be found which put it into Bondi gauge. This allows one to read off the second-order news function, which gives the $\sin^2\hat{\theta}$ part of the strong-field gravitational radiation pattern (1.2), in addition to allowing a further investigation of the new mass-loss formula described in Sec. VI of paper I. These results are presented and discussed in the following paper III.

Clearly a large amount of numerical work is involved in the computation of the integrals giving the $O(1)$ radiative part of $\hat{d}^{(2)}+\hat{e}^{(2)}$ at null infinity, as a function of ξ or τ. We do not discuss this numerical work here; a somewhat detailed treatment is given in Ref. [14]. Nevertheless we should remark on two of the difficulties which must be overcome numerically. First, the logarithmic terms in $\hat{d}^{(2)}+\hat{e}^{(2)}$ near null infinity must be carefully subtracted off numerically, leaving the $O(1)$ part which carries the information about gravitational waves. Second, the separate contributions from the surface term $(d^{(2)}+e^{(2)})_{\text{surf}}$ of Eq. (5.19) and from the volume contribution to $d^{(2)}+e^{(2)}$ grow exponentially at late times, the exponential terms canceling each other in the complete $d^{(2)}+e^{(2)}$. This requires very high accuracy in the computation when ξ is only moderately large and positive. This possibility for exponential behavior, already mentioned in Sec. III of paper I, may actually be realized in some of the higher-order metric perturbations, as will be discussed in paper III.

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