High-energy unitarity of gravitation and strings

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It is known that the behavior of a four-point string amplitude at large center-of-mass energy $\sqrt{s}$ and fixed momentum transfer $q=\sqrt{-t}$ is not perturbative. We study this region of phase space by summing multiple Reggeized graviton exchange in the eikonal approximation in D space-time dimensions. It is argued that the eikonal sum is at least representative of the summation of the leading powers of s in a string theory. The masslessness and high spin of the (Reggeized) graviton determine the character of the result. For $\kappa^2 sq^{D-4}\lesssim 1$ (κ is the gravitational coupling), the eikonal amplitude is dominated by single Reggeized graviton exchange. The amplitude in the region $\kappa^2 sq^{D-4}\gg 1$ is quite nonperturbative in character: simple Regge behavior and the Froissart bound are violated, and the amplitude does not satisfy a fixed-momentum-transfer dispersion relation. Although order by order the amplitude exhibits in q^2 the exponential decrease of Regge behavior, the final amplitude has only power-law falloff dependent on the number of space-time dimensions but independent of the Regge slope. The unitarity of the partial-wave projections of the eikonal amplitude is also studied. It is demonstrated that for $D\geq 4$ noncompact dimensions, the partial-wave amplitudes are bounded as $s\to\infty$ only for large values of angular momentum, $l\gtrsim x_0\sqrt{s}$, where x_0 is the dominant value of the impact parameter. A heuristic argument is presented that the eikonal approximation is successful in unitarizing Reggeized graviton exchange as $t/s\to 0$ in four dimensions but not in higher dimensions.

I. INTRODUCTION

The unitarity of closed-string amplitudes at high energy appears to be a challenging and interesting issue. While well behaved at fixed angles $O(e^{-s})$, closed-string tree amplitudes grow faster than s for momentum transfers $|q^2|\lesssim 1/\ln s$ effectively because of single Reggeized graviton exchange. At least in more than six dimensions, the infrared behavior is sufficiently suppressed to allow for a straightforward formulation of the appropriate partial-wave (PW) unitarity conditions for the exact amplitudes. The rapid growth at small momentum transfer occurs over a sufficiently large interval that the tree amplitudes do not satisfy the PW unitarity conditions $|a_l(s)|\leq 1$ for a fixed partial-wave number.[1] This situation is in contrast with the old Fermi theory of weak interactions. There, the tree amplitudes do not satisfy the relevant PW unitarity conditions (for the exact amplitudes) because of growth in the amplitudes proportional to $G_F s$ at fixed angles; the strong coupling of an essentially short-distance region of momentum transfer, indicated at the tree level, is associated with nonrenormalizable divergences at one loop. In a closed-string theory, the strong coupling at small momentum transfers,[2] indicated at the tree level, is not expected to be associated with ultraviolet divergences at finite order. Nevertheless, the region of momentum transfer over which the "bad" growth occurs increases at each order; therefore, it is a nontrivial matter to determine the full range in momentum transfer that is affected and the exact behavior of amplitudes at high energies.

The PW unitarity conditions (when applicable) should be satisfied by any exact four-body amplitude of a theory, if the theory is to be consistent. If the partial-wave projection of a tree-level amplitude does not satisfy the bounds on the exact amplitude, then at a minimum one must sum to all orders to obtain an amplitude that satisfies the bounds. The perturbative unitarity of the loop expansion guarantees that an exact amplitude is unitary if the perturbation series is sufficiently well behaved. However, perturbative unitarity or the formal Hermiticity of some Hamiltonian alone is not sufficient to guarantee the unitarity of the theory. Fermi theory presents an extreme case in this regard. The precise conditions on the perturbation series for a sensible restoration of unitarity are not known, in general, for a quantum field theory.

We are led then to investigate the behavior of the summation of string loop amplitudes. It will be evident that Reggeized graviton exchange with its incumbent high-energy growth arising from Regge intercept 2 and its low-momentum-transfer singularity present special problems in this context. Of course, a complete summation of the string loop expansion is a formidable task at this time. However, a more modest goal is the summation to all orders of the leading power growth in s.

In this work we sum direct and cross exchanges of Reggeized gravitons by a semiclassical eikonal approximation. Our treatment can be motivated both by analogy to previous studies and by the soft ultraviolet behavior of string amplitudes. It was suggested in the 1960s that Regge intercepts greater than 1 are not compatible with unitarity.[3,4] However, Cheng and Wu[5] demonstrated within the context of a leading-logarithm approximation based on a field theory with neutral massive vector mesons that unitarity is restored for the example of the light-by-light tower amplitude which has Regge intercept greater than 1. Their technique involved the relativistic eikonal representation based upon the leading contributions at high energy; it is well known that outside of the leading-term approximation the relativistic eikonal approximation fails.[6] Since these investigations contained the exchange of massive particles, their results lead to amplitudes that satisfy the Froissart bound.[4] It is natural, then, to investigate high-energy unitarity in string theory by a similar technique. In the context of string theory we will argue that the eikonal is an approximation to a leading power summation in s, which may be exact in some regards.

At fixed momentum transfer and high s we find an asymptotic behavior that is inherently nonperturbative. The amplitude has a fixed power behavior dependent on the number of space-time dimensions and independent of the Regge slope, in contrast with the exponential falloff of Regge behavior present at finite orders. This nonperturbative behavior at fixed momentum transfer suggests, but does not necessarily imply, that there might be difficulties out at fixed angles where string amplitudes are alleged to be soft. We also investigate generally the question of the unitarity of the amplitudes. Since we are dealing with amplitudes involving the exchange of a massless particle, namely, the graviton, we cannot expect the Froissart bound to be satisfied. Instead, we rely on the PW unitarity conditions to study the unitarity of our result. It is shown that the eikonal amplitude is unitary only for large values of the angular momentum in four or more dimensions. Nevertheless, the eikonalization is more successful in unitarizing single-graviton exchange in four dimensions than in (noncompact) higher dimensions.

The organization of the paper is the following. In Sec. II the eikonal method is reviewed and motivated for our application to string theory. In Sec. III the high-energy behavior of the eikonal amplitude is calculated. In Sec. IV we discuss the unitarity of the eikonal amplitude. A number of technicalities are left to the Appendixes.

II. EIKONAL METHOD

We would like to investigate whether or not the summation of the string loop expansion successfully restores unitarity. Since a complete summation of the string loop expansion is a formidable task, a more modest goal is the summation to all orders of the leading power growth in s. We will work with the eikonal series for Reggeized graviton exchange, and argue that at least in some regards it is a good approximation to the leading power series in a string theory.

The eikonal approximation is usually associated with quantum field theory. However, there is reason to believe that it should be useful in the context of string theory as well. First, note that at large s and small momentum transfer a closed-string tree amplitude is described easily as the exchange of massless Reggeized particles with the Reggeized graviton dominating, e.g.,

$$M=-2\kappa^2\beta_g(t)s^{2+\alpha' t}+O(s^{1+\alpha' t}) ,$$

where $\beta_g(t)\to 1/t$ as $t\to 0$. As the momentum transfer increases beyond the fixed-t region the simplest version of this picture breaks down, but not before the amplitude is exponentially small. It is natural to expect that the leading powers of s in higher orders come from the iteration of the source of the worst growth at tree level, i.e., soft Reggeized graviton exchange. The eikonal series is good in describing the iterated exchange of soft particles.

We examine the scattering of two particles, taken to be scalars (dilatons) for simplicity, by multiple Reggeized graviton exchange. To preserve gauge invariance, all possible direct and cross exchanges are included. Consider the limit of large s and fixed momentum transfer q. It is most convenient to work in a center-of-mass frame in which the momentum transfer is approximately transverse, $q^2=-\mathbf{q}_\perp^2$ (see Appendix A for more details). Then the Nth term in the eikonal series is

$$M_N=-i2s\frac{i^N}{N!}(\kappa^2 s)^N(2\pi)^{(2-D)(N-1)} \times \int d\mathbf{k}_\perp(1)\cdots d\mathbf{k}_\perp(N)\delta^{D-2}\left[\mathbf{q}_\perp-\sum_{j=1}^{N}\mathbf{k}_\perp(j)\right] \times\prod_{l=1}^{N}\frac{F[\mathbf{k}_\perp^2(l),s]}{\mathbf{k}_\perp^2(l)} . \tag{1}$$

Here κ is normalized so that in D space-time dimensions the coupling of a soft graviton to a fast scalar of momentum p is $i2\sqrt{2}\kappa p^\mu p^\nu$, where μ and ν specify the polarization of the graviton. A minimal form for F is $F[\mathbf{k}_\perp^2,s]=s^{-\alpha'\mathbf{k}_\perp^2}$, which corresponds to a form factor for tree-level Reggeized graviton exchange at small momentum transfer. More generally, F can be taken to be a four-point tree amplitude containing the graviton Regge trajectory divided by $2\kappa^2 s^2/\mathbf{k}_\perp^2$. The difference between the two will be unimportant.

The approximate expression, Eq. (1), can be motivated by arguments largely analogous to those presented for the relativistic eikonal approximation in particle field theory.[7] We consider graphs of the type illustrated in Fig. 1. The Reggeization of the gravitons implies that the dominant contributions arise when the momenta flowing through the graviton lines (dashed lines) are all soft. The net momentum transfer to a heavy line at any point is then assumed to be small, an assumption which is certainly safe at fixed order. Under this assumption the propagators of the heavy lines may be linearized. The sum of the resulting loop amplitudes of order κ^{2N} factorizes into the form of Eq. (1) in the relevant kinematic region. Graphs in which Reggeized gravitons in-

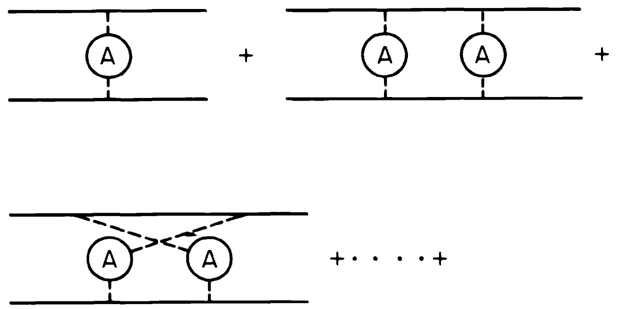

FIG. 1. Multiple exchange of Born graph in all possible ways.

teract with each other should be relatively suppressed because soft graviton self-interactions are weak. The neglect of higher intermediate states in the s channel (inelasticity) could well be justified since we are interested in the small-momentum-transfer region. In four dimensions only elastic scattering is singular in the infrared region and these assumptions for the relativistic eikonal approximation are plausible. Inelasticity in intermediate states typically leads to extra powers of $\alpha' k^2$ in the integrand due to gauge invariance. However, in higher-dimensional space-time the graphs are not infrared singular and string excitations in the s channel may contribute in leading order as well. It is important to note that, perhaps up to logarithms, Eq. (1) exhibits the leading dependence on s expected from general considerations of Regge theory with trajectories of decreasing slope and increasing intercept.[2] The general, expected dependence, $s^{\alpha_N(t)}$, where $\alpha_N(t)=N\alpha(t/N^2)-N+1$, is realized with linear trajectories: $\alpha(t)=2+\alpha' t$.

After summing over N the relativistic eikonal amplitude in terms of an impact-parameter representation is obtained:

$$M(s,q^2)=-2is\int d^{D-2}\mathbf{x}_\perp e^{i\mathbf{q}_\perp\cdot\mathbf{x}_\perp}\left(e^{i\kappa^2 s A(|\mathbf{x}_\perp|,s)}-1\right)\,. \tag{2}$$

Here A is the Fourier transform of $F[\mathbf{k}_\perp^2,s]/\mathbf{k}_\perp^2$ over the transverse dimensions. The result is manifestly unitary in impact-parameter space. At sufficiently large angular momentum (l), the impact parameter $x_\perp=|\mathbf{x}_\perp|$ and $2l/\sqrt{s}$ can be identified. Therefore, the amplitude should be unitary at sufficiently large l. However, the eikonal approximation was motivated at fixed q and some direct indication of unitarity at fixed momentum transfer is desirable as well. In Appendix A it is argued that the eikonal series is perturbatively unitary outside of the fixed-angle region if the eikonal approximation is self-consistent. Even if the eikonal series is not precisely the leading series in a string theory, it is a sum which reasonably can be expected to be s-channel unitary and which maintains many of the relevant, characteristic features expected in a string calculation.

It is remarkable that string theory enables us to make this analysis in a somewhat credible way. In previous theories of gravity, with their incumbent ill-behaved ultraviolet properties, this discussion would not have been a possibility. Actually, our results rely only weakly on the details of string theory and are true in any theory of gravity with convergent enough ultraviolet behavior such that the manipulations leading to Eq. (1) are valid.

There is an important caveat in this whole investigation; namely, we are investigating the high-energy limit when $s \gg M_{PL}^2$. Consequently, implicit assumptions are being made concerning the consistency and existence of a theory of gravity at these energies.

III. ASYMPTOTICS OF THE EIKONAL AMPLITUDE

An elementary calculation from Eq. (2) gives

$$\begin{aligned} M=&-i2(2\pi)^{(D-2)/2} s q_\perp^{-(D-4)/2}\\ &\times\int_0^\infty x_\perp^{(D-2)/2} dx_\perp J_{(D-4)/2}(x_\perp q_\perp)\\ &\times\left(e^{i\kappa^2 s A(x_\perp,s)}-1\right)\,, \end{aligned} \tag{3}$$

where $J_{(D-4)/2}(y)$ is a Bessel function of order $(D-4)/2$. The behavior of the amplitude for large s will depend strongly on the asymptotic behavior of A for large $x_\perp$.

A satisfactory form for A corresponds to single Reggeized graviton exchange:

$$A(x_\perp,s)=\int\frac{d^{D-2}\mathbf{k}_\perp}{(2\pi)^{D-2}}e^{-i\mathbf{k}_\perp\cdot\mathbf{x}_\perp}\frac{F(\mathbf{k}_\perp^2,s)}{\mathbf{k}_\perp^2}\,, \tag{4}$$

where

$$F(\mathbf{k}_\perp^2,\mathrm{s})=-\mathrm{s}^{-\alpha'\mathbf{k}_\perp^2}\beta_g(-\mathbf{k}_\perp^2)\mathbf{k}_\perp^2$$

is a form factor that gives the damping at large $\mathbf{k}_\perp^2$ characteristic of Regge behavior. In the regions $x_\perp\to\infty$ and $x_\perp\to 0$, A behaves as

$$\begin{aligned} \lim_{x_\perp\to\infty} A(x_\perp,s)=&\tfrac{1}{4}\pi^{-(D-2)/2}\Gamma\left[\frac{D-4}{2}\right]\frac{1}{x_\perp^{D-4}}\\ &+O\!\left(e^{-x_\perp^2/(\alpha'\ln s)}\right)\,,\\ \lim_{x_\perp\to 0} A\simeq&\,1\,. \end{aligned} \tag{5}$$

These results apply for $D>4$ where an infrared cutoff is unnecessary. Some details are presented in Appendix B. For $D=4$ we obtain

$$\lim_{x_\perp\to\infty} A(x_\perp,s)=-\frac{1}{2\pi}\ln(x_\perp\lambda_{\mathrm{IR}})\,, \tag{6}$$

where we have equated $1/(D-4)$ with $-\ln\lambda_{\mathrm{IR}}$, ignoring some irrelevant constants. In general, the dependence of A on s is weak. From Eqs. (5) and (6) it is evident that the integral of Eq. (3) is only convergent for $D>5$. The amplitudes in $D=4$ and 5 will be defined by continuation.

We proceed with an asymptotic, large-s expansion of Eq. (3) which is quite nontrivial. The asymptotic behavior depends on the size of the dimensionless quantity $\kappa^2\tau$, where $\tau=sq_\perp^{D-4}$. First we demonstrate that the Born approximation is valid in the region $\kappa^2\tau\lesssim 1$. We rewrite the eikonal amplitude, adding and subtracting the Born term:

$$M=-2i(2\pi)^{(D-2)/2}sq_\perp^{2-D/2}\int_0^\infty x_\perp^{(D-2)/2}dx_\perp J_{(D-4)/2}(x_\perp q_\perp)[e^{i\kappa^2 sA(x_\perp,s)}-1-i\kappa^2 sA(x_\perp,s)]+\frac{2\kappa^2 s^2}{q_\perp^2}F(s,q_\perp^2)\ . \qquad (7)$$

The first term is nonsingular as $q_\perp\to 0$ for $D>6$. The leading term in the expansion of the Bessel function for small argument yields a contribution which is independent of $q_\perp$. Using the asymptotic properties of A, Eq. (5), it is straightforward to show for large s the term independent of $q_\perp$ is proportional to $s^{1+(D-2)/(D-4)}$. Then

$$M=\frac{2\kappa^2 s^2}{q_\perp^2}F(q_\perp^2,s)[1+O((\kappa^2\tau)^{2/(D-4)})]\ . \qquad (8)$$

Through case-by-case study the result holds for $D=5$ and 6 as well. Therefore, the Born term dominates out to a region of size $\kappa^2 sq_\perp^{D-4}\sim 1$.

To investigate the amplitude's behavior outside of the interval $0<q_\perp\lesssim(\kappa^2 s)^{1/(4-D)}$, we proceed with a stationary phase analysis since we expect rapid oscillations of the integrand in Eq. (3) except where the phase is stationary.[8] A stationary phase point of A will lead to a contribution to the amplitude that can grow no faster than $\sqrt{s}$. Therefore, we search for a stationary phase point where the oscillations of the Bessel function beat against the phase $\kappa^2 sA$; this can only occur at large values of $x_\perp$ as A approaches 0. Use the asymptotic expansion of the Bessel function in Eq. (3),

$$J_{(D-4)/2}(y)=\left[\frac{2}{\pi y}\right]^{1/2}\left[\cos\left[y-\frac{\pi}{4}(D-3)\right]-\frac{(D-4)^2-1}{8y}\sin\left[y-\frac{\pi}{4}(D-3)\right]+O\left[\frac{1}{y^2}\right]\right]\ . \qquad (9)$$

For our definition of couplings, it is clear that only the exponential in the leading term of Eq. (9) proportional to e^{iy} will lead to a stationary phase point. After a rescaling of Eq. (3) and after use of Eq. (9), Eq. (3) becomes

$$M\simeq 2e^{-i(D-1)\pi/4}(2\pi)^{(D-3)/2}sq_\perp^{-(D-2)}\lambda^{(D-1)/2}\times\int_0^\infty du\, u^{(D-3)/2}(e^{i\lambda f(u)}-1)\ . \qquad (10)$$

Here $\lambda=c(\kappa^2\tau)^{1/(D-3)}$ is a large parameter,

$$c=\left[\frac{D-4}{4}\pi^{-(D-2)/2}\Gamma\left[\frac{D-4}{2}\right]\right]^{1/(D-3)}$$

and

$$f(u)=u+u^{4-D}\frac{1}{D-4}\ .$$

An analysis with the stationary-phase technique yields

$$\lim_{\kappa^2\tau\to\infty}M=2ie^{-i(D\pi/4)}\frac{(2\pi c)^{(D-2)/2}}{\sqrt{D-3}}\times\frac{\kappa^2 s^2}{q_\perp^2}(\kappa^2\tau)^{-(D-4)/[2(D-3)]}\times[e^{i\lambda f(u_0)}+O(\lambda^{-1/2})]\ , \qquad (11)$$

where u_0 is the point of stationary phase: $u_0=1$. The stationary-phase analysis is slightly unconventional because the term "-1" in the integrand of Eq. (10) is necessary for good convergence of the integral at large u. A careful examination of the integral shows that indeed the contribution from the region of $u\gg 1$ is relatively suppressed by a factor of $O(1/\sqrt{\lambda})$, as is the contribution of the -1 term for $u\lesssim 1$.

The result in four dimensions is defined by the limit $D\to 4$. The only quantity in Eq. (11) singular in the limit is $f(u_0)$. Again, as in Eq. (6), we interpret the pole, $1/(D-4)$, as the logarithm of an infrared cutoff $\lambda_{\rm IR}$. Then, in four dimensions,

$$\lim_{\kappa^2 s\to\infty}M=-2i\frac{\kappa^2 s^2}{q_\perp^2}\left[\frac{q_\perp^2}{\lambda_{\rm IR}^2}\right]^{i\kappa^2 s/(4\pi)}\times\exp\left\{-i\frac{\kappa^2 s}{2\pi}\left[\ln\left[\frac{\kappa^2 s}{2\pi}\right]-1\right]\right\}\ . \qquad (12)$$

Note that the limit $\kappa^2\tau\to\infty$ is simply the limit $\kappa^2 s\to\infty$; the form holds for all $q_\perp$.

The same result at sufficiently small t can be obtained from the work of Cheng and Wu on QED in four dimensions.[5] They worked with an explicit infrared cutoff. Using the substitution $e^2\to-\kappa^2 s$ one can convert their result for the sum of soft-photon exchanges into the above result for the sum of soft (Reggeized) graviton exchanges. The presence of the phase $(\lambda_{\rm IR}^2)^{-i\kappa^2 s/(4\pi)}$ was argued for long ago by Weinberg on general grounds.[9] The magnitude of the amplitude is simply that of single non-Reggeized graviton exchange. The reader will recall that this circumstance is similar to the case of Coulomb scattering, which is dominated by single-photon exchange even when the Coulomb phase is taken into account in the scattering wave function.

There are a number of comments to be made concerning Eqs. (11) and (12). Remarkably, although Regge behavior is present order by order, it is absent after summation. The amplitudes exhibit only power-law falloff in $q_\perp$ in a way which depends on the dimension of space-time but is independent of the Regge slope. On the other hand, the amplitudes are power bounded in s in the physical region; for fixed $q_\perp$ the amplitudes do not grow faster than s^2 for real s in any dimension. However, at fixed $q_\perp$ the amplitudes do grow faster than s, which is not in accord with the Froissart bound. The Froissart bound, though, is not applicable both because of the masslessness of the graviton and because the amplitudes are not power bounded in the complex s plane. The amplitudes do not satisfy fixed-t dispersion relations. These relations are quite different than those given by Cheng and Wu[4] where the Froissart[5] bound is saturated by an absorptive amplitude with finite-range interactions. The nonperturbative nature of the result in $D>4$ is underscored by the feature that the amplitude is proportional to κ^2 raised to a fractional power. These results are attributable to the high-energy growth and infrared singularity of Reggeized graviton exchange. They suggest that the all-orders, leading, high-energy behavior of amplitudes in a perturbatively finite theory of gravitation is fixed solely by the dimension of space-time and the gravitational coupling constant.

A few remarks on the limitations of the results should be made. The singularities in Eq. (11) at $D=3$ are spurious because the stationary-phase analysis is not valid in three dimensions. This case must be treated separately. The behavior at fixed angles is not reliable. Nevertheless, it is surprising that the sum exhibits only power-law falloff at fixed angles while at fixed order there is exponential decrease. Apparently, in this instance it is preferable for the large momentum transfer to be shared in many soft exchanges rather than a few hard ones.

The amplitudes carry special phase information. The particular form of the phases and power-law falloff in $q_\perp$ will be important in the next section.

IV. UNITARITY

In this section the unitarity of the relativistic eikonal amplitude, Eq. (2), is analyzed. As mentioned previously, it is demonstrated in Appendix A that the eikonal series is perturbatively unitary outside of the fixed-angle region as long as the eikonal approximation is self-consistent. However, even if it applies, perturbative unitarity in this sense does not guarantee the unitarity of the PW amplitudes in the eikonal approximation because of possibly large contributions from fixed angles. The unitary conditions, $|a_l(s)|\le 1$, for the PW amplitudes are examined using stationary-phase techniques. Although for large l we find that the unitarity conditions are satisfied, for small l they are not. In the eikonal approximation there is not enough phase oscillation or strong enough decrease in momentum transfer to produce unitary in all partial waves. While the trajectory $\alpha_N(t)$ is less than 1 at some large spacelike t, the region of "bad" growth in s is pushed out to successively larger t. In fact, the source of the worst unitarity violation is traced to the behavior of the eikonal amplitude at fixed angles. We will also consider the asymptotic behavior of the partially integrated amplitudes, obtained by integrating only out to a limited region of momentum transfer. We find for small l that only in four dimensions are the partially integrated amplitudes bounded as $s\to\infty$.

The PW projection of the eikonal amplitude can be treated in a very general way. Firstly, recall the expansion and projection formulas given in terms of Gegenbauer polynomials:[1]

$$M(s,t)=\lambda_D s^{2-D/2}\sum_{l=0}^{\infty}\frac{1}{N_l^\nu}C_l^\nu(1)C_l^\nu(\cos\theta)a_l(s)\ , \tag{13}$$

$$a_l(s)=\frac{s^{D/2-2}}{\lambda_D C_l^\nu(1)}\int_0^\pi d\theta(\sin\theta)^{D-3}C_l^\nu(\cos\theta)M(s,t)\ .$$

Here

$$\lambda_D=2\Gamma\left[\frac{D}{2}-1\right](16\pi)^{D/2-1}\ ,$$

$$N_l^\nu=\frac{2^{1-2\nu}\pi\Gamma(l+2\nu)}{\Gamma(l+1)\Gamma^2(\nu)(\nu+l)}\ ,$$

and

$$C_l^\nu(1)=\frac{\Gamma(l+2\nu)}{\Gamma(l+1)\Gamma(2\nu)}$$

with $\nu=(D-3)/2$.

If the eikonal formula, Eq. (2), is substituted into Eq. (13), the angular integration can be performed with the aid of a formula due to Sonnine (see Ref. 10):

$$\int_0^\pi d\theta(\sin\theta)^{(D-2)/2}C_l^{(D-3)/2}(\cos\theta)J_{(D-4)/2}\left[\frac{x\sqrt{s}}{2}\sin\theta\right]=i^l\left[\frac{4\pi}{x\sqrt{s}}\right]^{1/2}C_l^\nu(0)J_{l+(D-3)/2}\left[\frac{x\sqrt{s}}{2}\right]\ , \tag{14}$$

where

$$C_l^\nu(0)=\tfrac{1}{2}[1+(-)^l](-)^{l/2}\frac{\Gamma\left[\frac{l}{2}+\nu\right]}{\Gamma\left[\frac{l}{2}+1\right]\Gamma(\nu)}\ .$$

Then the PW amplitudes can be written exactly in terms of an integral over impact parameter.

$$a_l(s)=-i^{l+1}\frac{(4\pi)^{(D-1)/2}}{\lambda_D}\frac{C_l^\nu(0)}{C_l^\nu(1)}s^{(D-1)/4}\times\int_0^\infty dx\,x^{(D-3)/2}J_{l+(D-3)/2}\left[\frac{x\sqrt{s}}{2}\right]\times(e^{i\kappa^2 sA(x,s)}-1)\ . \tag{15}$$

We next study the extent to which the PW amplitudes are bounded (as $s\to\infty$). It is again appropriate to employ stationary-phase techniques in order to examine the asymptotic behavior of $a_l(s)$ at large s. Using the standard asymptotics of the Bessel function in Eq. (9) we obtain

$$a_l(s) \simeq -i^{l+1} \frac{(4\pi)^{(D-1)/2}}{\sqrt{\pi}\lambda_D} \frac{C_l^\nu(0)}{C_l^\nu(1)} s^{(D-2)/4}$$

$$\times \int_0^\infty dx\, x^{D/2-2}$$

$$\times \exp\left\{ i \left[\frac{x\sqrt{s}}{2} - \frac{\pi}{2}\left[l + \frac{D-2}{2} \right] \right] \right\}$$

$$\times (e^{i\kappa^2 s A(x,s)} - 1) \; . \quad (16)$$

An analysis along the same lines as that of the last section yields the result that the PW amplitudes at large s are of order

$$a_l(s) \sim s^{(D-2)/4} C_l^\nu(0)/C_l^\nu(1) \; .$$

Consider first the case when l is large. Then

$$C_l^\nu(0)/C_l^\nu(1) \sim l^{-(D-3)/2}$$

and

$$a_l(s) = O(l^{-(D-3)/2} s^{(D-2)/4}) \quad (l \text{ large}) \; . \quad (17)$$

Equation (17) implies that for $l \gtrsim s^{(D-2)/[2(D-3)]}$ the PW amplitudes are bounded. Actually, for $l \gg 1$ the asymptotic expansion of the Bessel function $J_l(y)$ is as in Eq. (9) only as long as $y \gtrsim l$; if $y \lesssim l$ the Bessel function is exponentially small.[10] Denote the stationary phase point of Eq. (16) as $x_0(s) \sim s^{1/[2(D-3)]}$. When l is not too large, l less than

$$\sqrt{s}\, x_0(s) \sim s^{(D-2)/[2(D-3)]} \; ,$$

the stationary phase point is in the region where the Bessel function oscillates and the above form is good in leading order. Otherwise, $a_l(s)$ is smaller than indicated because there is actually no stationary phase point. Thus, the PW unitarity conditions indeed are satisfied for $l \gtrsim \sqrt{s}\, x_0(s)$. This type of result is to be expected[11] because for such l's the long-range peripheral nature of graviton exchange is sampled.

In contrast, the PW amplitudes at fixed l and large s are not bounded.

$$a_l(s) = O(s^{(D-2)/4}) \quad (l \text{ fixed}) \; . \quad (18)$$

In Appendix C we demonstrate for the $l=0$ partial wave that growth of this order arises only from the contribution of the fixed-angle region. The eikonal amplitude cannot be expected to be accurate or unitary at fixed angles. A proper treatment of the fixed-angle region involves short-distance physics and more detailed information about string multiloop amplitudes than is available presently.[12]

The eikonal approximation was motivated in Sec. II as a technique to unitarize Reggeized graviton exchange in, e.g., the fixed-momentum-transfer region. As argued in Appendix A the relativistic eikonal amplitude should be unitary outside of the fixed-angle region if the eikonal approximation is self-consistent. It is desirable then to have some test of unitarity outside of the fixed-angle region. The usual check of unitarity at fixed momentum transfer is the Froissart bound; however, as remarked above, the Froissart bound is not applicable to the case at hand. If the theory is to be well behaved in the ultraviolet, then the fixed-angle region probably should not be so strongly coupled that some partial wave receives an unbounded contribution from the fixed-angle region. Typically, one expects the amplitude to fall on the average as t decreases from 0 out to the fixed-angle region. It then seems difficult for distinct regions of momentum transfer to make unbounded contributions to some partial wave which in fact cancel. Therefore, as a nonrigorous check of unitarity, we examine the boundedness of the amplitude if it is integrated out to a particular region of momentum transfer, as opposed to a complete PW projection. We will refer to this quantity as the partially integrated amplitude,

$$a_l(s,t_0) = [1+(-)^l] \frac{2^{D-3} s^{D/2-3}}{\lambda_D C_l^\nu(1)}$$

$$\times \int_{t_0}^0 dt \left[-\frac{t}{s} - \frac{t^2}{s^2} \right]^{D/2-2}$$

$$\times C_l^\nu(1+2t/s) M(s,t) \; . \quad (19)$$

Crossing symmetry for $M(s,t)$ has been assumed, so that $a_l(s) = a_l(s, t_0 = -s/2)$. Heuristically, the amplitude in a region of momentum transfer about t_0 will be considered to violate unitarity if that region makes an unbounded contribution to $a_l(s,t_0)$ for some l. We will focus on $a_0(s,t_0)$ below.

We can now obtain a heuristic understanding of the observation of Sec. III that single Reggeized graviton exchange dominates only in the interval $0 \le \kappa^2\tau \lesssim 1$, where $\tau \simeq s(-t)^{D/2-2}$. If $M(s,t) \sim \kappa^2 s^{2+\alpha' t}/t$, then, for small t_0, $a_0(s,t_0) \sim \kappa^2 \tau_0$. Consequently, $a_0(s,t_0)$ would become unbounded outside of this small interval in t if the amplitude still were to be dominated by single exchange.

It is shown in Sec. III that the eikonal amplitude in the region $\kappa^2\tau \gg 1$ differs markedly from single Reggeized graviton exchange. It is interesting to investigate whether the amplitudes of Eqs. (11) and (12) are unitary in the heuristic sense explained above. At first sight the growth faster than s of the amplitudes in the fixed-t region would appear to imply that $a_0(s,t_0)$ would be unbounded for t_0 fixed, in any dimension. However, the phase oscillations are very important in this context. Looking back at Eq. (11), the integration over t necessary to construct $a_0(s,t_0)$ effectively brings down a power of $(\kappa^2 s)^{-1/(D-3)}$ from the phase in the leading term. It is not obvious that the subleading terms of Eq. (11) do not dominate in the calculation of $a_0(s,t_0)$. However, it is shown in Appendix D that the correct leading result for $a_0(s,t_0)$ is given just by using the leading form of the eikonal amplitude. Substituting Eq. (11) into Eq. (19) gives the result that, for t_0 outside of the fixed-angle region, but $\kappa^2\tau_0 \gg 1$,

$$a_0(s,t_0) = O((\kappa^2\tau_0)^{(D-4)/[2(D-3)]}) \; . \quad (20)$$

In order to make sense out of the continuation of this result to $D=4$, it is necessary to make the (reasonable) assumption that after the S matrix is defined properly in four dimensions, it is possible to perform a PW analysis in essentially the usual way. Then in four dimensions for t_0 outside of the fixed-angle region,

$$a_0(s,t_0)\simeq\exp\left\{-i\frac{\kappa^2 s}{2\pi}\left[\ln\left[\frac{\kappa^2 s}{2\pi}\right]-1\right]\right\}\times\lim_{\epsilon\to 0}\left[\exp\left[i\frac{\kappa^2 s}{4\pi}\ln(\epsilon^2/\lambda_{\rm IR}^2)\right]-\exp\left[i\frac{\kappa^2 s}{4\pi}\ln(-t_0/\lambda_{\rm IR}^2)\right]\right] . \tag{21}$$

Presumably, the ill-defined oscillatory nature of $a_0(s,t_0)$ as $\epsilon\to 0$ would be amended in a more careful treatment.

The eikonal approximation has successfully unitarized Reggeized graviton exchange outside of the fixed-angle region apparently only in four dimensions even though the argument of Appendix A did not distinguish explicitly four dimensions. A possible explanation is the following. The behavior of the eikonal series appears to be determined by high orders. In four dimensions there is an infrared enhancement when the cumulative momentum transfer is small if consistent with the external kinematics, because of the infrared logarithmic singularities of the graphs. On the other hand, for $D>4$ there is no such enhancement. Each interaction tends to transfer momentum of order $1/\sqrt{\alpha'}$. In large orders, the accumulation of the individually small momentum kicks apparently can become large with high probability.

V. CONCLUSIONS

Perhaps the most striking conclusion of this investigation is the nonperturbative character of eikonalized amplitude. In particular, simple Regge behavior is violated as well as the Froissart bound and fixed-momentum-transfer dispersion relations. In fact the J-plane singularity of our results is an essential singularity due to an accumulation point of moving Regge cuts at $J=\infty$. There are suggestive but certainly inconclusive indications that the behavior of the string loop expansion in the fixed-angle region is nonperturbative as well; this issue is also a challenging and interesting one. The particular infrared properties of four dimensions appear to facilitate the unitarization of Reggeized graviton exchange. Therefore, it is attractive to speculate that unitarity is the general principle that allows for a consistent theory of gravity in four dimensions and excludes this possibility in higher noncompact dimensions.

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APPENDIX A

The perturbative unitarity of the eikonal series is examined in this appendix. The term of order N in the series is given in Eq. (1). To clarify the notation, write

$$M_N=\langle n' \mid M_N \mid n \rangle . \tag{A1}$$

Here $\mid n' \rangle$ and $\mid n \rangle$ are two-body states describing elastic scattering. Each two-body state defines an axis in the center-of-mass frame, e.g., $\mid n \rangle$ defines $\hat{\mathbf{n}}$. Transverse is defined relative to the axis $\hat{\mathbf{n}}$, so that the momentum transfer is approximately transverse outside of the fixed-angle region. Perturbative elastic unitarity at order N of the eikonal series would imply the equality of the two quantities

$$\frac{1}{16(2\pi)^{D-2}}(s/4)^{D/2-2}\sum_{j=1}^{N-1}\int d\Omega_n \langle n \mid M_j \mid f \rangle^* \times\langle n \mid M_{N-j} \mid i \rangle \tag{A2}$$

and

$$\mathrm{Im}\langle f \mid M_N \mid i \rangle . \tag{A2$'$}$$

Here Ω_n are the solid angles specifying the axis $\hat{\mathbf{n}}$. The equality cannot be exact, but should hold to a good approximation if two conditions are met.

The first condition is an external one, that the angle between $\hat{\mathbf{i}}$ and $\hat{\mathbf{f}}$ should not be fixed as the center-of-mass energy goes to infinity. If this condition is not satisfied, then the axes used to define transverse for the various matrix elements cannot be essentially identical. Therefore, the eikonal amplitude cannot be expected to be unitary when describing fixed-angle scattering.

The second condition is relevant then for scattering outside of the fixed-angle region. It is the dynamical condition that the dominant intermediate states are those for which $\hat{\mathbf{n}}$ is nearly parallel to $\hat{\mathbf{i}}$ and $\hat{\mathbf{f}}$. Again this is required for the relative consistency of the notion of transverse in (A2) and (A2$'$). Furthermore, if this condition is met than the integral over solid angle can be approximated by

$$\int d\Omega_n \simeq (s/4)^{1-D/2}\int d^{D-2}\mathbf{p}_\perp ,$$

where the integral over transverse momentum is unrestricted. The approximate equality of (A2) and (A2$'$) follows. Note that the second condition should be met when considering scattering outside of the fixed-angle region if the eikonal approximation is self-consistent.

APPENDIX B

In this appendix we examine the detailed behavior of the amplitude A in Eq. (4). Quite generally, A can be written as

$$A=\int\frac{d^{D-2}k}{(2\pi)^{D-2}}e^{-ik\cdot x}\frac{F(k^2,s)}{k^2} , \tag{B1}$$

where $F(k^2,s)$ is of the form $-k^2\sum\beta(-k^2)s^{\alpha(k^2)}$. The sum is over the various Regge trajectories contributing to the string tree amplitude. It is obvious that

$$\nabla^2 A(x,s) = -\tilde{F}(x,s)$$

$$= -\int \frac{d^{D-2}k}{(2\pi)^{D-2}} F(k^2,s)e^{-ik\cdot x} . \tag{B2}$$

This equation can be solved by Green's function techniques:

$$A(x,s) = c' \int d^{D-2}x' \frac{1}{|x-x'|^{D-4}} \tilde{F}(x',s) , \tag{B3}$$

where c' is the constant

$$\tfrac{1}{4}\pi^{-(D-2)/2}\Gamma\left[\frac{D-4}{2}\right] .$$

It then follows for nonsingular $\tilde{F}$ that $A(x,s)$ is $\mathcal{O}(1)$ as $x \to 0$. In addition, if $\tilde{F}$ is a member of L_1, it then follows that $A(x,s)$ is $\mathcal{O}(x^{4-D})$ as $x \to \infty$. These results are certainly satisfied by string tree graphs with their incumbent soft high-energy behavior. For the particular case of the form factor in Eq. (4),

$$F(k^2,s) \simeq -k^2\beta_g(-k^2)s^{-\alpha' k^2} ,$$

$$\tilde{F}(x,s) \simeq \exp(-x^2/4b) ,$$

where

$$b = \alpha' \ln s + \frac{d}{dk^2}\ln[k^2\beta_g(k^2)]\bigg|_{k^2=0} .$$

APPENDIX C

Although the analysis of the full PW amplitudes for fixed l sketched in Sec. IV is concise, it does not elucidate how various regions of t contribute to the overall violation of the PW unitarity conditions. In this appendix, the asymptotic result of Eq. (18) for $a_0(s)$ including the prefactor is compared with the $l=0$ projection of the stationary-phase point amplitude, the leading term of Eq. (11). The two results are shown to agree. The second method of calculation makes it clear that the region of t in which the worst violation of unitarity occurs is the region about $t=-s/2$

The $l=0$ projection of the stationary-phase amplitude is

$$a_{0,\mathrm{SP}}(s) = \frac{2^{D-2}s^{D/2-3}}{\lambda_D} \times \int_{-s/2}^{0} dt \left[-\frac{t}{s}-\frac{t^2}{s^2}\right]^{D/2-2} \times M_{\mathrm{SP}}(s,q_1^2) , \tag{C1}$$

where $M_{\mathrm{SP}}(s,q_1^2)$ is given by the leading term in Eq. (11), and $-q_1^2 = t + t^2/s$. After the change of variables $z = 1+2t/s$,

$$a_{0,\mathrm{SP}}(s) = d_D \int_0^1 dz(1-z^2)^{-1+k(k+1)/(2k+1)} \times \exp\left[i\left[\frac{2k+1}{2k}\right]\xi(1-z^2)^{k/(2k+1)}\right] , \tag{C2}$$

where $k=(D-4)/2$,

$$\xi = 2^{2k/(2k+1)}c\kappa^{2/(2k+1)}s^{1/2+1/[2(2k+1)]}$$

[c is given after Eq. (10)], and

$$d_D = -i\frac{4}{\lambda_D}e^{-ik\pi/2}2^{2(k+1)^2/(2k+1)}\frac{(2\pi)^{k+1}}{\sqrt{2k+1}} \times c^{k+1}(\kappa^2)^{(k+1)/(2k+1)}s^{(k+1)^2/(2k+1)} . \tag{C3}$$

The integral has a good stationary-phase point at the boundary, $z=0$. Evaluation of the contribution of the stationary-phase point gives the result

$$a_{0,\mathrm{SP}}(s) = 2(-4\pi i)^{(D-1)/2}e^{i[(D-3)/(D-4)]\xi} \times \left[\frac{2}{D-3}\right]^{1/2}\frac{c^{(D-3)/2}}{\lambda_D}\kappa s^{(D-2)/4} . \tag{C4}$$

This is precisely the result given in Eq. (18) if the constants in that analysis are kept. Note that the point $z=0$ corresponds to scattering at 90°.

APPENDIX D

Below, a calculation of $a_0(s,t_0)$ for t_0 outside the fixed-angle region is sketched in which the errors can be controlled. It is difficult to give explicitly the leading errors in a stationary-phase analysis. Therefore, it is advantageous to organize the calculation of $a_0(s,t_0)$ so that only one integration is performed using stationary-phase techniques and it is the last one.

The calculation begins with a rewriting of Eq. (19). Use a representation of the Bessel function due to Hankel (see Ref. 10):

$$J_k(y) = \tfrac{1}{2}[H_k^{(1)}(y)+H_k^{(2)}(y)] ,$$

$$H_k^{(1)}(y) = \left[\frac{2}{\pi y}\right]^{1/2}\frac{\exp\left[i\left[y-\frac{k\pi}{2}-\frac{\pi}{4}\right]\right]}{\Gamma(k+\frac{1}{2})} \times \int_0^\infty du\, e^{-u}u^{k-1/2}\left[1+\frac{iu}{2y}\right]^{k-1/2} , \tag{D1}$$

where $H_k^{(2)}(y)$ is given by the substitution of $-i$ for i in the above. Substituting Eqs. (3) and (D1) into Eq. (19) gives $a_0(s,t_0)$ as a threefold integral. The integration over impact parameter $x_\perp$ should be performed last, by stationary-phase techniques. By familiar arguments the dominant stationary-phase point should exist at large $x_\perp$. Therefore, the other two integrations only need to be performed in the limit of large $x_\perp$. The required identity is

$$\frac{1}{\Gamma(k+\frac{1}{2})}\int_0^\infty du\, e^{-u}u^{k-1/2}\int_{t_0}^0 dt(-t)^{(D-5)/4}e^{ix_\perp\sqrt{-t}}\left[1+\frac{iu}{2x_\perp\sqrt{-t}}\right]^{(D-5)/2}$$
$$\to -2\frac{i}{x_\perp}(-t_0)^{(D-3)/4}\left\{e^{ix_\perp\sqrt{-t_0}}\left[1+O\left[\frac{1}{x_\perp\sqrt{-t_0}}\right]\right]+O\left[\left[\frac{1}{x_\perp\sqrt{-t_0}}\right]^{(D-3)/2}\right]\right\}. \quad \text{(D2)}$$

The result of performing the integration over impact parameter in the stationary-phase approximation is

$$a_0(s,t_0)\simeq e^{-iD\pi/4}\frac{2^D}{\lambda_D}\frac{(2\pi)^{(D-2)/2}}{\sqrt{D-3}}c^{(D-4)/2}(\kappa^2\tau_0)^{(D-4)/[2(D-3)]}\exp\left[i\frac{D-3}{D-4}c(\kappa^2\tau_0)^{1/(D-3)}\right]. \quad \text{(D3)}$$

Here c is given following Eq. (10). This is precisely the result in Eq. (20) if the prefactor there is evaluated.

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