The characteristic development of trapped surfaces

Paul N. DemmieAllen I. Janis

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The characteristic development of trapped surfaces*

Paul N. Demmie

University of Pittsburgh at Johnstown, Johnstown, Pennsylvania 15904

Allen I. Janis

University of Pittsburgh, Pittsburgh, Pennsylvania 15213

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Conditions are found that are sufficient to insure the development of trapped surfaces in space-times whose metrics satisfy the Einstein field equations in vacuum or in the presence of a massless scalar field. These conditions involve a topological requirement that a certain two-surface be compact and inequalities that must be satisfied by certain pieces of the characteristic data determining these space-times. It is shown that a particular piece of data playing an important role in these inequalities is related to angular momentum.

1. INTRODUCTION

The concept of a trapped surface—a compact, spacelike two-surface having the property that all null geodesics meeting it orthogonally converge locally to the future—was introduced by Penrose[1] as a characterization of gravitational collapse that has proceeded beyond the point of no return. Under rather general conditions, if an object collapses sufficiently far that trapped surfaces develop in the region surrounding this object, then the spacetime containing the object must be singular.[2] Therefore, an important problem in the theory of gravitational collapse is the determination of the conditions under which trapped surfaces develop.

An important step toward solving this problem in the case of empty space-times was taken by Pajerski and Newman[3] (PN). Exploiting the property of the Schwarzschild space-time that the region containing trapped surfaces is separated from those that do not contain trapped surfaces by a nondiverging null hypersurface, they generalized the Schwarzschild space-time by considering a class of space-times each containing a nondiverging null hypersurface and determining the restrictions on the characteristic data for which trapped surfaces develop. The present work generalizes their work not only to Einstein-scalar space-times, but also to a larger class of empty space-times.

In Sec. 2 the formalism used in this investigation will be presented. This formalism was found to be particularly useful since it provides for a convenient characterization of trapped surfaces. In Sec. 3 the formalism presented in Sec. 2 will be used to determine all Einstein-scalar space-times containing a nondiverging null hypersurface. That these space-times are more general than those obtained in PN follows not only from their being Einstein-scalar space-times rather than empty space-times, but also from their dependence on an arbitrary function that was required to vanish in PN. Evidence suggesting that this function is related to angular momentum will be presented in Sec. 3. Also the characteristic data for these space-times will be determined, examined for restrictions placed on them in order that trapped surfaces develop, and discussed there. In Sec. 4 the results of Sec. 3 will be generalized and it will be established that there exist space-times more general than those containing a nondiverging null hypersurface that also contain trapped surfaces. Finally, in Sec. 5 the results of Secs. 3 and 4 will be summarized and discussed.

2. THE FORMALISM

The Newman–Penrose (NP) formalism[4] was found to be particularly useful for this investigation of the characteristic development of trapped surfaces. This formalism requires introducing into the tangent space at each point of the spacetime a null tetrad system,[5]

$$\left\{D = \frac{\ell^\mu \partial}{\partial x^\mu}, \quad \Delta = \frac{n^\mu \partial}{\partial x^\mu}, \quad \delta = \frac{m^\mu \partial}{\partial x^\mu}, \quad \bar{\delta} = \frac{\bar{m}^\mu \partial}{\partial x^\mu}\right\}, \tag{2.1}$$

consisting of two real null vectors, D and Δ, and a pair of complex null vectors, δ and $\bar{\delta}$, formed from two real, orthonormal, spacelike vectors, s_1 and s_2, as

$$\delta = (s_1 + is_2)/\sqrt{2},$$

and satisfying the orthonormality conditions

$$\begin{aligned} &\ell_\mu n^\mu = -m_\mu \bar{m}^\mu = 1, \\ &\ell_\mu \ell^\mu = n_\mu n^\mu = m_\mu m^\mu = \bar{m}_\mu \bar{m}^\mu = 0, \\ &\ell_\mu m^\mu = \ell_\mu n^\mu = n_\mu m^\mu = n_\mu \bar{m}^\mu = 0. \end{aligned} \tag{2.2}$$

The components $g^{\mu\nu}$ of the contravariant metric are found from (2.2) to be[6]

$$g^{\mu\nu} = 2\ell(^\mu n^\nu) - 2m(^\mu \bar{m}^\nu). \tag{2.3}$$

The formalism then provides a set of partial differential equations equivalent to the Einstein field equations for the determination of the $g^{\mu\nu}$. These equations are given in terms of the five independent physical components of the Weyl tensor $C_{\mu\nu\rho\sigma}$[7]

$$\begin{aligned} \Psi_0 &= -C_{\mu\nu\rho\sigma}\ell^\mu m^\nu \ell^\rho m^\sigma, \\ \Psi_1 &= -C_{\mu\nu\rho\sigma}\ell^\mu n^\nu \ell^\rho m^\sigma, \\ \Psi_2 &= -C_{\mu\nu\rho\sigma}\bar{m}^\mu n^\nu \ell^\rho m^\sigma, \\ \Psi_3 &= -C_{\mu\nu\rho\sigma}\bar{m}^\mu n^\nu \ell^\rho n^\sigma, \\ \Psi_4 &= -C_{\mu\nu\rho\sigma}\bar{m}^\mu n^\nu \bar{m}^\rho n^\sigma, \end{aligned} \tag{2.4}$$

the six independent physical components of the trace-free Ricci tensor $R_{\mu\nu}$,

$$\begin{aligned} \Phi_{00} &= -\tfrac{1}{2}R_{\mu\nu}\ell^\mu \ell^\nu = \bar{\Phi}_{00}, \\ \Phi_{01} &= -\tfrac{1}{2}R_{\mu\nu}\ell^\mu m^\nu = \bar{\Phi}_{10}, \\ \Phi_{02} &= -\tfrac{1}{2}R_{\mu\nu}m^\mu m^\nu = \bar{\Phi}_{20}, \\ \Phi_{11} &= -\tfrac{1}{4}R_{\mu\nu}(\ell^\mu n^\nu + m^\mu \bar{m}^\nu) = \bar{\Phi}_{11}, \\ \Phi_{12} &= -\tfrac{1}{2}R_{\mu\nu}n^\mu m^\nu = \bar{\Phi}_{21}, \\ \Phi_{22} &= -\tfrac{1}{2}R_{\mu\nu}n^\mu n^\nu = \bar{\Phi}_{22}, \end{aligned} \tag{2.5}$$

the Riemann scalar R, and the twelve spin coefficients,[8]

$$\begin{aligned}
\kappa &= \ell_{\mu;\nu} m^\mu \ell^\nu, & \nu &= -n_{\mu;\nu}\overline{m}^\mu n^\nu,\\
\rho &= \ell_{\mu;\nu} m^\mu \overline{m}^\nu, & \mu &= -n_{\mu;\nu}\overline{m}^\mu m^\nu,\\
\sigma &= \ell_{\mu;\nu} m^\mu m^\nu, & \lambda &= -n_{\mu;\nu}\overline{m}^\mu \overline{m}^\nu,\\
\tau &= \ell_{\mu;\nu} m^\mu n^\nu, & \pi &= -n_{\mu;\nu}\overline{m}^\mu \ell^\nu,\\
\alpha &= \tfrac{1}{2}(\ell_{\mu;\nu} n^\mu \overline{m}^\nu - m_{\mu;\nu}\overline{m}^\mu \overline{m}^\nu),\\
\beta &= \tfrac{1}{2}(\ell_{\mu;\nu} n^\mu m^\nu - m_{\mu;\nu}\overline{m}^\mu m^\nu),\\
\gamma &= \tfrac{1}{2}(\ell_{\mu;\nu} n^\mu n^\nu - m_{\mu;\nu}\overline{m}^\mu n^\nu),\\
\epsilon &= \tfrac{1}{2}(\ell_{\mu;\nu} n^\mu \ell^\nu - m_{\mu;\nu}\overline{m}^\mu \ell^\nu).
\end{aligned} \tag{2.6}$$

Before exhibiting the NP equations, a class of null tetrad systems appropriate for this investigation will be given. This class of null tetrad systems, which consists of those systems associated in a particular way with a class of null coordinate systems, will simplify these equations somewhat.

In a space-time it is possible to introduce at least locally a family of null hypersurfaces given by $u =$ const, where u is a scalar function satisfying

$$g^{\mu\nu}u_{,\mu}u_{,\nu} = 0. \tag{2.7}$$

Let the x^0 coordinate be u. Then choose the first member of the null tetrad system, D, so that

$$\ell_\mu = u_{,\mu}. \tag{2.8}$$

That D is null and hypersurface orthogonal follows from (2.7) and (2.8), respectively. Therefore D is tangent to a family of null geodesics. Let the x^1 coordinate be r where r is an affine parameter for D. Then

$$D = \frac{dx^\mu}{dr}\frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial r}.$$

These properties of D imply that

$$\kappa = 0 = (\epsilon + \overline{\epsilon}), \qquad \rho = \overline{\rho}, \qquad \tau = \overline{\alpha} + \beta. \tag{2.9}$$

Finally let the x^m coordinates label the null geodesics in the $u =$ constant hypersurface. In this manner a null coordinate system[9]

$$\{u, r, x^m\} \tag{2.10}$$

and an associated null vector D are given locally in a space-time. The most general null tetrad system containing D and preserving the orthonormality conditions (2.2) is $\{D, \Delta, \delta, \overline{\delta}\}$, where

$$D = \partial/\partial r, \tag{2.11a}$$

$$\Delta = \frac{\partial}{\partial u} + \frac{U\partial}{\partial r} + \frac{X^m\partial}{\partial x^m}, \tag{2.11b}$$

$$\delta = \frac{\omega\partial}{\partial r} + \frac{\xi^m\partial}{\partial x^m}. \tag{2.11c}$$

The null coordinate system (2.10) and associated null tetrad system (2.11) are not unique. (2.10) could be replaced by any coordinate system in which the coordinate conditions

$$g^{0\mu} = \delta^\mu_1 \tag{2.12}$$

hold. Also (2.11) could be replaced by any null tetrad system related to it by null rotations about D,

$$\begin{aligned}
\tilde{D} &= D,\\
\tilde{\Delta} &= \Delta + a\overline{\delta} + \overline{a}\delta + a\overline{a}D,\\
\tilde{\delta} &= \delta + aD,
\end{aligned} \tag{2.13}$$

where a is complex, and/or spatial rotations,

$$\tilde{D} = D, \qquad \tilde{\Delta} = \Delta, \qquad \text{and } \tilde{\delta} = e^{iC}\delta, \tag{2.14}$$

where C is real, since the conditions (2.2) and (2.8) are preserved by these transformations.

Much of the ambiguity in the null tetrad system can be removed by choosing (2.11c) in a particular way. Under (2.13) with $a = -\omega$, δ becomes

$$\tilde{\delta} = \delta + aD = \xi^m \frac{\partial}{\partial x^m}.$$

Therefore, the ambiguity between (2.11) and any null tetrad system related to it by null rotations about D can be eliminated by choosing

$$\omega = 0 \tag{2.15}$$

in (2.11c). Furthermore, under (2.14), $(\epsilon - \overline{\epsilon})$ becomes

$$(\epsilon \,\tilde{-}\, \overline{\epsilon}) = (\epsilon - \overline{\epsilon}) - iDC.$$

Therefore, the ambiguity between (2.11) and any null tetrad system related to it by spatial rotations can be reduced to those rotations with $C = C(u, x^m)$ by choosing ϵ real. This choice and (2.9) imply that

$$\epsilon = 0. \tag{2.16}$$

With (2.15) and (2.16) adopted, it has been established that:

In a spacetime there exists a class of null coordinate systems such that any one of these coordinate systems

$$\{u, r, x^m\} \tag{2.17}$$

satisfies the coordinate conditions (2.12) and has associated with it a particular null tetrad system $\{D, \Delta, \delta, \overline{\delta}\}$ with

$$D = \frac{\partial}{\partial r}, \tag{2.18a}$$

$$\Delta = \frac{\partial}{\partial u} + U\frac{\partial}{\partial r} + X^m\frac{\partial}{\partial x^m}, \tag{2.18b}$$

$$\delta = \xi^m\frac{\partial}{\partial x^m}, \tag{2.18c}$$

which satisfies the orthonormality conditions (2.2), is unique up to spatial rotations (2.14) with $C = C(u, x^m)$, and has spin coefficients satisfying (2.9) and (2.16).

From (2.1), (2.3), and (2.18) the components $g^{\mu\nu}$ of the contravariant metric are

$$\begin{aligned}
&g^{0\mu} = g^\mu_1, \qquad g^{11} = 2U, \qquad g^{1m} = X^m,\\
&g^{mn} = -(\xi^m\overline{\xi}^n + \overline{\xi}^m\xi^n).
\end{aligned} \tag{2.19}$$

With the null tetrad system (2.18) chosen, the NP equations will now be exhibited[10] in three classes: the commutator equations applied to the coordinates, the spin coefficient equations, and the spin-coefficient form of the Bianchi identities. The commutator equations applied to the coordinates imply that the spin coefficients τ, π, and μ satisfy

$$\tau = \bar{\pi}, \quad \mu = \bar{\mu} \tag{2.20}$$

and the metric variables U, X^m, and ξ^m satisfy

$$D\xi^m = \rho\xi^m + \sigma\bar{\xi}^m, \tag{2.21a}$$

$$DX^m = 2(\bar{\tau}\xi^m + \tau\bar{\xi}^m), \tag{2.21b}$$

$$DU = -(\gamma + \bar{\gamma}), \tag{2.21c}$$

$$\delta\bar{\xi}^m - \bar{\delta}\xi^m = (\bar{\alpha} - \beta)\bar{\xi}^m + (\bar{\beta} - \alpha)\xi^m, \tag{2.21d}$$

$$\Delta\xi^m - \delta X^m = -(\mu + \bar{\gamma} - \gamma)\xi^m - \bar{\lambda}\bar{\xi}^m, \tag{2.21e}$$

$$\nu = -\bar{\delta}U. \tag{2.21f}$$

With (2.9), (2.16), and (2.20) satisfied, the spin-coefficient equations are

$$D\rho = \rho^2 + \sigma\bar{\sigma} + \Phi_{00}, \tag{2.22a}$$

$$D\sigma = 2\rho\sigma + \Psi_0, \tag{2.22b}$$

$$D\tau = 2\rho\tau + 2\sigma\bar{\tau} + \Psi_1 + \Phi_{01}, \tag{2.22c}$$

$$D\alpha = (\alpha + \bar{\tau})\rho + \beta\bar{\sigma} + \Phi_{10}, \tag{2.22d}$$

$$D\beta = \rho\beta + (\alpha + \bar{\tau})\sigma + \Psi_1, \tag{2.22e}$$

$$D\gamma = 2\tau\alpha + 2\bar{\tau}\beta + \tau\bar{\tau} + \Psi_2 - \tfrac{1}{24}R + \Phi_{11}, \tag{2.22f}$$

$$D\lambda - \overline{\delta\tau} = \rho\lambda + \bar{\sigma}\mu + \bar{\tau}^2 + (\alpha - \bar{\beta})\bar{\tau} + \Phi_{20}, \tag{2.22g}$$

$$D\mu - \delta\bar{\tau} = \rho\mu + \sigma\lambda + \tau\bar{\tau} - (\bar{\alpha} - \beta)\bar{\tau} + \Psi_2 + \tfrac{1}{12}R, \tag{2.22h}$$

$$\delta\rho - \bar{\delta}\sigma = \rho\tau - (3\alpha - \bar{\beta})\sigma - \Psi_1 + \Phi_{01}, \tag{2.22i}$$

$$\delta\alpha - \bar{\delta}\beta = \rho\mu - \sigma\lambda + \alpha\bar{\alpha} + \beta\bar{\beta} - 2\alpha\beta - \Psi_2 + \tfrac{1}{24}R + \Phi_{11}, \tag{2.22j}$$

$$\delta\lambda - \bar{\delta}\mu = \mu\bar{\tau} + (\bar{\alpha} - 3\beta)\lambda - \Psi_3 + \Phi_{21}, \tag{2.22k}$$

$$\Delta\tau - D\bar{\nu} = -2\mu\tau - 2\overline{\tau\lambda} + (\gamma - \bar{\gamma})\tau - \bar{\Psi}_3 - \Phi_{12}, \tag{2.22l}$$

$$\Delta\lambda - \bar{\delta}\nu = (\bar{\gamma} - 3\gamma - 2\mu)\lambda + (3\alpha + \bar{\beta})\nu - \Psi_4, \tag{2.22m}$$

$$\Delta\mu - \delta\nu = -\mu^2 - \lambda\bar{\lambda} - (\gamma + \bar{\gamma})\mu + 2\beta\nu + \overline{\nu\tau} - \Phi_{22}, \tag{2.22n}$$

$$\Delta\beta - \delta\gamma = -\mu\tau + \sigma\nu + (\gamma - \bar{\gamma} - \mu)\beta - \alpha\bar{\lambda} - \Phi_{12}, \tag{2.22o}$$

$$\Delta\sigma - \delta\tau = -\mu\sigma - \rho\bar{\lambda} - 2\beta\tau + (3\gamma - \bar{\gamma})\sigma - \Phi_{02}, \tag{2.22p}$$

$$\Delta\rho - \bar{\delta}\tau = -\mu\rho - \sigma\lambda - 2\alpha\tau + (\gamma - \bar{\gamma})\rho - \Psi_2 - \tfrac{1}{12}R, \tag{2.22q}$$

$$\Delta\alpha - \bar{\delta}\gamma = \rho\nu - (\tau + \beta)\lambda + (\bar{\gamma} - \gamma - \mu)\alpha - \Psi_3; \tag{2.22r}$$

and the spin-coefficient form of the Bianchi identities are

$$D\Psi_1 - \bar{\delta}\Psi_0 = 4\rho\Psi_1 - (4\alpha - \bar{\tau})\Psi_0 + D\Phi_{01} - \delta\Phi_{00} - 2\rho\Phi_{01} + \tau\Phi_{00} - 2\sigma\Phi_{10}, \tag{2.23a}$$

$$\Delta\Psi_0 - \delta\Psi_1 = (4\gamma - \mu)\Psi_0 - 2(2\tau + \beta)\Psi_1 + 3\sigma\Psi_2 - D\Phi_{02} + \delta\Phi_{01} + \rho\Phi_{02} + 2\bar{\alpha}\Phi_{01} - \bar{\lambda}\Phi_{00} + 2\sigma\Phi_{11}, \tag{2.23b}$$

$$\begin{aligned} D\Psi_2 - \bar{\delta}\Psi_1 = {} & 3\rho\Psi_2 + 2\bar{\beta}\Psi_1 - \lambda\Psi_0 + \tfrac{2}{3}D\Phi_{11} - \tfrac{2}{3}\delta\Phi_{10} + \tfrac{1}{3}\bar{\delta}\Phi_{01} - \tfrac{1}{3}\Delta\Phi_{00} - \tfrac{2}{3}\rho\Phi_{11} - \tfrac{2}{3}(2\bar{\tau} + \alpha)\Phi_{01} \\ & - \tfrac{4}{3}\beta\Phi_{10} - \tfrac{2}{3}\sigma\Phi_{20} + \tfrac{1}{3}\bar{\sigma}\Phi_{02} + \tfrac{1}{3}(\mu + 2\gamma + 2\bar{\gamma})\Phi_{00}, \end{aligned} \tag{2.23c}$$

$$\begin{aligned} \Delta\Psi_1 - \delta\Psi_2 = {} & \nu\Psi_0 + 2(\gamma - \mu)\Psi_1 - 3\tau\Psi_2 + 2\sigma\Psi_3 - \tfrac{2}{3}D\Phi_{12} + \tfrac{2}{3}\delta\Phi_{11} - \tfrac{1}{3}\bar{\delta}\Phi_{02} + \tfrac{1}{3}\Delta\Phi_{01} - \tfrac{1}{3}\bar{\nu}\Phi_{00} \\ & - \tfrac{2}{3}\gamma\Phi_{01} - \tfrac{2}{3}\bar{\lambda}\Phi_{10} + 2\tau\Phi_{11} + \tfrac{1}{3}(4\alpha + \bar{\tau})\Phi_{02} + \tfrac{2}{3}\sigma\Phi_{21}, \end{aligned} \tag{2.23d}$$

$$\begin{aligned} D\Psi_3 - \bar{\delta}\Psi_2 = {} & 2\rho\Psi_3 + 3\bar{\tau}\Psi_2 - 2\lambda\Psi_1 + \tfrac{1}{3}D\Phi_{21} - \tfrac{1}{3}\delta\Phi_{20} + \tfrac{2}{3}\bar{\delta}\Phi_{11} - \tfrac{2}{3}\Delta\Phi_{10} + \tfrac{2}{3}\nu\Phi_{00} - \tfrac{2}{3}\lambda\Phi_{01} \\ & - \tfrac{4}{3}\bar{\gamma}\Phi_{10} - 2\bar{\tau}\Phi_{11} - \tfrac{1}{3}(4\beta + \tau)\Phi_{20} + \tfrac{2}{3}\bar{\sigma}\Phi_{12}, \end{aligned} \tag{2.23e}$$

$$\begin{aligned} \Delta\Psi_2 - \delta\Psi_3 = {} & 2\nu\Psi_1 - 3\mu\Psi_2 - 2\bar{\alpha}\Psi_3 + \sigma\Psi_4 - \tfrac{1}{3}D\Phi_{22} + \tfrac{1}{3}\delta\Phi_{21} - \tfrac{2}{3}\bar{\delta}\Phi_{12} + \tfrac{2}{3}\Delta\Phi_{11} - \tfrac{2}{3}\nu\Phi_{01} \\ & - \tfrac{2}{3}\bar{\nu}\Phi_{10} + \tfrac{2}{3}\mu\Phi_{11} + \tfrac{2}{3}\lambda\Phi_{02} - \tfrac{1}{3}\bar{\lambda}\Phi_{20} + \tfrac{4}{3}\alpha\Phi_{12} + \tfrac{2}{3}(\beta + 2\tau)\Phi_{21} - \tfrac{1}{3}\rho\Phi_{22}, \end{aligned} \tag{2.23f}$$

$$D\Psi_4 - \bar{\delta}\Psi_3 = \rho\Psi_4 + 2(\alpha + 2\bar{\tau})\Psi_3 - 3\lambda\Psi_2 + \bar{\delta}\Phi_{21} - \Delta\Phi_{20} + 2\nu\Phi_{10} - 2\lambda\Phi_{11} - (2\gamma - 2\bar{\gamma} + \mu)\Phi_{20} - 2(\bar{\tau} - \alpha)\Phi_{21} + \bar{\sigma}\Phi_{22}, \tag{2.23g}$$

$$\Delta\Psi_3 - \delta\Psi_4 = 3\nu\Psi_2 - 2(\gamma + 2\mu)\Psi_3 + (4\beta - \tau)\Psi_4 - \bar{\delta}\Phi_{22} + \Delta\Phi_{21} - 2\nu\Phi_{11} - \bar{\nu}\Phi_{20} + 2\lambda\Phi_{12} + 2(\gamma + \mu)\Phi_{21} - \bar{\tau}\Phi_{22}, \tag{2.23h}$$

$$D\Phi_{11} - \delta\Phi_{10} - \bar{\delta}\Phi_{01} + \Delta\Phi_{00} + \tfrac{1}{8}DR = 2(\gamma + \bar{\gamma} - \mu)\Phi_{00} - (2\alpha + \bar{\tau})\Phi_{01} - (2\bar{\alpha} + \tau)\Phi_{10} + 4\rho\Phi_{11} + \bar{\sigma}\Phi_{02} + \sigma\Phi_{20}, \tag{2.23i}$$

$$D\Phi_{12} - \delta\Phi_{11} - \bar{\delta}\Phi_{02} + \Delta\Phi_{01} + \tfrac{1}{8}\delta R = (2\gamma - 3\mu)\Phi_{01} + \bar{\nu}\Phi_{00} - \bar{\lambda}\Phi_{10} + 2(\bar{\beta} - \alpha)\Phi_{02} + 3\rho\Phi_{12} + \sigma\Phi_{21}, \tag{2.23j}$$

$$D\Phi_{22} - \delta\Phi_{21} - \bar{\delta}\Phi_{12} + \Delta\Phi_{11} + \tfrac{1}{8}\Delta R = \nu\Phi_{01} + \bar{\nu}\Phi_{10} - 4\mu\Phi_{11} - \lambda\Phi_{02} - \bar{\lambda}\Phi_{20} + (\bar{\tau} + 2\bar{\beta})\Phi_{12} + (\tau + 2\beta)\Phi_{21} + 2\rho\Phi_{22}. \tag{2.23k}$$

The class of space-times investigated were the Einstein-scalar space-times. An Einstein-scalar space-time is a space-time whose Einstein tensor $G_{\mu\nu}$ satisfies the Einstein field equations[11]

$$G_{\mu\nu} = -T_{\mu\nu} \tag{2.24}$$

with

$$T_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - \tfrac{1}{2}(g^{\rho\sigma}\phi_{,\rho}\phi_{,\sigma})g_{\mu\nu}, \tag{2.25}$$

where ϕ is a massless scalar field satisfying the equation

$$(g^{\mu\nu}\phi,_\nu);_\mu = 0. \tag{2.26}$$

From (2.24) and (2.25) it is easily seen that the Ricci tensor for an Einstein-scalar space-time is

$$R_{\mu\nu} = -\phi,_\mu\phi,_\nu. \tag{2.27}$$

The scalar field equation (2.26) can be expressed in spin-coefficient form as

$$\eta^{ab}(\phi,_{ab} - \gamma^c_{ab}\phi,_c) = 0, \tag{2.28}$$

where $\phi,_a$ and $\phi,_{ab}$ are the physical components of $\phi,_\mu$ and $(\phi,_a);_\mu$, respectively,

$$(\eta_{ab}) = (\eta^{ab}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \tag{2.29}$$

is the null form of the Minkowski metric, and the $\gamma_{abc} = z_{a\mu;\nu}z^\mu_b z^\nu_c$ with $(z^\mu_a) = (\ell^\mu, n^\mu, m^\mu, \overline{m}^\mu)$ are the Ricci rotation coefficients. With (2.1), (2.6), (2.9), and (2.16), Eq. (2.28) becomes

$$(D\Delta + \mu D - \rho\Delta - \bar\delta\delta - 2\bar\beta\delta - \tau\bar\delta)\phi = 0. \tag{2.30}$$

Not only was the spin-coefficient form of the scalar field equation used in this investigation, but also the spin-coefficient form of the optical scalars.[12] The spin-coefficient form of the divergence, rotation, and shear of a vector $k = k^\mu \dfrac{\partial}{\partial x^\mu}$ can be shown to be[13]

$$d(k) = \tfrac{1}{2}\eta^{ab}(k_{a,b} - \gamma_{acb}k^c), \tag{2.31a}$$

$$r(k) = [\tfrac{1}{2}(k_{[a,b]} - \gamma_{[a|c|b]}k^c)(k^{a,b} - \gamma^{adb}k_d)]^{1/2}, \tag{2.31b}$$

$$s(k) = [\tfrac{1}{2}(k_{(a,b)} - \gamma_{(a|c|b)}k^c)(k^{a,b} - \gamma^{adb}k_d) - d^2]^{1/2}, \tag{2.31c}$$

respectively.

The formalism presented here provides not only for the determination of the metric variables, spin coefficients, physical Weyl tensor components, and scalar field of an Einstein-scalar spacetime, but also for a convenient characterization of trapped surfaces. To discover this characterization, consider the spacelike two-surface

$$S_{(u,r)} = \{(u, r, x^m): u \text{ and } r \text{ are constant}\}.$$

In order for $S_{(u,r)}$ to be a trapped surface, it must be compact and have the property that all null geodesics meeting it orthogonally coverge locally to the future. A vector tangent to one of these null geodesics must coincide with either D or Δ on $S_{(u,r)}$. That which coincides with D must be D, since a geodesic is specified uniquely by a point and a direction in the tangent space at that point. Therefore, from (2.31a) its divergence must be $-\rho$. However, the divergence of the null vector that coincides with Δ on $S_{(u,r)}$ is not so easily determined since

$$n^\mu;_\nu n^\nu = -(\gamma + \bar\gamma)n^\mu + \bar\nu m^\mu + \nu \overline{m}^\mu$$

implies that Δ is not everywhere tangent to a null geodesic unless $\nu = 0$. For the general, case where $\nu \neq 0$, the divergence of this vector can be calculated directly from (2.31a). Let k be this vector parameterized with an affine parameter. Then the components of k, k^μ, can be expressed as

$$k_\mu = S,_\mu$$

where S is a solution to $g^{\mu\nu}S,_\mu S,_\nu = 0$.

From this it follows that

$$k = (\Delta S)D + (DS)\Delta + (\delta S)\bar\delta + (\bar\delta S)\delta.$$

$k = \Delta$ on $S_{(u,r)}$ implies that $DS = 1$, $\delta S = 0 = \bar\delta S$ on $S_{(u,r)}$. Therefore, from (2.31a) the divergence of k on $S_{(u,r)}$, θ_0, is

$$\theta_0 = D\Delta S + \tfrac{1}{2}(\delta\bar\delta S + \bar\delta\delta S) + \tfrac{1}{2}(\mu + \bar\mu),$$

where the derivatives are evaluated on $S_{(u,r)}$. Now

$$D\Delta S = D(k_\mu n^\mu) = k_{\mu;\nu}n^\mu\ell^\nu + n_{\mu;\nu}k^\mu\ell^\nu.$$

Since $n^\mu{}_{;\nu}\ell^\nu = -\tau\overline{m}^\mu - \bar\tau m^\mu$, it follows that $n_{\mu;\nu}n^\mu\ell^\nu = 0$. Also since $k_\mu = S_{,\mu}$, it follows that $k_{\mu;\nu}n^\mu\ell^\nu = k_{\mu;\nu}\ell^\mu n^\nu$. This vanishes on $S_{(u,r)}$ since $k = \Delta$ there and k is tangent to a null geodesic parameterized with an affine parameter. Therefore, $D\Delta S = 0$ on $S_{(u,r)}$. Also $\delta\bar\delta S = 0 = \bar\delta\delta S$ on $S_{(u,r)}$ since δ and $\bar\delta$ are differentiations in $S_{(u,r)}$ over which $\bar\delta S$ and δS are constant. $D\Delta S = 0 = \delta\bar\delta S = \bar\delta\delta S$ on $S_{(u,r)}$ and $\mu = \bar\mu$ imply that $\theta_0 = \mu$. With the determination of θ_0, it has been established that:

The spacelike two-surface

$$S_{(u,r)} = \{(u, r, x^m): u \text{ and } r \text{ are constant}\}$$

is a trapped surface if and only if it is compact and everywhere on it the spin coefficients ρ and μ satisfy

$$\rho > 0 \quad\text{and}\quad \mu < 0. \tag{2.32}$$

3. SPACE-TIMES CONTAINING A NONDIVERGING NULL HYPERSURFACE

The possibility that there exist nonspherically symmetric Einstein-scalar spacetimes containing both a nondiverging null hypersurface and trapped surfaces was investigated.[14] This investigation began by obtaining the metric variables, spin coefficients, physical Weyl tensor components, and scalar field of an Einstein-scalar space-time containing a nondiverging null hypersurface. Then the characteristic data for this space-time were determined and examined for restrictions placed on them for which trapped surfaces develop.

The problem of determining all Einstein-scalar space-times containing a nondiverging null hypersurface was solved using the formalism presented in Sec. 2. For any one of these space-times with a particular null coordinate system (2.17) and associated null tetrad system (2.18) introduced in it, the main conditions that were adopted are that the nondiverging null hypersurface is given by $u = 0$ and the metric variables, spin coefficients, physical Weyl tensor components, and scalar field are analytic functions of (u, r, x^m) in the region $\{(u, r, x^m)\}$. Subject to these conditions, the NP equations involving D and the scalar field equation (2.30) yielded these quantities in terms of a set of functions of (x^m) given on the spacelike two-surface

$$S_0 = \{(u, r, x^m): u = 0 = r\}.$$

Some of these functions had conditions placed on them in order that the coordinate and tetrad systems chosen initially be specified up to scale transformations.

$$\tilde{u}=Au, \quad \tilde{r}=A^{-1}r, \quad \tilde{x}^m=x^m, \\ \tilde{D}=A^{-1}D, \quad \tilde{\Delta}=A\Delta, \quad \tilde{\delta}=\delta; \tag{3.1}$$

whereas others were determined by the remaining NP equations. The functions that remained constitute the characteristic data for this space–time.

The condition that the nondiverging null hypersurface is given by $u=0$ implies that

$$\rho(0,r,x^m)=0. \tag{3.2}$$

This and Eq. (2.22a) imply that

$$\sigma(0,r,x^m)=0 \quad \text{and} \quad \phi(0,r,x^m)=\phi_0(x^m),$$

where $\phi_0(x^m)$ is an arbitrary function of (x^m). These and Eq. (2.22b) imply that

$$\Psi_0(0,r,x^m)=0.$$

$\rho=0=\sigma$ on $u=0$ and Eq. (2.21a) imply that

$$\xi^m(0,r,x^n)=\xi_0^m(x^n),$$

where the ξ_0^m are arbitrary functions of (x^n).

The presentation of subsequent results will be simplified considerably by further specification of (ξ_0^m). Consider the contravariant metric induced on S_0,

$$-(\xi_0^m\bar{\xi}_0^n+\bar{\xi}_0^m\xi_0^n)\frac{\partial}{\partial x^m}\otimes\frac{\partial}{\partial x^n}.$$

Since this is a two-metric, it can be made conformally flat by some transformation[15]

$$\tilde{u}=u, \quad \tilde{r}=r, \ \tilde{x}^m=\tilde{x}^m(x^n).$$

Therefore, it may be taken to be

$$-P\bar{P}\frac{\partial}{\partial z}\otimes\frac{\partial}{\partial \bar{z}}, \tag{3.3}$$

where $z=(x^2-ix^3)/\sqrt{2}$ and P is an arbitrary function of (x^m). With this choice (ξ_0^m) becomes

$$(\xi_0^m)=(P,iP)/\sqrt{2}.$$

Now consider the spatial rotations (2.14) with

$$C(u,x^m)=C_0(x^m)+C_1(x^m)u+\cdots.$$

Under this transformation δ becomes

$$\tilde{\delta}=[\exp(iC)]\delta=\frac{1}{\sqrt{2}}[\exp(iC_0)]P\frac{\partial}{\partial z}+\cdots.$$

Therefore, if P is required to be real, then δ is specified up to spatial rotations with $C_0=0$. Once the contravariant metric induced on S_0 is chosen to be (3.3) with P real, it is convenient to introduce the differential operators $\eth$ and $\bar{\eth}$[16] as

$$\eth\eta=P^{1-s}\frac{\partial}{\partial z}(P^s\eta) \quad \text{and} \quad \bar{\eth}\eta=P^{1+s}\frac{\partial}{\partial\bar{z}}(P^{-s}\eta),$$

where η is a quantity of spin weight s, that is, a quantity that transforms as

$$\tilde{\eta}=[\exp(is\psi)]\eta$$

under the transformation

$$\tilde{\delta}_0=[\exp(i\psi)]\delta_0 \quad \text{where } \delta_0=\frac{1}{\sqrt{2}}P\frac{\partial}{\partial z}.$$

The metric variables, spin coefficients, physical Weyl tensor components, and scalar field obtained thus far on $u=0$ were calculated using (3.2) and the NP equations expressed in a particular null coordinate system (2.17) and associated null tetrad system (2.18). Similarly, one can obtain on $u=0$: Ψ_1 from (2.22i), τ from (2.22c), α and β from (2.22d), (2.22e), (2.21d), and (2.9), X^m from (2.21b), Ψ_2 from (2.22j), γ from (2.22f), U from (2.21c), ν from (2.20), λ from (22.22g), and μ from (2.22h). However, (2.17) and (2.18), each being but one of the class given in Sec. 2, are not uniquely specified. By considering the coordinate transformation between (2.17) and any coordinate system $\{u,r,x^m\}$ satisfying the coordinate conditions (2.12), where

$$\tilde{u}=A_0(r,x^m)+A_1(r,x^m)u+\cdots, \\ \tilde{r}=B_0(r,x^m)+B_1(r,x^m)u+\cdots, \\ \tilde{x}=Y_0^m(r,x^n)+Y_1^m(r,x^n)u+\cdots,$$

and the tetrad transformation between (2.18) and any tetrad system $\{\tilde{D},\tilde{\Delta},\tilde{\delta},\bar{\tilde{\delta}}\}$ related to (2.18) by spatial rotations (2.14), where

$$C(u,x^m)=C_1(x^m)u+C_2(x^m)u^2+\cdots,$$

and then considering the effects of these coordinate and tetrad transformations on certain metric variables and spin coefficients, it can be established that[14]:

The null coordinate system (2.17) and associated null tetrad system (2.18) can be specified up to scale transformations (3.1) by imposing the conditions

$$[\xi^m(0,0,x^n)]=(P,iP)/\sqrt{2}, \tag{3.4a}$$

where P is real,

$$\mu(0,0,x^m)=0, \tag{3.4b}$$

$$X^m(u,0,x^n)=0, \tag{3.4c}$$

$$\tau_0(x^m)=\tau(0,0,x^m)=i\eth g, \tag{3.4d}$$

where g is real and an analytic function of $(z,\bar{z})$ with spin weight zero,

$$U(u,0,x^m)=0, \tag{3.4e}$$

and

$$\gamma(u,0,x^m)=0. \tag{3.4f}$$

After Ψ_1 through μ are obtained on $u=0$ as previously indicated, only Ψ_3 and Ψ_4 remain to be determined there. They can be calculated from Eqs. (2.22k) and (2.23g), respectively, once $\Delta\phi(0,r,x^m)$ is calculated from Eq. (2.30). After this is done, the metric variables, spin coefficients, physical Weyl tensor components, and scalar field will be known on $u=0$. Once they are known on $u=0$, they can be determined off $u=0$ in a straightforward but tedious manner from the NP equations and the scalar field equation. By doing this, subject to the conditions (3.4), it can be established that:

An Einstein-scalar space-time containing a nondiverging null hypersurface has over the region $\{(u,r,x^m)\}$ metric variables

$$U = (\tfrac{1}{2}K - 3\tau_0\bar{\tau}_0 - \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0)r^2 + (\dot{U}_2 r^2 + \dot{U}_3 r^3)u + \cdots, \quad (3.5a)$$

$$X^m = 2(\bar{\tau}_0\xi_0^m + \tau_0\bar{\xi}_0^m)r + (\dot{X}_1^m r + \dot{X}_2^m r^2)u + \cdots, \quad (3.5b)$$

$$\xi^m = \xi_0^m + \{-\bar{\lambda}_0\bar{\xi}_0^m + [(\tfrac{1}{2}K - \tau_0\bar{\tau}_0 - \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0)\xi_0^m + (\eth\tau_0/\sqrt{2} - \tau_0^2 - \tfrac{1}{4}(\eth\phi_0)^2)\bar{\xi}_0^m]r\}u + \cdots; \quad (3.5c)$$

nonzero spin coefficients

$$\rho = (\tfrac{1}{2}K - \tau_0\bar{\tau}_0 - \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0)u + \cdots, \quad (3.6a)$$

$$\sigma = (\eth\tau_0/\sqrt{2} - \tau_0^2 - \tfrac{1}{4}(\eth\phi_0)^2)u + \cdots, \quad (3.6b)$$

$$\mu = -(\tfrac{1}{2}K - \tau_0\bar{\tau}_0 - \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0)r + \{(-\lambda_0\bar{\lambda}_0 - \tfrac{1}{2}\dot{\phi}_0^2) + \dot{\mu}_1 r + \dot{\mu}_2 r^2\}u + \cdots, \quad (3.6c)$$

$$\lambda = \{\lambda_0 + (\bar{\tau}_0^2 + \bar{\eth}\bar{\tau}_0/\sqrt{2} + \tfrac{1}{4}(\bar{\eth}\phi_0)^2)r\} + (\dot{\lambda}_0 + \dot{\lambda}_1 r + \dot{\lambda}_2 r^2)u + \cdots, \quad (3.6d)$$

$$\tau = \tau_0 + (\dot{\tau}_0 + \dot{\tau}_1 r)u + \cdots, \quad (3.6e)$$

$$\nu = (1/\sqrt{2})\bar{\eth}(-\tfrac{1}{2}K + 3\tau_0\bar{\tau}_0 + \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0)r^2 + (\dot{\nu}_2 r^2 + \dot{\nu}_3 r^3)u + \cdots, \quad (3.6f)$$

$$\alpha = (\alpha_0 + \tfrac{1}{2}\bar{\tau}_0) + (\dot{\alpha}_0 + \dot{\alpha}_1 r)u + \cdots, \quad (3.6g)$$

$$\beta = (-\bar{\alpha}_0 + \tfrac{1}{2}\tau_0) + (\dot{\beta}_0 + \dot{\beta}_1 r)u + \cdots, \quad (3.6h)$$

$$\gamma = (-\tfrac{1}{2}K + 3\tau_0\bar{\tau}_0 + \bar{\eth}\tau_0/\sqrt{2} + \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0 + 2\alpha_0\tau_0 - 2\bar{\alpha}_0\bar{\tau}_0)r + (\dot{\gamma}_1 r + \dot{\gamma}_2 r^2)u + \cdots; \quad (3.6i)$$

physical Weyl tensor components

$$\Psi_0 = \ddot{\Psi}_0 u^2 + \cdots, \quad (3.7a)$$

$$\Psi_1 = (-\eth K/2\sqrt{2} + \tfrac{3}{2}\tau_0 K + \tfrac{1}{2}\eth\bar{\eth}\tau_0 - 3\tau_0\bar{\eth}\tau_0/\sqrt{2} + \bar{\eth}\phi_0\eth^2\phi_0/4\sqrt{2} - \tfrac{1}{2}\tau_0\eth\phi_0\bar{\eth}\phi_0)u + \cdots, \quad (3.7b)$$

$$\Psi_2 = (-\tfrac{1}{2}K + \bar{\eth}\tau_0/\sqrt{2} + \tfrac{1}{6}\eth\phi_0\bar{\eth}\phi_0) + (\dot{\Psi}_2^0 + \dot{\Psi}_2^1 r)u + \cdots, \quad (3.7c)$$

$$\Psi_3 = \{(-\eth\lambda_0/\sqrt{2} - \tau_0\lambda_0 + \dot{\phi}_0\bar{\eth}\phi_0/2\sqrt{2}) + (-\bar{\eth}K/2\sqrt{2} - \tfrac{3}{2}\bar{\tau}_0 K + \tfrac{1}{2}\bar{\eth}^2\tau_0 + 3\bar{\tau}_0\bar{\eth}\tau_0/\sqrt{2} + \eth\phi_0\bar{\eth}^2\phi_0/4\sqrt{2} + \tfrac{1}{2}\bar{\tau}_0\eth\phi_0\bar{\eth}\phi_0)r\} + (\dot{\Psi}_3^0 + \dot{\Psi}_3^1 r + \dot{\Psi}_3^2 r^2)u + \cdots, \quad (3.7d)$$

$$\Psi_4 = (\Psi_4^0 + \Psi_4^1 r + \Psi_4^2 r^2) + (\dot{\Psi}_4^0 + \dot{\Psi}_4^1 r + \dot{\Psi}_4^2 r^2 + \dot{\Psi}_4^3 r^3)u + \cdots; \quad (3.7e)$$

and scalar field

$$\phi = \phi_0 + \{\dot{\phi}_0 + (\tfrac{1}{2}\bar{\eth}\eth\phi_0 - \bar{\tau}_0\eth\phi_0/\sqrt{2} - \tau_0\bar{\eth}\phi_0/\sqrt{2})r\}u + \cdots; \quad (3.8)$$

where $K = \bar{\eth}\eth \ln P$, $\alpha_0 = (1/2\sqrt{2})\eth \ln P$, and

$\dot{U}_2, \dot{U}_3, \dot{X}_1^m, \dot{X}_2^m, \dot{\mu}_1, \dot{\mu}_2, \dot{\lambda}_0, \dot{\lambda}_1, \dot{\lambda}_2, \dot{\tau}_0, \dot{\tau}_1, \dot{\nu}_2, \dot{\nu}_3, \dot{\alpha}_0, \dot{\alpha}_1, \dot{\beta}_1, \dot{\beta}_2, \dot{\gamma}_1, \dot{\gamma}_2, \ddot{\Psi}_0, \dot{\Psi}_2^0, \dot{\Psi}_2^1, \dot{\Psi}_3^0, \dot{\Psi}_3^1, \dot{\Psi}_3^2, \dot{\Psi}_4^1, \dot{\Psi}_4^2, \dot{\Psi}_4^1, \dot{\Psi}_4^2$ and $\dot{\Psi}_4^3$

are functions of (x^m) that can be given explicitly[14] in terms of the arbitrary functions of (x^m),

$$P, \tau_0, \lambda_0, \Psi_4^0, \dot{\Psi}_4^0, \phi_0, \text{ and } \dot{\phi}_0. \quad (3.9)$$

The quantities given by (3.5) through (3.8) are determined up to terms linear in u by specifying (3.9). Terms involving higher powers of u can similarly be obtained using Eqs. (2.21), (2.22), and (2.23), and (2.30), but they will depend on additional arbitrary functions of (x^m). To find these functions, consider Eqs. (2.23g) and (2.30). Equation (2.23g) determines $\Psi_4(u,r,x^m)$ but only up to an arbitrary function $\Psi_4(u,0,x^m)$. Moreover, this function cannot be determined from the remaining NP equations. The scalar field equation (2.30) determines $\Delta\phi(u,r,x^m)$ but only up to an arbitrary function which from (2.18b) is $\dot{\phi}(u,0,x^m)$. Moreover, this function cannot be determined from any of the NP equations. Since $\Psi_4(u,0,x^m)$ and $\dot{\phi}(w,0,x^m)$ are the only quantities in addition to (3.9) that are not determined by the NP equations or the scalar field equation, it has been established that:

The metric variables, spin coefficients, physical Weyl tensor components, and scalar field of an Einstein-scalar space-time containing a nondiverging null hypersurface are completely determined in the region $\{(u,r,x^m)\}$ from Eqs. (2.21), (2.22), (2.23), and (2.30) by specifying the arbitrary functions

$$P(x^m),$$
$$\tau_0(x^m) = \tau(0,0,x^m), \quad \lambda_0(x^m) = \lambda(0,0,x^m), \quad (3.10)$$
$$\Psi_4(u,0,x^m), \quad \text{and} \quad \phi(u,0,x^m).$$

The arbitrary functions (3.10) constitute the characteristic data for an Einstein-scalar spacetime containing a nondiverging null hypersurface. Before these data are discussed, the restrictions placed on them for which trapped surfaces develop will be determined. From (2.32) the spacelike two-surface $S_{(u,r)}$ is a trapped surface if and only if it is compact and everywhere on it $\rho > 0$ and $\mu < 0$. Since for fixed u and r the mapping

$$f_{(u,r)}: S_0 \to S_{(u,r)} \quad \text{where } f_{(u,r)}(0,0,x^m) = (u,r,x^m)$$

is a homeomorphism of S_0 onto $S_{(u,r)}$, $S_{(u,r)}$ is compact if and only if S_0 is compact. From (3.6a) and (3.6c) it is seen that if everywhere on S_0, the functions K, τ_0, and ϕ_0 satisfy

$$\tfrac{1}{2}K - \tau_0\bar{\tau}_0 - \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0 > 0, \quad (3.11)$$

then for any positive value of r, r_0, there exists a sufficiently small value of u, u_0, such that $\rho(u,r,x^m) > 0$ and $\mu(u,r,x^m) < 0$ for $0 < u \le u_0$ and $0 < r \le r_0$. Furthermore, since on $u = 0$ (3.5) through (3.8) are polynomials in r whose coefficients are analytic functions of (x^m), r_0 can be taken arbitrarily large on $u = 0$. With this it has been established that:

In the region $\{(u,r,x^m)\}$ of an Einstein-scalar space-time containing a nondiverging null hypersurface $u = 0$, trapped surfaces develop to the future of the $r > 0$ branch of this hypersurface if S_0 is compact and everywhere on it K, τ_0, and ϕ_0 satisfy

$$\tfrac{1}{2}K - \tau_0\bar{\tau}_0 - \tfrac{1}{4}\delta\phi_0\bar{\delta}\phi_0 > 0. \tag{3.12}$$

This result establishes the existence of nonspherically symmetric Einstein-scalar space–times that contain both a nondiverging null hypersurface and trapped surfaces.

In order to better understand (3.12), the characteristic data (3.10) will now be discussed. The function P is the most important of (3.10) for the development of trapped surfaces, since unless the spacelike two-surface with induced covariant metric,

$$-P^{-2}dz \otimes d\bar{z},$$

has strictly positive Gaussian curvature, $K = \bar{\delta}\delta \ln P$, there is no possibility of satisfying (3.11). In the case of a spherically symmetric space–time it is known that S_0 is a two-sphere with $K > 0$.[14] Although in the case of an arbitrary space–time, S_0 may be chosen to be a two-sphere, there do exist other compact two-surfaces with strictly positive Gaussian curvature and hence this choice is not imperative.

The function τ_0 is also very important for the development of trapped surfaces, since even if $K > 0$ and even in vacuum, the magnitude of τ_0 could be sufficiently large that (3.11) is violated. This possibility suggests that there may exist a relationship between τ_0 and angular momentum. Such a relationship can be obtained in the case of the linearized Kerr space–time, whose metric depends on two parameters m and a, where m is the mass and a is the angular momentum per unit mass. The components $(g^{\mu\nu})$ of the contravariant metric of this space–time are obtained relative to $\{u, r, \theta, \Phi\}$ from the components of the contravariant metric of the Kerr space–time,[17]

$$\frac{1}{(r^2 + a^2\cos^2\theta)} \times \begin{bmatrix} -a^2\sin^2\theta & (r^2+a^2) & 0 & -a \\ (r^2+a^2) & -(r^2-2mr+a^2) & 0 & a \\ 0 & 0 & -1 & 0 \\ -a & a & 0 & -\csc^2\theta \end{bmatrix},$$

by neglecting all terms that are quadratic in a. Therefore,

$$(g^{\mu\nu}) = \frac{1}{r^2}\begin{bmatrix} 0 & r^2 & 0 & -a \\ r^2 & -(r^2-2mr) & 0 & a \\ 0 & 0 & -1 & 0 \\ -a & a & 0 & -\csc^2\theta \end{bmatrix}.$$

Since $(g^{\mu\nu})$ does not satisfy the coordinate conditions (2.12), it is necessary to transform to a coordinate system $\{\tilde{u}, \tilde{r}, \tilde{\theta}, \tilde{\Phi}\}$ in which (2.12) is satisfied by $(\tilde{g}^{\mu\nu})$. This is accomplished by the coordinate transformation

$$\tilde{u} = -\exp(-u/4m), \quad \tilde{r} = 4m(r-2m)\exp(u/4m),$$
$$\tilde{\theta} = \theta, \quad \tilde{\Phi} = \Phi - \frac{a}{r} - (a/4m^2)u,$$

whose inverse transformation is

$$u = -4m\ln(-\tilde{u}), \quad r = (8m^2 - \tilde{u}\tilde{r})/4m,$$
$$\theta = \tilde{\theta}, \quad \Phi = \tilde{\Phi} + \frac{4am}{(8m^2 - \tilde{u}\tilde{r})} - \frac{a}{m}\ln(-\tilde{u}).$$

Under this transformation $(g^{\mu\nu})$ becomes

$$(\tilde{g}^{\mu\nu}) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 2\tilde{r}^2(8m^2 - \tilde{u}\tilde{r})^{-1} & 0 & \tilde{g}^{13} \\ 0 & 0 & -r^{-2} & 0 \\ 0 & \tilde{g}^{13} & 0 & -r^{-2}\csc^2\tilde{\theta} \end{bmatrix},$$

where

$$\tilde{g}^{13} = -a\left(\frac{1}{4m^2r} + \frac{1}{2mr^2} + \frac{1}{r^3}\right)\tilde{r}.$$

The null tetrad system that yields $(\tilde{g}^{\mu\nu})$ is $\{\tilde{D}, \tilde{\Delta}, \tilde{\delta}, \bar{\tilde{\delta}}\}$, where

$$\tilde{D} = \frac{\partial}{\partial\tilde{r}},$$

$$\tilde{\Delta} = \frac{\partial}{\partial\tilde{u}} + \frac{\tilde{r}^2}{(8m^2 - \tilde{u}\tilde{r})}\frac{\partial}{\partial\tilde{r}} - a\left(\frac{1}{4m^2r} + \frac{1}{2mr^2} + \frac{1}{r^3}\right)\tilde{r}\frac{\partial}{\partial\tilde{\Phi}},$$

$$\tilde{\delta} = \frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\tilde{\theta}} + i\csc\tilde{\theta}\frac{\partial}{\partial\tilde{\Phi}}\right).$$

The Lie brackets of this null tetrad system yield after a straightforward calculation the nonzero spin coefficients

$$\rho = \frac{\tilde{u}}{(8m^2 - \tilde{u}\tilde{r})}, \quad \tilde{\mu} = -\frac{8m^2\tilde{r}}{(8m^2 - \tilde{u}\tilde{r})^2},$$

$$\tilde{\tau} = -\frac{ia\sin\tilde{\theta}}{2\sqrt{2}}\left[\left(\frac{1}{4m^2} + \frac{1}{2mr} + \frac{1}{r^2}\right) + \left(\frac{1}{4m^2r} + \frac{1}{mr^2} + \frac{3}{r^3}\right)\tilde{u}\tilde{r}\right],$$

$$\tilde{\alpha} = -\cot\tilde{\theta}/2\sqrt{2}r + \tfrac{1}{2}\bar{\tilde{\tau}}, \quad \tilde{\beta} = \cot\tilde{\theta}/2\sqrt{2}r + \tfrac{1}{2}\tilde{\tau},$$

$$\tilde{\gamma} = -\frac{(8m^2 - \tilde{u}\tilde{r}/2)\tilde{r}}{(8m^2 - \tilde{u}\tilde{r})^2}.$$

The null tetrad system, these spin coefficients, and Eqs. (2.22b) and (2.22m) imply that the only nonzero physical Weyl tensor components are Ψ_1, Ψ_2, and Ψ_3. Furthermore, the metric variables and spin coefficients imply that the nonzero characteristic data for linearized Kerr space–time are

$$(\tilde{\xi}^n_0) = (1, i\csc\tilde{\theta})/2\sqrt{2}m \quad \text{and} \quad \tilde{\tau}_0 = -3ia\sin\tilde{\theta}/8\sqrt{2}m^2.$$

Additional evidence that τ_0 is related to angular momentum can be given by considering the propagation of the null tetrad system (2.18) along the generators of $u = 0$. A tetrad system is normally said to be propagated without rotation along a timelike curve if and only if it is Fermi propagated[18] along this curve, which in the case of a timelike geodesic is equivalent to being parallelly propagated. If this notion is extended to null geodesics, then it can be said that the null tetrad system (2.18) is propagated without rotation along the generators of $u = 0$ if and only if it is parallelly propagated along them. From (2.6) it can be shown that

$$n^\mu{}_{;\nu}\ell^\nu = \bar{\tau}m^\mu + \tau\bar{m}^\mu \quad \text{and} \quad m^\mu{}_{;\nu}\ell^\nu = \tau\ell^\mu.$$

Therefore Δ, δ, and $\bar{\delta}$ are parallelly propagated along the generators of $u = 0$ if and only if $\tau_0 = 0$.

Like τ_0, ϕ_0 is important for the development of trapped surfaces, since even if $K > 0$ and $\tau_0 = 0$, the magnitude of $\eth\phi_0$ could be sufficiently large that (3.11) is violated.

While the metric and scalar field are determined on $u = 0$ by specifying P, τ_0 and ϕ_0, they are determined in a neighborhood of $u = 0$ only by specifying $\lambda(0,0,x^m)$, $\Psi_4(u,0,x^m)$, and $\dot{\phi}(u,0,x^m)$. Additional significance of $\lambda(0,0,x^m)$, $\Psi_4(u,0,x^m)$, and $\dot{\phi}(u,0,x^m)$ can be shown by considering the $r = 0$ hypersurface. The general $r =$ constant hypersurface has a normal k where

$$k = k^\mu \frac{\partial}{\partial x^\mu} = g^{\mu\nu} r_{,\nu} \frac{\partial}{\partial x^\mu} = \delta_0^\mu \frac{\partial}{\partial x^\mu} = \frac{\partial}{\partial u}.$$

From this the $r =$ const hypersurface is spacelike, null, or timelike according to the sign of

$$g_{\mu\nu} k^\mu k^\nu = -2U$$

being positive, zero, or negative respectively on $r =$ const. The condition (3.4e) implies that the $r = 0$ hypersurface is null. From (3.4c) and (3.4e)

$$\Delta = \partial/\partial u$$

on $r = 0$. Therefore, the $r = 0$ hypersurface is a null hypersurface generated by null geodesics each with Δ as its tangent vector and each parameterized with an affine parameter u. From (2.31) the vectors tangent to these generators have divergence $\mu(u,0,x^m)$, zero rotation, and shear $\lambda(u,0,x^m)$. Equations (2.22m) and (2.22n), conditions (3.4), and (2.20) imply that

$$\begin{aligned} \dot{\mu}(u,0,x^m) &= -\mu^2(u,0,x^m) - \lambda(u,0,x^m)\bar{\lambda}(u,0,x^m) \\ &\quad - \tfrac{1}{2}\dot{\phi}^2(u,0,x^m), \\ \mu(0,0,x^m) &= 0, \\ \dot{\lambda}(u,0,x^m) &= -2\mu(u,0,x^m)\lambda(u,0,x^m) - \Psi_4(u,0,x^m), \\ \lambda(0,0,x^m) &= \lambda_0. \end{aligned} \tag{3.13}$$

Therefore through these $\lambda_0, \Psi_4(u,0,x^m)$, and $\dot{\phi}(u,0,x^m)$ determine $\mu(u,0,x^m)$ and $\lambda(u,0,x^m)$ and hence the optical properties of the generators of $r = 0$.

4. GENERAL SPACE-TIMES CONTAINING TRAPPED SURFACES

The possibility that there exist Einstein-scalar space-times more general than those containing a nondiverging null hypersurface that also contain trapped surfaces was investigated.[14] This investigation began by obtaining the metric variables, spin coefficients, physical Weyl tensor components, and scalar field of an Einstein-scalar space–time. Then the characteristic data for this space–time were determined and examined for restrictions placed on them for which trapped surfaces develop.

The metric variables, spin coefficients, physical Weyl tensor components, and scalar field of an Einstein-scalar space–time were obtained using the formalism presented in Sec. 2. To accomplish this, the main condition adopted was that, in a particular null coordinate system (2.17) and associated null tetrad system (2.18), these quantities are analytic functions of (u, r, x^m) in the region $\{(u,r,x^m)\}$. Since the procedure involved in obtaining these quantities is exactly that employed in Sec. 3, only the results will be stated here. By using the techniques of Sec. 3 subject to the conditions (3.4), it can be established that:

An Einstein-scalar space–time has over the region $\{(u,r,x^m)\}$

metric variables

$$\begin{aligned} U = \{[&\tfrac{1}{2}K - 3\tau_0\bar{\tau}_0 + (\sigma_0\lambda_0 + \bar{\sigma}_0\bar{\lambda}_0) - \tfrac{1}{2}\phi_1\dot{\phi}_0 \\ &- \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0]r^2 + \cdots\} + (\cdots)u + \cdots, \end{aligned} \tag{4.1a}$$

$$\begin{aligned} X^m = \{&2(\bar{\tau}_0\xi_0{}^m + \tau_0\bar{\xi}_0{}^m)r + \cdots\} + \{2(\eth\lambda_0/\sqrt{2} - 2\tau_0\lambda_0 \\ &- \dot{\phi}_0\bar{\eth}\phi_0/\sqrt{2})\xi_0^m + \overline{\eth\lambda}_0/\sqrt{2} - 2\bar{\tau}_0\bar{\lambda}_0 \\ &- \dot{\phi}_0\eth\phi_0)\bar{\xi}_0^m]r + \cdots\}u + \cdots, \end{aligned} \tag{4.1b}$$

$$\begin{aligned} \xi^m = \{&\xi_0^m + (\rho_0\xi_0^m + \sigma_0\bar{\xi}_0^m)r + \cdots\} + (-\bar{\lambda}_0\bar{\xi}_0^m \\ &+ \cdots)u + \cdots; \end{aligned} \tag{4.1c}$$

nonzero spin coefficients

$$\begin{aligned} \rho = \{&\rho_0 + (\rho_0^2 + \sigma_0\bar{\sigma}_0 + \tfrac{1}{2}(\phi_1)^2)r + \cdots\} + \{(\tfrac{1}{2}K - \tau_0\bar{\tau}_0 \\ &- \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0) + \cdots\}u + \cdots, \end{aligned} \tag{4.2a}$$

$$\begin{aligned} \sigma = \{&\sigma_0 + (2\rho_0\sigma_0 + \Psi_0^0)r + \cdots\} + \{(-\rho_0\bar{\lambda}_0 + \eth\tau_0/\sqrt{2} - \tau_0^2 \\ &- \tfrac{1}{4}(\eth\phi_0)^2) + \cdots\}u + \cdots, \end{aligned} \tag{4.2b}$$

$$\begin{aligned} \mu = \{&-(\tfrac{1}{2}K - \tau_0\bar{\tau}_0 - \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0)r + \cdots\} \\ &+ \{(-\lambda_0\bar{\lambda}_0 - \tfrac{1}{2}(\dot{\phi}_0)^2)r + \cdots\} + \cdots, \end{aligned} \tag{4.2c}$$

$$\begin{aligned} \lambda = \{&\lambda_0 + (\rho_0\lambda_0 + \overline{\eth\tau}_0/\sqrt{2} + \bar{\tau}_0^2 + \tfrac{1}{4}(\bar{\eth}\phi_0)^2)r + \cdots\} \\ &+ (-\Psi_4^0 + \cdots)u + \cdots, \end{aligned} \tag{4.2d}$$

$$\begin{aligned} \tau = \{&\tau_0 + (3\rho_0\tau_0 + \sigma_0\bar{\tau}_0 - \eth\rho_0/\sqrt{2} + \bar{\eth}\sigma_0/\sqrt{2} \\ &+ \phi_1\eth\phi_0)r + \cdots\} \\ &+ \{(\overline{\eth\lambda}_0/\sqrt{2} - \bar{\tau}_0\bar{\lambda}_0 - \dot{\phi}_0\eth\phi_0/\sqrt{2}) + \cdots\}u + \cdots, \end{aligned} \tag{4.2e}$$

$$\nu = -\delta U, \tag{4.2f}$$

$$\begin{aligned} \alpha = \{&(\alpha_0 + \tfrac{1}{2}\bar{\tau}_0) + (\tfrac{3}{2}\rho_0\bar{\tau}_0 + \tfrac{1}{2}\bar{\sigma}_0\tau_0 + \phi_1\bar{\eth}\phi_0/2\sqrt{2} \\ &+ \rho_0\alpha_0 - \bar{\sigma}_0\bar{\alpha}_0)r + \cdots\} + (\cdots)u + \cdots, \end{aligned} \tag{4.2g}$$

$$\begin{aligned} \beta = \{&(-\bar{\alpha}_0 + \tfrac{1}{2}\tau_0) + (\tfrac{3}{2}\rho_0\tau_0 + \tfrac{1}{2}\sigma_0\bar{\tau}_0 - \eth\rho_0/\sqrt{2} \\ &+ \bar{\eth}\sigma_0/\sqrt{2} + \phi_1\eth\phi_0/2\sqrt{2} + \sigma_0\alpha_0 - \rho_0\bar{\alpha}_0)r + \cdots\} \\ &+ (\cdots)u + \cdots, \end{aligned} \tag{4.2h}$$

$$\begin{aligned} \gamma = \{&(-\tfrac{1}{2}K + 3\tau_0\bar{\tau}_0 + \bar{\eth}\tau_0/\sqrt{2} - \sigma_0\lambda_0 - \bar{\sigma}_0\bar{\lambda}_0 \\ &+ \tfrac{1}{2}\phi_1\dot{\phi}_0 + \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0 + 2\alpha_0\tau_0 - 2\bar{\alpha}_0\bar{\tau}_0)r \\ &+ \cdots\} + (\cdots)u + \cdots; \end{aligned} \tag{4.2i}$$

physical Weyl tensor components

$$\Psi_0 = (\Psi_0^0 + \Psi_0^1 r + \cdots) + (\cdots)u + \cdots, \tag{4.3a}$$

$$\Psi_1 = (\cdots) + (\cdots)u + \cdots, \tag{4.3b}$$

$$\begin{aligned} \Psi_2 = \{&(-\tfrac{1}{2}K + \bar{\eth}\tau_0/\sqrt{2} - \sigma_0\lambda_0 + \tfrac{1}{6}\eth\phi_0\bar{\eth}\phi_0 \\ &+ \tfrac{1}{6}\phi_1\dot{\phi}_0) + \cdots\} + (\cdots)u + \cdots, \end{aligned} \tag{4.3c}$$

$$\begin{aligned} \Psi_3 = \{&(-\eth\lambda_0/\sqrt{2} - \tau_0\lambda_0 + \dot{\phi}_0\bar{\eth}\phi_0/2\sqrt{2}) + \cdots\} \\ &+ \{\cdots\}u + \cdots, \end{aligned} \tag{4.3d}$$

$$\Psi_4 = (\Psi_4^0 + \cdots) + (\dot{\Psi}_4^0 + \cdots)u + \cdots; \tag{4.3e}$$

and scalar field

$$\begin{aligned} \phi = (&\phi_0 + \phi_1 r + \cdots) + \{\dot{\phi}_0 + (\rho_0\dot{\phi}_0 + \tfrac{1}{2}\bar{\eth}\eth\phi_0 \\ &- \tau_0\bar{\eth}\phi_0/\sqrt{2} - \bar{\tau}_0\eth\phi_0/\sqrt{2})r + \cdots\}u + \cdots; \end{aligned} \tag{4.4}$$

where $K = \bar{\eth}\eth \ln P$, $\alpha_0 = (1/2\sqrt{2})\bar{\eth} \ln P$, and

$$P, \rho_0, \sigma_0, \tau_0, \lambda_0, \Psi_0^0, \Psi_0^1, \Psi_4^0, \dot{\Psi}_4^0, \phi_0, \phi_1, \text{ and } \dot{\phi}_0 \qquad (4.5)$$

are arbitrary functions of (x^m).

The terms displayed in (4.1) through (4.4) depend on the arbitrary functions (4.5). Terms involving higher powers of u and r can be obtained using Eqs. (2.21), (2.22), (2.23), and (2.30), but they will depend on additional arbitrary functions of (x^m). By reasoning similar to that used to obtain (3.10), it can be shown that:

The metric variables, spin coefficients, physical Weyl tensor components, and scalar field of an Einstein-scalar space–time are completely determined from Eqs. (2.21), (2.22), (2.23), and (2.30) in the region $\{(u, r, x^m)\}$ by specifying the arbitrary functions

$$P(x^m), \rho_0(x^m), \sigma_0(x^m), \tau_0(x^m), \lambda_0(x^m), \Psi_0(0, r, x^m), \Psi_4(u, 0, x^m), \phi(0, r, x^m), \phi(u, 0, x^m). \qquad (4.6)$$

The arbitrary functions (4.6) constitute the characteristic data for an Einstein-scalar space–time. The functions $P(x^m), \tau_0(x^m), \lambda_0(x^m), \Psi_4(u, 0, x^m), \phi(u, 0, x^m)$ were already discussed in Sec. 3. The remaining functions $\rho_0(x^m), \sigma_0(x^m), \Psi_0(0, r, x^m)$, and $\phi(0, r, x^m)$ through Eqs. (2.22a) and (2.22b) determine the divergence of D on $u = 0$, $\rho(0, r, x^m)$, and the shear of D on $u = 0, \sigma(0, r, x^m)$. Hence these additional functions determine the optical properties of the generators of $u = 0$.

Of (4.6) it is P, ρ_0, τ_0, and ϕ_0 that are important for the development of trapped surfaces; since from (4.2a) and (4.2c), if everywhere on S_0

$$\rho_0 \geq 0 \quad \text{and} \quad \tfrac{1}{2}K - \tau_0\bar{\tau}_0 - \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0 > 0,$$

then there exist sufficiently small positive real numbers u_0 and r_0 such that $\rho > 0$ and $\mu < 0$ for $0 < u \leq u_0$ and $0 < r \leq r_0$. Therefore by (2.32) the two-surface $S_{(u,r)}$ for $0 < u \leq u_0$ and $0 < r \leq r_0$ are trapped surfaces if and only if they are compact, which was shown in Sec. 3 to be equivalent to S_0 being compact. With this it has been established that:

The region $\{(u, r, x^m)\}$ of an Einstein-scalar space–time contains trapped surfaces $S_{(u,r)}$ for some range of u and r, $0 < u \leq u_0$ and $0 < r \leq r_0$, if S_0 is compact and everywhere on it ρ_0, K, τ_0 and ϕ_0 satisfy

$$\rho_0 \geq 0 \quad \text{and} \quad \tfrac{1}{2}K - \tau_0\bar{\tau}_0 - \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0 > 0. \qquad (4.7)$$

This result establishes the existence of Einstein-scalar space–times more general than those containing a nondiverging null hypersurface that also contain trapped surfaces.

5. SUMMARY AND CONCLUSIONS

An investigation of the characteristic development of trapped surfaces in Einstein-scalar space–times, of which the empty space–times are a special case, was discussed in this paper. After presenting the formalism used in this investigation in Sec. 2, the characteristic development of trapped surfaces in Einstein-scalar space–times containing a nondiverging null hypersurface was considered in Sec. 3. The main result of this section is (3.12), which states that in a region $\{(u, r, x^m)\}$ of an Einstein-scalar space–time containing a nondiverging null hypersurface $u = 0$, trapped surfaces develop to the future of the $r = 0$ branch of this hypersurface if the spacelike two-surface $S_0 = \{(u, r, x^m) : u = 0 = r\}$ is compact and everywhere on it the characteristic data $P(x^m)$, $\tau_0(x^m)$, and $\phi_0(x^m)$ satisfy (3.11), $\tfrac{1}{2}K - \tau_0\bar{\tau}_0 - \tfrac{1}{4}\eth\phi_0\bar{\eth}\phi_0 > 0$, where $K = \bar{\eth}\eth \ln P$ is the Gaussian curvature of S_0. If (3.11) is to be satisfied at all, then K must be strictly positive. This suggests that the collapse of an object maintaining a cylindrical or toroidal shape during collapse cannot result in the formation of trapped surfaces, since a cylinder has zero Gaussian curvature[19] and a torus has a region of negative Gaussian curvature.[19] Indeed, it has been shown that at least one cylindrical collapse model results in complete collapse without the formation of trapped surfaces.[20] The possibility that the magnitude of τ_0 could be sufficiently large that (3.11) is not satisfied even if $K > 0$, and even in vacuum, suggests that τ_0 is related to angular momentum. The role of τ_0 in the criteria for the existence of trapped surfaces, even in vacuum, and the evidence presented in support of its interpretation in terms of angular momentum are considered to be important results of this investigation. With the presence of angular momentum and asymmetries in the scalar field indicated by τ_0 and $\eth\phi_0$ respectively, (3.12) emphasizes the importance of these quantities in determining whether or not trapped surfaces develop in Einstein-scalar space–times containing a nondiverging null hypersurface. The metric variables, spin coefficients, physical Weyl tensor components, and scalar field for these space–times are determined in some neighborhood of $u = 0$ by the characteristic data (3.10) through (3.5), (3.6), (3.7), and (3.8), respectively. The data $\lambda_0(x^m), \psi_4(u, 0, x^m)$, and $\dot{\phi}(u, 0, x^m)$ have the additional significance of determining the optical properties of the $r = 0$ null hypersurface through (3.13).

In Sec. 4 the existence of Einstein-scalar space–times more general than those containing a nondiverging null hypersurface that also contain trapped surfaces was established. The main result of this section is that in a region $\{(u, r, x^m)\}$ of an Einstein-scalar space–time, trapped surfaces develop to the future in some neighborhood of S_0 if S_0 is compact and everywhere on it the characteristic data $\rho_0(x^m), P(x^m), \tau_0(x^m)$, and $\phi_0(x^m)$ satisfy (3.11) and $\rho_0 \geq 0$. The metric variables, spin coefficients, physical Weyl tensor components, and scalar field are determined in some neighborhood of S_0 by the characteristic data (4.6) through (4.1), (4.2), (4.3), and (4.4), respectively. The additional characteristic data for these space–times, $\rho_0(x^m), \sigma_0(x^m), \Psi_0(0, r, x^m)$, and $\phi(0, r, x^m)$, were shown to determine the optical properties of the generators of the $u = 0$ null hypersurface through (2.22a) and (2.22b).

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[1] R. Penrose, Phys. Rev. Lett. **14**, 57 (1965).
[2] See, for example, S. W. Hawking and R. Penrose, Proc. Roy. Soc. (London) A **314**, 529 (1970).
[3] D. W. Pajerski and E. T. Newman, J. Math. Phys. **12**, 1929 (1971).
[4] E. T. Newman and R. Penrose, J. Math. Phys. 3, 566 (1962).

[5]Greek indices and the lower-case Latin indices a through d range and sum over the values $(0,1,2,3)$ and the remaining lower case Latin indices range and sum over the values $(2, 3)$.

[6]The parentheses denote symmetrization: $A_{(\mu\nu)} = \frac{1}{2}(A_{\mu\nu} + A_{\nu\mu})$; and the brackets denote antisymmetrization: $A_{[\mu\nu]} = \frac{1}{2}(A_{\mu\nu} - A_{\nu\mu})$.

[7]The physical components $T_{a_1 \ldots a_s}{}^{b_1 \ldots b_t}$ relative to a tetrad system $\{z_a^\mu \partial/\partial x^\mu\}$ of a tensor $T_{\mu_1 \ldots \mu_s}{}^{\nu_1 \ldots \nu_t}$ are given by $T_{a_1 \ldots a_s}{}^{b_1 \ldots b_t} = z_{a_1}{}^{\mu_1} \ldots z_{a_s}{}^{\mu_s} z_{\nu_1}{}^{b_1} \ldots z_{\nu_t}{}^{b_t} T_{\mu_1 \ldots \mu_s}{}^{\nu_1 \ldots \nu_t}$.

[8]A comma denotes partial differentiation and a semicolon denotes covariant differentiation.

[9]$\{x^\mu\}$ is the coordinate system; $\{(x^\mu)\}$ is the region consisting of all points of the manifold covered by $\{x^\mu\}$, and (x^μ) is a point in the region $\{(x^\mu)\}$.

[10]These equations can also be found in S. W. Hawking, J. Math. Phys. **9**, 598 (1968). Misprints in Hawking's Eqs. (3.32) and (3.41) are corrected in Eqs. (2.23b) and (2.23k), respectively.

[11]Units are employed in which the gravitational constant and the speed of light are of unit magnitude.

[12]R. Sachs, Proc. Roy. Soc. (London) A **264**, 309 (1961).

[13]The subscript notations $(a|c|b)$ and $[a|c|b]$ denote symmetrization and antisymmetrization, respectively, with respect to a and b.

[14]P. Demmie, Ph.D. thesis (University of Pittsburgh, 1971).

[15]L. P. Eisenhart, *An Introduction to Differential Geometry* (Princeton U.P., Princeton, New Jersey, 1947), pp. 261–65.

[16]E. T. Newman and R. Penrose, J. Math. Phys. **7**, 863 (1966).

[17]This metric follows from that given in E. Newman *et al.*, J. Math. Phys. **6**, 918 (1965), by setting $e = 0$. It should be emphasized that by linearized it is meant linearized in a.

[18]See, for example, J. L. Synge, *Relativity: The General Theory* (North-Holland, Amsterdam, 1960), Chap. III, Sec. 8.

[19]See, for example, B. O'Neill, *Elementary Differential Geometry* (Academic, New York, 1966), pp. 231–36.

[20]K. S. Thorne, "Nonspherical Gravitational Collapse—A Short Review," (preprint, 1971).