

# Gravitational shock waves for Schwarzschild and Kerr black holes

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The metrics of gravitational shock waves for a Schwarzschild black hole in Schwarzschild coordinates and for a Kerr black hole in Boyer-Lindquist coordinates are derived. The Kerr metric is discussed for two cases: the case of a Kerr black hole moving parallel to the rotational axis, and moving perpendicular to the rotational axis. Then, two properties from the derived metrics are investigated: the shift of a null coordinate and the refraction angle crossing the gravitational shock wave. Astrophysical applications for these metrics are briefly discussed.

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## I. INTRODUCTION

The metrics of black holes which are derived from the vacuum solution of Einstein's equation have been investigated for many years [1]. But the space-time structure of *moving* black holes have not been studied as much. Recently, several papers have been published investigating this problem. In particular, the problem of gravitational waves which are emitted when two black holes encounter each other, and of the scattering of elementary particles with Planck scale energies are discussed [2,3]. In this paper, we investigate the metric when a black hole (Schwarzschild or Kerr) is moving at the limit of light velocity.

Let us consider the case of a Schwarzschild black hole. When a black hole is moving relative to an observer far from the black hole, the metric derived for the rest system of the black hole should be Lorentz transformed to that of the observer's system. Generally, the metric of a black hole is not invariant for Lorentz transformations, so the transformed metric has a different structure from that for the black hole system. If a black hole is moving at the limit of light velocity in the  $z$  direction (Lorentz  $\gamma$  factor  $\rightarrow \infty$ ), the metric is changed to the form

$$\lim_{\gamma \rightarrow \infty} ds^2 \rightarrow -du dv + dx^2 + dy^2 - A(\rho)\delta(u)du^2, \quad (1.1)$$

where we put  $\rho^2 = x^2 + y^2$ ,  $u = t - z$ , and  $v = t + z$ . It is clear from this form that this metric describes a space-time which has two flat half-spaces  $u > 0$  and  $u < 0$  and the axially symmetric plane gravitational wave concentrated on the interface  $u = 0$ . Moreover, the infinite Lorentz boost has changed the algebraic type of the Weyl tensor from the type  $D$  to the type  $N$  for the limiting metric (1.1). The corresponding curvature tensor van-

ishes everywhere except at the surface  $u = 0$ , where its nonzero components have singularities of  $\delta(u)$ . Such a metric is often called "the gravitational shock wave" [2,4,5].

Aichelburg and Sexl [6] (AS) have obtained the shock wave metric by performing an infinite Lorentz transformation of the metric of a Schwarzschild black hole for isotropic coordinates. Their result is

$$A(\rho) = 4p \ln \rho^2. \quad (1.2)$$

Here,  $p = \gamma M$  and only the leading term in  $\gamma$  is retained. Also in this paper, we take into account only the leading order of  $\gamma$  in  $A(\rho)$ .

In Sec. II, we calculate the metric of a Schwarzschild black hole in Schwarzschild coordinates when the infinite Lorentz transformation is performed. Then, the limit of  $\gamma \rightarrow \infty$  is carried out by the method of Loustó and Sánchez [7]: first, carry out a Lorentz boost with a finite  $\gamma$ . Next, integrate the result by  $u$ . Then, take the limit  $\gamma \rightarrow \infty$ , and finally, differentiate it by  $u$ . With these procedures, the metric of the gravitational shock wave of the form of (1.1) is obtained. In this method, it is not necessary to take the limit of  $M \rightarrow 0$  like Aichelburg and Sexl did, so that we can examine the space-time structure near the region  $\rho \sim 2M$ , the Schwarzschild horizon.

In Sec. III, we seek for a metric of a Kerr black hole in Boyer-Lindquist coordinates after taking the infinite Lorentz transformation in the same way. We find that the metric for the case of a black hole moving parallel to the rotational axis ( $z$  axis) is different from that for perpendicular motion. In Sec. III A, the parallel case is investigated in detail. After obtaining the metric in general, we ask for the metric for the special case of the maximally rotating black hole  $M \rightarrow a$ , where  $a$  is the angular momentum per unit mass. In this case, the event horizon is at  $\rho = a$ , equal to the case of a stationary black hole. On the other hand, the metric far from the horizon almost coincides with the metric of Schwarzschild type given by Sec. II. In Sec. III B, the metric for moving in

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the perpendicular direction (we take the  $x$  direction) is calculated. To see the maximal effect about a Kerr black hole, we restrict our discussion only on the equatorial plane (i.e.,  $z=0$ ). The point we wish to emphasize is that the metric depends on the sign of  $y$  (i.e., perpendicular to the moving direction and the rotating direction). This dependence comes from  $g_{t\phi}$  component of the Kerr metric. The metric for  $M \rightarrow a$  is also given as before.

Next, two properties of the obtained metrics are investigated. There are general properties for the gravitational shock wave for the metric of (1.1) [4], that is, the discontinuity and the refraction of a geodesic. For a geodesic which crosses the shock wave located at  $u=0$ , the discontinuity  $\Delta v$  at  $u=0$  is given by (see Appendix A)

$$\Delta v = -2A(\rho)|_{\rho=\rho_0}, \quad (1.3)$$

where  $\rho_0$  is the value of  $\rho$  when the geodesic reaches  $u=0$ . And as illustrated in Fig. 1, the refraction angle  $\Delta\phi$  is given by

$$\tan(\Delta\phi) = \frac{1}{2} \frac{dA(\rho)}{d\rho} \Big|_{\rho=\rho_0}. \quad (1.4)$$

These  $\Delta v$  and  $\Delta\phi$  are calculated for the metrics obtained in Secs. II and III. In particular, when a Kerr black hole moves in the  $x$  direction, the refraction angle depends on the sign of  $y$ . This is to be attributed to the dragging effect of the Kerr black hole.

The gravitational shock wave metric is not only the academic interest of theoretical physics, but also is applicable to the actual astrophysics. In fact, this metric was employed by D'Eath [2] to solve the problem of the scattering of two ultrarelativistic black holes which are approaching in parallel. He assumed that the radiation from a head-on collision of two black holes of equal masses is fairly isotropic and concluded that the efficiency of the transformation of the gravitational energy into radiation is close to 25%. On the other hand, the

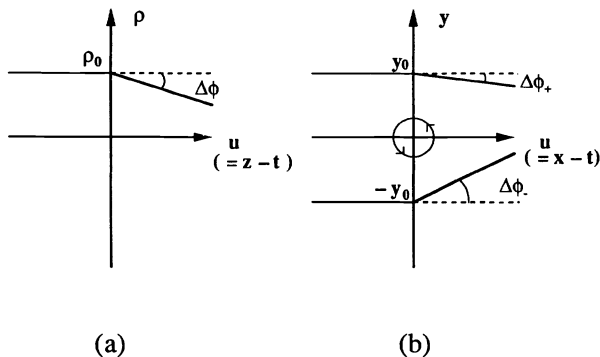


FIG. 1. The refraction angle,  $\Delta\phi$ , of a null geodesic when the light is injected perpendicular to the shock wave. (a) The Schwarzschild case and the Kerr case moving parallel to the rotational axis. The black hole is moving in the  $z$  direction. (b) The Kerr case moving perpendicular to the axis. The black hole is moving in the  $x$  direction. An arrow of the circle near the center represents the rotational direction of the Kerr black hole.

upper limit of the efficiency would be found on the basis of Hawking's theorem and is equal to  $\sim 25\%$  [8]. It is very interesting that these two values coincide.

The gravitational shock waves have also attracted interest in recent years since they shed light on modeling quantum scattering processes at very high energies, where gravitational effects dominate [3].

In Sec. IV, we consider as an astrophysical application, the effect which is analogous to that of the gravitational lens. When there is a static supermassive black hole between an observer and a source which is receding by a high redshift, the refraction of light rays radiated from the source is calculated. In that case, the metric of the gravitational shock wave is dealt with as the first-order approximation of the astrophysical body which is running at very large velocity. However, the most distant quasar observed at present has  $\gamma \sim 3$ , so the result is only approximate in nature.

Section V is devoted to conclusions and discussions.

We use the natural unit of  $c = G = 1$ , throughout.

## II. GRAVITATIONAL SHOCK WAVES FOR SCHWARZSCHILD BLACK HOLES

We consider a Schwarzschild metric in Schwarzschild coordinates [1]:

$$\begin{aligned} ds_S^2 &= -(1-2M/\bar{r})d\bar{t}^2 + (1-2M/\bar{r})^{-1}d\bar{r}^2 \\ &\quad + \bar{r}^2 d\bar{\theta}^2 + \bar{r}^2 \sin^2 \bar{\theta} d\bar{\phi}^2 \\ &= ds_M^2 + \Delta S_S^2, \end{aligned} \quad (2.1)$$

where

$$\begin{aligned} ds_M^2 &= -d\bar{t}^2 + d\bar{r}^2 + \bar{r}^2 d\bar{\theta}^2 + \bar{r}^2 \sin^2 \bar{\theta} d\bar{\phi}^2, \\ \Delta S_S^2 &= (2M/\bar{r})d\bar{t}^2 + (2M/\bar{r})/(1-2M/\bar{r})d\bar{r}^2. \end{aligned}$$

We set  $\bar{r}^2 = \bar{x}^2 + \bar{y}^2 + \bar{z}^2$ , and  $M$  is the mass of the black hole.  $ds_M^2$  is the metric of the Minkowski space-time, and  $\Delta S_S^2$  represents the surplus term. Now, perform a Lorentz transformation to the system where the black hole is moving in the  $z$  direction with the velocity  $v$  close to 1:

$$t = \gamma(\bar{t} + v\bar{z}), \quad z = \gamma(\bar{z} + v\bar{t}), \quad x = \bar{x}, \quad y = \bar{y}, \quad (2.2)$$

where

$$\gamma = (1-v^2)^{-1/2}.$$

As  $ds_M^2$  of (2.1) is invariant for Lorentz transformations, only  $\Delta S_S^2$  is transformed according to (2.2). Putting  $u = t - z$ , and  $v = t + z$ , old coordinates,  $\bar{x}$ 's, are changed as follows:

$$\begin{aligned} \bar{t} &\rightarrow \gamma u, \\ \bar{z} &\rightarrow -\gamma u, \\ \bar{r}^2 &\rightarrow \gamma^2 u^2 + \rho^2, \\ d\bar{r} &\rightarrow \frac{\gamma u du}{\sqrt{u^2 + \rho^2}} + O(1/\gamma), \end{aligned} \quad (2.3)$$

where

$$\rho^2 = x^2 + y^2, \quad \tilde{\rho} = \rho/\gamma.$$

As mentioned in Sec. I,  $g_{uu}$  is calculated to  $O(\gamma)$ .  $\Delta S_S^2$  is then given by

$$\Delta S_S^2 = 2p \left[ \frac{1}{\sqrt{u^2 + \tilde{\rho}^2}} + \frac{u^2}{\sqrt{(u^2 + \tilde{\rho}^2)^3 - 2\tilde{M}(u^2 + \tilde{\rho}^2)}} \right] du^2, \quad (2.4)$$

where we set the energy  $p = \gamma M$  and  $\tilde{M} = M/\gamma$ .

Taking the limit of  $v \rightarrow 1$  (namely  $\gamma \rightarrow \infty$ ),  $\Delta S_S^2$  behaves like the metric of the gravitational shock wave. To see it, consider the first term of on the right-hand side (RHS) of (2.4). The integration of this term by  $u$  is given by

$$\int \frac{du}{\sqrt{u^2 + \tilde{\rho}^2}} = \ln(u + \sqrt{u^2 + \tilde{\rho}^2}). \quad (2.5)$$

Now the limit of  $\gamma \rightarrow \infty$  is taken:

$$\int \frac{u^2 du}{\sqrt{(u^2 + \tilde{\rho}^2)^3 - 2\tilde{M}(u^2 + \tilde{\rho}^2)}} = \frac{\rho}{2M} \arctan \left[ \frac{u}{\tilde{\rho}} \right] + \ln \frac{u - \tilde{\rho} + \sqrt{u^2 + \tilde{\rho}^2}}{u + \tilde{\rho} - \sqrt{u^2 + \tilde{\rho}^2}} - 2 \left[ \frac{\rho^2}{4M^2} - 1 \right]^{1/2} \arctan \left[ \left( \frac{\rho + 2M}{\rho - 2M} \right)^{1/2} \frac{\sqrt{u^2 + \tilde{\rho}^2} - \tilde{\rho}}{u} \right]. \quad (2.9)$$

Equation (2.4) for  $\gamma \rightarrow \infty$  is now given by

$$\lim_{\gamma \rightarrow \infty} \Delta S_S^2 \rightarrow 4p \left[ \frac{1}{|u|} - \left[ \ln \rho^2 - \frac{\rho}{4M} \left[ \pi - 4 \left[ 1 - \frac{4M^2}{\rho^2} \right]^{1/2} \arctan \left[ \frac{\rho + 2M}{\rho - 2M} \right]^{1/2} \right] \right] \delta(u) \right] du^2. \quad (2.10)$$

Introducing a new coordinate

$$dv' = dv - 4pdu/|u|, \quad (2.11)$$

the gravitational shock wave metric for a Schwarzschild black hole has the final form (dash on  $v'$  is omitted)

$$\lim_{\gamma \rightarrow \infty} ds_S^2 \rightarrow -du dv + dx^2 + dy^2 - 4p \left[ \ln \rho^2 - \frac{\rho}{4M} \left[ \pi - 4 \left[ 1 - \frac{4M^2}{\rho^2} \right]^{1/2} \arctan \left[ \frac{\rho + 2M}{\rho - 2M} \right]^{1/2} \right] \right] \delta(u) du^2. \quad (2.12)$$

This is the exact result.

We now consider the nature of this metric in various limits. When  $\rho \gg 2M$ , (2.12) is reduced to

$$\lim_{\gamma \rightarrow \infty} ds_S^2 \sim -du dv + dx^2 + dy^2 - 4p \left[ \ln \rho^2 + 1 - \frac{\pi}{2} \frac{M}{\rho} + \frac{4}{3} \frac{M^2}{\rho^2} + O \left( \frac{M^3}{\rho^3} \right) \right] \delta(u) du^2. \quad (2.13)$$

Further, we take the limit  $M \rightarrow 0$  and execute a scale transformation  $\hat{x}^i = e^{x^i}$ . Then (2.13) is equal to the Aichelburg-Sexl metric (AS metric hereafter). As mentioned in Sec. I, the coefficient of  $g_{uu}$  represents the shift  $\Delta v$  of  $v$  crossing  $u=0$ . In Fig. 2 the bold line shows the value of  $\Delta v$  near  $\rho \sim M$ . It is to be noted that  $\Delta v=0$  at the event horizon  $\rho=2M$ .

From (1.1), (1.4), and (2.12), we can calculate the refraction angle  $\Delta\phi$  of a null geodesic crossing the shock wave. The result is depicted in Fig. 3 by the bold line. It is to be noted that the refraction angle becomes  $90^\circ$  for  $\rho=2M$ . This is natural because the event horizon of a

$$\lim_{\gamma \rightarrow \infty} \int \frac{du}{\sqrt{u^2 + \tilde{\rho}^2}} = \theta(u) \ln(2u) + \theta(-u) [\ln \tilde{\rho}^2 - \ln(-2u)]. \quad (2.6)$$

Differentiating by  $u$ , we obtain (with redundant terms removed)

$$\frac{d}{du} \left[ \lim_{\gamma \rightarrow \infty} \int \frac{du}{\sqrt{u^2 + \tilde{\rho}^2}} \right] = \frac{1}{|u|} - \ln \rho^2 \delta(u). \quad (2.7)$$

Similarly, the second term is calculated:

$$\begin{aligned} \frac{d}{du} \left[ \lim_{\gamma \rightarrow \infty} \int \frac{u^2 du}{\sqrt{(u^2 + \tilde{\rho}^2)^3 - 2\tilde{M}(u^2 + \tilde{\rho}^2)}} \right] \\ = \frac{1}{|u|} - \left[ \ln \rho^2 - \frac{\rho}{2M} \pi + 4 \left[ \frac{\rho^2}{4M^2} - 1 \right]^{1/2} \right. \\ \left. \times \arctan \left[ \frac{\rho + 2M}{\rho - 2M} \right]^{1/2} \right] \delta(u), \quad (2.8) \end{aligned}$$

where use has been made of the formula

Schwarzschild black hole is at  $\rho=2M$ , and the event horizon is invariant under the Lorentz transformation for the  $z$  direction. The metric of (2.12) represents the space-time structured for all region of  $\rho$  compared with AS metric, which describes only for  $\rho \gg M$ .

### III. GRAVITATIONAL SHOCK WAVE FOR KERR BLACK HOLES

From above discussions, it is apparent that the gravitational shock wave solution can be calculated for any sort of stationary black hole metrics. In this section we apply

$-\Delta v / 8 p$

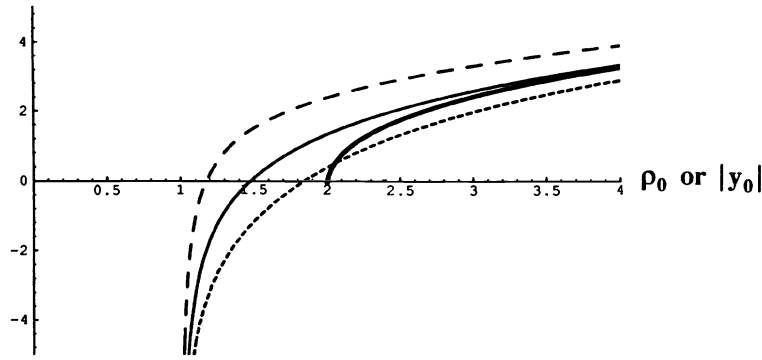


FIG. 2. The shift  $\Delta v$  crossing  $u=0$ . We put  $M=1$ . The bold line indicates the Schwarzschild case, and the thin solid line indicates the Kerr case moving parallel to the rotational axis, where the limit  $M \rightarrow a$  is taken. The dashed line and the dotted line represent the Kerr case moving perpendicular to the axis, where the limit  $M \rightarrow a$  is taken. The dashed line indicates  $y > 0$ , and the dotted line indicates  $y < 0$ .

this method for a Kerr black hole. We consider a Kerr black hole in Boyer-Lindquist coordinates [1]:

$$ds_K^2 = - \left[ 1 - \frac{2M\bar{r}}{\Sigma} \right] d\bar{t}^2 - \frac{4M\bar{r}a\sin^2\bar{\theta}}{\Sigma} d\bar{t}d\bar{\phi} + \frac{\Sigma}{\Delta} d\bar{r}^2 + \Sigma d\bar{\theta}^2 + \left[ \bar{r}^2 + a^2 + \frac{2M\bar{r}a^2\sin^2\bar{\theta}}{\Sigma} \right] \sin^2\bar{\theta} d\bar{\phi}^2, \quad (3.1)$$

where

$$\Delta = \bar{r}^2 - 2M\bar{r} + a^2, \\ \Sigma = \bar{r}^2 + a^2 \cos^2\bar{\theta}.$$

The metric (3.1) is divided into the Minkowski metric  $ds_M^2$  and the other parts:

$$ds_K^2 = ds_M^2 + \Delta S_u^2 + \Delta S_{t\phi}^2 + \Delta S_r^2 + \Delta S_{\theta\theta}^2 + \Delta S_{\phi\phi}^2, \quad (3.2)$$

where

$$\Delta S_u^2 = \frac{2M\bar{r}}{\bar{r}^2 + a^2 \cos^2\bar{\theta}} d\bar{t}^2, \\ \Delta S_{t\phi}^2 = - \frac{4M\bar{r}a\sin^2\bar{\theta}}{\bar{r}^2 + a^2 \cos^2\bar{\theta}} d\bar{t}d\bar{\phi},$$

$$\Delta S_r^2 = \frac{2M\bar{r} - a^2 \sin^2\bar{\theta}}{\bar{r}^2 - 2M\bar{r} + a^2} d\bar{r}^2,$$

$$\Delta S_{\theta\theta}^2 = a^2 \cos^2\bar{\theta} d\bar{\theta}^2,$$

$$\Delta S_{\phi\phi}^2 = \left[ a^2 + \frac{2M\bar{r}a^2 \sin^2\bar{\theta}}{\bar{r}^2 + a^2 \cos^2\bar{\theta}} \right] \sin^2\bar{\theta} d\bar{\phi}^2.$$

Here, we discuss two cases: First, for the motion parallel to the rotational axis of the Kerr black hole, and second, perpendicular. The rotational axis is taken in the  $z$  direction.

#### A. Motion parallel to the rotational axis

First, we investigate a Kerr black hole which moves parallel to the rotational axis. As in Sec. II, Lorentz boost of (2.2) is performed for the metric (3.2). In leading orders of  $\gamma$ , the old variables are transformed to

$$d\bar{\theta} \rightarrow \frac{-\bar{\rho} du}{u^2 + \bar{\rho}^2}, \\ d\bar{\phi} \rightarrow \frac{x dy - y dx}{\rho^2},$$

and

$\tan(\Delta\phi) / 2 p$

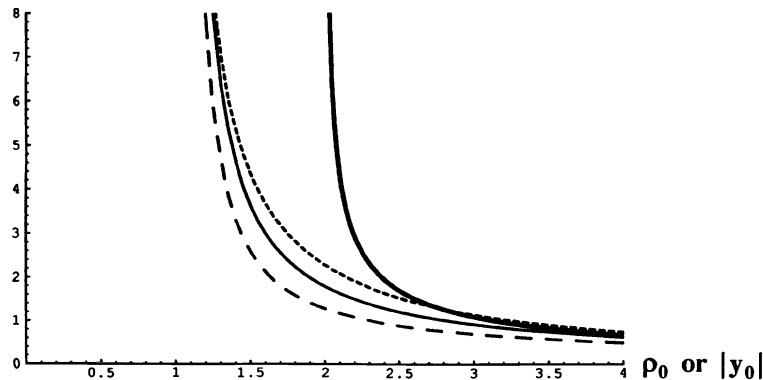


FIG. 3. The refraction angle,  $\Delta\phi$ , crossing  $u=0$  where the light ray is injected perpendicular to the shock wave. Notations are the same as in Fig. 2.

$$\begin{aligned}\sin\bar{\theta} &= \frac{\bar{\rho}}{\sqrt{u^2 + \bar{\rho}^2}}, \\ \cos\bar{\theta} &= \frac{-u}{\sqrt{u^2 + \bar{\rho}^2}}.\end{aligned}\quad (3.3)$$

Below, we retain only the terms of  $g_{uu} = O(\gamma)$ . Because  $\lim \Delta S_{t\phi}^2 \rightarrow O(1)$  and  $\lim \Delta S_{\phi\phi}^2 \rightarrow O(1/\gamma)$ , these terms can be neglected.

$\Delta S_{tt}^2$  of (3.2) is transformed as follows:

$$\Delta S_{tt}^2 \rightarrow \frac{2p\sqrt{(u^2 + \bar{\rho}^2)^3}}{u^4 + (2\bar{\rho}^2 + \bar{a}^2)u^2 + \bar{\rho}^4} du^2, \quad (3.4)$$

where we put  $\bar{a} = a/\gamma$ . This is integrated using (B1) of Appendix B, then the limit  $\gamma \rightarrow \infty$  is taken, and finally, it is differentiated. The result is

$$\lim_{\gamma \rightarrow \infty} \Delta S_{tt}^2 \rightarrow 2p \left[ \frac{1}{|u|} - \left[ \ln \rho^2 + \frac{1}{\sqrt{2}\alpha} \left\{ \sqrt{\alpha+1} \ln \frac{\sqrt{\alpha+1} + \sqrt{2}}{\sqrt{\alpha+1} - \sqrt{2}} - 2\sqrt{\alpha-1} \arctan \left[ \frac{2}{\alpha-1} \right]^{1/2} \right\} \right] \right] \delta(u) du^2, \quad (3.5)$$

where

$$\alpha = \frac{\sqrt{4\rho^2 + a^2}}{a}.$$

On the other hand,  $\Delta S_{rr}^2$  is converted into the following two divided terms by the Lorentz transformation:

$$\begin{aligned}\Delta S_{rr}^2 &\rightarrow \frac{2pu^2 du^2}{[u^2 + \bar{\rho}^2 + \bar{a}^2 - 2\bar{M}\sqrt{u^2 + \bar{\rho}^2}]\sqrt{u^2 + \bar{\rho}^2}} - \frac{a^2 \bar{\rho}^2 u^2 du^2}{[u^2 + \bar{\rho}^2 + \bar{a}^2 - 2\bar{M}\sqrt{u^2 + \bar{\rho}^2}](u^2 + \bar{\rho}^2)^2}, \\ &= \Delta S_{rr1}^2 + \Delta S_{rr2}^2,\end{aligned}\quad (3.6)$$

where we put  $\bar{M} = M/\gamma$ . Applying (B2), we find the final form of  $\Delta S_{rr1}^2$  by the same procedures:

$$\lim_{\gamma \rightarrow \infty} \Delta S_{rr1}^2 \rightarrow 2p \left[ \frac{1}{|u|} - \left\{ \ln \rho^2 - \frac{4}{\beta - \xi} [\sqrt{\beta}(1 + \xi) \arctan(1/\sqrt{\beta}) - \sqrt{\xi}(1 + \beta) \arctan(1/\sqrt{\xi})] \right\} \delta(u) \right] du^2, \quad (3.7)$$

where

$$\begin{aligned}\beta &= \frac{\rho^2 - a^2 + 2\rho\sqrt{M^2 - a^2}}{\rho^2 + a^2 + 2M\rho}, \\ \xi &= \frac{\rho^2 - a^2 - 2\rho\sqrt{M^2 - a^2}}{\rho^2 + a^2 + 2M\rho}.\end{aligned}$$

Similarly,  $\Delta S_{rr2}^2$  becomes

$$\begin{aligned}\lim_{\gamma \rightarrow \infty} \Delta S_{rr2}^2 &\rightarrow -\frac{4p\rho}{Ma^4} \left\{ (a^4 + 2a^2\rho^2 - 8\rho^2 M^2)\pi/8 - a^2 M\rho + \frac{1}{4\sqrt{M^2 - a^2}} \right. \\ &\quad \left. \times [(\eta\sqrt{\beta} - \theta/\sqrt{\beta}) \arctan(1/\sqrt{\beta}) - (\eta\sqrt{\xi} - \theta/\sqrt{\xi}) \arctan(1/\sqrt{\xi})] \right\} \delta(u) du^2,\end{aligned}\quad (3.8)$$

where

$$\begin{aligned}\eta &= (4\rho M^2 - a^2\rho - 2a^2 M)(\rho^2 + a^2 + 2\rho M), \\ \theta &= (4\rho M^2 - a^2\rho + 2a^2 M)(\rho^2 + a^2 - 2\rho M).\end{aligned}$$

By the Lorentz transformation,  $\Delta S_{\theta\theta}^2$  becomes

$$\Delta S_{\theta\theta}^2 \rightarrow \frac{a^2 \bar{\rho}^2 u^2 du^2}{(u^2 + \bar{\rho}^2)^3}, \quad (3.9)$$

and using (B4), after taking the limit  $\gamma \rightarrow \infty$  is

$$\lim_{\gamma \rightarrow \infty} \Delta S_{\theta\theta}^2 \rightarrow 4p \left[ \frac{\pi a^2}{32\rho M} \right] \delta(u) du^2. \quad (3.10)$$

Putting (3.4)–(3.10) altogether, and carrying out the coordinate transformation, the exact gravitational shock wave solution for a Kerr black hole moving parallel to the rotational axis is given by

$$\lim_{\gamma \rightarrow \infty} ds_K^2 \rightarrow -du dv + dx^2 + dy^2 - 4p [\Delta g_1 + \Delta g_2 + \Delta g_3 + \Delta g_4] \delta(u) du^2, \quad (3.11)$$

where

$$\begin{aligned}\Delta g_1 &= \ln \rho^2 + \frac{1}{2\sqrt{2}\alpha} \left[ \sqrt{\alpha+1} \ln \frac{\sqrt{\alpha+1} + \sqrt{2}}{\sqrt{\alpha+1} - \sqrt{2}} - 2\sqrt{\alpha-1} \arctan \left[ \frac{2}{\alpha-1} \right]^{1/2} \right], \\ \Delta g_2 &= -\frac{2}{\beta-\xi} \left[ \sqrt{\beta}(1+\xi) \arctan(1/\sqrt{\beta}) - \sqrt{\xi}(1+\beta) \arctan(1/\sqrt{\xi}) \right], \\ \Delta g_3 &= \frac{\rho}{Ma^4} \left\{ (a^4 + 2a^2\rho^2 - 8\rho^2 M^2) \pi / 8 - a^2 M \rho + \frac{1}{4\sqrt{M^2 - a^2}} \right. \\ &\quad \times [(\eta\sqrt{\beta} - \theta/\sqrt{\beta}) \arctan(1/\sqrt{\beta}) - (\eta\sqrt{\xi} - \theta/\sqrt{\xi}) \arctan(1/\sqrt{\xi})] \left. \right\}, \\ \Delta g_4 &= -\frac{\pi a^2}{32\rho M}.\end{aligned}$$

Assume there are no naked singularities. Then a black hole with  $a=M$  is the maximally rotating black hole. We now investigate the properties of (3.11) in the limit of  $M \rightarrow a$ . In this limit, (3.11) has the form:

$$\begin{aligned}\lim_{M \rightarrow a} ds_K^2 &\rightarrow -du dv + dx^2 + dy^2 \\ &- 4p \left\{ \ln \rho^2 + 1 + \frac{1}{2\sqrt{2}} \frac{1}{\alpha} \left[ \sqrt{\alpha+1} \ln \frac{\sqrt{\alpha+1} + \sqrt{2}}{\sqrt{\alpha+1} - \sqrt{2}} - 2\sqrt{\alpha-1} \arctan \left[ \frac{2}{\alpha-1} \right]^{1/2} \right] \right. \\ &\quad \left. + \frac{3\rho^4 - 2a^2\rho^2 - 2a^4}{a^3\sqrt{\rho^2 - a^2}} \arctan \left[ \frac{\rho+a}{\rho-a} \right]^{1/2} + \left[ \frac{\rho}{8a} - \frac{3\rho^3}{4a^3} - \frac{a}{32\rho} \right] \pi - \frac{3\rho^2}{2a^2} \right\} \delta(u) du^2.\end{aligned}\quad (3.12)$$

The shift  $\Delta v$  for  $\rho \sim a$  is shown in Fig. 2 by the thin solid line. Notice that the shift is divergent at  $\rho=a$ , the event horizon of a Kerr black hole for  $M \rightarrow a$ . On the other hand, when  $\rho \gg a$ ,

$$\lim_{M \rightarrow a} ds_K^2 \sim -du dv + dx^2 + dy^2 - 4p \left[ \ln \rho^2 + 1 - \frac{\pi a}{2\rho} + \left[ \frac{4}{3} - \frac{4}{5} \right] \frac{a^2}{\rho^2} + O\left[ \frac{a^3}{\rho^3} \right] \right] \delta(u) du^2. \quad (3.13)$$

Compared with (2.13) of Schwarzschild type, (3.13) is slightly different in the order of  $O(a^2/\rho^2)$ . Thus, far from the hole, the metric of the Kerr-type hole moving in the  $z$  direction is not distinguished from that of the Schwarzschild type one.

The refraction angle derived from (3.13) is presented in Fig. 3 by the thin solid line. As mentioned in Sec. II, the refraction angle is  $90^\circ$  at  $\rho=M$ , the event horizon of the Kerr black hole.

### B. The motion perpendicular to the rotational axis

In this section we study the metric in a reference system moving perpendicular to the rotational axis of a Kerr black hole. The moving direction is taken as the  $x$  direction, and the rotational axis is set as the  $z$  direction. We discuss only in the equatorial plane ( $\theta = \pi/2$ ,  $\dot{\theta} = 0$ ).

The boost in the  $x$  direction at the speed  $v$  gives

$$t = \gamma(\bar{t} + v\bar{x}), \quad x = \gamma(\bar{x} + v\bar{t}), \quad y = \bar{y}, \quad (3.14)$$

where

$$\gamma = (1 - v^2)^{-1/2},$$

and  $z$  is always set to zero. We put  $u = t - x$  and  $v = t + x$  as the null coordinates. Then, old coordinates are transformed as

$$\begin{aligned}\bar{t} &\rightarrow \gamma u, \\ \bar{x} &\rightarrow -\gamma u, \\ \bar{r}^2 &\rightarrow \gamma^2 u^2 + y^2,\end{aligned}\quad (3.15)$$

and the infinitesimal lengths  $d\bar{r}$  and  $d\bar{\phi}$  are transformed as (in the leading order of  $\gamma$ )

$$\begin{aligned}d\bar{r} &\rightarrow \frac{\gamma u du}{\sqrt{u^2 + y^2}}, \\ d\bar{\phi} &\rightarrow \frac{1}{\gamma} \frac{y du}{u^2 + y^2}.\end{aligned}\quad (3.16)$$

Making use of (3.15) and (3.16), (3.2) is changed under the Lorentz transformation as

$$\begin{aligned}ds_K^2 &\rightarrow -du dv + dy^2 + \left[ \frac{2p}{\sqrt{u^2 + y^2}} - \frac{1}{\gamma} \frac{4May}{\sqrt{(u^2 + y^2)^3}} + \frac{2pu^2}{[u^2 + y^2 + a^2 - 2\tilde{M}\sqrt{u^2 + y^2}]\sqrt{u^2 + y^2}} \right. \\ &\quad \left. - \frac{a^2 u^2}{[u^2 + y^2 + a^2 - 2\tilde{M}\sqrt{u^2 + y^2}](u^2 + y^2)} + \frac{2pa^2 y^2}{\gamma^4 (u^2 + y^2 + a^2) \sqrt{(u^2 + y^2)^3}} \right] du^2.\end{aligned}\quad (3.17)$$

Now we take the limit  $\gamma \rightarrow \infty$ . Using the integration formulas of (3.2), (B2), (B5), (B6), and (B7), the metric calculated by the same method of Sec. II and Sec. III A becomes

$$\lim_{\gamma \rightarrow \infty} ds_K^2 \rightarrow -du dv + dy^2 - 4p [\delta g_1 + \delta g_2 + \delta g_3 + \delta g_4] \delta(u) du^2, \quad (3.18)$$

where

$$\begin{aligned} \delta g_1 &= \ln|y|^2 - \frac{2}{\beta' - \zeta'} [\sqrt{\beta'}(1 + \zeta') \arctan(1/\sqrt{\beta'}) - \sqrt{\zeta'}(1 + \beta') \arctan(1/\sqrt{\zeta'})], \\ \beta' &= \frac{y^2 - a^2 + 2|y|\sqrt{M^2 - a^2}}{y^2 + a^2 + 2M|y|}, \\ \zeta' &= \frac{y^2 - a^2 - 2|y|\sqrt{M^2 - a^2}}{y^2 + a^2 + 2M|y|}, \\ \delta g_2 &= \frac{2|y|a^2}{M\{|y|(\sqrt{M^2 - a^2} + M) + a^2\}} \left\{ \frac{1}{\beta' - 1} \left[ \frac{\pi}{4} - \frac{1}{\sqrt{\beta'}} \arctan(1/\sqrt{\beta'}) \right] \right. \\ &\quad \left. + \frac{\zeta'}{\zeta' - \beta'} \left[ \frac{1}{\sqrt{\zeta'}} \arctan(1/\sqrt{\zeta'}) - \frac{1}{\sqrt{\beta'}} \arctan(1/\sqrt{\beta'}) \right] \right\}, \\ \delta g_3 &= \frac{2a}{y}, \\ \delta g_4 &= -1 - \frac{y^2}{2a\sqrt{y^2 + a^2}} \ln \frac{\sqrt{y^2 + a^2} - a}{\sqrt{y^2 + a^2} + a}. \end{aligned}$$

Here, the important point to be noted is that  $\delta g_3$  depends on the sign of  $y$  explicitly. In other words, when the magnitudes of  $y$ 's are equal with opposite signs, the shift  $\Delta v$ 's and the refraction angles are different. As the term is derived from  $g_{t\phi} dt d\phi$  of the Kerr metric, the effect of the space-time dragging would be the cause.

Equation (3.18) is the general result. Now we discuss the properties of it for special cases. Take the limit  $M \rightarrow a$ , i.e., the fast rotating black hole:

$$\begin{aligned} \lim_{M \rightarrow a} \left[ \lim_{\gamma \rightarrow \infty} ds_K^2 \right] &\rightarrow -du dv + dy^2 \\ &\quad - 4p \left[ \ln|y|^2 - \frac{1}{2} + \frac{y^2 - 2a^2}{a\sqrt{y^2 - a^2}} \arctan \left( \frac{|y| + a}{|y| - a} \right)^{1/2} - \frac{|y|}{4a} \pi \right. \\ &\quad \left. + \frac{2a}{y} - \frac{y^2}{2a\sqrt{y^2 + a^2}} \ln \frac{\sqrt{y^2 + a^2} - a}{\sqrt{y^2 + a^2} + a} \right] \delta(u) du^2. \end{aligned} \quad (3.19)$$

Consider the case of  $|y| \gg a$ , far from the hole:

$$ds_K^2|_{|y| \gg a} \sim -du dv + dy^2 - 4p \left[ \ln|y|^2 + 1 - \frac{3\pi}{8} \left[ \frac{a}{|y|} \right] + 2 \left[ \frac{a}{y} \right] - \frac{4}{3} \left[ \frac{a^2}{y^2} \right] + O \left[ \frac{a^3}{y^3} \right] \right] \delta(u) du^2. \quad (3.20)$$

On the other hand, in the case of  $|y| \sim a$ , the shift  $\Delta v$  is shown in Fig. 2.

The refraction angle is depicted in Fig. 3. In Fig. 4, the ratio of the refraction angle of the negative sign of  $y$  to that of the positive sign is represented. This figure indicates that the refraction for  $y < 0$  is larger than that for  $y > 0$ .

#### IV. ASTROPHYSICAL APPLICATIONS

In this section, we discuss shortly possible applications of our results to some astrophysical phenomena. The metrics obtained above are the first-order approximate equations of Lorentz factor  $\gamma$ . Therefore, the metrics are exact in the limit  $\gamma \rightarrow \infty$ . If the  $\gamma$  factor of an object is

$\sim 10$ , the metric of the gravitational shock wave is applied to the object with sufficient reliability. Quasars which are observed in recent years have, however,  $\sim 2-3$ , so that discussions for quasars below would be qualitative in nature.

The metric can be applied to the problem of gravitational waves which are radiated by the collision of two black holes and the scattering problem of particle physics at energies of Planck scale. These problems have also been calculated by using the AS metric [2,3]. It will be very interesting when the result calculated by our metric is compared with that by the AS metric, because our case can be traced even for the region of  $p \sim M$ . This is now under investigation, and will be published in the next paper.

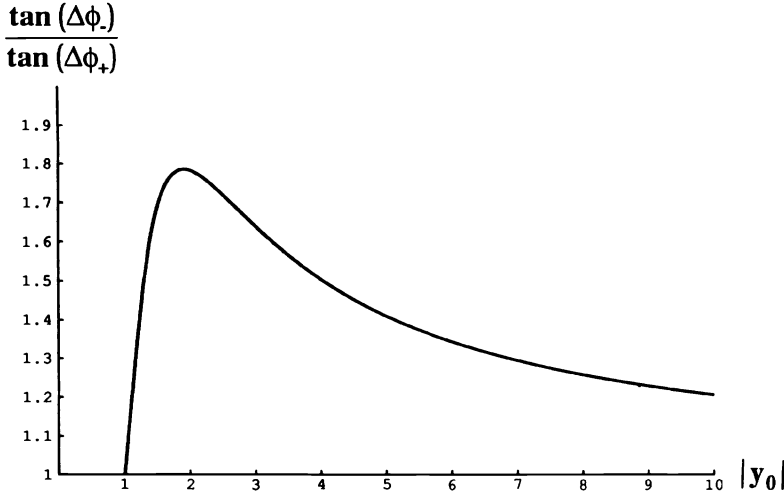


FIG. 4. The ratio of the refraction angle of the negative sign of  $y$  to that of the positive sign.

Let us return to our main subject. The main point of us compared with others is that black holes are moving.

Figure 5 shows the effect of the gravitational lens. The middle object represents a supermassive black hole, and the left side object is the light source (i.e., a quasar for example). The observer at the right will see two or many images, because the light rays are refracted near the black hole. The light source is, in fact, receding with a Lorentz factor  $\gamma$  from us caused by the expansions of the Universe.

Let us discuss in the rest frame of the source. In this frame, the black hole moves in the right direction with the Lorentz factor  $\gamma$ . For a Schwarzschild black hole, the light radiated from the source is refracted by the angle  $\Delta\phi_S$  (2.13):

$$\Delta\hat{\phi}_S \sim \frac{4p}{\rho_0} \left[ 1 + \frac{\pi}{2} \frac{M}{\rho_0} - \frac{8}{3} \frac{M^2}{\rho_0^2} + O\left(\frac{M^3}{\rho_0^3}\right) \right], \quad (4.1)$$

where  $\rho_0 \gg M$ . In the observer's frame, the refraction angle  $\Delta\phi_S$  is given from  $\Delta\hat{\phi}_S$  by the Lorentz transformation as  $\Delta\phi_S \sim \Delta\hat{\phi}_S/\gamma$ . Then,  $\Delta\phi_S$  is given by

$$\Delta\phi_S \sim \frac{4M}{\rho_0} \left[ 1 + \frac{\pi}{2} \frac{M}{\rho_0} - \frac{8}{3} \frac{M^2}{\rho_0^2} + O\left(\frac{M^3}{\rho_0^3}\right) \right]. \quad (4.2)$$

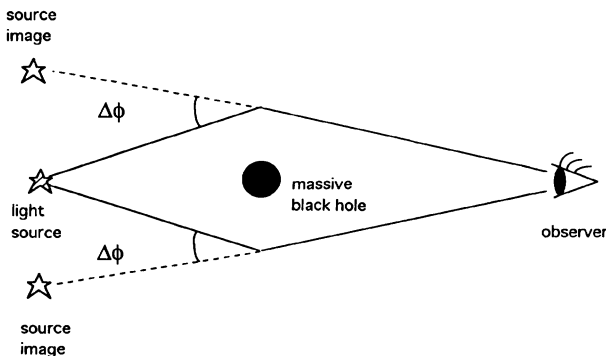


FIG. 5. The effect of the gravitational lens.

We would like to emphasize that  $p$  of (2.13) is replaced by  $M$ . The naive value of  $\Delta\phi_S$  is  $\sim 4M/\rho_0$ , so that the effect of  $\gamma$  appears in the order of  $M^2/\rho_0^2$ .

Similarly, when the black hole is of the Kerr type, we can calculate the refraction angle  $\Delta\phi_{Kz}$  and  $\Delta\phi_{Kx}$  of the parallel motion to the rotational axis from (3.14), and of the perpendicular motion from (3.21), respectively:

$$\Delta\phi_{Kz} \sim \frac{4M}{\rho_0} \left[ 1 + \frac{\pi}{2} \frac{M}{\rho_0} - \left[ \frac{8}{3} - \frac{8}{5} \right] \frac{M^2}{\rho_0^2} + O\left(\frac{M^3}{\rho_0^3}\right) \right], \quad (4.3)$$

$$\Delta\phi_{Kx} \sim \frac{4M}{y_0} \left[ 1 + \left[ \frac{3\pi}{8} - 2 \right] \frac{a}{y_0} + \frac{8}{3} \frac{M^2}{y_0^2} + O\left(\frac{M^3}{y_0^3}\right) \right]. \quad (4.4)$$

On the other hand, in the near region  $\rho_0 \sim a$ , or  $y_0 \sim a$ ,  $p$  of the vertical axis in Fig. 3 is to be replaced by  $M$  as discussed above.

## V. CONCLUSION AND DISCUSSION

We have derived the metrics of a moving Schwarzschild black hole and a moving Kerr black hole by Lorentz transformations. The Schwarzschild metric is calculated in the Schwarzschild coordinates, not in the isotropic coordinates, and the Kerr metric in the Boyer-Lindquist coordinates. For the Kerr black hole, we have calculated the metrics for two cases: the parallel and the perpendicular cases. For both cases, obtained metrics are place-fronted shock waves and the geometries of them have structures of two portions of the Minkowski types for  $u < 0$  and  $u > 0$ , with null shock surfaces at  $u = 0$  with warps. The metrics can be investigated in the region near the black hole at  $u = 0$ , because we do not put  $M \rightarrow 0$  which was required for the AS metric. Next, we have discussed the shift,  $\Delta v$ , of a null coordinate  $v$  crossing  $u = 0$ , and have obtained the refraction angle from our metric.

Our main results are (i) the exact formulas for the



metrics of Schwarzschild and Kerr black holes moving with a large Lorentz factor  $\gamma$  are obtained, (ii) the shift  $\Delta v$ 's of the null coordinates crossing the gravitational shock at  $u=0$  for a Kerr black hole is derived, both for parallel and perpendicular motions, (iii) the refraction angle of null geodesics passing by a Kerr black hole is obtained and it is found that it depends on the sign of  $y$ , and (iv) the gravitational lens effect by a moving black holes manifests in the second order of  $M/\rho$ , compared with static cases.

This is the first step to the physics of moving black holes. Nature waits for our endeavors to look into her mysterious secrets.

#### APPENDIX A: GEODESICS OF THE METRIC (1.1)

For the metric (1.1), we put  $x = \rho \cos \phi$  and  $y = \rho \sin \phi$ . Then, it has the form

$$\lim_{\gamma \rightarrow \infty} ds^2 \rightarrow -du dv + d\rho^2 + \rho^2 d\phi^2 - A(\rho) \delta(u) du^2. \quad (\text{A1})$$

The null geodesics derived from (A1) satisfy

$$\ddot{u} = 0, \quad (\text{A2})$$

$$\frac{d}{d\lambda}(\rho^2 \dot{\phi}) = 0, \quad (\text{A3})$$

$$\frac{d}{d\lambda}[\dot{v} + 2A(\rho)\dot{u}\delta(u)] = 0, \quad (\text{A4})$$

$$\ddot{\rho} - \rho \dot{\phi}^2 + \frac{1}{2} A'(\rho) \dot{u}^2 \delta(u) = 0, \quad (\text{A5})$$

where dash and dot denote the derivatives with respect to  $\rho$  and the affine parameter  $\lambda$ , respectively.

From (A2) we can put

$$u = \lambda, \quad (\text{A6})$$

without loss of generality. Then, (A4) means

$$v = -2A(\rho)|_{\rho=\rho_0} \theta(u), \quad (\text{A7})$$

where inessential constants have been neglected. The shift  $\Delta v$  of the geodesic  $v$  crossing  $u=0$  is now given by

$$\Delta v = -2A(\rho)|_{\rho=\rho_0}, \quad (\text{A8})$$

(A3) represents the conservation of the angular momentum:

$$\rho^2 \dot{\phi} = \text{const} \equiv L. \quad (\text{A9})$$

Equation (A5) is written by now as

$$\ddot{\rho} - \frac{L^2}{\rho^3} + \frac{1}{2} A'(\rho) \delta(u) = 0. \quad (\text{A10})$$

In (A10), if the last term were absent, it would give the straight geodesic,  $\rho \cos \phi = \rho_0$ . Since we are interested in the behavior of the geodesic crossing the shock  $u=0$ , we neglect the second term of (A10). The solution is now given by

$$\dot{\rho} = -\frac{1}{2} A'(\rho) \Big|_{\rho=\rho_0} \theta(u). \quad (\text{A11})$$

From (A11), the refraction angle  $\Delta \phi$  is given

$$\tan(\Delta \phi) = \frac{1}{2} \frac{dA(\rho)}{d\rho} \Big|_{\rho=\rho_0}. \quad (\text{A12})$$

#### APPENDIX B: INTEGRATION FORMULAS

The integration formulas are

$$\int \frac{\sqrt{(u^2 + \bar{\rho}^2)^3} du}{u^4 + (2\bar{\rho}^2 + \bar{a}^2)u^2 + \bar{\rho}^4} = \ln \frac{\sqrt{u^2 + \bar{\rho}^2} + u}{\bar{\rho}} - \frac{1}{2\bar{\mathcal{A}}} \left\{ \left[ \frac{\bar{\mathcal{A}} + 1}{2} \right]^{1/2} \ln \frac{\sqrt{(\bar{\mathcal{A}} + 1)(u^2 + \bar{\rho}^2)} + \sqrt{2}u}{\sqrt{(\bar{\mathcal{A}} + 1)(u^2 + \bar{\rho}^2)} - \sqrt{2}u} \right. \\ \left. - 2 \left[ \frac{\bar{\mathcal{A}} - 1}{2} \right]^{1/2} \arctan \frac{\sqrt{2}u}{\sqrt{(\bar{\mathcal{A}} - 1)(u^2 + \bar{\rho}^2)}} \right\}, \quad (\text{B1})$$

where tilde denotes the division by  $\gamma$ , and

$$\bar{\mathcal{A}} = \frac{\sqrt{4\rho^2 + a^2}}{a}.$$

$$\int \frac{u^2 du}{[u^2 + \bar{\rho}^2 + \bar{a}^2 - 2\bar{M}\sqrt{u^2 + \bar{\rho}^2}]\sqrt{u^2 + \bar{\rho}^2}} \\ = \ln \frac{1+q}{1-q} + \frac{2}{\bar{\mathcal{B}} - \bar{\mathcal{C}}} [\sqrt{\bar{\mathcal{B}}}(1 + \bar{\mathcal{C}}) \arctan(q/\sqrt{\bar{\mathcal{B}}}) - \sqrt{\bar{\mathcal{C}}}(1 + \bar{\mathcal{B}}) \arctan(q/\sqrt{\bar{\mathcal{C}}})], \quad (\text{B2})$$

where

$$q = \frac{\sqrt{u^2 + \bar{\rho}^2} - \bar{\rho}}{u},$$

$$\bar{\mathcal{B}} = \frac{\rho^2 - a^2 + 2\rho\sqrt{M^2 - a^2}}{\rho^2 + a^2 + 2M\rho},$$

$$\begin{aligned}
\mathcal{C} &= \frac{\rho^2 - a^2 - 2\rho\sqrt{M^2 - a^2}}{\rho^2 + a^2 + 2M\rho} . \\
&\int \frac{u^2 du}{[u^2 + \bar{\rho}^2 + \bar{a}^2 - 2\bar{M}\sqrt{u^2 + \bar{\rho}^2}](u^2 + \bar{\rho}^2)^2} \\
&= \frac{4\gamma^4}{\rho(\rho^2 + a^2 + 2M\rho)} \left[ \mathcal{D}\arctan q + \mathcal{E} \frac{q}{1+q^2} + \mathcal{F} \frac{q}{(1+q^2)^2} + \frac{1}{\mathcal{B} - \mathcal{C}} \right. \\
&\quad \left. \times \{ (\mathcal{G}\sqrt{\mathcal{B}} - \mathcal{H}/\sqrt{\mathcal{B}})\arctan(q/\sqrt{\mathcal{B}}) - (\mathcal{G}\sqrt{\mathcal{C}} - \mathcal{H}/\sqrt{\mathcal{C}})\arctan(q/\sqrt{\mathcal{C}}) \} \right] , \tag{B3}
\end{aligned}$$

where  $q$ ,  $\mathcal{B}$ , and  $\mathcal{C}$  are the same as (B2),

$$\mathcal{D} = (a^4 + 2a^2\rho^2 - 8\rho^2 M^2)(\rho^2 + a^2 + 2\rho M)/4a^6 ,$$

$$\mathcal{E} = (a^2 - 4M\rho)(\rho^2 + a^2 + 2\rho M)/4a^4 ,$$

$$\mathcal{F} = -(\rho^2 + a^2 + 2\rho M)/2a^2 ,$$

$$\mathcal{G} = \rho(4\rho M^2 - a^2\rho - 2a^2\rho^2)(\rho^2 + a^2 + 2\rho M)/2a^6 ,$$

$$\mathcal{H} = \rho(4\rho M^2 - a^2\rho + 2a^2\rho^2)(\rho^2 + a^2 - 2\rho M)/2a^6 .$$

$$\int \frac{u^2 du}{(u^2 + \bar{\rho}^2)^3} = \frac{1}{8} \left\{ \frac{1}{\bar{\rho}^3} \arctan(u/\bar{\rho}) + \frac{1}{\bar{\rho}^2} \frac{u}{u^2 + \bar{\rho}^2} - \frac{2u}{(u^2 + \bar{\rho}^2)^2} \right\} . \tag{B4}$$

$$\int \frac{du}{\sqrt{(u^2 + \bar{y}^2)^3}} = \frac{u}{\bar{y}^2 \sqrt{u^2 + \bar{y}^2}} . \tag{B5}$$

$$\begin{aligned}
&\int \frac{u^2 du}{[u^2 + \bar{y}^2 + \bar{a}^2 - 2\bar{M}\sqrt{u^2 + \bar{y}^2}](u^2 + \bar{y}^2)^2} \\
&= \frac{4\gamma|y|}{a^2 + |y|(M + \sqrt{M^2 - a^2})} \left\{ \frac{1}{\mathcal{J} - 1} \left[ \arctan w - \frac{1}{\sqrt{\mathcal{J}}} \arctan(w/\sqrt{\mathcal{J}}) \right] \right. \\
&\quad \left. - \frac{\mathcal{H}}{\mathcal{J} - \mathcal{H}} \left[ \frac{1}{\sqrt{\mathcal{H}}} \arctan(w/\sqrt{\mathcal{H}}) - \frac{1}{\sqrt{\mathcal{J}}} \arctan(w/\sqrt{\mathcal{J}}) \right] \right\} , \tag{B6}
\end{aligned}$$

where

$$\begin{aligned}
w &= \frac{\sqrt{u^2 + \bar{y}^2} - |y|}{u} , \\
\mathcal{J} &= \frac{y^2 - a^2 + 2|y|\sqrt{M^2 - a^2}}{y^2 + a^2 + 2M|y|} , \tag{B7}
\end{aligned}$$

$$\begin{aligned}
\mathcal{H} &= \frac{y^2 - a^2 - 2|y|\sqrt{M^2 - a^2}}{y^2 + a^2 + 2M|y|} . \\
&\int \frac{du}{(u^2 + \bar{y}^2 + \bar{a}^2)\sqrt{(u^2 + \bar{y}^2)^3}} = \frac{1}{\bar{y}^2 \bar{a}^2} \left\{ \frac{u}{\sqrt{u^2 + \bar{y}^2}} + \frac{y^2}{2a\sqrt{y^2 + a^2}} \ln \frac{\sqrt{y^2 + a^2}\sqrt{u^2 + \bar{y}^2} - au}{\sqrt{y^2 + a^2}\sqrt{u^2 + \bar{y}^2} + au} \right\} . \tag{B8}
\end{aligned}$$

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- [1] For a review see, e.g., S. Chandrasekhar, *The Mathematical Theory of Black Holes* (Clarendon Press, Oxford, 1983).  
[2] P. D. D'Eath, Phys. Rev. D **18**, 990 (1978); P. D. D'Eath and P. N. Payne, *ibid.* **46**, 658 (1992); **46**, 675 (1992); **46**, 694 (1992).  
[3] For example see, G. 't Hooft, Phys. Lett. B **198**, 61 (1987); H. Verlinde and E. Verlinde, Nucl. Phys. **B371**, 246 (1992).  
[4] G. 't Hooft and T. Dray, Nucl. Phys. **B253**, 173 (1985).  
[5] I. D. Novikov and V. P. Frolov, *Physics of Black Holes*

- (Kluwer Academic, Dordrecht, 1989).  
[6] P. C. Aichelburg and R. Ú. Sexl, Gen. Relativ. Gravit. **2**, 303 (1971).  
[7] C. O. Loustó and N. Sánchez, Int. J. Mod. Phys. A **5**, 915 (1989); Phys. Lett. B **232**, 462 (1989); Nucl. Phys. **B355**, 231 (1991); **B383**, 377 (1992).  
[8] R. Penrose, Seminar at Cambridge University, 1974 (unpublished); L. A. Smarr *et al.*, Phys. Rev. D **14**, 2443 (1976).