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67. Algebraic graph theory

NORMAN BIGGS
Lecturer in Mathematics
Royal Holloway College, University of London

Algebraic Graph Theory

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Contents

Preface

It is a pleasure for me to acknowledge the help which I have received during the preparation of this book. A preliminary draft of the manuscript was read by Dr R. J. Wilson, and his detailed comments resulted in substantial changes and improvements. I was then fortunate to be able to rely upon the expert assistance of my wife for the production of a typescript. Ideas and helpful criticisms were offered by several friends and colleagues, among them G. de Barra, R. M. Damerell, A. D. Gardiner, R. K. Guy, P. McMullen and J. W. Moon. The general editor of the Cambridge Mathematical Tracts, Professor C. T. C. Wall, was swift and perceptive in his appraisal, and his comments were much appreciated. The staff of the Cambridge University Press maintained their usual high standard of courtesy and efficiency throughout the process of publication.

During the months January–April 1973, when the final stages of the writing were completed, I held a visiting appointment at the University of Waterloo, and my thanks are due to Professor W. T. Tutte for arranging this. In addition, I owe a mathematical debt to Professor Tutte, for he is the author of the two results, Theorems 13.9 and 18.6, which I regard as the most important in the book. I should venture the opinion that, were it not for his pioneering work, these results would still be unknown to this day.

NORMAN BIGGS

Waterloo, Canada

March 1973

1. *Introduction*

This book is concerned with the use of algebraic techniques in the study of graphs. We aim to translate properties of graphs into algebraic properties and then, using the results and methods of algebra, to deduce theorems about graphs.

The exposition which we shall give is not part of the modern functorial approach to topology, despite the claims of those who hold that, since graphs are one-dimensional spaces, graph theory is merely one-dimensional topology. By that definition, algebraic graph theory would consist only of the homology of 1-complexes. But the problems dealt with in graph theory are more delicate than those which form the substance of algebraic topology, and even if these problems can be generalized to dimensions greater than one, there is usually no hope of a general solution at the present time. Consequently, the algebra used in algebraic graph theory is largely unrelated to the subject which has come to be known as homological algebra.

This book is not an introduction to graph theory. It would be to the reader's advantage if he were familiar with the basic concepts of the subject, for example, as they are set out in the book by R. J. Wilson entitled *Introduction to graph theory*. However, for the convenience of those readers who do not have this background, we give brief explanations of important standard terms. These explanations are usually accompanied by a reference to Wilson's book (in the form [W, p. 99]), where further details may be found. In the same way, some concepts from permutation-group theory are accompanied by a reference [B, p. 99] to the author's book *Finite groups of automorphisms*. Both these books are described fully at the end of this chapter.

A few other books are also referred to for results which may be unfamiliar to some readers. In such cases, the result required is necessary for an understanding of the topic under discussion, so that the reference is given in full, enclosed in square brackets, where it is needed. Other references, of a supplementary nature, are given in parentheses in the form (Smith 1971) or Smith (1971). In such cases, the full reference may be found in the bibliography at the end of the book.

The tract is in three parts, each of which is further subdivided into a number of short chapters. Within each chapter, the major definitions and results are labelled using the decimal system.

The first part deals with the applications of linear algebra and matrix theory to the study of graphs. We begin by introducing the adjacency matrix of a graph; this matrix completely determines the graph, and its spectral properties are shown to be related to properties of the graph. For example, if a graph is regular, then the eigenvalues of its adjacency matrix are bounded in absolute value by the valency of the graph. In the case of a line graph, there is a strong lower bound for the eigenvalues.

Another matrix which completely describes a graph is the incidence matrix of the graph. This matrix represents a linear mapping which, in modern language, determines the homology of the graph; however, the sophistication of this language obscures the underlying simplicity of the situation. The problem of choosing a basis for the homology of a graph is just that of finding a fundamental system of circuits, and we solve this problem by using a spanning tree in the graph. At the same time we study the cutsets of the graph. These ideas are then applied to the systematic solution of network equations, a topic which supplied the stimulus for the original theoretical development.

We then investigate various formulae for the number of spanning trees in a graph, and apply these formulae to several well-known families of graphs. The first part of the book ends with results which are derived from the expansion of certain determinants, and which illuminate the relationship between a graph and the characteristic polynomial of its adjacency matrix.

The second part of the book deals with the problem of colouring the vertices of a graph in such a way that adjacent vertices have different colours. The least number of colours for which such a colouring is possible is called the chromatic number of the graph, and we begin by investigating some connections between this number and the eigenvalues of the adjacency matrix of the graph.

The algebraic technique for counting the colourings of a graph is founded on a polynomial known as the chromatic polynomial. We first discuss some simple ways of calculating this polynomial, and show how these can be applied in several important cases. Many important properties of the chromatic polynomial of a graph stem from its connection with the family of subgraphs of the graph, and we show how the chromatic polynomial can be expanded in terms of subgraphs. From our first (additive) expansion another (multiplicative) expansion can be derived, and the latter depends upon a very restricted class of subgraphs. This leads to efficient methods for approximating the chromatic polynomials of large graphs.

A completely different kind of expansion relates the chromatic polynomial to the spanning trees of a graph; this expansion has several remarkable features and leads to new ways of looking at the colouring problem, and some new properties of chromatic polynomials.

The third part of the book is concerned with symmetry and regularity properties. A symmetry property of a graph is related to the existence of automorphisms – that is, permutations of the vertices which preserve adjacency. A regularity property is defined in purely numerical terms. Consequently, symmetry properties induce regularity properties, but the converse is not necessarily true.

We first study the elementary properties of automorphisms, and explain the connection between the automorphisms of a graph and the eigenvalues of its adjacency matrix. We then introduce a hierarchy of symmetry conditions which can be imposed on a graph, and proceed to investigate their consequences. The condition that all vertices be alike (under the action of the group of automorphisms) turns out to be rather a weak one, but a slight strengthening of it leads to highly non-trivial conclusions. In fact, under certain conditions, there is an absolute bound to the level of symmetry which a graph can possess.

A new kind of symmetry property, called distance-transitivity, and the consequent regularity property, called distance-regularity, are then introduced. We return to the methods of linear algebra to derive strong constraints upon the existence of graphs with these properties. Finally, these constraints are applied to the problem of finding minimal regular graphs whose valency and girth are given.

At the end of each chapter there are some supplementary results and examples, labelled by the number of the chapter and a letter (as, for example, 9A). The reader is warned that these results are variable in difficulty and in kind. Their presence allows the inclusion of a great deal of material which would otherwise have interrupted the mainstream of the exposition, or would have had to be omitted altogether.

We end this introductory chapter by describing the few ways in which we differ from the terminology of Wilson's book.

In this book, a *general graph* Γ consists of three things: a finite set $V\Gamma$ of *vertices*, a finite set $E\Gamma$ of *edges*, and an *incidence* relation between vertices and edges. If v is a vertex, e is an edge, and (v, e) is a pair in the incidence relation, then we say that v is *incident* with e, and e is *incident* with v. Each edge is incident with either one vertex (in which case it is a *loop*) or two vertices.

If each edge is incident with two vertices, and no two edges are incident with the same pair of vertices, then we say that Γ is a *simple graph* or briefly, a *graph*. In this case, $E\Gamma$ can be identified with a subset of the set of (unordered) pairs of vertices, and we shall always assume that this identification has been made. We shall deal mainly with graphs (that is, simple graphs), except in Part Two, where it is sometimes essential to consider general graphs.

If v and w are vertices of a graph Γ, and $e = \{v, w\}$ is an edge of Γ, then we say that e *joins* v and w, and that v and w are the *ends* of e. The number of edges of which v is an end is called the *valency* of v.

We consider two kinds of subgraph of a general graph Γ. An *edge-subgraph* of Γ is constructed by taking a subset S of $E\Gamma$ together with all vertices of Γ incident in Γ with some edge belonging to S. A *vertex-subgraph* of Γ is constructed by taking a subset U of $V\Gamma$ together with all edges of Γ which are incident in Γ only with vertices belonging to U. In both cases the incidence relation in the subgraph is inherited from the incidence relation in Γ. We shall use the notation $\langle S\rangle_\Gamma$, $\langle U\rangle_\Gamma$ for these subgraphs, and usually, when the context is clear, the subscript reference to Γ will be omitted.

Further new terminology and notation will be defined when it is required.

Basic references

R. J. Wilson. *Introduction to graph theory* (Oliver and Boyd, Edinburgh, 1972).

N. L. Biggs. *Finite groups of automorphisms*, London Math. Society Lecture Notes Series, No. 6 (Cambridge University Press, 1971).

PART ONE

Linear algebra in graph theory

2. *The spectrum of a graph*

We begin by defining a matrix which will play an important role in many parts of this book. We shall suppose that Γ is a graph whose vertex-set $V\Gamma$ is the set $\{v_1, v_2, \ldots, v_n\}$; as explained in Chapter 1, we shall take $E\Gamma$ to be a subset of the set of unordered pairs of elements of $V\Gamma$. If $\{v_i, v_j\}$ is an edge, then we say that v_i and v_j are *adjacent*.

DEFINITION 2.1 The *adjacency matrix* of Γ is the $n \times n$ matrix $\mathbf{A} = \mathbf{A}(\Gamma)$, over the complex field, whose entries a_{ij} are given by

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } v_j \text{ are adjacent;} \\ 0 & \text{otherwise.} \end{cases}$$

It follows directly from the definition that $\mathbf{A}$ is a real symmetric matrix, and that the trace of $\mathbf{A}$ is zero. Since the rows and columns of $\mathbf{A}$ correspond to an arbitrary labelling of the vertices of Γ, it is clear that we shall be interested primarily in those properties of the adjacency matrix which are invariant under permutations of the rows and columns. Foremost among such properties are the spectral properties of $\mathbf{A}$.

Suppose that λ is an eigenvalue of $\mathbf{A}$. Then, since $\mathbf{A}$ is real and symmetric, λ is real, and the multiplicity of λ as a root of the characteristic equation $\det(\lambda\mathbf{I} - \mathbf{A}) = 0$ is equal to the dimension of the space of eigenvectors corresponding to λ.

DEFINITION 2.2 The *spectrum* of a graph Γ is the set of numbers which are eigenvalues of $\mathbf{A}(\Gamma)$, together with their multiplicities as eigenvalues of $\mathbf{A}(\Gamma)$. If the distinct eigenvalues of $\mathbf{A}(\Gamma)$ are $\lambda_0 > \lambda_1 > \ldots > \lambda_{s-1}$, and their multiplicities are $m(\lambda_0), m(\lambda_1), \ldots, m(\lambda_{s-1})$, then we shall write

$$\operatorname{Spec} \Gamma = \begin{pmatrix} \lambda_0 & \lambda_1 & \ldots & \lambda_{s-1} \\ m(\lambda_0) & m(\lambda_1) & \ldots & m(\lambda_{s-1}) \end{pmatrix}.$$

For example, the *complete graph* K_n has n vertices, and each distinct pair are adjacent [W, p. 16]. Thus, the graph K_4 has adjacency matrix

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix},$$

and an easy calculation shows that the spectrum of K_4 is:

$$\operatorname{Spec} K_4 = \begin{pmatrix} 3 & -1 \\ 1 & 3 \end{pmatrix}.$$

We shall often refer to the eigenvalues of $\mathbf{A}(\Gamma)$ as the *eigenvalues of* Γ. Also, the characteristic polynomial of $\mathbf{A}(\Gamma)$ will be denoted by $\chi(\Gamma; \lambda)$, and referred to as the *characteristic polynomial of* Γ.

Let us suppose that the characteristic polynomial of Γ is

$$\chi(\Gamma; \lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + c_3\lambda^{n-3} + \ldots + c_n.$$

Then the coefficients c_i can be interpreted as sums of principal minors of $\mathbf{A}$, and this leads to the following simple result.

PROPOSITION 2.3 *Using the notation given above, we have:*

(1) $c_1 = 0$;
(2) $-c_2$ *is the number of edges of* Γ;
(3) $-c_3$ *is twice the number of triangles in* Γ.

Proof For each $i \in \{1, 2, \ldots, n\}$, the number $(-1)^i c_i$ is the sum of those principal minors of $\mathbf{A}$ which have i rows and columns. Thus:

(1) Since the diagonal elements of $\mathbf{A}$ are all zero, $c_1 = 0$.

(2) A principal minor with two rows and columns, and which has a non-zero entry, must be of the form

$$\begin{vmatrix} 0 & 1 \\ 1 & 0 \end{vmatrix}.$$

There is one such minor for each pair of adjacent vertices of Γ, and each has value -1. Hence $(-1)^2 c_2 = -|E\Gamma|$, giving the result.

(3) There are essentially three possibilities for non-trivial principal minors with three rows and columns:

$$\begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{vmatrix}, \quad \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{vmatrix},$$

and, of these, the only non-zero one is the last (whose value is 2). This principal minor corresponds to three mutually adjacent vertices in Γ, and so we have the required description of c_3. □

These elementary results indicate that the characteristic polynomial of a graph is a typical object of the kind one considers in algebraic theory: it is an algebraic construction which contains graphical information. Proposition 2.3 is just a pointer, and we shall obtain a more comprehensive result on the coefficients of the characteristic polynomial in Chapter 7.

Suppose $\mathbf{A}$ is the adjacency matrix of a graph Γ. Then the set of polynomials in $\mathbf{A}$, with complex coefficients, forms an algebra under the usual matrix operations. This algebra has finite dimension as a complex vector space.

DEFINITION 2.4 The *adjacency algebra* of a graph Γ is the algebra of polynomials in the adjacency matrix $\mathbf{A} = \mathbf{A}(\Gamma)$. We shall denote the adjacency algebra of Γ by $\mathscr{A}(\Gamma)$.

Since every element of the adjacency algebra is a linear combination of powers of $\mathbf{A}$, we can obtain results about $\mathscr{A}(\Gamma)$ from a study of these powers. We define a *walk* of length l in Γ, joining v_i to v_j, to be a finite sequence of vertices of Γ,

$$v_i = u_0, u_1, \ldots, u_l = v_j,$$

such that u_{t-1} and u_t are adjacent for $1 \leqslant t \leqslant l$. (If u_{t-1} and u_{t+1} are distinct, $1 \leqslant t \leqslant l-1$, then we say that the walk is a *path*.)

LEMMA 2.5 *The number of walks of length l in Γ, joining v_i to v_j, is the entry in position (i, j) of the matrix $\mathbf{A}^l$.*

Proof The result is true for $l = 0$ (since $\mathbf{A}^0 = \mathbf{I}$) and for $l = 1$ (since $\mathbf{A}^1 = \mathbf{A}$ is the adjacency matrix). Suppose that the result is true for $l = L$. From the identity

$$(\mathbf{A}^{L+1})_{ij} = \sum_{h=1}^{n} (\mathbf{A}^L)_{ih} a_{hj},$$

we deduce that $(\mathbf{A}^{L+1})_{ij}$ is the number of walks of length $L+1$ joining v_i to v_j, whence the result for all l follows by induction. □

In a *connected* graph every pair of vertices may be joined by a walk. The number of edges traversed in the shortest walk joining v_i and v_j is called the *distance* in Γ between v_i and v_j and is denoted by $\partial(v_i, v_j)$ [W, p. 31]. The maximum value of the distance function in Γ is called the *diameter* of Γ.

PROPOSITION 2.6 *Let Γ be a connected graph with adjacency algebra $\mathscr{A}(\Gamma)$ and diameter d. Then the dimension of $\mathscr{A}(\Gamma)$ is at least $d+1$.*

Proof Let x and y be vertices of Γ such that $\partial(x, y) = d$, and suppose

$$x = w_0, w_1, \ldots, w_d = y$$

is a path of length d. Then, for each $i \in \{1, 2, \ldots, d\}$, there is at least one walk of length i, but no shorter walk, joining w_0 to w_i. Consequently, $\mathbf{A}^i$ has a non-zero entry in a position where the corresponding entries of $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \ldots, \mathbf{A}^{i-1}$ are zero; and it follows that $\mathbf{A}^i$ is not linearly dependent on $\{\mathbf{I}, \mathbf{A}, \ldots, \mathbf{A}^{i-1}\}$. We deduce that $\{\mathbf{I}, \mathbf{A}, \ldots, \mathbf{A}^d\}$ is a linearly independent set in $\mathscr{A}(\Gamma)$, and the proposition is proved. □

There is a close connection between the adjacency algebra and the spectrum of Γ. If the adjacency matrix has s distinct eigenvalues, then (since it is a real symmetric matrix), its minimum polynomial has degree s, and consequently the dimension of the adjacency algebra is equal to s. We remark that there is no real loss of generality in restricting our attention to connected graphs, since the spectrum of a disconnected graph is the union of the spectra of its connected components. We have obtained the following bounds for the number of distinct eigenvalues.

COROLLARY 2.7 *A connected graph with n vertices and diameter d has at least $d+1$, and at most n, distinct eigenvalues.* □

One of the major topics of the last part of this book is the study of graphs which have the minimal number of distinct eigenvalues. These graphs have startling regularity properties.

2A *A criterion for connectedness* A graph Γ with n vertices is connected if and only if $(\mathbf{A}+\mathbf{I})^{n-1}$ has no zero entries, where $\mathbf{A}$ is the adjacency matrix of Γ.

2B *Cospectral graphs* Two non-isomorphic graphs are said to be *cospectral* if they have the same eigenvalues with the same multiplicities. Many examples of this phenomenon are cited in the survey by Cvetković (1971). In particular, there are two cospectral connected graphs (shown in Fig. 1) with 6 vertices; their characteristic polynomial is $\lambda^6 - 7\lambda^4 - 4\lambda^3 + 7\lambda^2 + 4\lambda - 1$. (See also Chapter 7.)

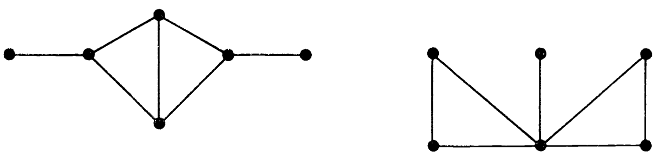


Fig. 1

2C *The walk generating function* Let Γ be a graph with vertex-set $\{v_1, v_2, \ldots, v_n\}$, and let $\nu_{ij}(r)$ denote the number of walks of length r in Γ joining v_i to v_j. If we write $\mathbf{N}$ for the matrix whose entries are

$$(\mathbf{N})_{ij} = \sum_{r=1}^{\infty} \nu_{ij}(r)\, t^r,$$

then $\mathbf{N} = (\mathbf{I} - t\mathbf{A})^{-1}$, where $\mathbf{A}$ is the adjacency matrix of Γ.

2D *A bound for the eigenvalues* Let Γ be a graph with n vertices and m edges, and suppose the eigenvalues of Γ are $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n$. Then $\Sigma\lambda_i = 0$, $\Sigma\lambda_i^2 = 2m$. Applying the Cauchy–Schwarz inequality to $(\lambda_2, \ldots, \lambda_n)$ and $(1, 1, \ldots, 1)$ leads to the bound

$$\lambda_1 \leqslant \left(\frac{2m(n-1)}{n}\right)^{\frac{1}{2}}.$$

3. *Regular graphs and line graphs*

In this chapter we discuss graphs which possess some kinds of combinatorial regularity, and whose spectra, in consequence, have distinctive features. A graph is said to be *regular* of valency k (or k-valent) if each of its vertices has valency k. This is the most obvious kind of combinatorial regularity, and it has interesting consequences.

PROPOSITION 3.1 *Let Γ be a regular graph of valency k. Then:*

(1) *k is an eigenvalue of Γ.*
(2) *If Γ is connected, then the multiplicity of k is one.*
(3) *For any eigenvalue λ of Γ, we have $|\lambda| \leqslant k$.*

Proof (1) Let $\mathbf{u} = [1, 1, \ldots, 1]^t$; then if $\mathbf{A}$ is the adjacency matrix of Γ we have $\mathbf{Au} = k\mathbf{u}$, so that k is an eigenvalue of Γ.

(2) Let $\mathbf{x} = [x_1, x_2, \ldots, x_n]^t$ denote any non-zero vector for which $\mathbf{Ax} = k\mathbf{x}$, and suppose x_j is an entry of $\mathbf{x}$ having the largest absolute value. Since $(\mathbf{Ax})_j = kx_j$, we have

$$\Sigma' x_i = kx_j,$$

where the summation is over those k vertices v_i which are adjacent to v_j. By the maximal property of x_j, it follows that $x_i = x_j$ for all these vertices. If Γ is connected we may proceed successively in this way, eventually showing that all entries of $\mathbf{x}$ are equal. Thus $\mathbf{x}$ is a multiple of $\mathbf{u}$, and the space of eigenvectors associated with the eigenvalue k has dimension one.

(3) Suppose that $\mathbf{Ay} = \lambda\mathbf{y}$, $\mathbf{y} \neq \mathbf{0}$, and let y_j denote an entry of $\mathbf{y}$ which is largest in absolute value. By the same argument as in (2), we have $\Sigma' y_i = \lambda y_j$, and so

$$|\lambda|\,|y_j| = |\Sigma' y_i| \leqslant \Sigma' |y_i| \leqslant k\,|y_j|.$$

Thus $|\lambda| \leqslant k$, as required. □

The adjacency algebra of a regular connected graph also has a distinctive property, related to the results of Proposition 3.1. Let $\mathbf{J}$ denote the matrix each of whose entries is $+1$. Then, if $\mathbf{A}$ is the adjacency matrix of a regular graph of valency k, we have $\mathbf{AJ} = \mathbf{JA} = k\mathbf{J}$. This is the point of departure for the following result.

PROPOSITION 3.2 (Hoffman 1963) *The matrix* $\mathbf{J}$ *belongs to the adjacency algebra* $\mathscr{A}(\Gamma)$ *if and only if* Γ *is a regular connected graph.*

Proof Suppose $\mathbf{J} \in \mathscr{A}(\Gamma)$. Then $\mathbf{J}$ is, by definition of $\mathscr{A}(\Gamma)$, a polynomial in $\mathbf{A}$; consequently $\mathbf{AJ} = \mathbf{JA}$. Now if $k^{(i)}$ denotes the valency of the vertex v_i, then $(\mathbf{AJ})_{ij} = k^{(i)}$ and $(\mathbf{JA})_{ij} = k^{(j)}$, so that all the valencies are equal and Γ is regular. Further, if Γ were disconnected we could find two vertices with no walks in joining them, so that the corresponding entry of $\mathbf{A}^l$ would be zero for all $l \geqslant 0$. Then every polynomial in $\mathbf{A}$ would have a zero entry, contradicting the fact that $\mathbf{J} \in \mathscr{A}(\Gamma)$. Thus Γ is connected.

Conversely, suppose Γ is connected and regular of valency k. Then, by part (1) of Proposition 3.1, k is an eigenvalue of Γ, and so the minimum polynomial of $\mathbf{A}$ is of the form $(\lambda - k)q(\lambda)$. This means that $\mathbf{A}q(\mathbf{A}) = kq(\mathbf{A})$, that is, each column of $q(\mathbf{A})$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue k. By part (2) of Proposition 3.1, this means that each column of $q(\mathbf{A})$ is a multiple of $\mathbf{u}$, and since $q(\mathbf{A})$ is a symmetric matrix, it is a multiple of $\mathbf{J}$. Thus $\mathbf{J}$ is a polynomial in $\mathbf{A}$. $\square$

COROLLARY 3.3 *Let* Γ *be a regular connected graph with* n *vertices, and let the distinct eigenvalues of* Γ *be* $k > \lambda_1 > \ldots > \lambda_{s-1}$. *Then if* $q(\lambda) = \Pi(\lambda - \lambda_i)$, *where the product is over the range* $1 \leqslant i \leqslant s-1$, *we have*

$$\mathbf{J} = \left(\frac{n}{q(k)}\right) q(\mathbf{A}).$$

Proof It follows from the proof of Proposition 3.2 that $q(\mathbf{A}) = \alpha\mathbf{J}$, for some constant α. Now the eigenvalues of $q(\mathbf{A})$ are $q(k)$ and $q(\lambda_i)$ $(1 \leqslant i \leqslant s-1)$, and all these except $q(k)$ are zero. The only non-zero eigenvalue of $\alpha\mathbf{J}$ is αn, hence $\alpha = q(k)/n$. $\square$

There is a special class of regular graphs whose spectra can be found by means of an old technique in matrix theory. As this class contains several well-known families of graphs, we shall briefly review the relevant theory.

An $n \times n$ matrix $\mathbf{S}$ is said to be a *circulant matrix* if its entries satisfy $s_{ij} = s_{1,j-i+1}$, where the subscripts are reduced modulo n and lie in the set $\{1, 2, \ldots, n\}$. In other words, row i of $\mathbf{S}$ is obtained from the first row of $\mathbf{S}$ by a cyclic shift of $i-1$ steps, and so any circulant matrix is determined by its first row. Let $\mathbf{W}$ denote the circulant matrix whose first row is $[0, 1, 0, \ldots, 0]$, and let $\mathbf{S}$ denote a general circulant matrix whose first row is $[s_1, s_2, \ldots, s_n]$. Then a straightforward calculation shows that

$$\mathbf{S} = \sum_{j=1}^{n} s_j \mathbf{W}^{j-1}.$$

Since the eigenvalues of $\mathbf{W}$ are $1, \omega, \omega^2, \ldots, \omega^{n-1}$, where $\omega = \exp(2\pi i/n)$, it follows that the eigenvalues of $\mathbf{S}$ are

$$\lambda_r = \sum_{j=1}^{n} s_j \omega^{(j-1)r}, \quad r = 0, 1, \ldots, n-1.$$

DEFINITION 3.4 A *circulant graph* is a graph Γ whose adjacency matrix $\mathbf{A}(\Gamma)$ is a circulant matrix.

It follows, from the fact the adjacency matrix is a symmetric matrix with zero entries on the main diagonal, that, if the first row of the adjacency matrix of a circulant graph is $[a_1, a_2, \ldots, a_n]$, then $a_1 = 0$ and $a_i = a_{n-i+2}$ $(2 \leqslant i \leqslant n)$.

PROPOSITION 3.5 *Suppose* $[0, a_2, \ldots, a_n]$ *is the first row of the adjacency matrix of a circulant graph* Γ. *Then the eigenvalues of* Γ *are*

$$\lambda_r = \sum_{j=2}^{n} a_j \omega^{(j-1)r}, \quad r = 0, 1, \ldots, n-1.$$

Proof This result follows directly from the expression for the eigenvalues of a circulant matrix. □

We notice that the condition $a_i = a_{n-i+2}$ ensures that the eigenvalues of a circulant graph are real; and that the n eigenvalues given by the formula of Proposition 3.5 are not necessarily all distinct.

We give three important examples of this technique. First, the complete graph K_n is a circulant graph, and the first row of its adjacency matrix is $[0, 1, 1, \ldots, 1]$. Since $1+\omega^r+\ldots+\omega^{(n-1)r}=0$ for $r\in\{1, 2, \ldots, n-1\}$, it follows from Proposition 3.5 that the spectrum of K_n is:

$$\operatorname{Spec} K_n = \begin{pmatrix} n-1 & -1 \\ 1 & n-1 \end{pmatrix}.$$

Our second example is the *circuit graph* C_n, whose adjacency matrix is a circulant matrix with first row $[0, 1, 0, \ldots, 0, 1]$. In the notation of Proposition 3.5, the eigenvalues are $\lambda_r = 2\cos(2\pi r/n)$, but these numbers are not all distinct and the complete description of the spectrum is:

$$\text{(if } n \text{ is odd)} \quad \operatorname{Spec} C_n = \begin{pmatrix} 2 & 2\cos 2\pi/n & \ldots & 2\cos(n-1)\pi/n \\ 1 & 2 & \ldots & 2 \end{pmatrix},$$

$$\text{(if } n \text{ is even)} \operatorname{Spec} C_n = \begin{pmatrix} 2 & 2\cos 2\pi/n & \ldots & 2\cos(n-2)\pi/n & -2 \\ 1 & 2 & \ldots & 2 & 1 \end{pmatrix}.$$

A third family of circulant graphs which we shall encounter is the family H_s of *hyperoctahedral graphs* [B, p. 101]. The graph H_s may be constructed by removing s disjoint edges from K_{2s}; consequently H_s is the complete multipartite graph $K_{2,2,\ldots,2}$, as defined on p. 37. It is also a circulant graph, and the first row of its adjacency matrix is $[a_1, \ldots, a_{2s}]$, where each entry is 1, except for a_1 and a_{s+1} which are zero. It follows that the eigenvalues of H_s are

$$\lambda_0 = 2s-2, \quad \lambda_r = -1-\omega^{rs} \quad (1 \leqslant r \leqslant 2s-1),$$

where $\omega^{2s} = 1$ and $\omega \neq 1$. Consequently,

$$\operatorname{Spec} H_s = \begin{pmatrix} 2s-2 & 0 & -2 \\ 1 & s & s-1 \end{pmatrix}.$$

The *line graph* $L(\Gamma)$ of a graph Γ is constructed by taking the edges of Γ as vertices of $L(\Gamma)$, and joining two vertices in $L(\Gamma)$ whenever the corresponding edges in Γ have a common vertex [W, p. 15]. The spectra of line graphs have been investigated extensively, particularly by Hoffman (1969) and his associates, and we now outline the basic results in this field.

We shall continue to suppose that Γ has n vertices, labelled $v_1, v_2, \ldots, v_n$, and we shall need to label the edges of Γ similarly; that is $E\Gamma = \{e_1, e_2, \ldots, e_m\}$. For this chapter only we define an $n \times m$ matrix $\mathbf{X} = \mathbf{X}(\Gamma)$ as follows:

$$(\mathbf{X})_{ij} = \begin{cases} 1 & \text{if } v_i \text{ and } e_j \text{ are incident,} \\ 0 & \text{otherwise.} \end{cases}$$

LEMMA 3.6 *Suppose that Γ and $\mathbf{X}$ are as above. Let $\mathbf{A}$ denote the adjacency matrix of Γ and $\mathbf{A}_L$ the adjacency matrix of $L(\Gamma)$. Then:*

(1) $\mathbf{X}^t\mathbf{X} = \mathbf{A}_L + 2\mathbf{I}_m$;

(2) *if Γ is regular of valency k, then* $\mathbf{X}\mathbf{X}^t = \mathbf{A} + k\mathbf{I}_n$.

(The subscripts denote the sizes of the identity matrices.)

Proof (1) We have

$$(\mathbf{X}^t\mathbf{X})_{ij} = \sum_{l=1}^{n} (\mathbf{X})_{li}\,(\mathbf{X})_{lj},$$

from which it follows that $(\mathbf{X}^t\mathbf{X})_{ij}$ is the number of vertices v_l of Γ which are incident with both the edges e_i and e_j. The required result is now a consequence of the definitions of $L(\Gamma)$ and $\mathbf{A}_L$.

(2) This part is proved by a similar counting argument. □

PROPOSITION 3.7 *If λ is an eigenvalue of a line graph $L(\Gamma)$, then $\lambda \geqslant -2$.*

Proof The matrix $\mathbf{X}^t\mathbf{X}$ is positive-semidefinite, since for each vector $\mathbf{z}$ we have $\mathbf{z}^t\mathbf{X}^t\mathbf{X}\mathbf{z} = \|\mathbf{X}\mathbf{z}\|^2 \geqslant 0$. Thus the eigenvalues of $\mathbf{X}^t\mathbf{X}$ are non-negative. But $\mathbf{A}_L = \mathbf{X}^t\mathbf{X} - 2\mathbf{I}_m$, so the eigenvalues of $\mathbf{A}_L$ are not less than -2. □

The condition that all eigenvalues of a graph be not less than -2 is a restrictive one, but it is not sufficient to characterize line graphs. For, each hyperoctahedral graph H_s satisfies the condition, but these graphs are not line graphs. However, there are only a few known examples of regular graphs which have least eigenvalue -2 and which are not either line graphs or hyperoctahedral graphs (Seidel 1968, see § 3E).

When Γ is a regular graph of valency k, its line graph $L(\Gamma)$ is regular of valency $2k-2$; this is a connection between the maximum eigenvalues of Γ and $L(\Gamma)$. In fact this connection can be extended to all eigenvalues, by means of the following relationship between the characteristic polynomials of Γ and $L(\Gamma)$.

THEOREM 3.8 (Sachs 1967) *If Γ is a regular graph of valency k with n vertices and $m = \frac{1}{2}nk$ edges, then*

$$\chi(L(\Gamma); \lambda) = (\lambda+2)^{m-n} \chi(\Gamma; \lambda+2-k).$$

Proof We shall use the notation and results of Lemma 3.6. Let us define two partitioned matrices with $n+m$ rows and columns as follows:

$$\mathbf{U} = \begin{bmatrix} \lambda \mathbf{I}_n & -\mathbf{X} \\ \mathbf{0} & \mathbf{I}_m \end{bmatrix}, \quad \mathbf{V} = \begin{bmatrix} \mathbf{I}_n & \mathbf{X} \\ \mathbf{X}^t & \lambda \mathbf{I}_m \end{bmatrix}.$$

We then have

$$\mathbf{UV} = \begin{bmatrix} \lambda \mathbf{I}_n - \mathbf{XX}^t & \mathbf{0} \\ \mathbf{X}^t & \lambda \mathbf{I}_m \end{bmatrix}, \quad \mathbf{VU} = \begin{bmatrix} \lambda \mathbf{I}_n & \mathbf{0} \\ \lambda \mathbf{X}^t & \lambda \mathbf{I}_m - \mathbf{X}^t\mathbf{X} \end{bmatrix}.$$

Since $\det \mathbf{UV} = \det \mathbf{VU}$ we deduce that

$$\lambda^m \det(\lambda \mathbf{I}_n - \mathbf{XX}^t) = \lambda^n \det(\lambda \mathbf{I}_m - \mathbf{X}^t\mathbf{X}).$$

Thus we may argue as follows:

$$\begin{aligned} \chi(L(\Gamma); \lambda) &= \det(\lambda \mathbf{I}_m - \mathbf{A}_L) \\ &= \det((\lambda+2)\mathbf{I}_m - \mathbf{X}^t\mathbf{X}) \\ &= (\lambda+2)^{m-n} \det((\lambda+2)\mathbf{I}_n - \mathbf{XX}^t) \\ &= (\lambda+2)^{m-n} \det((\lambda+2-k)\mathbf{I}_n - \mathbf{A}) \\ &= (\lambda+2)^{m-n} \chi(\Gamma; \lambda+2-k). \quad \square \end{aligned}$$

In other words, if the spectrum of Γ is

$$\operatorname{Spec} \Gamma = \begin{pmatrix} k & \lambda_1 & \dots & \lambda_s \\ 1 & m_1 & \dots & m_s \end{pmatrix},$$

then the spectrum of $L(\Gamma)$ is

$$\operatorname{Spec} L(\Gamma) = \begin{pmatrix} 2k-2 & k-2+\lambda_1 & \dots & k-2+\lambda_s & -2 \\ 1 & m_1 & \dots & m_s & m-n \end{pmatrix}.$$

For example, the line graph $L(K_t)$ is sometimes called the *triangle graph* and denoted by Δ_t. It can be described by saying that its vertices correspond to the $\frac{1}{2}t(t-1)$ pairs of numbers from the set $\{1, 2, \ldots, t\}$, two vertices being adjacent whenever the corresponding pairs have just one common member. From the known spectrum of K_t, and Theorem 3.8, we have

$$\operatorname{Spec} \Delta_t = \begin{pmatrix} 2t-4 & t-4 & -2 \\ 1 & t-1 & \frac{1}{2}t(t-3) \end{pmatrix}.$$

3*A* *The complement of a regular graph* Let Γ be a regular connected graph with valency k and n vertices, and let Γ^c denote its complement [W, p. 20]. Then

$$(\lambda+k+1)\,\chi(\Gamma^c; \lambda) = (-1)^n(\lambda-n+k+1)\,\chi(\Gamma; -\lambda-1).$$

We can deduce from this that, if $k-n$ is an eigenvalue of Γ, then Γ^c is not connected.

3*B* *Petersen's graph* The complement of the line graph of K_5 is known as Petersen's graph [W, p. 16]. We shall denote this graph by the symbol O_3, as it is the trivalent member of the family $\{O_k\}$ of *odd graphs*, to be defined later (§ 8E). We have

$$\operatorname{Spec} O_3 = \begin{pmatrix} 3 & 1 & -2 \\ 1 & 5 & 4 \end{pmatrix}.$$

3*C* *The Möbius ladders* The Möbius ladder M_h is a trivalent graph with $2h$ vertices ($h \geqslant 3$). It is constructed from the circuit graph C_{2h} by adding new edges joining each pair of opposite vertices. The eigenvalues of M_h are the numbers

$$\lambda_j = 2\cos \pi j/h + (-1)^j \quad (0 \leqslant j \leqslant 2h-1).$$

3*D* *Regular graphs characterized by their spectra* (*a*) The spectrum of the triangle graph $\Delta_t = L(K_t)$ is given above. If Γ is a graph for which $\operatorname{Spec}\Gamma = \operatorname{Spec}\Delta_t$ ($t \neq 8$), then $\Gamma = \Delta_t$. In the case $t = 8$, there are three exceptional graphs, not isomorphic with Δ_8, but having the same spectrum as Δ_8 (Chang 1959, Hoffman 1960).

(*b*) If Γ is a graph for which $\operatorname{Spec}\Gamma = \operatorname{Spec} L(K_{a,a})$ ($a \neq 4$), where $K_{a,a}$ is the complete bipartite graph, then $\Gamma = L(K_{a,a})$. In the case $a = 4$ there is one exceptional graph; this graph is depicted in Fig. 2 (Shrikhande 1959).

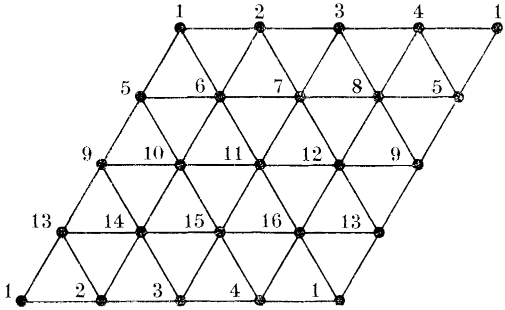


Fig. 2

3*E* *Regular graphs whose least eigenvalue is* -2 The following regular graphs having least eigenvalue -2 are described by Seidel (1968). They are neither line graphs nor hyper-octahedral graphs.

(*a*) Petersen's graph.

(*b*) A 5-valent graph with 16 vertices (Seidel 1968, p. 296).

(*c*) A 16-valent graph with 27 vertices (*ibid.*, p. 296).

(*d*) The exceptional graphs mentioned in § 3D.

4. The homology of graphs

This chapter is devoted to some linear algebra which is of fundamental importance in graph theory. The material is quite well-known, but we shall make some innovations in our notation and mode of presentation.

Let $\mathbb{C}$ denote the field of complex numbers, and let X be any finite set. Then the set of all functions from X to $\mathbb{C}$ has the structure of a finite-dimensional vector space; if $f: X \to \mathbb{C}$ and $g: X \to \mathbb{C}$, then the vector space operations are defined by the rules

$$(f+g)(x) = f(x)+g(x), \quad (\alpha f)(x) = \alpha f(x) \quad (x \in X, \alpha \in \mathbb{C}).$$

The dimension of this vector space is equal to the number of members of X.

DEFINITION 4.1 The *vertex-space* $C_0(\Gamma)$ of a graph Γ is the vector space of all functions from $V\Gamma$ to $\mathbb{C}$. The *edge-space* $C_1(\Gamma)$ of Γ is the vector space of all functions from $E\Gamma$ to $\mathbb{C}$.

We shall suppose throughout this chapter that $V\Gamma = \{v_1, v_2, \ldots, v_n\}$ and $E\Gamma = \{e_1, e_2, \ldots, e_m\}$; thus $C_0(\Gamma)$ is a vector space of dimension n and $C_1(\Gamma)$ is a vector space of dimension m. Any function η: $V\Gamma \to \mathbb{C}$ can be represented by a column vector

$$\mathbf{y} = [y_1, y_2, \ldots, y_n]^t,$$

where $y_i = \eta(v_i)$ $(1 \leqslant i \leqslant n)$. This representation corresponds to choosing as a basis for $C_0(\Gamma)$ the set of functions $\{\omega_1, \omega_2, \ldots, \omega_n\}$, defined by

$$\omega_i(v_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

In a similar way, we may choose the basis $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$ for $C_1(\Gamma)$ defined by

$$\epsilon_i(e_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise;} \end{cases}$$

and hence represent a function $\xi\colon E\Gamma \to \mathbb{C}$ by a column vector $\mathbf{x} = [x_1, x_2, \ldots, x_m]^t$ such that $x_i = \xi(e_i)$ $(1 \leqslant i \leqslant m)$. The bases $\{\omega_1, \omega_2, \ldots, \omega_n\}$ and $\{\epsilon_1, \epsilon_2, \ldots, \epsilon_m\}$ will be referred to as the *standard bases* for $C_0(\Gamma)$ and $C_1(\Gamma)$.

We now introduce a useful device. For each edge $e_\alpha = \{v_\sigma, v_\tau\}$ of Γ, we shall choose one of v_σ, v_τ to be the *positive* end of e_α, and the other one to be the *negative* end. We refer to this procedure by saying that Γ has been given an *orientation*. It will appear that, although this device is employed in the proofs of several results, the results themselves are independent of it.

DEFINITION 4.2 The *incidence matrix* $\mathbf{D}$ of Γ, with respect to a given orientation of Γ, is the $n \times m$ matrix (d_{ij}) whose entries are

$$d_{ij} = \begin{cases} +1 & \text{if } v_i \text{ is the positive end of } e_j, \\ -1 & \text{if } v_i \text{ is the negative end of } e_j, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that $\mathbf{D}$ is the representation, with respect to the standard bases, of a linear mapping from $C_1(\Gamma)$ to $C_0(\Gamma)$. This mapping will be called the *incidence mapping*, and be denoted by D. For each $\xi\colon E\Gamma \to \mathbb{C}$ the function $D\xi\colon V\Gamma \to \mathbb{C}$ is defined by

$$D\xi(v_i) = \sum_{j=1}^{m} d_{ij}\,\xi(e_j) \quad (1 \leqslant i \leqslant n).$$

The rows of the incidence matrix correspond to the vertices of Γ, and its columns correspond to the edges of Γ; each column contains just two non-zero entries, $+1$ and -1, representing the positive and negative ends of the corresponding edge.

For the remainder of this chapter we shall let c denote the number of connected components of Γ.

PROPOSITION 4.3 *The incidence matrix* $\mathbf{D}$ *of* Γ *has rank* $n-c$.

Proof The incidence matrix can be written in the partitioned form

$$\begin{bmatrix} \mathbf{D}^{(1)} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{(2)} & \ldots & \mathbf{0} \\ \vdots & \vdots & & \vdots \\ \mathbf{0} & \mathbf{0} & \ldots & \mathbf{D}^{(c)} \end{bmatrix},$$

by a suitable labelling of the vertices and edges of Γ. The matrix $\mathbf{D}^{(i)}$ $(1 \leqslant i \leqslant c)$ is the incidence matrix of a component $\Gamma^{(i)}$ of Γ, and we shall show that the rank of $\mathbf{D}^{(i)}$ is $n_i - 1$, where $n_i = |V\Gamma^{(i)}|$. The required result then follows, by addition.

Let $\mathbf{d}_j$ denote the row of $\mathbf{D}^{(i)}$ corresponding to the vertex v_j of $\Gamma^{(i)}$. Since there is just one $+1$ and just one -1 in each column of $\mathbf{D}^{(i)}$, it follows that the sum of the rows of $\mathbf{D}^{(i)}$ is the zero row vector, and the rank of $\mathbf{D}^{(i)}$ is at most $n_i - 1$. Suppose we have a linear relation $\Sigma \alpha_j \mathbf{d}_j = \mathbf{0}$, where the summation is over all rows of $\mathbf{D}^{(i)}$, and not all the coefficients α_j are zero. Choose a row $\mathbf{d}_k$ for which $\alpha_k \neq 0$; this row has non-zero entries in those columns corresponding to the edges incident with v_k. For each such column, there is just one other row $\mathbf{d}_l$ with a non-zero entry in that column, and in order that the given linear relation should hold, we must have $\alpha_l = \alpha_k$. Thus, if $\alpha_k \neq 0$, then $\alpha_l = \alpha_k$ for all vertices v_l adjacent to v_k. Since $\Gamma^{(i)}$ is connected, it follows that all coefficients α_j are equal, and so the given linear relation is just a multiple of $\Sigma \mathbf{d}_j = \mathbf{0}$. Consequently, the rank of $\mathbf{D}^{(i)}$ is $n_i - 1$. □

The following definition, which is for a general graph Γ with n vertices, m edges, and c components, will be applied (in this part of the book) to simple graphs only.

DEFINITION 4.4 The *rank* of Γ and the *co-rank* of Γ are, respectively

$$r(\Gamma) = n - c; \quad s(\Gamma) = m - n + c.$$

We now investigate the kernel of the incidence mapping D, and its relationship with graph-theoretical properties of Γ.

Let Q be a set of edges such that the edge-subgraph $\langle Q \rangle$ is a circuit graph. We say that Q is a *circuit* in Γ; the two possible cyclic orderings of the vertices of $\langle Q \rangle$ induce two possible *circuit-orientations* of the edges of Q. Let us choose one of these circuit-orientations, and define a function ξ_Q in $C_1(\Gamma)$ as follows. We put $\xi_Q(e) = +1$ if e belongs to Q and its circuit-orientation coincides with its orientation in Γ, $\xi_Q(e) = -1$ if e belongs to Q and its circuit-orientation is the reverse of its orientation in Γ, while if e is not in Q we put $\xi_Q(e) = 0$.

THEOREM 4.5 *The kernel of the incidence mapping D of Γ is a vector space whose dimension is equal to the co-rank of Γ. If Q is a circuit in Γ, then ξ_Q belongs to the kernel of D.*

Proof Since the rank of D is $n-c$, and the dimension of $C_1(\Gamma)$ is m, it follows that the kernel of D has dimension $m-n+c = s(\Gamma)$.

With respect to the standard bases for $C_1(\Gamma)$ and $C_0(\Gamma)$, we may take D to be the incidence matrix, and ξ_Q to be represented by a column vector $\mathbf{x}_Q$. Now $(\mathbf{D}\mathbf{x}_Q)_i$ is the inner product of the row $\mathbf{d}_i$ of $\mathbf{D}$ and the vector $\mathbf{x}_Q$. If v_i is not incident with any edges of Q, then this inner product is zero; if v_i is incident with some edges of Q, then it is incident with precisely two edges, and the choice of signs in the definition of ξ_Q implies that the inner product is again zero. Thus $\mathbf{D}\mathbf{x}_Q = \mathbf{0}$, and ξ_Q belongs to the kernel of D. □

If ρ and σ are two elements of the edge-space of Γ (that is, functions from $E\Gamma$ to $\mathbb{C}$), then we may define the inner product

$$(\rho, \sigma) = \sum_{e \in E\Gamma} \rho(e)\,\sigma(e).$$

When ρ and σ are represented by coordinate vectors, with respect to the standard basis of $C_1(\Gamma)$, this inner product corresponds to the usual inner product of vectors.

DEFINITION 4.6 The *circuit-subspace* of Γ is the kernel of the incidence mapping of Γ. The *cutset-subspace* of Γ is the orthogonal complement of the circuit-subspace in $C_1(\Gamma)$, with respect to the inner product defined above.

The first part of this definition is justified by the result of Theorem 4.5, which says that vectors representing circuits belong to the circuit-subspace. We now proceed to justify the second part of the definition.

Let $V\Gamma = V_1 \cup V_2$ be a partition of $V\Gamma$ into non-empty disjoint subsets. If the set H of edges of Γ which have one end in V_1 and one end in V_2 is non-empty, then we say that H is a *cutset* in Γ; the edge-subgraph $\langle E\Gamma - H\rangle$ has more components than Γ. We may choose one of the two possible *cutset-orientations* for H, by specifying that one of V_1, V_2 contains the positive ends of all edges in H, while the other contains the negative ends. We now define a function ξ_H in $C_1(\Gamma)$ by putting $\xi_H(e) = +1$ if e belongs to H and its cutset-orientation coincides with its orientation in Γ, $\xi_H(e) = -1$ if e belongs to H and its cutset-orientation is the reverse of its orientation in Γ, and $\xi_H(e) = 0$ if e is not in H.

PROPOSITION 4.7 *The cutset-subspace of Γ is a vector space whose dimension is equal to the rank of Γ. If H is a cutset in Γ, then ξ_H belongs to the cutset-subspace.*

Proof Since the dimension of the circuit-subspace is $m-n+c$, its orthogonal complement, the cutset-subspace, has dimension $n-c = r(\Gamma)$.

If H is a cutset in Γ, we have $V\Gamma = V_1 \cup V_2$, where V_1 and V_2 are disjoint and non-empty, and H consists precisely of those edges which have one end in V_1 and one end in V_2. Thus, if $\mathbf{x}_H$ is the column vector representing ξ_H, we have

$$\mathbf{x}_H^t = \pm\tfrac{1}{2}\Big[\sum_{v_i \in V_1} \mathbf{d}_i - \sum_{v_i \in V_2} \mathbf{d}_i\Big],$$

where $\mathbf{d}_i$ is the row of the incidence matrix corresponding to v_i. The sign on the right-hand side of this equation depends only on which of the two possible cutset-orientations has been chosen for H. Now if $\mathbf{Dz} = \mathbf{0}$, then $\mathbf{d}_i\mathbf{z} = 0$ for each $v_i \in V$, and so we deduce that $\mathbf{x}_H^t\mathbf{z} = 0$. In other words, ξ_H belongs to the orthogonal complement of the circuit-subspace, and, by definition, this is the cutset-subspace. □

The proof of Proposition 4.7 indicates one way of choosing a basis $\{\xi_1, \xi_2, \ldots, \xi_{n-c}\}$ for the cutset-subspace of Γ. The set of edges incident with a vertex v_j of Γ forms a cutset whose representative vector is $\mathbf{d}_j^t$. If, for each component $\Gamma^{(i)}$ $(1 \leqslant i \leqslant c)$ of Γ, we delete one row of $\mathbf{D}$ corresponding to a vertex in $\Gamma^{(i)}$, then, the remaining $n-c$ rows are linearly independent. Furthermore, the transpose of any vector $\mathbf{x}_H$, representing a cutset H, can be expressed as a linear combination of these $n-c$ rows by using the equation displayed in the proof of Proposition 4.7, and the fact that the sum of the rows corresponding to each component is zero.

This basis has the desirable property that each member represents an actual cutset, rather than a 'linear combination' of cutsets. It is, however, rather clumsy to work with, and in the next chapter we shall investigate a more elegant procedure which has the added advantage that it provides a basis for the circuit-subspace as well.

We end this chapter by proving a simple relationship between the incidence matrix and the adjacency matrix of Γ.

PROPOSITION 4.8 *Let* $\mathbf{D}$ *be the incidence matrix (with respect to some orientation) of a graph* Γ, *and let* $\mathbf{A}$ *be the adjacency matrix of* Γ. *Then*

$$\mathbf{D}\mathbf{D}^t = \mathbf{\Delta} - \mathbf{A},$$

where $\mathbf{\Delta}$ *is the diagonal matrix whose diagonal entry* $(\mathbf{\Delta})_{ii}$ *is the valency of the vertex* v_i $(1 \leqslant i \leqslant n)$.

Consequently, $\mathbf{D}\mathbf{D}^t$ *is independent of the orientation given to* Γ.

Proof $(\mathbf{D}\mathbf{D}^t)_{ij}$ is the inner product of the rows $\mathbf{d}_i$ and $\mathbf{d}_j$ of $\mathbf{D}$. If $i \neq j$, then these rows have a non-zero entry in the same column if and only if there is an edge joining v_i and v_j. In this case, the two non-zero entries are $+1$ and -1, so that $(\mathbf{D}\mathbf{D}^t)_{ij} = -1$. Similarly, $(\mathbf{D}\mathbf{D}^t)_{ii}$ is the inner product of $\mathbf{d}_i$ with itself, and, since the number of entries ± 1 in $\mathbf{d}_i$ is equal to the valency of v_i, the result follows. □

4*A* *Planar graphs and duality* Let Γ be a planar connected graph and Γ^* its geometric-dual [W, p. 72]. If Γ is given an orientation and $\mathbf{D}$ is the incidence matrix of Γ, then Γ^* can be given an orientation so that its incidence matrix $\mathbf{D}^*$ satisfies:

(*a*) $\operatorname{rank}(\mathbf{D}) + \operatorname{rank}(\mathbf{D}^*) = |E\Gamma|$;

(*b*) $\mathbf{D}^*\mathbf{D}^t = \mathbf{0}$.

4*B* *The co-boundary mapping* The linear mapping from $C_0(\Gamma)$ to $C_1(\Gamma)$ defined (with respect to the standard bases) by $\mathbf{x} \mapsto \mathbf{D}^t\mathbf{x}$ is sometimes called the *co-boundary mapping* for Γ. The kernel of the co-boundary mapping is a vector space of dimension c, and the image of the co-boundary mapping is the cutset-subspace of Γ.

4*C* *Flows* An element ϕ of the circuit-subspace of Γ is often called a *flow* on Γ. The *support* of ϕ, written $S(\phi)$, is the set of edges e for which $\phi(e) \neq 0$; a subset S of $E\Gamma$ is a *minimal support* if $S = S(\phi)$ for some flow ϕ, and the only flow whose support is properly contained in S is the zero flow.

(*a*) The set of flows with a given minimal support (together with the zero flow) forms a one-dimensional space.

(*b*) A minimal support is a circuit.

(*c*) If ϕ is a flow whose support is minimal, then $|\phi(e)|$ is constant on $S(\phi)$.

4*D* *Integral flows* The flow ϕ is *integral* if each $\phi(e)$ is an integer ($e \in E\Gamma$); it is *primitive* if $S(\phi)$ is minimal and each $\phi(e)$ is 0, 1, or -1. We say that the flow θ *conforms* to the flow χ if $S(\theta) \subseteq S(\chi)$ and $\theta(e)\chi(e) > 0$ for e in $S(\theta)$.

(*a*) For a given integral flow ϕ there is a primitive flow which conforms to ϕ.

(*b*) Any integral flow ϕ is the sum of integer multiples of primitive flows, each of which conforms to ϕ (Tutte 1956).

4*E* *Modular flows* Suppose the entries 0, 1, -1 of $\mathbf{D}$ are taken to be elements of the ring $\mathbb{Z}_u$ of residue classes of integers modulo u. A *flow mod u* on Γ is a vector $\mathbf{x}$ with components in $\mathbb{Z}_u$ for which $\mathbf{Dx} = \mathbf{0}$, where $\mathbf{0}$ is the zero vector over $\mathbb{Z}_u$. The results of § 4D imply that if $\mathbf{x}$ is a given flow mod u, then there is an integral flow $\mathbf{y}$, each of whose components y_i satisfies

(*a*) $y_i \in x_i$,

(*b*) $-u < y_i < u$.

Consequently, if Γ has a flow mod u, then it has a flow mod $(u+1)$ (Tutte 1956).

5. *Spanning trees and associated structures*

The problem of finding bases for the circuit- and cutset-subspaces is of great practical and theoretical importance. It was originally solved by Kirchhoff (1847) in his studies of electrical networks, and we shall give a brief exposition of that topic at the end of the chapter.

We shall only consider graphs which are connected, for the circuit-subspace and the cutset-subspace of a disconnected graph are the direct sums of the corresponding spaces for the components. Throughout this chapter, Γ will denote a connected graph with n vertices and m edges, so that $r(\Gamma) = n-1$ and $s(\Gamma) = m-n+1$. We shall also assume that Γ has been given an orientation.

A *spanning tree* in Γ is an edge-subgraph of Γ which has $n-1$ edges and contains no circuits [W, p. 46]. We shall use the symbol T to denote both the spanning tree itself and its edge-set. The following simple results are of fundamental importance.

LEMMA 5.1 *Let T be a spanning tree in a connected graph Γ. Then:*

(1) *For each edge g of Γ which is not in T, there is a unique circuit in Γ containing g and edges of T.*

(2) *For each edge h of Γ which is in T, there is a unique cutset in Γ containing h and edges not in T.*

Proof The proofs of these two statements are elementary, and may be found in [W, p. 47]. □

We write cir (T, g) and cut (T, h) for the unique circuit and cutset whose existence is guaranteed by Lemma 5.1. We give cir (T, g) and cut (T, h) the circuit-orientation and cutset-orientation which coincide, on g and h, with the orientation in Γ. Then we have elements $\xi_{(T,g)}$ and $\xi_{(T,h)}$ of the edge-space $C_1(\Gamma)$; these elements are defined (in terms of the given circuit and cutset) as in Chapter 4.

THEOREM 5.2 *With the same hypotheses as in Lemma* 5.1, *we have:*

(1) *As g runs through the set $E\Gamma - T$, the $m-n+1$ elements $\xi_{(T,g)}$ form a basis for the circuit-subspace of* Γ.

(2) *As h runs through the set T, the $n-1$ elements $\xi_{(T,h)}$ form a basis for the cutset-subpace of* Γ.

Proof (1) Since the elements $\xi_{(T,g)}$ correspond to circuits, it follows from Theorem 4.5 that they belong to the circuit-subspace. They comprise a linearly independent set, because a given edge g in $E\Gamma - T$ belongs to cir(T, g) but to no other cir(T, g') for $g' \neq g$. Finally, since there are $m-n+1$ of these elements, and this is the dimension of the circuit-subspace, it follows that we have a basis.

(2) This is proved by arguments analogous to those used in the proof of the first part. □

We shall now put the foregoing ideas into a form which will show explicitly how circuits and cutsets can be deduced from the incidence matrix, by means of simple matrix operations. To do this, we shall require some properties of submatrices of the incidence matrix.

PROPOSITION 5.3 (Poincaré 1901) *Any square submatrix of the incidence matrix* $\mathbf{D}$ *of a graph* Γ *has determinant equal to* 0 *or* $+1$ *or* -1.

Proof Let $\mathbf{S}$ denote a square submatrix of $\mathbf{D}$. If every column of $\mathbf{S}$ has two non-zero entries, then these entries must be $+1$ and -1 and so (since each column has sum zero) $\mathbf{S}$ is singular and $\det \mathbf{S} = 0$. Also, if every column of $\mathbf{S}$ has no non-zero entries, then $\det \mathbf{S} = 0$.

The remaining case occurs when a column of $\mathbf{S}$ has precisely one non-zero entry. In this case we can expand $\det \mathbf{S}$ in terms of this column, obtaining $\det \mathbf{S} = \pm \det \mathbf{S}'$, where $\mathbf{S}'$ has one row and column fewer than $\mathbf{S}$. Continuing this process, we eventually arrive at either a zero determinant or a single entry of $\mathbf{D}$, and so the result is proved. □

PROPOSITION 5.4 *Let U be a subset of $E\Gamma$ with $|U| = n-1$. Let $\mathbf{D}_U$ denote an $(n-1) \times (n-1)$ submatrix of $\mathbf{D}$, consisting of the* *intersection of those $n-1$ columns of* $\mathbf{D}$ *corresponding to the edges in* U *and any set of* $n-1$ *rows of* $\mathbf{D}$. *Then* $\mathbf{D}_U$ *is non-singular if and only if the edge-subgraph* $\langle U \rangle$ *is a spanning tree of* Γ.

Proof Suppose $\langle U \rangle$ is a spanning tree of Γ. Then the submatrix $\mathbf{D}_U$ consists of $n-1$ rows of the incidence matrix $\mathbf{D}'$ of $\langle U \rangle$. Since $\langle U \rangle$ is connected, the rank of $\mathbf{D}'$ is $n-1$, and so $\mathbf{D}_U$ is non-singular.

Conversely, suppose $\mathbf{D}_U$ is non-singular. Then the incidence matrix $\mathbf{D}'$ of $\langle U \rangle$ has a non-singular $(n-1) \times (n-1)$ submatrix, and consequently the rank of $\mathbf{D}'$ is $n-1$. Since $|U| = n-1$, this means that the circuit-subspace of $\langle U \rangle$ has dimension zero, and so $\langle U \rangle$ is a spanning tree of Γ. □

We now suppose that

$$V\Gamma = \{v_1, v_2, \ldots, v_n\} \quad \text{and} \quad E\Gamma = \{e_1, e_2, \ldots, e_m\},$$

where the labelling has been chosen so that $e_1, e_2, \ldots, e_{n-1}$ are the edges of a given spanning tree T of Γ. The incidence matrix of Γ can be partitioned as follows:

$$\mathbf{D} = \begin{bmatrix} \mathbf{D}_T & \mathbf{D}_N \\ \multicolumn{2}{c}{\mathbf{d}_n} \end{bmatrix},$$

where $\mathbf{D}_T$ is non-singular, by Proposition 5.4, and the last row $\mathbf{d}_n$ is linearly dependent on the other rows.

Let $\mathbf{C}$ denote the matrix whose columns are the vectors representing the elements $\xi_{(T, e_j)}$ $(n \leqslant j \leqslant m)$ with respect to the standard basis of $C_1(\Gamma)$. Then $\mathbf{C}$ can be written in the partitioned form

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_T \\ \mathbf{I}_{m-n+1} \end{bmatrix}.$$

Since every column of $\mathbf{C}$ represents a circuit, and consequently belongs to the kernel of $\mathbf{D}$, we have $\mathbf{DC} = \mathbf{0}$. Thus

$$\mathbf{C}_T = -\mathbf{D}_T^{-1}\mathbf{D}_N.$$

In a similar fashion, the matrix $\mathbf{K}$ whose columns represent the elements $\xi_{(T, e_j)}$ $(1 \leqslant j \leqslant n-1)$ can be written in the form

$$\mathbf{K} = \begin{bmatrix} \mathbf{I}_{n-1} \\ \mathbf{K}_T \end{bmatrix}.$$

Since each column of $\mathbf{K}$ belongs to the orthogonal complement of the circuit-subspace, we have $\mathbf{CK}^t = \mathbf{0}$, that is $\mathbf{C}_T + \mathbf{K}_T^t = \mathbf{0}$. Thus

$$\mathbf{K}_T = (\mathbf{D}_T^{-1}\mathbf{D}_N)^t.$$

Our equations for $\mathbf{C}_T$ and $\mathbf{K}_T$ show how the basic circuits and cutsets associated with T can be deduced from the incidence matrix. We also have an algebraic proof of the following proposition.

PROPOSITION 5.5 *Let T be a spanning tree of Γ and let a and b be edges of Γ such that $a \in T$, $b \notin T$. Then*

$$b \in \operatorname{cut}(T, a) \Leftrightarrow a \in \operatorname{cir}(T, b).$$

Proof This result follows immediately from the definitions of $\mathbf{C}_T$ and $\mathbf{K}_T$, and the fact that $\mathbf{C}_T + \mathbf{K}_T^t = \mathbf{0}$. □

We end this chapter with a brief exposition of the solution of network equations; this application provided the stimulus for the development of the foregoing theory.

An *electrical network* is a graph Γ (with an orientation) which has certain physical characteristics, specified by two vectors in the edge-space of Γ. These vectors are the *current* vector $\mathbf{w}$ and the *voltage* vector $\mathbf{z}$. In general, they are time-dependent, and they are related by a linear equation $\mathbf{z} = \mathbf{Mw} + \mathbf{n}$, where $\mathbf{M}$ is a diagonal matrix whose entries may be differential or integral operators, or numbers. Further, $\mathbf{w}$ and $\mathbf{z}$ satisfy the equations

$$\mathbf{Dw} = \mathbf{0}, \quad \mathbf{C}^t\mathbf{z} = \mathbf{0},$$

which are known as *Kirchhoff's laws*. If we choose a spanning tree T in Γ, and partition $\mathbf{D}$ and $\mathbf{C}$ as before, then the same partition imposed on $\mathbf{w}$ and $\mathbf{z}$ gives

$$\mathbf{w} = \begin{bmatrix} \mathbf{w}_T \\ \mathbf{w}_N \end{bmatrix}, \quad \mathbf{z} = \begin{bmatrix} \mathbf{z}_T \\ \mathbf{z}_N \end{bmatrix}.$$

Now, from $\mathbf{Dw} = \mathbf{0}$ we have $\mathbf{D}_T\mathbf{w}_T + \mathbf{D}_N\mathbf{w}_N = \mathbf{0}$, and since $\mathbf{C}_T = -\mathbf{D}_T^{-1}\mathbf{D}_N$ it follows that

$$\mathbf{w}_T = \mathbf{C}_T\mathbf{w}_N \quad \text{and} \quad \mathbf{w} = \mathbf{Cw}_N.$$

In other words, all the entries of the current vector are determined by the entries corresponding to edges not in T.

Substituting in $\mathbf{z} = \mathbf{Mw}+\mathbf{n}$, and premultiplying by $\mathbf{C}^t$, we obtain

$$(\mathbf{C}^t\mathbf{MC})\,\mathbf{w}_N = -\mathbf{C}^t\mathbf{n}.$$

This equation determines $\mathbf{w}_N$, and consequently both $\mathbf{w}$ (from $\mathbf{w} = \mathbf{Cw}_N$) and $\mathbf{z}$ (from $\mathbf{z} = \mathbf{Mw}+\mathbf{n}$) in turn. Thus we have a systematic method of solving network equations, which distinguishes clearly between the essential unknowns and the redundant ones.

5*A* *Total unimodularity* A matrix is said to be *totally unimodular* if every square submatrix of it has determinant 0, 1, or -1. Proposition 5.3 states that $\mathbf{D}$ is totally unimodular; the matrices $\mathbf{C}$ and $\mathbf{K}$ are also totally unimodular.

5*B* *The inverse of* $\mathbf{D}_T$ Let T be a spanning tree for Γ and let $\mathbf{D}_T$ denote the corresponding $(n-1)\times(n-1)$ matrix occurring in the partition of $\mathbf{D}$. Then $(\mathbf{D}_T^{-1})_{ij} = \pm 1$ if the edge e_i occurs in the unique path in T joining v_j to v_n. Otherwise $(\mathbf{D}_T^{-1})_{ij} = 0$. This result is due to Branin (see Bryant 1967).

5*C* *The unoriented incidence matrix* Let $\mathbf{X}$ denote the matrix obtained from the incidence matrix $\mathbf{D}$ of Γ by replacing each entry ± 1 by $+1$. Then the graph Γ is bipartite [W, p. 17] if and only if $\mathbf{X}$ is totally unimodular.

5*D* *The inverse image* Let ω be an element of $C_0(\Gamma)$, where Γ is a connected graph. Then $D^{-1}(\omega)$ is non-empty if and only if

$$\sum_{v\in V\Gamma} \omega(v) = 0.$$

If this condition holds, then $D^{-1}(\omega)$ contains an integral vector if and only if ω is integral.

6. Complexity

Some of the oldest results in algebraic graph theory are formulae which give the numbers of spanning trees of certain graphs; many results (both old and new) of this kind can be found in the monograph written by Moon (1970). We shall show how such results can be derived from the techniques of the preceding chapters.

DEFINITION 6.1 The *complexity* of a graph Γ is the number of spanning trees of Γ. The complexity of Γ will be denoted by $\kappa(\Gamma)$.

Of course, if Γ is disconnected, then $\kappa(\Gamma) = 0$. Our first theorem is a classical formula which has been rediscovered many times; we need a preparatory lemma.

LEMMA 6.2 *Let* $\mathbf{D}$ *be the incidence matrix of a graph* Γ, *and let* $\mathbf{Q} = \mathbf{D}\mathbf{D}^t = \mathbf{\Delta} - \mathbf{A}$ *denote the matrix introduced in Proposition* 4.8. *Then the matrix of cofactors* (*adjugate*) *of* $\mathbf{Q}$ *is a multiple of* $\mathbf{J}$.

Proof Let n be the number of vertices of Γ. If Γ is disconnected, then

$$\operatorname{rank}(\mathbf{Q}) = \operatorname{rank}(\mathbf{D}) < n-1,$$

and so every cofactor of $\mathbf{Q}$ is zero. That is, $\operatorname{adj}\mathbf{Q} = \mathbf{0} = 0\mathbf{J}$.

If Γ is connected, then the rank of $\mathbf{Q}$ is $n-1$. Since

$$\mathbf{Q}\operatorname{adj}\mathbf{Q} = (\det\mathbf{Q})\,\mathbf{I} = \mathbf{0},$$

it follows that each column of $\operatorname{adj}\mathbf{Q}$ belongs to the kernel of $\mathbf{Q}$. But this kernel is a one-dimensional space, spanned by $\mathbf{u} = [1, 1, \ldots, 1]^t$. Thus, each column of $\mathbf{Q}$ is a multiple of $\mathbf{u}$; since $\mathbf{Q}$ is symmetric, so is $\operatorname{adj}\mathbf{Q}$, and all the multipliers must be equal. Hence, $\operatorname{adj}\mathbf{Q}$ is a multiple of $\mathbf{J}$. □

THEOREM 6.3 *With the above notation, we have*

$$\operatorname{adj}\mathbf{Q} = \kappa(\Gamma)\,\mathbf{J}.$$

Proof It is sufficient (because of Lemma 6.2) to show that one cofactor of $\mathbf{Q}$ is equal to $\kappa(\Gamma)$. Let $\mathbf{D}_0$ denote the matrix obtained from $\mathbf{D}$ by removing the last row; then $\det \mathbf{D}_0\mathbf{D}_0^t$ is a cofactor of $\mathbf{Q}$. Applying the Binet–Cauchy theorem [P. Lancaster, *Theory of matrices* (Academic Press, 1969), p. 38], we have

$$\det \mathbf{D}_0\mathbf{D}_0^t = \Sigma \det \mathbf{D}_U \det \mathbf{D}_U^t.$$

Here $\mathbf{D}_U$ denotes that square submatrix of $\mathbf{D}_0$ whose $n-1$ columns correspond to the edges in a subset U of $E\Gamma$, and the summation is over all possible choices of U. Now, by Proposition 5.4, $\det \mathbf{D}_U$ is non-zero if and only if the edge-subgraph $\langle U\rangle$ is a spanning tree for Γ, and then $\det \mathbf{D}_U$ takes the values ± 1. Since $\det \mathbf{D}_U^t = \det \mathbf{D}_U$, we have $\det \mathbf{D}_0\mathbf{D}_0^t = \kappa(\Gamma)$, and the result follows. □

For the complete graph K_n we have $\mathbf{A} = \mathbf{J}-\mathbf{I}$ and $\mathbf{\Delta} = (n-1)\mathbf{I}$, hence $\mathbf{Q} = n\mathbf{I}-\mathbf{J}$. A simple determinant manipulation on $n\mathbf{I}-\mathbf{J}$ with one row and column removed shows that $\kappa(K_n) = n^{n-2}$; this result was first derived by Cayley (1889).

We can dispense with the rather arbitrary procedure of removing one row and column from $\mathbf{Q}$, by means of the following result.

PROPOSITION 6.4 (Temperley 1964) *The complexity of a graph Γ with n vertices is given by the formula*

$$\kappa(\Gamma) = n^{-2}\det(\mathbf{J}+\mathbf{Q}).$$

Proof Since $n\mathbf{J} = \mathbf{J}^2$ and $\mathbf{J}\mathbf{Q} = \mathbf{0}$ we have the following equation:

$$(n\mathbf{I}-\mathbf{J})(\mathbf{J}+\mathbf{Q}) = n\mathbf{J}+n\mathbf{Q}-\mathbf{J}^2-\mathbf{J}\mathbf{Q} = n\mathbf{Q}.$$

Thus, taking adjugates and using Theorem 6.3, we can argue as follows, where $\kappa = \kappa(\Gamma)$:

$$\begin{aligned}
\operatorname{adj}(\mathbf{J}+\mathbf{Q})\operatorname{adj}(n\mathbf{I}-\mathbf{J}) &= \operatorname{adj} n\mathbf{Q},\\
\operatorname{adj}(\mathbf{J}+\mathbf{Q})\,n^{n-2}\mathbf{J} &= n^{n-1}\operatorname{adj}\mathbf{Q},\\
\operatorname{adj}(\mathbf{J}+\mathbf{Q})\,\mathbf{J} &= n\kappa\mathbf{J},\\
(\mathbf{J}+\mathbf{Q})\operatorname{adj}(\mathbf{J}+\mathbf{Q})\,\mathbf{J} &= (\mathbf{J}+\mathbf{Q})\,n\kappa\mathbf{J},\\
\det(\mathbf{J}+\mathbf{Q})\,\mathbf{J} &= n^2\kappa\mathbf{J}.
\end{aligned}$$

It follows that $\det(\mathbf{J}+\mathbf{Q}) = n^2\kappa$, as required. □

COROLLARY 6.5 *Let Γ be a k-valent graph with n vertices, and let*

$$\operatorname{Spec}\Gamma = \begin{pmatrix} k & \lambda_1 & \dots & \lambda_{s-1} \\ 1 & m_1 & \dots & m_{s-1} \end{pmatrix}.$$

Then the complexity of Γ is given by

$$\kappa(\Gamma) = n^{-1}\prod_{r=1}^{s-1}(k-\lambda_r)^{m_r} = n^{-1}\chi'(\Gamma; k),$$

where χ' denotes the derivative of the characteristic polynomial χ.

Proof In the case of a regular graph of valency k, the matrix $\mathbf{J}+\mathbf{Q}$ is simply $\mathbf{J}+k\mathbf{I}-\mathbf{A}$. Now, since $\mathbf{J}$ commutes with $\mathbf{A}$ (by Proposition 3.2), and the eigenvalues of $\mathbf{J}$ are n (once) and 0 ($n-1$ times), it follows that the eigenvalues of $\mathbf{J}+k\mathbf{I}-\mathbf{A}$ are n (once), and $k-\lambda_r$ (m_r times) for $1 \leqslant r \leqslant s-1$. The determinant of $\mathbf{J}+\mathbf{Q}$ is the product of these factors, and so the first equality follows from Proposition 6.4. The second equality follows from the fact that $k-\lambda$ is a simple factor of χ. $\square$

Later in this book, when we have developed sophisticated techniques for calculating the spectra of certain graphs, we shall be able to use Corollary 6.5 to write down the complexities of many well-known families of graphs.

For the moment, we shall consider applications of Corollary 6.5 in some simple, but important, cases. If Γ is a regular graph of valency k, then the characteristic polynomial of its line graph $L(\Gamma)$ is known (Theorem 3.8) in terms of that of Γ. If Γ has n vertices and m edges, so that $2m = nk$, then we have

$$\kappa(L(\Gamma)) = m^{-1}\chi'(L(\Gamma);\, 2k-2),$$
$$\kappa(\Gamma) = n^{-1}\chi'(\Gamma;\, k).$$

Differentiating the result of Theorem 3.8 and putting $\lambda = 2k-2$, we find

$$\chi'(L(\Gamma);\, 2k-2) = (2k)^{m-n}\chi'(\Gamma; k),$$

whence $$\kappa(L(\Gamma)) = 2^{m-n+1}k^{m-n-1}\kappa(\Gamma).$$

For example, the complexity of the triangle graph $\Delta_t = L(K_t)$ is

$$\kappa(\Delta_t) = 2^{\frac{1}{2}(t^2-3t+2)}(t-1)^{\frac{1}{2}(t^2-3t-2)}t^{t-2}.$$

The *complete multipartite graph* $K_{a_1, a_2, \ldots, a_s}$ has a vertex-set which is partitioned into s parts $A_1, A_2, \ldots, A_s$, where $|A_i| = a_i$ $(1 \leqslant i \leqslant s)$; two vertices are joined by an edge if and only if they belong to different parts. This graph is not regular (in general) but its *complement* (the graph with the same vertex-set and complementary edge-set) consists of regular connected components. The complexity of such graphs can be found by a modification of Proposition 6.4, due to Moon (1967). We shall denote the function $\det(\lambda \mathbf{I} + \mathbf{Q})$ by $\tau(\Gamma; \lambda)$.

PROPOSITION 6.6 (1) *If Γ is disconnected, then the τ function for Γ is the product of the τ functions for the components of Γ.*

(2) *If Γ is a regular graph of valency k, with characteristic polynomial χ, then $\tau(\Gamma; \lambda) = \chi(\Gamma; k+\lambda)$.*

(3) *If Γ^c is the complement of Γ, and Γ has n vertices, then*

$$\kappa(\Gamma) = (-1)^n n^{-2} \tau(\Gamma^c; -n).$$

Proof (1) This follows directly from the definition of τ.

(2) In this case, we have $\det(\lambda \mathbf{I} + \mathbf{Q}) = \det((\lambda + k)\mathbf{I} - \mathbf{A})$, whence the result.

(3) Let $\mathbf{Q}^c$ denote the $\mathbf{Q}$ matrix for Γ^c; then $\mathbf{Q} + \mathbf{Q}^c = n\mathbf{I} - \mathbf{J}$. Thus, using Proposition 6.4, we have

$$\begin{aligned} \kappa(\Gamma) &= n^{-2} \det(\mathbf{J} + \mathbf{Q}) = n^{-2} \det(n\mathbf{I} - \mathbf{Q}^c) \\ &= (-1)^n n^{-2} \tau(\Gamma^c; -n). \quad \square \end{aligned}$$

Consider the complete multipartite graph $K_{a_1, a_2, \ldots, a_s}$, where $a_1 + a_2 + \ldots + a_s = n$. The complement of this graph consists of s components isomorphic with $K_{a_1}, K_{a_2}, \ldots, K_{a_s}$, and, from the known spectra of complete graphs and part (2) of Proposition 6.6, we have

$$\tau(K_a; \lambda) = \chi(K_a; a-1+\lambda) = \lambda(\lambda + a)^{a-1}.$$

Consequently, applying parts (1) and (3) of Proposition 6.6,

$$\begin{aligned} \kappa(K_{a_1, a_2, \ldots, a_s}) &= (-1)^n n^{-2} \cdot (-n)(a_1 - n)^{a_1 - 1} \ldots (-n)(a_s - n)^{a_s - 1} \\ &= n^{s-2}(n - a_1)^{a_1 - 1} \ldots (n - a_s)^{a_s - 1}. \end{aligned}$$

This result was originally found (by different means) by Austin (1960).

We note the special cases:

$$\kappa(K_{a,b}) = a^{b-1}b^{a-1},$$
$$\kappa(H_s) = 2^{2s-2}s^{s-1}(s-1)^s.$$

6*A* *A bound for the complexity of a regular graph* If Γ is a connected regular graph having n vertices and valency k, then

$$\kappa(\Gamma) \leqslant \frac{1}{n}\left(\frac{nk}{n-1}\right)^{n-1}$$

with equality if and only if $\Gamma = K_n$. (Apply the inequality of the arithmetic and geometric means to Corollary 6.5.)

6*B* *Complexity of the Möbius ladders* The complexity of the trivalent Möbius ladder M_h (§ 3C), is given by the formulae

$$\kappa(M_h) = \frac{1}{2h}\prod_{j=1}^{2h-1}\left(3-(-1)^j - 2\cos\frac{\pi j}{h}\right)$$
$$= \frac{h}{2}\left[(2+\sqrt{3})^h + (2-\sqrt{3})^h\right] + h.$$

6C *Almost-complete graphs* Let Γ be a graph constructed by removing q disjoint edges from K_n ($n \geqslant 2q$). Then

$$\kappa(\Gamma) = n^{n-2}\left(1-\frac{2}{n}\right)^q.$$

In particular, taking $n = 2q$, we have the formula for the complexity of $\Gamma = H_q$.

6*D* *Complexity of planar duals* Let Γ, Γ^* be geometric-duals (as in § 4A) and let $\mathbf{D}$, $\mathbf{D}^*$ be the corresponding incidence matrices. Suppose Γ has n vertices, Γ^* has n^* vertices, and $|E\Gamma| = |E\Gamma^*| = m$; then $(n-1)+(n^*-1) = m$. If $\mathbf{D}_U$ is a square submatrix of $\mathbf{D}$, whose $n-1$ columns correspond to the edges of a subset U of $E\Gamma$, and U^* denotes the complementary subset of $E\Gamma^* = E\Gamma$, then $\mathbf{D}_U$ is non-singular if and only if $\mathbf{D}^*_{U^*}$ is non-singular. Consequently

$$\kappa(\Gamma) = \kappa(\Gamma^*).$$

6*E* *The octahedron and the cube* The graphs of the octahedron and the cube are depicted in [W, pp. 16, 17]. The octahedron is the graph $H_3 = K_{2,2,2}$, and the cube Q_3 is its planar dual. We have

$$\operatorname{Spec} H_3 = \begin{pmatrix} 4 & 0 & -2 \\ 1 & 3 & 2 \end{pmatrix}, \quad \operatorname{Spec} Q_3 = \begin{pmatrix} 3 & 1 & -1 & -3 \\ 1 & 3 & 3 & 1 \end{pmatrix}.$$

Hence $$\kappa(H_3) = \kappa(Q_3) = 384.$$

7. *Determinant expansions*

In this chapter we shall investigate the characteristic polynomial χ, and the polynomial τ introduced in Chapter 6, by means of determinant expansions.

We begin by considering the determinant of the adjacency matrix $\mathbf{A}$ of a graph Γ. The expansion which is useful here is the usual definition of a determinant: if $\mathbf{A} = (a_{ij})$, then

$$\det \mathbf{A} = \Sigma \operatorname{sgn}(\pi)\, a_{1,\pi 1}\, a_{2,\pi 2} \ldots a_{n,\pi n},$$

where the summation is over all permutations π of the set $\{1, 2, \ldots, n\}$. (We shall suppose, as before, that $V\Gamma = \{v_1, v_2, \ldots, v_n\}$ and that the rows and columns of $\mathbf{A}$ are labelled to conform with this notation.)

In order to express the quantities which appear in the above expansion in graph-theoretical terms, it is helpful to introduce a new definition.

DEFINITION 7.1 A *sesquivalent* graph is a simple graph, each component of which is regular and has valency 1 or 2. In other words, the components are single edges and circuits.

We notice that the co-rank of a sesquivalent graph is just the number of its components which are circuits.

PROPOSITION 7.2 (Harary 1962) *Let* $\mathbf{A}$ *be the adjacency matrix of a graph* Γ. *Then*

$$\det \mathbf{A} = \Sigma(-1)^{r(\Lambda)}\, 2^{s(\Lambda)},$$

where the summation is over all sesquivalent edge-subgraphs Λ *of* Γ *such that* $V\Lambda = V\Gamma$.

Proof Consider a term $\operatorname{sgn}(\pi)\, a_{1,\pi 1}\, a_{2,\pi 2} \ldots a_{n,\pi n}$ in the expansion of $\det \mathbf{A}$; this term vanishes if, for some $i \in \{1, 2, \ldots, n\}$, $a_{i,\pi i} = 0$; that is, if $\{v_i, v_{\pi i}\}$ is not an edge of Γ. In particular, the term vanishes if π fixes any symbol.

Thus, if the term corresponding to a permutation π is non-zero, then π can be expressed uniquely as the composition of disjoint cycles of length at least two. Each cycle (ij) of length two corresponds to the factors $a_{ij}a_{ji}$, which in turn signifies a single edge $\{v_i, v_j\}$ in Γ. Each cycle $(pqr \ldots t)$ of length greater than two corresponds to the factors $a_{pq}a_{qr} \ldots a_{tp}$, and signifies a simple circuit $\{v_p, v_q, \ldots, v_t\}$ in Γ. Consequently, each non-vanishing term in the determinant expansion gives rise to a sesquivalent edge-subgraph Λ of Γ, with $V\Lambda = V\Gamma$. Further, the sign of the permutation π is $(-1)^{N_e}$, where N_e is the number of even cycles in π. If there are c_l cycles of length l, then the equation $\Sigma l c_l = n$ shows that the number N_o of odd cycles is congruent to n modulo 2. Hence,

$$r(\Lambda) = n - (N_o + N_e) \equiv N_e \pmod 2,$$

and the sign of π is $(-1)^{r(\Lambda)}$.

Now each sesquivalent edge-subgraph Λ, having n vertices, gives rise to several permutations π for which the corresponding term in the determinant expansion does not vanish. The number of such π arising from a given Λ is $2^{s(\Lambda)}$, since for each circuit in Λ there are two ways of choosing the corresponding cycle in π. Thus each Λ contributes $(-1)^{r(\Lambda)} 2^{s(\Lambda)}$ to the determinant, and we have the result. □

The first proposition in this book (Proposition 2.3) gave a description of the first few coefficients of the characteristic polynomial of Γ, in terms of some small subgraphs of Γ. We shall now extend that result to all the coefficients. We shall suppose, as before, that

$$\chi(\Gamma; \lambda) = \lambda^n + c_1\lambda^{n-1} + c_2\lambda^{n-2} + \ldots + c_n.$$

PROPOSITION 7.3 *With the above notation,*

$$(-1)^i c_i = \Sigma(-1)^{r(\Lambda)} 2^{s(\Lambda)}, \quad 0 \leqslant i \leqslant n,$$

where the summation is over all sesquivalent edge-subgraphs Λ of Γ which have i vertices.

Proof The number $(-1)^i c_i$ is the sum of all principal minors of $\mathbf{A}$ which have i rows and columns. Each such minor is the determinant of the adjacency matrix of a vertex-subgraph of Γ with i vertices. Now, any sesquivalent edge-subgraph with

i vertices is contained in precisely one of these vertex-subgraphs, and so, by applying Proposition 7.2 to each minor, we obtain the required result. $\square$

The only sesquivalent graphs with fewer than four vertices are K_2, the graph with one edge, and K_3, the triangle; thus, we can immediately obtain the results of Proposition 2.3 from the general formula of Proposition 7.3. We can also use Proposition 7.3 to derive explicit expressions for the other coefficients, for example, c_4. Since the only sesquivalent graphs with four vertices are the circuit graph C_4 and the graph having two disjoint edges, it follows that

$$c_4 = n_1 - 2n_2,$$

where n_1 is the number of pairs of disjoint edges in Γ, and n_2 is the number of circuits of length four in Γ. This expression can be further simplified if Γ is regular, when n_1 can be expressed in terms of the valency.

As well as giving explicit expressions for the coefficients of the characteristic polynomial, the result of Proposition 7.3 throws some light on the problem of cospectral graphs (§ 2B). The fact that sesquivalent edge-subgraphs are rather loosely related to the structure of a graph, tends to explain why there are many pairs of non-isomorphic graphs having the same spectrum.

We now turn to an expansion of the function

$$\tau(\Gamma; \lambda) = \det(\lambda \mathbf{I} + \mathbf{Q}).$$

Although $\mathbf{Q}$ differs from $\mathbf{A}$ only in its diagonal entries, the ideas involved in this expansion are quite different from those which we have used to investigate the characteristic polynomial $\det(\lambda\mathbf{I} - \mathbf{A})$. One reason for this is that a principal submatrix of $\mathbf{Q}$ is (in general) not the $\mathbf{Q}$ matrix of a vertex-subgraph of Γ (the diagonal entries give the valencies in Γ, rather than in the subgraph).

We shall write

$$\tau(\Gamma; \lambda) = \det(\lambda\mathbf{I} + \mathbf{Q}) = \lambda^n + q_1\lambda^{n-1} + \ldots + q_{n-1}\lambda + q_n.$$

Then q_i $(1 \leqslant i \leqslant n)$ is the sum of those principal minors of $\mathbf{Q}$

which have i rows and columns, and, from elementary observations and the results of Chapter 6, we have

$$q_1 = 2\,|E\Gamma|, \quad q_{n-1} = n\kappa(\Gamma), \quad q_n = 0.$$

We shall find, for each coefficient q_i, an expression which subsumes these results. Our method is based on the expansion of a principal minor of $\mathbf{Q} = \mathbf{D}\mathbf{D}^t$ by means of the Binet–Cauchy theorem.

Let X be a non-empty subset of the vertex-set of Γ, and Y a non-empty subset of the edge-set of Γ. We shall denote by $\mathbf{D}(\ ,Y)$ the submatrix of the incidence matrix $\mathbf{D}$ of Γ consisting of those columns corresponding to edges in Y; we shall denote by $\mathbf{D}(X, Y)$ the submatrix of $\mathbf{D}(\ ,Y)$ consisting of those rows corresponding to vertices in X. The following lemma amplifies the results of Propositions 5.3 and 5.4.

LEMMA 7.4 *Let the notation be as above, with $|X| = |Y|$, and let V_0 denote the vertex-set of the edge-subgraph $\langle Y\rangle$. Then $\mathbf{D}(X, Y)$ is non-singular if and only if the following conditions are satisfied:*

(1) *X is a subset of V_0;*

(2) *$\langle Y\rangle$ contains no circuits;*

(3) *$V_0 - X$ contains precisely one vertex from each component of $\langle Y\rangle$.*

Proof Suppose that $\mathbf{D}(X, Y)$ is non-singular. If X were not a subset of V_0, then $\mathbf{D}(X, Y)$ would contain a row of zeros and would be singular; hence condition (1) holds. The matrix $\mathbf{D}(V_0, Y)$ is the incidence matrix of $\langle Y\rangle$, and if $\langle Y\rangle$ contains a circuit then $\mathbf{D}(V_0, Y)\,\mathbf{z} = \mathbf{0}$ for the non-zero vector $\mathbf{z}$ representing this circuit. Consequently, $\mathbf{D}(X, Y)\,\mathbf{z} = \mathbf{0}$ and $\mathbf{D}(X, Y)$ would be singular. Thus, condition (2) holds. Now it follows that the co-rank of $\langle Y\rangle$ is zero:

$$|Y| - |V_0| + c = 0,$$

where c is the number of components of $\langle Y\rangle$. Since $|X| = |Y|$ we have $|V_0 - X| = c$. If X contained all the vertices from some component of $\langle Y\rangle$, then the corresponding rows of $\mathbf{D}(X, Y)$ would sum to zero, and $\mathbf{D}(X, Y)$ would be singular. Thus $V_0 - X$ contains some vertices from each component of $\langle Y\rangle$, and since $|V_0 - X| = c$, it must contain precisely one vertex from each component. Condition (3) is verified.

The converse is proved by reversing the argument given above. □

A graph Φ whose co-rank is zero is often called a *forest*. We shall use the symbol $p(\Phi)$ to denote the product of the numbers of vertices in the components of Φ. In particular, if Φ is connected it is a *tree*, and we have $p(\Phi) = |V\Phi|$.

THEOREM 7.5 *The coefficients q_i of the polynomial $\tau(\Gamma; \lambda)$ are given by the formula*

$$q_i = \Sigma p(\Phi) \quad (1 \leqslant i \leqslant n),$$

where the summation is over all edge-subgraphs of Γ which have i edges and are forests.

Proof Let $\mathbf{Q}_X$ denote the principal submatrix of $\mathbf{Q}$ whose rows and columns correspond to the vertices in a subset X of $V\Gamma$. Then $q_i = \Sigma \det \mathbf{Q}_X$, where the summation is over all X with $|X| = i$. Using the notation of the preceding lemma, and the fact that $\mathbf{Q} = \mathbf{D}\mathbf{D}^t$, it follows from the Binet–Cauchy theorem that

$$\begin{aligned}\det \mathbf{Q}_X &= \Sigma \det \mathbf{D}(X, Y) \det \mathbf{D}(X, Y)^t \\ &= \Sigma (\det \mathbf{D}(X, Y))^2.\end{aligned}$$

This summation is over all subsets Y of $E\Gamma$ with $|Y| = |X| = i$. Thus,

$$q_i = \sum_{X,\, Y} (\det \mathbf{D}(X, Y))^2.$$

Now $(\det \mathbf{D}(X, Y))^2$ is either 0 or 1, by Proposition 5.3. Further, it takes the value 1 if and only if the three conditions of Lemma 7.4 hold. For each forest $\Phi = \langle Y \rangle$ there are $p(\Phi)$ ways of omitting one vertex from each component of Φ, and consequently there are $p(\Phi)$ summands equal to 1 in the expression for q_i. □

COROLLARY 7.6 *The complexity of a graph Γ is given by the formula*

$$\kappa(\Gamma) = n^{n-2} \Sigma p(\Phi) \left(-\frac{1}{n}\right)^{|E\Phi|},$$

where the summation is over all forests Φ which are edge-subgraphs of the complement of Γ.

Proof The result of part (3) of Proposition 6.6 expresses $\kappa(\Gamma)$ in terms of the τ function of Γ^c. Using the formula of Theorem 7.5 for the coefficients of τ, the stated result follows. □

The formula of Corollary 7.6 can be useful when the complement of Γ is relatively small; examples of this situation are given in §§6C and 7D.

In the case of a regular graph Γ, the relationship between τ and χ leads to an interesting consequence of Theorem 7.5.

PROPOSITION 7.7 *Let Γ be a regular graph of valency k, and let $\chi^{(i)}$ $(0 \leqslant i \leqslant n)$ denote the i-th derivative of the characteristic polynomial of Γ. Then*

$$\chi^{(i)}(\Gamma; k) = i!\,\Sigma p(\Phi),$$

where the summation is over all forests Φ which are edge-subgraphs of Γ with $|E\Phi| = n-i$.

Proof From part (2) of Proposition 6.6, we have

$$\tau(\Gamma; \lambda) = \chi(\Gamma; k+\lambda).$$

Now the Taylor expansion of χ at the value k is

$$\chi(\Gamma; k+\lambda) = \sum_{i=0}^{n} \chi^{(i)}(\Gamma; k)\frac{\lambda^i}{i!}.$$

Comparing this with $\tau(\Gamma; \lambda) = \Sigma q_{n-i}\lambda^i$, we have the result. □

We notice that the case $i = 1$ of Proposition 7.7 gives

$$\chi'(\Gamma; k) = q_{n-1} = n\kappa(\Gamma),$$

a result already mentioned in Chapter 6.

7A *Odd circuits*

Let $\chi(\Gamma; \lambda) = \Sigma c_{n-i}\lambda^i$ and suppose

$$c_3 = c_5 = \dots = c_{2r-1} = 0, \quad c_{2r+1} \neq 0.$$

Then the shortest circuit of odd length in Γ has length $2r+1$, and there are $-c_{2r+1}/2$ such circuits (Sachs 1964).

7*B* *The characteristic polynomial of a tree* Suppose that $\Sigma c_i \lambda^{n-i}$ is the characteristic polynomial of a tree with n vertices. Then the odd coefficients c_{2r+1} are zero, and the even coefficients c_{2r} are given by the following rule: the number of ways of choosing r disjoint edges in the tree is equal to $(-1)^r c_{2r}$.

7*C* *Cospectral trees* The result of § 7B facilitates the construction of pairs of trees which are cospectral (§ 2B). For example, there are two different trees with eight vertices and characteristic polynomial $\lambda^8 - 7\lambda^6 + 10\lambda^4$. Further, if we are given a tree T with n vertices, then the probability that T belongs to a cospectral pair tends to 1 as n tends to infinity (Schwenk 1973).

7*D* *The τ function of a star graph* For the star graph $K_{1,b}$ we can calculate τ from the formula of Theorem 7.5; the result is

$$\tau(K_{1,b}; \lambda) = \lambda(\lambda + b + 1)(\lambda + 1)^{b-1}.$$

Consequently, if Γ is the graph which is constructed by removing a star $K_{1,b}$ from K_n $(n > b+1)$, we have

$$\kappa(\Gamma) = n^{n-2}\left(1 - \frac{1}{n}\right)^{b-1}\left(1 - \frac{b+1}{n}\right).$$

PART TWO

Colouring problems

8. *Vertex-colourings and the spectrum*

The problem of assigning colours to the vertices of a graph in such a way that adjacent vertices have different colours is one of the oldest topics in graph theory. It is this problem which underlies the second part of our tract.

In this chapter we shall apply the techniques developed in Part One to the vertex-colouring problem, by means of certain inequalities involving the eigenvalues of a graph. We begin by stating the basic definitions.

DEFINITION 8.1 A *colour-partition* of a general graph Γ is a partition of $V\Gamma$ into subsets, called *colour-classes*,

$$V\Gamma = V_1 \cup V_2 \cup \dots \cup V_l,$$

where the subsets V_i $(1 \leqslant i \leqslant l)$ are non-empty and mutually disjoint, and each contains no pair of adjacent vertices. In other words, the vertex-subgraphs $\langle V_i \rangle$ $(1 \leqslant i \leqslant l)$ have no edges. The *chromatic number* of Γ, written $\nu(\Gamma)$, is the smallest natural number l for which such a partition is possible.

We note that if Γ has a loop, then it has a self-adjacent vertex, and consequently no colour-partitions. Also, if Γ has several edges joining the same pair of vertices then only one of these edges is relevant to the definition of a colour-partition, since this definition depends only on adjacency. Thus we can continue, for the moment, to deal with simple graphs. However, this is allowable only for the purposes of the present chapter; some of the constructions of later chapters require the introduction of general graphs.

An alternative approach to the definition of the chromatic number is to begin by definining a *vertex-colouring* of Γ to be an assignment of colours to the vertices of Γ, with the property that adjacent vertices are given different colours [W, p. 81]. Every vertex-colouring in which l colours are used gives rise to a colour-partition into l colour-classes.

If $\nu(\Gamma) = 1$, then Γ has no edges. If $\nu(\Gamma) = 2$ then Γ is often called a *bipartite* graph; since a circuit of odd length cannot be coloured with two colours, it follows that a bipartite graph contains no odd circuits. This observation leads to a distinctive spectral property of bipartite graphs.

PROPOSITION 8.2 *Suppose the bipartite graph Γ has an eigenvalue λ of multiplicity $m(\lambda)$. Then $-\lambda$ is also an eigenvalue of Γ, and $m(-\lambda) = m(\lambda)$.*

Proof The result of Proposition 7.3 expresses the characteristic polynomial of a graph Γ in terms of the sesquivalent subgraphs of Γ. If Γ is bipartite, then the remarks in the previous paragraph imply that Γ has no sesquivalent subgraphs with an odd number of vertices. Consequently, the characteristic polynomial of Γ has the form

$$\begin{aligned}\chi(\Gamma; z) &= z^n + c_2 z^{n-2} + c_4 z^{n-4} + \dots \\ &= z^\delta p(z^2),\end{aligned}$$

where $\delta = 0$ or 1, and p is a polynomial function. It follows that the eigenvalues, which are the zeros of χ, have the required property. $\square$

Proposition 8.2 can also be proved by more direct means.

The spectrum of the complete bipartite graph $K_{a,b}$ can be found in the following manner. We may suppose that the vertices of $K_{a,b}$ are labelled in such a way that its adjacency matrix is

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{J} \\ \mathbf{J}^t & \mathbf{0} \end{bmatrix},$$

where $\mathbf{J}$ is the $a \times b$ matrix having all entries $+1$. The matrix $\mathbf{A}$ has just two linearly independent rows, and so its rank is 2. Consequently, 0 is an eigenvalue of $\mathbf{A}$ with multiplicity $a+b-2$. The characteristic polynomial is thus of the form $z^{a+b-2}(z^2 + c_2)$, and (by Proposition 2.3) $-c_2$ is equal to ab, the number of edges of $K_{a,b}$. Hence

$$\operatorname{Spec} K_{a,b} = \begin{pmatrix} \sqrt{(ab)} & 0 & -\sqrt{(ab)} \\ 1 & a+b-2 & 1 \end{pmatrix}.$$

If $\nu(\Gamma) > 2$, then the spectrum of Γ does not have a distinctive property, as it does in the bipartite case. However, it is possible to make important deductions about the chromatic number from a knowledge of the extreme eigenvalues of Γ.

We need some notation. For any real symmetric matrix $\mathbf{M}$, we shall denote the maximum and minimum eigenvalues of $\mathbf{M}$ by $\lambda_{\max}(\mathbf{M})$ and $\lambda_{\min}(\mathbf{M})$; if $\mathbf{M}$ is the adjacency matrix of a graph Γ we shall also use the notation $\lambda_{\max}(\Gamma)$ and $\lambda_{\min}(\Gamma)$.

It follows from Proposition 8.2 that, for a bipartite graph Γ, we have $\lambda_{\min}(\Gamma) = -\lambda_{\max}(\Gamma)$; in fact the converse statement is also true, but its proof (if Γ is not regular) requires the Perron–Frobenius theory of matrices with non-negative entries [P. Lancaster, *Theory of matrices* (Academic Press, 1969), p. 286]. As we do not use the converse result, we shall not prove it. It should be noted that the same theory also gives the results of Proposition 3.1, but its use at that point would not elucidate the graph-theoretical features of the proof.

We do need one technique from matrix theory. Let $(\mathbf{x}, \mathbf{y})$ denote the inner product of the column vectors $\mathbf{x}$, $\mathbf{y}$. For any real $n \times n$ symmetric matrix $\mathbf{X}$, and any real non-zero $n \times 1$ column vector $\mathbf{z}$, we say that $(\mathbf{z}, \mathbf{Xz})/(\mathbf{z}, \mathbf{z})$ is the *Rayleigh quotient*, $R(\mathbf{X}; \mathbf{z})$. We have [P. Lancaster, *ibid.* p. 109]:

$$\lambda_{\max}(\mathbf{X}) \geqslant R(\mathbf{X}; \mathbf{z}) \geqslant \lambda_{\min}(\mathbf{X}) \quad \text{for all} \quad \mathbf{z} \neq \mathbf{0}.$$

This result is used in the proof of the next proposition.

PROPOSITION 8.3 (1) *If Λ is a vertex-subgraph of Γ, then*

$$\lambda_{\max}(\Lambda) \leqslant \lambda_{\max}(\Gamma); \quad \lambda_{\min}(\Lambda) \geqslant \lambda_{\min}(\Gamma).$$

(2) *If the greatest and least valencies among the vertices of Γ are $k_{\max}(\Gamma)$ and $k_{\min}(\Gamma)$, then*

$$k_{\max}(\Gamma) \geqslant \lambda_{\max}(\Gamma) \geqslant k_{\min}(\Gamma).$$

Proof (1) We may suppose that the vertices of Γ are labelled so that the adjacency matrix $\mathbf{A}$ of Γ has a leading principal submatrix $\mathbf{A}_0$, which is the adjacency matrix of Λ. Let $\mathbf{z}_0$ be chosen such that $\mathbf{A}_0\mathbf{z}_0 = \lambda_{\max}(\mathbf{A}_0)\mathbf{z}_0$ and $(\mathbf{z}_0, \mathbf{z}_0) = 1$. Further, let $\mathbf{z}$ be the column vector with $|V\Gamma|$ rows formed by adjoining zero entries to $\mathbf{z}_0$. Then

$$\lambda_{\max}(\mathbf{A}_0) = R(\mathbf{A}_0; \mathbf{z}_0) = R(\mathbf{A}; \mathbf{z}) \leqslant \lambda_{\max}(\mathbf{A}).$$

That is, $\lambda_{\max}(\Lambda) \leqslant \lambda_{\max}(\Gamma)$. The other inequality is proved similarly.

(2) Let $\mathbf{u}$ be the column vector each of whose entries is $+1$. Then, if $n = |V\Gamma|$ and $k^{(i)}$ is the valency of the vertex v_i, we have

$$R(\mathbf{A}; \mathbf{u}) = \frac{1}{n}\sum_{i,j} a_{ij} = \frac{1}{n}\sum_{i} k^{(i)} \geqslant k_{\min}(\Gamma),$$

since the mean valency exceeds the minimum valency. But the Rayleigh quotient $R(\mathbf{A}; \mathbf{u})$ is at most $\lambda_{\max}(\mathbf{A})$, hence

$$\lambda_{\max}(\Gamma) \geqslant k_{\min}(\Gamma).$$

Finally, let $\mathbf{x}$ be an eigenvector corresponding to the eigenvalue $\lambda_0 = \lambda_{\max}(\Gamma)$, and let x_j be a largest positive entry of $\mathbf{x}$. Then, by an argument similar to that used in Proposition 3.1, we have

$$\lambda_0 x_j = (\lambda_0 \mathbf{x})_j = \Sigma' x_i \leqslant k^{(j)} x_j \leqslant k_{\max}(\Gamma) x_j,$$

where the summation is over the vertices v_i adjacent to v_j. Thus $\lambda_0 \leqslant k_{\max}(\Gamma)$. $\square$

We shall now relate the chromatic number of Γ to the extreme eigenvalues of Γ. To begin, we need a lemma about critical graphs. We say that a graph Γ is *l-critical* if $\nu(\Gamma) = l$, and if, for all vertex-subgraphs Λ of Γ (except $\Lambda = \Gamma$), we have $\nu(\Lambda) < l$.

LEMMA 8.4 *Suppose Γ is a graph with chromatic number $l \geqslant 2$. Then Γ has an l-critical vertex-subgraph Λ, and every vertex of Λ has valency at least $l-1$ in Λ.*

Proof The set of all vertex-subgraphs of Γ is non-empty and contains some graphs (for example, Γ itself) whose chromatic number is l, and also some graphs (for example, those with one vertex) whose chromatic number is not l. Let Λ be a vertex-subgraph whose chromatic number is l, and which is minimal with respect to the number of vertices; then clearly Λ is l-critical. If $v \in V\Gamma$, then $\langle V\Gamma - v\rangle$ is a vertex-subgraph of Λ and has a vertex-colouring with $l-1$ colours. If the valency of v in Λ were less than $l-1$, then we could extend this vertex-colouring to Λ, contradicting the fact that $\nu(\Lambda) = l$. Thus the valency of v is at least $l-1$. $\square$

PROPOSITION 8.5 (Wilf 1967) *For any graph* Γ *we have*

$$\nu(\Gamma) \leqslant 1+\lambda_{\max}(\Gamma).$$

Proof We know, from Lemma 8.4, that there is a vertex-subgraph Λ of Γ with the following properties: $\nu(\Lambda) = \nu(\Gamma)$, and $k_{\min}(\Lambda) \geqslant \nu(\Gamma)-1$. Thus, using the inequalities of Proposition 8.3, we have

$$\nu(\Gamma) \leqslant 1+k_{\min}(\Lambda) \leqslant 1+\lambda_{\max}(\Lambda) \leqslant 1+\lambda_{\max}(\Gamma). \qquad \square$$

This inequality is a considerable improvement on the elementary inequality $\nu(\Gamma) \leqslant 1+k_{\max}(\Gamma)$, [W, p. 81]. For example:

$$k_{\max}(K_{a,b}) = \max(a,b), \quad \lambda_{\max}(K_{a,b}) = \sqrt{(ab)},$$

and the second number can be significantly smaller than the first. In cases like this, Proposition 8.5 is also an improvement on the result due to Brooks [W, p. 82] which is itself a refinement of the elementary inequality mentioned above.

Our next major result is complementary to the previous one, in that it provides a lower bound for the chromatic number. We require a preliminary lemma and its corollary.

LEMMA 8.6 *Let* $\mathbf{X}$ *be a real symmetric matrix, partitioned in the form*

$$\mathbf{X} = \begin{bmatrix} \mathbf{P} & \mathbf{Q} \\ \mathbf{Q}^t & \mathbf{R} \end{bmatrix},$$

where $\mathbf{P}$ *and* $\mathbf{R}$ *are square, and consequently symmetric, matrices. Then*

$$\lambda_{\max}(\mathbf{X})+\lambda_{\min}(\mathbf{X}) \leqslant \lambda_{\max}(\mathbf{P})+\lambda_{\max}(\mathbf{R}).$$

Proof Let $\lambda = \lambda_{\min}(\mathbf{X})$ and take an arbitrary $\epsilon > 0$. Then $\mathbf{X}^* = \mathbf{X}-(\lambda-\epsilon)\mathbf{I}$ is a positive-definite symmetric matrix, partitioned in the same way as $\mathbf{X}$, with

$$\mathbf{P}^* = \mathbf{P}-(\lambda-\epsilon)\mathbf{I}, \quad \mathbf{Q}^* = \mathbf{Q}, \quad \mathbf{R}^* = \mathbf{R}-(\lambda-\epsilon)\mathbf{I}.$$

Now a delicate application of the method of Rayleigh quotients to the matrix $\mathbf{X}^*$ leads to the result

$$\lambda_{\max}(\mathbf{X}^*) \leqslant \lambda_{\max}(\mathbf{P}^*) + \lambda_{\max}(\mathbf{R}^*).$$

(The proof of this result is given in [H. L. Hamburger and M. E. Grimshaw, *Linear transformations* (Cambridge University Press, 1956), p. 77].) Thus, in terms of $\mathbf{X}$, $\mathbf{P}$ and $\mathbf{Q}$, we have

$$\lambda_{\max}(\mathbf{X}) - (\lambda - \epsilon) \leqslant \lambda_{\max}(\mathbf{P}) - (\lambda - \epsilon) + \lambda_{\max}(\mathbf{R}) - (\lambda - \epsilon),$$

and since ϵ is arbitrary and $\lambda = \lambda_{\min}(\mathbf{X})$ we have the result. □

COROLLARY 8.7 *Let* $\mathbf{A}$ *be a real symmetric matrix, partitioned into* t^2 *submatrices* $\mathbf{A}_{ij}$ $(1 \leqslant i \leqslant t,\ 1 \leqslant j \leqslant t)$ *in such a way that the row and column partitions are the same; in other words, each diagonal sub-matrix* $\mathbf{A}_{ii}$ $(1 \leqslant i \leqslant t)$ *is square. Then*

$$\lambda_{\max}(\mathbf{A}) + (t-1)\,\lambda_{\min}(\mathbf{A}) \leqslant \sum_{i=1}^{t} \lambda_{\max}(\mathbf{A}_{ii}).$$

Proof We prove this result by induction on t. It is true when $t = 2$, by the lemma; suppose it is true when $t = T-1$. Let $\mathbf{A}$ be partitioned into T^2 submatrices, as in the statement, and let $\mathbf{B}$ denote $\mathbf{A}$ with the last row and column of submatrices deleted. By the lemma,

$$\lambda_{\max}(\mathbf{A}) + \lambda_{\min}(\mathbf{A}) \leqslant \lambda_{\max}(\mathbf{B}) + \lambda_{\max}(\mathbf{A}_{TT}),$$

and by the induction hypothesis,

$$\lambda_{\max}(\mathbf{B}) + (T-2)\,\lambda_{\min}(\mathbf{B}) \leqslant \sum_{i=1}^{T-1} \lambda_{\max}(\mathbf{A}_{ii}).$$

Now $\lambda_{\min}(\mathbf{B}) \geqslant \lambda_{\min}(\mathbf{A})$, as in the proof of Proposition 8.3. Thus, adding the two inequalities, we have the result for $t = T$, and the general result is true. □

We can now state a lower bound for the chromatic number.

THEOREM 8.8 (Hoffman 1970) *For any graph* Γ, *whose edge-set is non-empty,*

$$\nu(\Gamma) \geqslant 1 + \frac{\lambda_{\max}(\Gamma)}{-\lambda_{\min}(\Gamma)}.$$

Proof The vertex-set $V\Gamma$ can be partitioned into $\nu = \nu(\Gamma)$ colour-classes; consequently the adjacency matrix $\mathbf{A}$ of Γ can be partitioned into ν^2 submatrices as in the preceding corollary.

Now, in this case, the diagonal submatrices $\mathbf{A}_{ii}$ $(1 \leqslant i \leqslant \nu)$ consist entirely of zeros, and so $\lambda_{\max}(\mathbf{A}_{ii}) = 0$ $(1 \leqslant i \leqslant \nu)$. Applying Corollary 8.7 we have

$$\lambda_{\max}(\mathbf{A}) + (\nu - 1)\lambda_{\min}(\mathbf{A}) \leqslant 0.$$

But, if Γ has at least one edge, then $\lambda_{\min}(\mathbf{A}) = \lambda_{\min}(\Gamma) < 0$. The result now follows. $\square$

In cases where the spectrum of a graph is known, the inequality of Theorem 8.8 can be very useful. For example (as every geometer knows) there are 27 lines on a general cubic surface, and each line meets 10 other lines (Henderson 1912); if we represent this configuration by means of a graph in which vertices represent lines, and adjacent vertices represent skew lines, then we have a regular graph Σ with 27 vertices and valency 16. This is the graph mentioned in § 3E, and we shall compute the spectrum of Σ fully in Chapter 21. We shall prove that $\lambda_{\max}(\Sigma) = 16$, $\lambda_{\min}(\Sigma) = -2$, and so $\nu(\Sigma) \geqslant 1 + 16/2 = 9$, a result which would be very difficult to establish by direct means.

8*A* *Girth and chromatic number* (*a*) For any natural numbers $g \geqslant 3$ and $h \geqslant 2$ there is a graph with girth at least g and chromatic number h (Erdős 1959). (The *girth* of a graph is the length of its smallest circuit.)

(*b*) The smallest graph with girth 4 and chromatic number 4 is *Grötzsch's graph* (Fig. 3), which has 11 vertices.

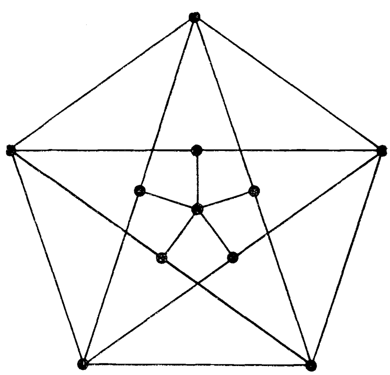


Fig. 3

8*B* *3-critical graphs* The only 3-critical graphs are the odd circuit graphs C_{2s+1}, $s \geqslant 1$ (Dirac 1952).

8*C* *The eigenvalues of a planar graph* Let Γ be a planar connected graph. Then

$$\lambda_{\max}(\Gamma) + 4\lambda_{\min}(\Gamma) \leqslant 0.$$

If the four-colour conjecture is true, then

$$\lambda_{\max}(\Gamma) + 3\lambda_{\min}(\Gamma) \leqslant 0.$$

8*D* *Another bound for the chromatic number* Let Γ be a regular graph of valency k with n vertices. In any colour-partition of Γ each colour-class has at most $n-k$ vertices; consequently

$$\nu(\Gamma) \geqslant \frac{n}{n-k}.$$

However, this bound is never better than that of Theorem 8.8 (Hoffman 1970).

8*E* *The odd graphs* Let k be a natural number greater than 1, and S a set of cardinality $2k-1$. The *odd graph* O_k is defined as follows: its vertices correspond to the subsets of S of cardinality $k-1$, and two vertices are adjacent if and only if the corresponding subsets are disjoint. (For example, $O_2 = K_3$, and O_3 is Petersen's graph.) O_k is a simple graph of valency k; its girth is 3 when $k = 2$, 5 when $k = 3$, and 6 for all $k \geqslant 4$. For the chromatic number of O_k we have

$$\nu(O_k) = 3 \quad \text{for all} \quad k \geqslant 2.$$

(See also §§ 17A, 20B and 21A.)

9. *The chromatic polynomial*

In this chapter we introduce a polynomial function which enumerates the vertex-colourings of a graph. We emphasize that we shall be dealing with general graphs, unless the contrary is stated explicitly.

DEFINITION 9.1 Let Γ be a general graph and u a complex number. For each natural number r, let $m_r(\Gamma)$ denote the number of distinct colour-partitions of $V\Gamma$ into r colour-classes, and define $u_{(r)}$ to be the complex number $u(u-1)(u-2)\dots(u-r+1)$. The *chromatic polynomial* of Γ is the polynomial

$$C(\Gamma; u) = \sum_r m_r(\Gamma)\, u_{(r)}.$$

PROPOSITION 9.2 *If s is a natural number, then $C(\Gamma; s)$ is the number of vertex-colourings of Γ using at most s colours.*

Proof Every vertex-colouring of Γ in which exactly r colours are used gives rise to a colour-partition of Γ into r colour-classes. Conversely, for each colour-partition into r colour-classes we can assign s colours to the colour-classes in $s(s-1)\dots(s-r+1)$ ways. Hence the number of vertex-colourings in which s colours are available is $\Sigma m_r(\Gamma)\, s_{(r)} = C(\Gamma; s)$. $\square$

The chromatic number $\nu(\Gamma)$ is the smallest natural number ν for which $C(\Gamma; \nu) \neq 0$. Thus the problem of finding the chromatic number of a graph is part of the general problem of locating the zeros of its chromatic polynomial.

Some simple properties of the chromatic polynomial follow directly from its definition: if Γ has n vertices, then $m_r(\Gamma) = 0$ when $r > n$, and $m_n(\Gamma) = 1$; hence $C(\Gamma; u)$ is a monic polynomial of degree n. Other results follow from Proposition 9.2 and the principle that a polynomial is uniquely determined by its values at the natural numbers. For instance, if Γ is disconnected, with two components Γ_1 and Γ_2, then we can colour the vertices of Γ_1

and Γ_2 independently. It follows that $C(\Gamma; s) = C(\Gamma_1; s)\, C(\Gamma_2; s)$ for any natural number s, and consequently $C(\Gamma; u)$ is identically equal to $C(\Gamma_1; u)\, C(\Gamma_2; u)$.

Since u is a factor of $u_{(r)}$ for all $r \geqslant 1$, we deduce that $C(\Gamma; 0) = 0$ for any general graph Γ. If Γ has c components, then the coefficients of $1 = u^0, u^1, \ldots, u^{c-1}$ are all zero, by virtue of the result on disconnected graphs in the previous paragraph. Also, if $E\Gamma \neq \varnothing$ then Γ has no vertex-colouring with just one colour, and so $C(\Gamma; 1) = 0$ and $u-1$ is a factor of $C(\Gamma; u)$.

The simplest example is the chromatic polynomial of the complete graph K_n. Since every vertex of K_n is adjacent to every other one, the numbers of colour-partitions are

$$m_1(K_n) = m_2(K_n) = \ldots = m_{n-1}(K_n) = 0; \quad m_n(K_n) = 1.$$

Hence $$C(K_n; u) = u(u-1)(u-2)\ldots(u-n+1).$$

There is a recursive technique which (theoretically) allows us to calculate the chromatic polynomial of any general graph in a finite number of steps. Suppose that Γ is a general graph and Γ has an edge e which is not a loop. The graph $\Gamma^{(e)}$ whose edge-set is $E\Gamma - \{e\}$ and whose vertex-set is $V\Gamma$ is said to be obtained by *deleting* e; while the graph $\Gamma_{(e)}$ constructed from $\Gamma^{(e)}$ by identifying the two vertices incident with e in Γ, is said to be obtained by *contracting* e. We note that $\Gamma^{(e)}$ has one edge fewer than Γ, and $\Gamma_{(e)}$ has one edge and one vertex fewer than Γ.

PROPOSITION 9.3 *In the notation which we have just introduced,*

$$C(\Gamma; u) = C(\Gamma^{(e)}; u) - C(\Gamma_{(e)}; u).$$

Proof Consider the vertex-colourings of $\Gamma^{(e)}$ for which s colours are available. These colourings fall into two disjoint sets: those in which the ends of e are coloured differently, and those in which the ends of e are coloured alike. The first set is in bijective correspondence with the colourings of Γ, and the second set is in bijective correspondence with the colourings of $\Gamma_{(e)}$. Hence $C(\Gamma^{(e)}; s) = C(\Gamma; s) + C(\Gamma_{(e)}; s)$ for each natural number s, and the result follows. $\square$

COROLLARY 9.4 *If T is a tree with n vertices then*

$$C(T;\, u) = u(u-1)^{n-1}.$$

Proof The result is true when $n = 1$. Any tree with $n \geqslant 2$ vertices has at least two monovalent vertices [W, p. 46]. Suppose the result is true when $n = N-1$, and let T be a tree with N vertices, e an edge of T incident with a monovalent vertex. Then $T^{(e)}$ has two components: an isolated vertex, and a tree with $N-1$ vertices; the latter component is $T_{(e)}$. Hence

$$C(T^{(e)};\, u) = uC(T_{(e)};\, u),$$

and by Proposition 9.3

$$C(T;\, u) = (u-1)\, C(T_{(e)};\, u) = u(u-1)^{N-1},$$

by the induction hypothesis. Hence the result is true when $n = N$, and for all n. □

Another application of the deletion and contraction method yields the chromatic polynomial of a circuit graph. If $n \geqslant 2$ the deletion of any edge from a circuit graph C_n results in a tree with n vertices, and the contraction of any edge results in a circuit graph C_{n-1}. Hence

$$C(C_n;\, u) = u(u-1)^{n-1} - C(C_{n-1};\, u).$$

Since $C(C_2;\, u) = u(u-1)$, we may solve the recursion given above to obtain

$$C(C_n;\, u) = (u-1)^n + (-1)^n\, (u-1).$$

We shall now describe two other useful techniques for calculating chromatic polynomials. The first is concerned with a form of product for graphs. As this concept is variously defined and named in the literature we give an explicit definition. Suppose Γ_1 and Γ_2 are two simple graphs; then $\Gamma_1 \circ \Gamma_2$ is the simple graph with vertex-set and edge-set given by

$$V(\Gamma_1 \circ \Gamma_2) = V\Gamma_1 \cup V\Gamma_2;$$

$$E(\Gamma_1 \circ \Gamma_2) = E\Gamma_1 \cup E\Gamma_2 \cup \{\{x, y\} \mid x \in V\Gamma_1, y \in V\Gamma_2\}.$$

In other words, $\Gamma_1 \circ \Gamma_2$ consists of disjoint copies of Γ_1 and Γ_2 with additional edges joining every vertex of Γ_1 to every vertex of Γ_2.

PROPOSITION 9.5 *If* $\Gamma = \Gamma_1 \circ \Gamma_2$ *then the numbers of colour-partitions for* Γ *are*

$$m_i(\Gamma) = \sum_{j+l=i} m_j(\Gamma_1)\, m_l(\Gamma_2).$$

Proof Since every vertex of Γ_1 is adjacent (in Γ) to every vertex of Γ_2, any colour-class of vertices in Γ is either a colour-class in Γ_1 or a colour-class in Γ_2. Hence the result. □

COROLLARY 9.6 *The chromatic polynomial of* $\Gamma_1 \circ \Gamma_2$ *is given by*

$$C(\Gamma_1 \circ \Gamma_2; u) = C(\Gamma_1; u) \circ C(\Gamma_2; u),$$

where the $\circ$ *operation on polynomials signifies that we write each polynomial in the form* $\Sigma m_i u_{(i)}$ *and multiply as if* $u_{(i)}$ *were the power* u^i. □

For example, $K_{3,3}$ is the product $N_3 \circ N_3$, where N_n is the graph with n vertices and no edges. From Corollary 9.6, we have

$$\begin{aligned} C(K_{3,3}; u) = u^3 \circ u^3 &= (u_{(3)} + 3u_{(2)} + u_{(1)}) \circ (u_{(3)} + 3u_{(2)} + u_{(1)}) \\ &= u_{(6)} + 6u_{(5)} + 11u_{(4)} + 6u_{(3)} + u_{(2)} \\ &= u^6 - 9u^5 + 36u^4 - 75u^3 + 78u^2 - 31u. \end{aligned}$$

The chromatic polynomials of all complete multipartite graphs can be found in this way.

The graphs $N_1 \circ \Gamma$ and $N_2 \circ \Gamma$ are called the *cone* and *suspension* of Γ, written $c\Gamma$ and $s\Gamma$ respectively.

PROPOSITION 9.7 (1) $C(c\Gamma; u) = uC(\Gamma; u-1)$.
(2) $C(s\Gamma; u) = u(u-1)\, C(\Gamma; u-2) + uC(\Gamma; u-1)$.

Proof (1) Let $C(\Gamma; u) = \Sigma m_i u_{(i)}$; then, using Corollary 9.6 and the fact that $u_{(i+1)} = u(u-1)_{(i)}$, we have

$$\begin{aligned} C(c\Gamma; u) = C(N_1 \circ \Gamma; u) &= u \circ C(\Gamma; u) \\ &= u_{(1)} \circ \Sigma m_i u_{(i)} \\ &= \Sigma m_i u_{(i+1)} \\ &= u \Sigma m_i (u-1)_{(i)} \\ &= uC(\Gamma; u-1). \end{aligned}$$

The second part is proved in a similar fashion, using the identity $u^2 = u_{(2)} + u_{(1)}$. □

Another useful technique in the calculation of chromatic polynomials applies to graphs of the kind described in the next definition.

DEFINITION 9.8 The general graph Γ is *quasi-separable* if there is a subset K of $V\Gamma$ such that the vertex-subgraph $\langle K\rangle$ is a complete graph and the vertex-subgraph $\langle V\Gamma - K\rangle$ is disconnected. Γ is *separable* if K is empty (in which case Γ itself is disconnected) or if $|K| = 1$ (in which case we say that the single vertex of K is a *cut-vertex*).

It follows that in a quasi-separable graph we have $V\Gamma = V_1 \cup V_2$, where $\langle V_1 \cap V_2\rangle$ is complete and there are no edges in Γ joining $V_1 - (V_1 \cap V_2)$ to $V_2 - (V_1 \cap V_2)$.

The smallest graph which is quasi-separable but not separable is shown in Fig. 4; the relevant sets are $V_1 = \{1, 2, 4\}$, $V_2 = \{2, 3, 4\}$.

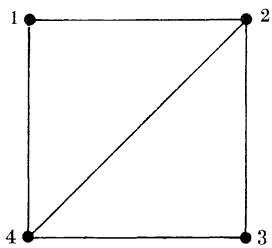


Fig. 4

PROPOSITION 9.9 *If the graph* Γ *is quasi-separable, then, with the notation of Definition* 9.8, *we have*

$$C(\Gamma; u) = \frac{C(\langle V_1\rangle; u)\, C(\langle V_2\rangle; u)}{C(\langle V_1 \cap V_2\rangle; u)}.$$

Proof If $V_1 \cap V_2$ is empty, then we make the convention that the denominator is 1, and the result is a consequence of the remarks about disconnected graphs following Proposition 9.2. Suppose that $\langle V_1 \cap V_2\rangle$ is a complete graph K_t, $t \geqslant 1$. Since Γ contains this complete graph, Γ has no vertex-colouring with fewer than t colours, and so $u_{(t)}$ is a factor of $C(\Gamma; u)$. For each natural number $s \geqslant t$, $C(\Gamma; s)/s_{(t)}$ is the number of ways of extending a given vertex-colouring of $\langle V_1 \cap V_2 \rangle$ to the whole of Γ, using at most s colours.

Also, both $\langle V_1 \rangle$ and $\langle V_2 \rangle$ contain the complete graph

$$K_t = \langle V_1 \cap V_2 \rangle,$$

and so $C(\langle V_i \rangle; s)/s_{(t)}$ $(i = 1, 2)$ has the same interpretation in these graphs. But there are no edges in Γ joining $V_1 - (V_1 \cap V_2)$ to $V_2 - (V_1 \cap V_2)$, so the extensions of a vertex-colouring of $\langle V_1 \cap V_2 \rangle$ to $\langle V_1 \rangle$ and to $\langle V_2 \rangle$ are independent. Hence

$$\frac{C(\Gamma; s)}{s_{(t)}} = \frac{C(\langle V_1 \rangle; s)}{s_{(t)}} \frac{C(\langle V_2 \rangle; s)}{s_{(t)}},$$

for all $s \geqslant t$, from which the result for all u follows. □

The formula of Proposition 9.9 is often useful in the hand calculation of the chromatic polynomials of small graphs. For instance, the chromatic polynomial of the graph shown in Fig. 4 is

$$\frac{u(u-1)(u-2)\,u(u-1)(u-2)}{u(u-1)} = u(u-1)(u-2)^2.$$

But much more significant is the theoretical application of Proposition 9.9 which will be explained in Chapter 12.

9*A* *The chromatic polynomial characterizes trees* If Γ is a simple graph with n vertices, and $C(\Gamma; u) = u(u-1)^{n-1}$, then Γ is a tree.

9*B* *The hyperoctahedral graphs* Let $p_s(u) = C(H_s; u)$, where H_s is the hyperoctahedral graph defined on p. 17. The chromatic polynomials $p_s(u)$ can be found from the recursion

$$p_1(u) = u^2,$$
$$p_s(u) = u(u-1)\,p_{s-1}(u-2) + u p_{s-1}(u-1) \quad \text{for} \quad s \geqslant 2.$$

9*C* *Wheels and pyramids* The cone of the circuit graph C_n is the *wheel* or *pyramid* W_n; the suspension of C_n is the *double pyramid* Π_n. The chromatic polynomials of these graphs are

$$C(W_n; u) = u(u-2)^n + (-1)^n u(u-2),$$
$$C(\Pi_n; u) = u(u-1)(u-3)^n + u(u-2)^n + (-1)^n u(u^2 - 3u + 1).$$

9*D* *Prisms and Möbius ladders* The *prism* T_h ($h \geqslant 3$) is a trivalent graph with $2h$ vertices $u_1, u_2, \ldots, u_h, v_1, v_2, \ldots, v_h$; the vertices $u_1, \ldots, u_h$ form a circuit of length h, as do the vertices $v_1, \ldots, v_h$ and the remaining edges are of the form $\{u_i, v_i\}$, $1 \leqslant i \leqslant h$. The Möbius ladders M_h were defined in § 3C. The chromatic polynomials of these graphs are:

$$C(T_h; u) = (u^2-3u+3)^h + (u-1)\{(3-u)^h + (1-u)^h\} + u^2 - 3u + 1;$$

$$C(M_h; u) = (u^2-3u+3)^h + (u-1)\{(3-u)^h - (1-u)^h\} - 1$$

(Biggs, Damerell and Sands, 1972).

10. Edge-subgraph expansions

The methods developed in the previous chapter for the calculation of chromatic polynomials are of two kinds. The method of deletion and contraction is a simple, if tedious, technique whereby the chromatic polynomial of any given graph can be calculated in a finite number of steps. The other two methods (Corollary 9.6 and Proposition 9.9) exploit special features of certain graphs. In this chapter we shall introduce a technique whose immediate importance is more theoretical than practical: we shall study the relationships between the edge-subgraphs of a graph and its chromatic polynomial. In later chapters it will appear that this technique can be refined in such a way as to make it practicable in specific cases.

We begin with a definition which will be used both in this chapter, and in Chapter 13.

DEFINITION 10.1 The *rank polynomial* of a general graph Γ is

$$R(\Gamma; x, y) = \sum_{S \subseteq E\Gamma} x^{r\langle S \rangle} y^{s\langle S \rangle},$$

where $r\langle S \rangle$ and $s\langle S \rangle$ are the rank and co-rank of the edge-subgraph $\langle S \rangle$ of Γ. If we write $R(\Gamma; x, y) = \Sigma \rho_{rs} x^r y^s$ then ρ_{rs} is the number of edge-subgraphs of Γ with rank r and co-rank s, and we say that the matrix (ρ_{rs}) is the *rank matrix* of Γ.

We notice that, since $r\langle S \rangle + s\langle S \rangle = |S|$ for all $S \subseteq E\Gamma$, we have $R(\Gamma; x, x) = (x+1)^{|E\Gamma|}$.

The chromatic polynomial is a partial evaluation of the rank polynomial. In our proof of this fact we shall develop a general form of expansion which has other interesting consequences.

Suppose that u is a natural number, and $[u]$ denotes the set $\{1, 2, \ldots, u\}$. Let $[u]^X$ denote the set of all functions $\xi: X \to [u]$. Then, for a general graph Γ, the set $[u]^{V\Gamma}$ contains some functions which are vertex-colourings of Γ with u colours available, and some functions which are not vertex-colourings since they violate the condition that adjacent vertices must be given different colours. In order to pick out the vertex-colourings we associate with each $\xi \in [u]^{V\Gamma}$ an indicator function $\hat{\xi}$: $E\Gamma \to \{0, 1\}$, defined as follows.

$$\hat{\xi}(e) = \begin{cases} 1 & \text{if there are vertices } v_1, v_2 \text{ incident with } e \\ & \qquad\qquad \text{such that } \xi(v_1) \neq \xi(v_2); \\ 0 & \text{otherwise.} \end{cases}$$

In particular, $\hat{\xi}(e) = 0$ if e is a loop. The following definition will provide the means of relating the rank polynomial and the chromatic polynomial.

DEFINITION 10.2 The *weight* of a general graph Γ is the function $W(\Gamma; t, u)$ of the complex number t and the natural number u given by

$$W(\Gamma; t, u) = u^{-|V\Gamma|} \sum_{\xi \in [u]^{V\Gamma}} \prod_{e \in E\Gamma} (\hat{\xi}(e) - t).$$

In other words, $W(\Gamma; t, u)$ is the mean value of the product $\Pi(\hat{\xi}(e) - t)$, taken over all ξ: $V\Gamma \to [u]$.

LEMMA 10.3 *If Γ is a general graph and u is a natural number, then*

$$C(\Gamma; u) = u^{|V\Gamma|} W(\Gamma; 0, u).$$

Proof Put $t = 0$ in the definition of W, and consider $\Pi\hat{\xi}(e)$. The product is zero unless $\hat{\xi}(e) = 1$ for all $e \in E\Gamma$, and this is so only if ξ is a vertex-colouring of Γ. Thus $u^{|V\Gamma|} W(\Gamma;\ 0, u)$ is the number of vertex-colourings of Γ using at most u colours. The result follows from Proposition 9.2. □

Since, for each fixed t, $u^{|V\Gamma|} W(\Gamma; t, u)$ is a polynomial function of u, we may suppose that $W(\Gamma; t, u)$ is defined by the same formula for all complex numbers u. Then Lemma 10.3 becomes an identity in the complex variable u. We shall apply this procedure without comment in the following exposition.

THEOREM 10.4 *Let Γ be a general graph, and t a complex number. The chromatic polynomial $C(\Gamma; u)$ can be expanded in terms of the weights of the edge-subgraphs of Γ, as follows:*

$$C(\Gamma; u) = t^{|E\Gamma|} u^{|V\Gamma|} \sum_{S \subseteq E\Gamma} W(\langle S \rangle; t, u) t^{-|S|}.$$

Proof For any natural number u we have, using Lemma 10.3 and the definition of W,

$$\begin{aligned} C(\Gamma;\,u) &= u^{|V\Gamma|}W(\Gamma;\,0,u) \\ &= \sum_{\xi\in[u]^{V\Gamma}} \prod_{e\in E\Gamma} \hat{\xi}(e) \\ &= \sum_{\xi\in[u]^{V\Gamma}} \prod_{e\in E\Gamma} \{(\hat{\xi}(e)-t)+t\} \\ &= \sum_{\xi\in[u]^{V\Gamma}} \sum_{S\subseteq E\Gamma} \prod_{e\in S} (\hat{\xi}(e)-t)\,t^{|E\Gamma-S|}. \end{aligned}$$

The last step is the expansion of the product in powers of t.

We now consider the double summation. For each $S \subseteq E\Gamma$ let $VS = V\langle S\rangle$; then any function from VS to $[u]$ is the restriction to VS of $u^{|V\Gamma-VS|}$ functions from $V\Gamma$ to $[u]$. Thus, inverting the double summation,

$$\begin{aligned} C(\Gamma;\,u) &= \sum_{S\subseteq E\Gamma} u^{|V\Gamma-VS|} \sum_{\xi\in[u]^{VS}} \prod_{e\in S} (\hat{\xi}(e)-t)\,t^{|E\Gamma-S|} \\ &= u^{|V\Gamma|}t^{|E\Gamma|} \sum_{S\subseteq E\Gamma} u^{-|VS|} \sum_{\xi\in[u]^{VS}} \prod_{e\in S} (\hat{\xi}(e)-t)\,t^{-|S|} \\ &= u^{|V\Gamma|}t^{|E\Gamma|} \sum_{S\subseteq E\Gamma} W(\langle S\rangle;\,t,u)\,t^{-|S|}. \quad \square \end{aligned}$$

COROLLARY 10.5 *The chromatic polynomial and the rank polynomial of a general graph* Γ *are related by the identity*

$$C(\Gamma;u) = u^{|V\Gamma|}R(\Gamma;\,-u^{-1},\,-1).$$

Proof Putting $t = 1$, and taking u to be a natural number, the expansion of Theorem 10.4 becomes

$$C(\Gamma;\,u) = u^{|V\Gamma|} \sum_{S\subseteq E\Gamma} W(\langle S\rangle;\,1,u).$$

Consider the product $\Pi(\hat{\xi}(e)-1)$ over all edges $e\in S$. If the product is non-zero, then ξ must be constant on each component of $\langle S\rangle$, and the value of the product is $(-1)^{|S|}$. If $\langle S\rangle$ has c components there are u^c such functions ξ; hence $W(\langle S\rangle;\,1,u)$, which is the mean value of the product over all $u^{|VS|}$ functions $\xi\colon VS\to[u]$, is given by

$$W(\langle S\rangle;\,1,u) = u^{-|VS|}(-1)^{|S|}u^c = (-u)^{-r\langle S\rangle}(-1)^{s\langle S\rangle}.$$

Thus
$$C(\Gamma;\,u) = u^{|V\Gamma|}R(\Gamma;\,-u^{-1},\,-1). \quad \square$$

The preceding expansion is derived by substituting $t = 1$ in the general expansion of Theorem 10.4; its usefulness stems from the fact that the resulting weights have an especially simple form. A different substitution, which leads to an expansion with several distinctive features, is described in § 10A.

The connection between the chromatic polynomial and the rank polynomial given by Corollary 10.5 has several important theoretical consequences; these were first studied by Birkhoff (1912) in the original paper on chromatic polynomials, and Whitney (1932*a*). Let us write the chromatic polynomial of a general graph Γ with n vertices in the form

$$C(\Gamma;\, u) = b_0 u^n + b_1 u^{n-1} + \ldots + b_{n-1} u + b_n,$$

where we have already remarked that $b_0 = 1$, and that $b_n = 0$ (provided that Γ is not the graph with no vertices and no edges). Using Corollary 10.5 we can express the coefficients b_i $(0 \leqslant i \leqslant n)$ in terms of the rank matrix (ρ_{rs}) of Γ. We have

$$\begin{aligned} C(\Gamma; u) = \sum_i b_i u^{n-i} &= u^n R(\Gamma;\, -u^{-1}, -1) \\ &= u^n \sum_{r,s} \rho_{rs}(-u)^{-r}(-1)^s \\ &= \sum_r \sum_s (-1)^{r+s} \rho_{rs} u^{n-r}. \end{aligned}$$

Equating coefficients of powers of u, and rearranging the signs, we have

$$(-1)^i b_i = \sum_j (-1)^j \rho_{ij}.$$

For example, the rank matrix of the graph $\Gamma = K_{3,3}$ (for which $m = |E\Gamma| = 9$ and $n = |V\Gamma| = 6$) is

$$\begin{bmatrix} 1 & & & & \\ 9 & & & & \\ 36 & & & & \\ 84 & 9 & & & \\ 117 & 45 & 6 & & \\ 81 & 78 & 36 & 9 & 1 \end{bmatrix}.$$

Here the rows are labelled by the values of the rank r from 0 to 5, and the columns are labelled by the values of the co-rank s from 0 to 4. We notice that each antidiagonal (sloping bottom left to top right) represents all edge-subgraphs with a fixed number t of edges, and consequently sums to the binomial coefficient $\binom{m}{t}$. This remark enables us to make a considerable simplification in the calculation of the rank matrix. We notice also that $\rho_{n-1,0}$ is just the complexity of Γ. Expressing the coefficients of the chromatic polynomial of $K_{3,3}$ in terms of the alternating sums of the rows of the rank matrix we find (cf. p. 60)

$$C(K_{3,3}; u) = u^6 - 9u^5 + 36u^4 - 75u^3 + 78u^2 - 31u.$$

PROPOSITION 10.6 *Let Γ be a simple graph of girth g, having m edges and η circuits of length g. Then, with the above notation for the coefficients of the chromatic polynomial of Γ, we have*

(1) $$(-1)^i b_i = \binom{m}{i} \quad \text{for} \quad i = 0, 1, \ldots, g-2;$$

(2) $$(-1)^{g-1} b_{g-1} = \binom{m}{g-1} - \eta.$$

Proof The only edge-subgraphs of Γ with rank $i \leqslant g-2$ must have co-rank zero, since Γ has no circuits with fewer than g edges. Thus, for all $i \leqslant g-2$, $\rho_{i0} = \binom{m}{i}$ and $\rho_{ij} = 0$ if $j > 0$. Further, the only edge-subgraphs of Γ with rank $g-1$ are the $\binom{m}{g-1}$ forests with $g-1$ edges (these have co-rank zero), and the η circuits with g edges (these have co-rank 1). Thus,

$$\rho_{g-1,0} = \binom{m}{i}, \quad \rho_{g-1,1} = \eta, \quad \rho_{g-1,j} = 0, \quad \text{if} \quad j > 1.$$

The result now follows from our expression for the coefficients of the chromatic polynomial. □

For a simple graph with n vertices and m edges, it follows, since g is at least 3, that the coefficient of u^{n-1} in the chromatic polynomial is $-m$.

We now prove the important theoretical result that the coefficients of the chromatic polynomial alternate in sign; that is,

$(-1)^i b_i$ is always positive. To do this, we employ a reduction of Corollary 10.5 which involves counting fewer subgraphs.

Let Γ be a simple graph whose edge-set $E\Gamma = \{e_1, e_2, \ldots, e_m\}$ is ordered by the natural ordering of subscripts. This ordering is to remain fixed throughout our discussion. A *broken circuit* in Γ is the result of removing the first edge from some circuit; in other words, it is a subset B of $E\Gamma$ such that for some edge e_l we have

(1) $B \cup \{e_l\}$ is a circuit in Γ;

(2) for each e_i in B, $i > l$.

The next proposition expresses the coefficients of the chromatic polynomial in terms of those edge-subgraphs which contain no broken circuits; clearly such subgraphs contain no circuits, and so they are forests.

PROPOSITION 10.7 (Whitney 1932*a*) *If* Γ *is a simple graph whose edge-set is ordered as above, and if* $C(\Gamma; u) = \Sigma b_i u^{n-i}$, *then* $(-1)^i b_i$ *is the number of edge-subgraphs of* Γ *which have* i *edges and contain no broken circuits.*

Proof Suppose $B_1, B_2, \ldots, B_t$ is a list of the broken circuits of Γ, in dictionary order† based on the ordering of $E\Gamma$. Let f_i $(1 \leqslant i \leqslant t)$ denote the edge which, when added to B_i, completes a circuit. The edges f_i are not necessarily all different, but, because of the way in which the broken circuits are ordered, it follows that f_i is not in B_j when $j \geqslant i$.

Define Σ_0 to be the set of edge-subgraphs of Γ containing no broken circuit, and Σ_h to be the set of edge-subgraphs containing B_h but not $B_{h+1}, B_{h+2}, \ldots, B_t$ $(1 \leqslant h \leqslant t)$. Then $\Sigma_0, \Sigma_1, \ldots, \Sigma_t$ is a partition of the set of all edge-subgraphs of Γ. We show that, in the expression

$$(-1)^i b_i = \Sigma(-1)^j \rho_{ij},$$

the total contribution from $\Sigma_1, \ldots, \Sigma_t$, on the right-hand side, is zero.

† In Whitney's treatment the last edge, rather than the first, is removed to form a broken circuit. Thus Whitney's ordering of broken circuits is not the dictionary ordering: 'any' comes before 'apt' in a dictionary, but not in Whitney. Whitney's proof is given wrongly in Ore's book on graph theory, for there the last edge is removed, but the ordering of broken circuits is lexicographic. The present treatment, removing the first edge, seems more natural, and we shall make a similar modification in Chapter 13.

For suppose S is a subset of $E\Gamma$ not containing f_h; then S contains B_h if and only if $S \cup \{f_h\}$ contains B_h. Further, S contains B_i $(i > h)$ if and only if $S \cup \{f_h\}$ contains B_i, since f_h is not in B_i. Thus if one of the edge-subgraphs $\langle S\rangle$, $\langle S \cup \{f_h\}\rangle$ is in Σ_h, then both are in Σ_h. They have the same rank, but their co-ranks differ by one, and so their contributions to the alternating sum cancel.

Consequently, we need only consider the contribution of Σ_0 to $\Sigma(-1)^j \rho_{ij}$. Since an edge-subgraph $\langle S\rangle$ in Σ_0 is a forest, it has co-rank $j = 0$ and rank $i = |S|$, whence the result. □

COROLLARY 10.8 *Let Γ be a simple graph with rank r. Then the coefficients of $C(\Gamma; u)$ alternate in sign. Precisely, in the notation of the preceding paragraphs, $(-1)^i b_i > 0$ for $i = 0, 1, \ldots, r$.*

Proof The characterization of Proposition 10.7 shows that $(-1)^i b_i \geqslant 0$ for $0 \leqslant i \leqslant n$. In order to obtain the strict inequality we must show that there is some edge-subgraph with i edges, containing no broken circuit, for $i = 0, 1, \ldots, r$. Suppose we successively remove edges from Γ in such a way that one circuit is destroyed at each stage; this process stops when we reach an edge-subgraph $\langle F\rangle$ of Γ with $|F| = r$ and $s\langle F\rangle = 0$. Let us order the edges of Γ so that the edges in F come first. Then $\langle F\rangle$ contains no broken circuit, and any subset of F generates an edge-subgraph containing no broken circuit. Thus we have produced the necessary edge-subgraphs, and the result follows. □

We recall that, at the beginning of Chapter 9, we used elementary observations to show that $b_i = 0$ if $i = n, n-1, \ldots, n-(c-1)$, where $n = |V\Gamma|$ and Γ has c components. That is, $b_i = 0$ if $i = r+1, \ldots, n$. Thus the coefficients of the chromatic polynomial alternate strictly, and then become zero.

10*A* *Nagle's form of the edge-subgraph expansion* With the notation of Theorem 10.4, put $t = 1-u^{-1}$, and let

$$N(\Gamma; u) = u^{|V\Gamma|} W(\Gamma; 1-u^{-1}, u).$$

Then we have:

(*a*) If Γ has an isthmus [W, p. 30], then $N(\Gamma; u) = 0$.

(*b*) If Γ_1 and Γ_2 are homeomorphic [W, p. 60], then

$$N(\Gamma_1; u) = N(\Gamma_2; u).$$

(*c*) The chromatic polynomial may be expanded as follows:

$$C(\Gamma; u) = \sum_{S \subseteq E\Gamma} u^{|V\Gamma - VS|}(1-u^{-1})^{|E\Gamma - S|} N(\langle S \rangle; u)$$

(Nagle 1971).

10*B* *Inequalities for the coefficients of the chromatic polynomial* If Γ is a simple connected graph with n vertices and m edges, and $C(\Gamma; u) = \Sigma b_i u^{n-i}$, then

$$\binom{n-1}{i} \leqslant (-1)^i b_i \leqslant \binom{m}{i}$$

(Read 1968, Meredith 1972).

10*C* *The icosahedron* Let I denote the graph which is the 1-skeleton of an icosahedron in $\mathbb{R}^3$.

$$\begin{aligned} C(I; u) = u(u-1)(u-2)(u-3) \\ \times (u^8 - 24u^7 + 260u^6 - 1670u^5 + 6999u^4 \\ - 19\,698u^3 + 36\,408u^2 - 40\,240u + 20\,170) \end{aligned}$$

(Whitney 1932*b*).

10*D* *The dodecahedron* Let D denote the 1-skeleton of a dodecahedron; D is the planar dual of I and is depicted in [W, p. 37]. Writing $C(D; u)$ in the form

$$-u(u-1)(u-2)\sum_{i=0}^{17} c_i(1-u)^i,$$

the coefficients $c_{17}, c_{16}, \ldots, c_0$ are:

1, 10, 56, 230, 759, 2112, 5104, 10 912, 20 890, 36 052,

56 048, 77 702, 94 118, 96 556, 80 332, 50 648, 21 302, 4412

(Sands 1972).

11. *The logarithmic transformation*

In this chapter and the next one we shall investigate expansions of the chromatic polynomial which involve relatively few subgraphs in comparison with the expansion of Chapter 10.

The present chapter is devoted to a transformation of the general expansion of Theorem 10.4; this transformation enables us to disregard all those edge-subgraphs which are separable. The original idea appears in the work of Whitney (1932*b*), and it has been independently rediscovered by several people, including Tutte (1967) and a group of physicists who interpret the method as a 'linked-cluster expansion' (Baker 1971). The simplified version given here is based on a paper by the present author (Biggs 1973).

We begin with some remarks about separable and non-separable graphs. Let Γ be a separable graph in which (as in Definition 9.8) $V\Gamma$ is the union of two non-empty subsets V_1, V_2, with the properties that $|V_1 \cap V_2| \leqslant 1$ and $\langle V\Gamma - (V_1 \cap V_2)\rangle$ is disconnected. If $E_1 = E\langle V_1\rangle$, $E_2 = E\langle V_2\rangle$, then it is easy to see that

$$E_1 \cup E_2 = E\Gamma, \quad E_1 \cap E_2 = \varnothing.$$

It follows that every general graph is the union of its maximal non-separable edge-subgraphs, called *blocks*, together with a set (possibly empty) of isolated vertices, which are sometimes called *degenerate blocks*. We shall make the convention that a degenerate block is not a block, although the opposite convention is equally tenable.

If the graph Γ is non-separable, then it has just one block, which is the whole of Γ; it is also the edge-subgraph $\langle E\Gamma\rangle$. (Throughout the rest of this chapter, pointed brackets refer to edge-subgraphs, unless the contrary is explicitly stated.)

We shall set up our theory in a general framework. Let Y be a function defined for all general graphs, taking positive real values, and having the following properties:

(1) $Y(\Gamma) = 1$ if Γ has no edges;

(2) $Y(\Gamma)$ is the product of the numbers $Y(B)$ as B runs through the blocks of Γ.

Let X be the real-valued function defined in terms of Y by

(3) $X(\Gamma) = \sum_{S \subseteq E\Gamma} Y\langle S \rangle.$

LEMMA 11.1 *Let Γ be a separable graph, where $V\Gamma = V_1 \cup V_2$ as in Definition 9.8. Then, for a function X defined as above, we have*

$$X(\Gamma) = X(\Gamma_1)\, X(\Gamma_2),$$

where Γ_i denotes the vertex-subgraph $\langle V_i \rangle$ $(i = 1, 2)$.

Proof Suppose that $S \subseteq E\Gamma$, and that E_1 and E_2 are the edge-sets of Γ_1 and Γ_2. If $S_1 = S \cap E_1$ and $S_2 = S \cap E_2$, then $S = S_1 \cup S_2$ and $S_1 \cap S_2 = \varnothing$. Thus the blocks of $\langle S \rangle_\Gamma$ are the blocks of $\langle S_1 \rangle_{\Gamma_1}$ together with the blocks of $\langle S_2 \rangle_{\Gamma_2}$, and

$$Y\langle S \rangle_\Gamma = Y\langle S_1 \rangle_{\Gamma_1}\, Y\langle S_2 \rangle_{\Gamma_2}.$$

(If either, or both, of S_1, S_2 are empty, this equation remains true.) Consequently,

$$X(\Gamma) = \sum_{S \subseteq E\Gamma} Y\langle S \rangle_\Gamma = \sum_{S_1 \subseteq E_1} \sum_{S_2 \subseteq E_2} Y\langle S_1 \rangle_{\Gamma_1}\, Y\langle S_2 \rangle_{\Gamma_2} = X(\Gamma_1) X(\Gamma_2). \quad \square$$

We shall now transform the sum $X(\Gamma)$ into a product; this is accomplished by the introduction of logarithmic functions, as in the following definition.

DEFINITION 11.2 Let (X, Y) be a pair of functions satisfying the conditions (1) and (2), and the relationship (3) stated above. Then the *logarithmic transformation* of the pair (X, Y) is the pair of functions $(\tilde{X}, \tilde{Y})$ defined by

$$\tilde{X}(\Gamma) = (-1)^{|E\Gamma|} \sum_{S \subseteq E\Gamma} (-1)^{|S|} \log X\langle S \rangle, \qquad \tilde{Y}(\Gamma) = \exp \tilde{X}(\Gamma).$$

PROPOSITION 11.3 *Let Γ be a general graph. If Γ has no edges, or if Γ is separable and has no isolated vertices, then $\tilde{X}(\Gamma) = 0$.*

Proof If $E\Gamma$ is empty then, by following the sequence of definitions, $Y(\Gamma) = 1$, $X(\Gamma) = 1$, $\tilde{X}(\Gamma) = 0$.

Suppose that Γ has no isolated vertices and is separable. We shall continue with the notation of Lemma 11.1. Then, for $S \subseteq E\Gamma$, we have

$$X\langle S\rangle = X\langle S_1\rangle X\langle S_2\rangle,$$

and so $\log X\langle S\rangle = \log X\langle S_1\rangle + \log X\langle S_2\rangle$. Consequently,

$$\begin{aligned}(-1)^{|E\Gamma|}\tilde{X}(\Gamma) &= \sum_{S\subseteq E\Gamma} (-1)^{|S|} \log X\langle S\rangle \\ &= \sum_{S_1\subseteq E_1} \sum_{S_2\subseteq E_2} (-1)^{|S_1|+|S_2|} (\log X\langle S_1\rangle + \log X\langle S_2\rangle) \\ &= [\sum_{S_1\subseteq E_1} (-1)^{|S_2|} \log X\langle S_1\rangle \sum_{S_2\subseteq E_2} (-1)^{|S_2|}] \\ &\quad + [\sum_{S_2\subseteq E_2} (-1)^{|S_2|} \log X\langle S_2\rangle \sum_{S_1\subseteq E_1} (-1)^{|S_1|}].\end{aligned}$$

Now, since Γ has no isolated vertices, both E_1 and E_2 are non-empty. But the sum $\Sigma(-1)^{|A|}$, over all subsets A of a set B, is zero if B is non-empty. Hence the entire sum above is zero, and we have the result. □

THEOREM 11.4 *Let Γ be a non-separable graph, and let $(\tilde{X}, \tilde{Y})$ denote the logarithmic transformation of the pair (X, Y). Then we have a multiplicative expansion*

$$X(\Gamma) = \prod_{S\subseteq E\Gamma} \tilde{Y}\langle S\rangle,$$

in which $\tilde{Y}$ is equal to 1 (and so may be ignored) for separable edge-subgraphs of Γ.

Proof We require to prove that

$$\log X(\Gamma) = \sum_{S\subseteq E\Gamma} \tilde{X}\langle S\rangle,$$

from which the theorem follows by taking exponentials. Now, from the definition of $\tilde{X}$,

$$\sum_{S\subseteq E\Gamma} \tilde{X}\langle S\rangle = \sum_{S\subseteq E\Gamma} (-1)^{|S|} \sum_{R\subseteq S} (-1)^{|R|} \log X\langle R\rangle,$$

and $\langle R\rangle$ as an edge-subgraph of $\langle S\rangle$ is identical with $\langle R\rangle$ as an edge-subgraph of Γ. Writing $Y = S - R$ the right-hand side becomes

$$\begin{aligned}&\sum_{R\subseteq E\Gamma} \sum_{Y\subseteq E\Gamma - R} (-1)^{|R|+|Y|} (-1)^{|R|} \log X\langle R\rangle \\ &\qquad = \sum_{R\subseteq E\Gamma} \log X\langle R\rangle \sum_{Y\subseteq E\Gamma - R} (-1)^{|Y|}.\end{aligned}$$

The inner sum is non-zero only when $E\Gamma - R = \varnothing$; that is, when $R = E\Gamma$. Thus the expression reduces to $\log X\langle E\Gamma\rangle = \log X(\Gamma)$, as required. Finally, a separable edge-subgraph either has no edges, or it has no isolated vertices; consequently Proposition 11.3 shows that $\tilde{Y}\langle S\rangle = \exp \tilde{X}\langle S\rangle$ is equal to 1 for separable edge-subgraphs. □

We now apply the general theory of the logarithmic transformation to the chromatic polynomial, by means of the expansion in terms of weights introduced in Chapter 10.

We take our function Y to be

$$Y(\Gamma) = t^{-|E\Gamma|}W(\Gamma; t, u),$$

where t and u are taken to be fixed positive real numbers, for the moment. Then, by Theorem 10.4, the corresponding function X is

$$X(\Gamma) = u^{-|V\Gamma|}t^{-|E\Gamma|}C(\Gamma; u).$$

In the case of the additive expansion of Theorem 10.4, we can obtain basically different expansions by giving different values to the parameter t. Remarkably, when the logarithmic transformation is applied to this pair (X, Y) the resulting multiplicative expansion is independent of t. This we now prove.

LEMMA 11.5 *Let (X, Y) denote the particular pair of functions defined above, and let $(\tilde{X}, \tilde{Y})$ be the logarithmic transformation of this pair. If Γ is a graph with more than one edge, then $\tilde{Y}(\Gamma)$ is independent of t. If Γ is the graph having just one edge incident with two distinct vertices ($\Gamma = K_2$), then $\tilde{Y}(\Gamma) = t^{-1}(1 - u^{-1})$.*

Proof From the definitions, we have

$$\begin{aligned}\tilde{X}(\Gamma) &= (-1)^{|E\Gamma|}\Sigma(-1)^{|S|}\log X\langle S\rangle \\ &= (-1)^{|E\Gamma|}\Sigma(-1)^{|S|}\log [u^{-|VS|}t^{-|S|}C(\langle S\rangle; u)].\end{aligned}$$

That part of this expression which depends (apparently) on t, is:

$$\Sigma(-1)^{|S|}\log t^{-|S|} = (\log t) \sum_{S\subseteq E\Gamma} (-1)^{|S|+1}|S|.$$

Now this sum is zero, unless $|E\Gamma| = 1$. Thus we have the result for graphs with more than one edge, and the result for $\Gamma = K_2$ follows by explicit calculation. □

Proposition 11.6 *There is a function q, defined for all non-separable graphs having more than one edge, such that the chromatic polynomial of a general graph Γ has a multiplicative expansion*

$$C(\Gamma; u) = u^{|V\Gamma|}(1-u^{-1})^{|E\Gamma|}\,\Pi\, q(\Lambda; u),$$

where product is taken over all those non-separable edge-subgraphs Λ of Γ which have more than one edge.

Proof We have seen that, if (X, Y) is the pair defined by

$$Y(\Gamma) = t^{-|E\Gamma|}W(\Gamma; t, u), \quad X(\Gamma) = u^{-|V\Gamma|}t^{-|E\Gamma|}\,C(\Gamma; u),$$

then $\tilde{Y}(K_2) = t^{-1}(1-u^{-1})$, and $\tilde{Y}(\Gamma)$ is independent of t if $|E\Gamma| > 1$. We now apply the result of Theorem 11.4, writing $q(\Lambda; u) = \tilde{Y}(\Lambda)$ if $|E\Lambda| > 1$; this gives

$$u^{-|V\Gamma|}t^{-|E\Gamma|}C(\Gamma; u) = [t^{-1}(1-u^{-1})]^{|E\Gamma|}\,\Pi q(\Lambda; u).$$

Thus the terms in t cancel, and we have the result. □

The functions $q(\Gamma; u)$ can be found explicitly for certain standard graphs. We give only one example, as these functions will be superseded in the next chapter.

For the circuit graph C_n, the only edge-subgraph occurring in the product is C_n itself; hence

$$C(C_n; u) = u^n(1-u^{-1})^n\, q(C_n; u).$$

By a result of Chapter 9, the left-hand side is

$$(u-1)^n + (-1)^n(u-1),$$

so that

$$q(C_n; u) = 1 + \frac{(-1)^n}{(u-1)^{n-1}}.$$

In the following chapter we shall develop a version of Proposition 11.6 in which the number of subgraphs involved is reduced still further. We shall also show how to overcome an apparent circularity in the use of Proposition 11.6 to calculate chromatic polynomials. This logical difficulty arises from the fact that the right-hand side of the multiplicative expansion for $C(\Gamma; u)$ contains a term $q(\Gamma; u)$, and we have, as yet, no way of finding

$q(\Gamma; u)$ without prior knowledge of $C(\Gamma; u)$. It will be shown that this seemingly fundamental objection can be surmounted by means of a few simple observations.

11*A* *Crossed circuits* Let C_n^+ denote a graph constructed from the circuit graph C_n by the addition of one edge joining two distinct vertices which are non-adjacent in C_n. Then

$$q(C_n^+; u) = \{q(C_n; u)\}^{-1}.$$

11*B* *Theta graphs* Let $\Theta_{r,s,t}$ denote the graph consisting of two vertices joined by three disjoint paths of length r, s, and t. $\Theta_{r,s,t}$ has $n = r+s+t-1$ vertices and $r+s+t$ edges, and

$$q(\Theta_{r,s,t}; u) = \frac{1-(1-u)^{r-n}-(1-u)^{s-n}-(1-u)^{t-n}+(2-u)(1-u)^{-n}}{(1-(1-u)^{r-n})(1-(1-u)^{s-n})(1-(1-u)^{t-n})}$$

(Baker 1971).

11*C* *The multiplicative expansion of the rank polynomial* If $Y(\Gamma) = x^{r(\Gamma)}y^{s(\Gamma)}$ then $X(\Gamma) = R(\Gamma; x, y)$, and the logarithmic transformation applied to the pair (X, Y) leads to a multiplicative expansion

$$R(\Gamma; x, y) = (1+x)^{|E\Gamma|} \Pi\, Y(\Lambda; x, y),$$

where the product is over all non-separable edge-subgraphs Λ of Γ with $|E\Lambda| > 1$ (Tutte 1967).

12. *The vertex-subgraph expansion*

The multiplicative expansion of the chromatic polynomial can be modified in such a way that the edge-subgraphs are replaced by vertex-subgraphs. This procedure has two advantages. First, the number of vertex-subgraphs is usually much smaller than the number of edge-subgraphs; and second, the factors which replace the q factors (in the notation of Proposition 11.6) turn out to be trivial for a wider class of graphs.

The result of Proposition 11.6 is an identity involving (as we shall see) polynomials and rational functions only. Accordingly we can assume that the variable u is a complex variable, with the usual convention that the poles of rational functions need not be mentioned explicitly.

The formal details of the transition to vertex-subgraphs are quite straightforward. For any non-separable general graph Λ define

$$Q(\Lambda; u) = \Pi q(\Delta; u),$$

where the product is over the set of edge-subgraphs Δ of Λ for which $V\Delta = V\Lambda$.

PROPOSITION 12.1 *The chromatic polynomial has a multiplicative expansion*

$$C(\Gamma; u) = u^{|V\Gamma|}(1-u^{-1})^{|E\Gamma|}\,\Pi Q(\Lambda; u),$$

where the product is over all non-separable vertex-subgraphs of Γ *having more than one edge.*

Proof The factors which appear in Proposition 11.6 can be grouped in such a way that each group contains those edge-subgraphs of Γ which have a given vertex-set. This grouping of factors corresponds precisely to that given in the definition of Q and the resulting expression for C. For, each edge-subgraph Δ of Γ is an edge-subgraph of exactly one vertex-subgraph Λ of Γ

such that $V\Delta = V\Lambda$; conversely, each edge-subgraph of Λ is an edge-subgraph of Γ. $\square$

The circuit graph C_n has just one non-separable edge-subgraph (with n vertices), which is C_n itself. Thus

$$Q(C_n; u) = q(C_n; u) = 1 + \frac{(-1)^n}{(u-1)^{n-1}}.$$

A crucial fact in the application of Proposition 12.1 is that the Q function always takes the same general form as in this example. In order to prove this we must look again at the machinery of Chapter 11.

Recall that q was defined to be the function $\tilde{Y}$ of the pair $(\tilde{X}, \tilde{Y})$, related by means of the logarithmic transformation to the following pair (X, Y):

$$X(\Gamma) = u^{-|V\Gamma|}C(\Gamma; u), \quad Y(\Gamma) = W(\Gamma; 1, u),$$

where we have put $t = 1$ in our original definition.

We can eliminate the logarithmic and exponential functions from the definition of $\tilde{Y}$, writing $\tilde{Y}$ directly in terms of X:

$$\tilde{Y}(\Gamma) = \prod_{S \subseteq E\Gamma} \{X\langle S\rangle\}^{\epsilon(S)}, \quad \epsilon(S) = (-1)^{|E\Gamma - S|}.$$

In our case, this gives

$$q(\Gamma; u) = \prod_{S \subseteq E\Gamma} \{u^{-|VS|}C(\langle S\rangle; u)\}^{\epsilon(S)}.$$

This formula shows that q is a rational function. In fact, it is possible to be more precise: it can be shown that

$$q(\Gamma; u) = 1 + \frac{\nu(u)}{\delta(u)},$$

where ν and δ are polynomials whose degrees satisfy

$$\deg\delta - \deg\nu \geqslant |V\Gamma| - 1.$$

The only satisfactory proof is given by Tutte (1967), and it involves the application of 'tree mappings' to the product formula for q. As the exposition of this concept would take up a great deal of time and space, we shall refer the reader to Tutte's paper for details.

Accepting this result, we can prove the same thing for Q.

PROPOSITION 12.2 *Let Γ be a non-separable graph. Then $Q(\Gamma; u)$ may be written in the form*

$$Q(\Gamma; u) = 1 + \frac{\nu(u)}{\delta(u)},$$

where ν and δ are polynomials such that $\deg\delta - \deg\nu \geqslant |V\Gamma| - 1$.

Proof The function Q is defined to be the product of functions q, over a set of graphs with the same number of vertices. Thus the result for q implies the result for Q. $\square$

We are now in a position to overcome the circularity mentioned at the end of the previous chapter. It is possible, using Proposition 12.2, to calculate both $C(\Gamma; u)$ and $Q(\Gamma; u)$ provided only that we know the Q functions for all *proper* vertex-subgraphs of Γ; that is, those vertex-subgraphs with fewer vertices than Γ. For we can write

$$C(\Gamma; u) = P(u)\, Q(\Gamma; u)$$

where $P(u)$ is a product of rational functions, corresponding to the proper vertex-subgraphs; consequently $P(u)$ may be assumed known. Now, since the initial factor in $P(u)$ is u^n (where $n = |V\Gamma|$), and all subsequent factors have the form $1+\rho(u)$ (where ρ is a rational function whose denominator has degree greater than its numerator), we have

$$P(u) = u^n + \alpha_1 u^{n-1} + \ldots + \alpha_{n-1} u + \alpha_n + \alpha_{n+1} u^{-1} + \ldots.$$

But, from Proposition 12.2, the function $Q(\Gamma\colon u)$ can be written

$$Q(\Gamma; u) = 1 + \beta_0 u^{-n+1} + \beta_1 u^{-n} + \ldots,$$

so that multiplying $P(u)$ by this expression does not alter the coefficients of $u^n, u^{n-1}, \ldots, u^2$ in $P(u)$. Thus the polynomial part of $P(u)$ is a correct expression for $C(\Gamma; u)$, except for the coefficients of u and 1. But these coefficients in $C(\Gamma; u)$ can be determined, simply by noting that if $|E\Gamma| > 1$, then $u(u-1)$ is a factor of $C(\Gamma; u)$. It follows that both $C(\Gamma; u)$ and $Q(\Gamma; u)$ are determined by the known function $P(u)$.

An example will elucidate this argument. Take $\Gamma = K_4$; then the only proper vertex-subgraphs of Γ having more than one edge are the four copies of $K_3 = C_3$. Thus

$$\begin{aligned} C(K_4;\, u) &= u^4(1-u^{-1})^6 \left\{1 - \frac{1}{(u-1)^2}\right\}^4 Q(K_4;\, u) \\ &= \frac{u^2(u-2)^4}{(u-1)^2}\, Q(K_4;\, u). \end{aligned}$$

Dividing $(u-1)^2$ into $u^2(u-2)^4$ gives $u^4 - 6u^3 + 11u^2 - \ldots$, and so $C(K_4;\, u) = u^4 - 6u^3 + 11u^2 - au + b$. Since $u(u-1)$ is a factor of $C(K_4;\, u)$ it follows that

$$a = 6, \quad b = 0 \quad \text{and} \quad C(K_4; u) = u(u-1)(u-2)(u-3).$$

We can now find $Q(K_4;\, u)$ by substituting back, leading to

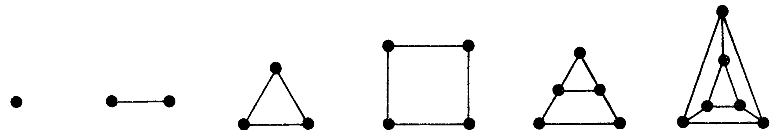
$$Q(K_4; u) = 1 - \frac{2u-3}{u(u-2)^3}.$$

The technique which we have just described and exemplified has an important theoretical consequence: we can calculate chromatic polynomials merely by counting vertex-subgraphs, without knowing any C and Q functions in advance. Precisely, suppose $\Lambda_1, \Lambda_2, \ldots, \Lambda_l$ is a list of the isomorphism types of non-separable vertex-subgraphs of Γ. (We shall suppose that $K_1 = \Lambda_1$ and $K_2 = \Lambda_2$ are included, for the sake of uniformity, and $\Gamma = \Lambda_l$.) Then we define a matrix $\mathbf{N} = (n_{ij})$, by putting n_{ij} equal to the number of vertex-subgraphs of Λ_i which are isomorphic with Λ_j. We may suppose that the list has been ordered in such a way that $\mathbf{N}$ is a triangular matrix each of whose diagonal entries is $+1$.

PROPOSITION 12.3 *The matrix* $\mathbf{N}$, *defined above, completely determines the chromatic polynomial of* Γ.

Proof We know the C and Q function for all the graphs with at most three vertices. Now, suppose we know the C and Q functions for the vertex-subgraphs of Γ with at most t vertices; then we can find the C and Q functions for each vertex-subgraph with $t+1$ vertices, by using the technique previously explained. Thus, recursive use of this procedure leads eventually to the chromatic polynomial of Γ. □

For example, the following is a complete list of the relevant isomorphism types of vertex-subgraphs of $\Gamma = \Lambda_6$ (Fig. 5).



Λ_1 Λ_2 Λ_3 Λ_4 Λ_5 Λ_6

Fig. 5

The **N** matrix for Γ is:

$$\begin{bmatrix} 1 & & & & & \\ 2 & 1 & & & & \\ 3 & 3 & 1 & & & \\ 4 & 4 & 0 & 1 & & \\ 5 & 6 & 1 & 1 & 1 & \\ 6 & 9 & 2 & 3 & 6 & 1 \end{bmatrix}.$$

Suppose that we have completed the calculations for subgraphs with at most four vertices. The C and Q functions for these graphs are:

	Λ_2	Λ_3	Λ_4
C:	$u(u-1)$	$u(u-1)(u-2)$	$u(u-1)(u^2-3u+3)$
Q:	$\dfrac{u-1}{u}$	$\dfrac{u(u-2)}{(u-1)^2}$	$\dfrac{u(u^2-3u+3)}{(u-1)^3}$

The remainder of the calculation proceeds as follows. We have $C(\Lambda_5; u) = P_5(u)\, Q(\Lambda_5; u)$, where

$$P_5(u) = u^5 \left(\frac{u-1}{u}\right)^6 \left(\frac{u(u-2)}{(u-1)^2}\right) \left(\frac{u(u^2-3u+3)}{(u-1)^3}\right)$$
$$= u(u-1)(u-2)(u^2-3u+3).$$

Remarkably, this is exact; that is,

$$C(\Lambda_5;\, u) = u(u-1)(u-2)(u^2-3u+3),$$

and $$Q(\Lambda_5;\, u) = 1.$$

At the next stage, we have $C(\Lambda_6; u) = P_6(u)\,Q(\Lambda_6; u)$, where

$$P_6(u) = u^6\left(\frac{u-1}{u}\right)^9 \left(\frac{u(u-2)}{u-1}\right)^2 \left(\frac{u(u^2-3u+3)}{(u-1)^3}\right)^3 (1)^6$$
$$= u^6 - 9u^5 + 34u^4 - 67u^3 + 67u^2 - \dots.$$

Consequently, $C(\Gamma; u) = u^6 - 9u^5 + 34u^4 - 67u^3 + 67u^2 - 26u$.

One noteworthy feature of the preceding calculation is that $Q(\Lambda_5; u) = 1$, although Λ_5 is a non-separable graph. This means that we could have ignored Λ_5 completely, both in setting up the matrix **N** and in the subsequent calculations. The next proposition describes a large class of graphs Γ for which $Q(\Gamma; u) = 1$.

PROPOSITION 12.4 (Baker 1971) *If the graph Γ is quasi-separable* (*Definition* 9.8) *then* $Q(\Gamma; u) = 1$.

Proof We prove this result by induction on the number of vertices of Γ. The result is true for all quasi-separable graphs with at most four vertices. For this set contains only one graph (the graph of Fig. 4, p. 61) which is not in fact separable, and our claim can be readily checked for that graph.

Suppose that the result is true for all quasi-separable graphs with at most L vertices, and let Γ be a quasi-separable graph with $L+1$ vertices. We have $V\Gamma = V_1 \cup V_2$, where $\langle V_1 \cap V_2\rangle$ is complete and $\langle V\Gamma - (V_1 \cap V_2)\rangle$ is disconnected.

Now, the expansion of Proposition 12.1 can be written in the form

$$C(\Gamma; u) = P(u)\,Q(\Gamma; u),$$

where $P(u)$ is a product of factors corresponding to the proper non-separable vertex-subgraphs of Γ. If U is any proper subset of $V\Gamma$ for which $U \nsubseteq V_1$ and $U \nsubseteq V_2$, then $\langle U\rangle$ is a quasi-separable graph, because of the expression $V\langle U\rangle = (V_1 \cap U) \cup (V_2 \cap U)$. Consequently, in this case $Q(\langle U\rangle; u) = 1$, by the induction hypothesis.

Thus the product $P(u)$ contains non-trivial terms corresponding to the subsets of V_1 and the subsets of V_2. However, a subset of $V_1 \cap V_2$ occurs just once, rather than twice. It follows that

$$P(u) = \frac{C(\langle V_1\rangle; u)\,C(\langle V_2\rangle; u)}{C(\langle V_1 \cap V_2\rangle; u)}.$$

But Proposition 9.9 tells us that $C(\Gamma; u)$ is also equal to this expression, hence $Q(\Gamma; u) = 1$ and the induction step is verified. The result follows. $\square$

We notice that the graph Λ_5, in the example preceding the proposition, is in fact quasi-separable, and so its anomalous behaviour is explained.

We now state, as a theorem, the culmination of the theory of Chapters 10–12.

THEOREM 12.5 *The chromatic polynomial of a graph can be found from a knowledge of those of its vertex-subgraphs which are not quasi-separable.*

Proof This theorem is the conjunction of Propositions 12.1 and 12.4. $\square$

The above result is of great practical importance in approximating the chromatic polynomials of large graphs. We shall illustrate this idea by reference to the infinite square lattice graph [W, p. 39], which we denote by Υ.

The relevant extension of the chromatic polynomial is the function

$$C_\infty(\Upsilon; u) = \lim_{s\to\infty} \{C(\Upsilon_s; u)\}^{1/|V\Upsilon_s|},$$

where Υ_s is a sequence of vertex-subgraphs of Υ, chosen in such a way that the limit exists for a range of values of u. In this case we choose Υ_s to be a square portion of Υ with s^2 vertices, so that Υ_s has approximately $2s^2$ edges. The only other vertex-subgraphs of Υ_s which are not quasi-separable and have fewer than 9 vertices are the copies of C_4 (approximately s^2 in number) and the copies of C_8 (also approximately s^2 in number). Hence

$$C_\infty(\Upsilon; u) \approx u(1-u^{-1})^2\left(1+\frac{1}{(u-1)^3}\right)\left(1+\frac{1}{(u-1)^7}\right).$$

The exact value of $C_\infty(\Upsilon; 3)$ is known (Lieb 1967); it is $(4/3)^{\frac{3}{2}} = 1.540\ldots$. Our approximation gives $1.512,\ldots$, and for larger values of u there is reason to believe that the approximation is even better.

12*A* *The Q function for complete graphs* If

$$\eta(i) = (-1)^{n-1-i}\binom{n-1}{i},$$

then $$Q(K_n; u) = \prod_{0 \leqslant i \leqslant n-1} (u-i)^{\eta(i)}.$$

12*B* *The first non-trivial coefficient in q and Q* If Γ is non-separable and has n vertices and m edges, then the coefficient of $u^{-(n-1)}$ in the expansion of $q(\Gamma; u)$ in descending powers of u, is equal to $(-1)^m$. The corresponding coefficient in $Q(\Gamma; u)$ is therefore $\Sigma(-1)^{|E\Lambda|}$, where the summation is over all non-separable edge-subgraphs Λ of Γ for which $V\Lambda = V\Gamma$ (Tutte 1967).

12*C* *Chromatic powers* Let $\sigma_m(\Gamma)$ denote the sum of the mth powers of the zeros of $C(\Gamma; u)$. Suppose that

$$\log Q(\Lambda; u) = -\sum_{j=1}^{\infty} \frac{c_j(\Lambda)}{ju^j},$$

where the expansion is valid for $|u|$ sufficiently large. Then, if $n(\Gamma, \Lambda)$ denotes the number of vertex-subgraphs of Γ which are isomorphic with Λ, we have

$$\sigma_m(\Gamma) = \Sigma n(\Gamma, \Lambda)\, c_m(\Lambda),$$

where the summation is over all isomorphism classes of non-quasi-separable graphs (Tutte 1967).

13. The Tutte polynomial

An expansion of a kind completely different from that discussed in the preceding chapters is that which relates the rank polynomial to the spanning trees of a graph. This expansion is the subject of the present chapter.

The definition of the rank polynomial depends upon the assignment of the ordered pair (rank, co-rank) of non-negative integers to each edge-subgraph; we shall call such an assignment a *bigrading* of the set of edge-subgraphs. If Γ is connected, the set of edge-subgraphs whose bigrading is $(r(\Gamma), 0)$ is just the set of spanning trees of Γ. We shall introduce a new bigrading of edge-subgraphs, which has the remarkable property that, if it is given only for the spanning trees of Γ, then the entire rank polynomial of Γ is determined.

Our procedure is based initially upon an ordering of the edge-set $E\Gamma$, although a consequence of our main result is the fact that this arbitrary ordering is essentially irrelevant. Another consequence of the main result is an expansion of the chromatic polynomial in terms of spanning trees; this will be the subject of Chapter 14.

We now fix some hypotheses and conventions which will remain in force throughout this chapter. The graph Γ is a connected general graph, and $E\Gamma$ has a fixed total ordering denoted by $\leqslant$. If $X \subseteq E\Gamma$ we shall use the symbol X (rather than $\langle X \rangle$) to denote the corresponding edge-subgraph of Γ, and the singleton sets $\{x\} \subseteq E\Gamma$ will be denoted by x instead of $\{x\}$. The rank of Γ will be denoted by r_0; thus $r_0 = r(\Gamma) = |V\Gamma| - 1$.

If $X \subseteq E\Gamma$ and $x \in E\Gamma - X$, then the rank of $X \cup x$ is either $r(X)$ or $r(X)+1$, and in the latter case we say that x is *independent* of X. Now if $r(X) \neq r_0$, there will certainly be some edges of Γ which are independent of X, and we shall denote the first of these (in the ordering $\leqslant$) by $\lambda(X)$. We note that, since

$$r(Y)+s(Y) = |Y| \quad \text{for all} \quad Y \subseteq E\Gamma,$$

we have the equations

$$r(X \cup \lambda(X)) = r(X)+1, \quad s(X \cup \lambda(X)) = s(X).$$

Similarly, if $s(X) \neq 0$ then there are some edges x for which $s(X-x) = s(X)-1$, and we denote the first of these by $\mu(X)$. We have

$$r(X-\mu(X)) = r(X), \quad s(X-\mu(X)) = s(X)-1.$$

DEFINITION 13.1 The λ *operator* on subsets of $E\Gamma$ assigns to each set $X \subseteq E\Gamma$ the set X^λ derived from X by successively adjoining the edges $\lambda(X), \lambda(X \cup \lambda(X)), \ldots,$ until no further increase in the rank is possible. The μ *operator* takes X to the set X^μ, which is derived from X by successively removing the edges $\mu(X), \mu(X-\mu(X)), \ldots,$ until no further decrease in the co-rank is possible.

We notice the following properties of the λ and μ operators.

$$X \subseteq X^\lambda, \quad r(X^\lambda) = r_0, \qquad s(X^\lambda) = s(X).$$
$$X^\mu \subseteq X, \quad r(X^\mu) = r(X), \quad s(X^\mu) = 0.$$

In what follows we shall exploit the obvious similarity between the two operators by giving proofs only for one of them. Our first lemma says that the edges which must be added to an edge-subgraph A to get A^λ, can be added in any order. (The notation $x < y$ will mean $x \leqslant y$ and $x \neq y$.)

LEMMA 13.2 *If $A \subseteq B \subseteq A^\lambda$, then $B^\lambda = A^\lambda$.*

Proof If $A = A^\lambda$, the result is trivial. Suppose

$$A^\lambda - A = X = \{x_1, x_2, \ldots, x_t\},$$

where $x_1 < x_2 < \ldots < x_t$, and let $B = A \cup Y$, where $Y \subseteq X$. If $Y = X$, then $B = A^\lambda$ and $B^\lambda = A^{\lambda\lambda} = A^\lambda$. If $Y \neq X$, let x_a be the first edge in $X - Y$. Then for any edge x independent of B, we have x independent of $A \cup \{x_1, \ldots, x_{a-1}\}$ (which is contained in B), and so $x_a \leqslant x$ since x_a is the first edge independent of

$$A \cup \{x_1, \ldots, x_{a-1}\}.$$

But x_a itself is certainly independent of B, since when we add the edges in X to A, the rank must increase by exactly one at each step. Thus $x_a = \lambda(B)$, and by successively repeating the argument with $B' = B \cup \lambda(B)$, $B'' = B' \cup \lambda(B')$, ..., we have the result. $\square$

LEMMA 13.3 *If $A \subseteq B$ and $r(B) \neq r_0$, then $\lambda(B) \in A^\lambda$.*

Proof Since $r(B) \neq r_0$, there is a first edge $\lambda(B)$ independent of B, and consequently independent of A. Suppose $\lambda(B)$ is not in A^λ. Then each edge x in $A^\lambda - A$ must satisfy $x < \lambda(B)$, and so x is not independent of B; also, since $A \subseteq B$, no edge in A is independent of B. Thus all edges in A^λ are not independent of B, and $r(B) = r(A^\lambda) = r_0$. This is a contradiction, so our hypothesis was false and $\lambda(B)$ is in A^λ. $\square$

We note the analogous properties of the μ operator:

(1) $$A^\mu \subseteq B \subseteq A \Rightarrow B^\mu = A^\mu;$$

(2) $$B \subseteq A \quad \text{and} \quad s(B) \neq 0 \quad \Rightarrow \quad \mu(B) \notin A^\mu.$$

Our next definition introduces a new bigrading of the subsets of E. The relationship between this bigrading and the λ and μ operators will lead to our main result.

DEFINITION 13.4 Let X be a subset of $E\Gamma$. An edge e in $E\Gamma - X$ is said to be *externally active* with respect to X if $\mu(X \cup e) = e$. An edge f in X is said to be *internally active* with respect to X if $\lambda(X - f) = f$. The number of edges which are externally (internally) active with respect to X is called the *external* (*internal*) *activity* of X.

We shall denote the sets of edges which are externally and internally active with respect to X by X^ϵ and X^ι respectively, and we shall use the notation

$$X^+ = X \cup X^\epsilon, \quad X^- = X - X^\iota.$$

These concepts may be clarified by studying them in the case of a spanning tree; in that case they are related to the systems of basic circuits and cutsets which we introduced in Chapter 5.

PROPOSITION 13.5 *For any spanning tree T of Γ we have:*

(1) *the edge e is externally active with respect to T if and only if e is the first edge (in the ordering $\leqslant$) of* cir (T, e);

(2) *the edge f is internally active with respect to T if and only if f is the first edge (in the ordering $\leqslant$) of* cut (T, f).

Proof (1) By definition, e is externally active if and only if e is the first edge whose removal decreases the co-rank of $T \cup e$. But $T \cup e$ contains just one circuit, which is cir (T, e), and any edge whose removal decreases the co-rank must belong to this circuit.

(2) This is proved by a parallel argument. $\square$

DEFINITION 13.6 The *Tutte polynomial* of a connected graph Γ, with respect to an ordering $\leqslant$ of $E\Gamma$ is defined as follows. Suppose t_{ij} is the number of spanning trees of Γ whose internal activity is i and whose external activity is j. Then the Tutte polynomial is

$$T(\Gamma, \leqslant; x, y) = \Sigma\, t_{ij} x^i y^j.$$

Remarkably, it will appear (as a consequence of our main theorem) that T is independent of the chosen ordering.

We shall investigate the relationship between the concepts just defined and the following diagram of operators:

$$\begin{tikzcd} \mathscr{A} \arrow[r, "\lambda"] \arrow[d, "\mu"'] & \mathscr{B} \arrow[d, "\mu"] \\ \mathscr{C} \arrow[r, "\lambda"] & \mathscr{D} \end{tikzcd}$$

Here $\mathscr{A}$, $\mathscr{B}$, $\mathscr{C}$, $\mathscr{D}$ denote respectively: all subsets of E, subsets Z with $r(Z) = r_0$, subsets W with $s(W) = 0$, subsets T with $r(T) = r_0$ and $s(T) = 0$ (that is, spanning trees). In fact, the diagram is commutative, although we shall not need this result (§ 13A).

PROPOSITION 13.7 *Let X be any subset in the image of the λ operator, so that $r(X) = r_0$ and $X^\lambda = X$. Then*

$$X = Y^\lambda \Leftrightarrow X^- \subseteq Y \subseteq X.$$

Proof Suppose $X = Y^\lambda$. Then $Y \subseteq Y^\lambda = X$, so $Y \subseteq X$. If f is an edge of X^-, then certainly f is in $X = Y^\lambda$. If f were in $Y^\lambda - Y$, then by Lemma 13.2, $\lambda(Y^\lambda - f) = f$; but this means that f is internally active with respect to $X = Y^\lambda$, contradicting $f \in X^-$. Thus f is in Y, and $X^- \subseteq Y$.

Suppose $X^- \subseteq Y \subseteq X$. If $X = Y$, then we have $X = X^\lambda = Y^\lambda$. Now if $f \in X - Y$, then f is internally active with respect to X and so $\lambda(X-f) = f$. From $Y \subseteq X-f$ we have (by Lemma 13.3) $\lambda(X-f) \in Y^\lambda$, that is, $f \in Y^\lambda$. Since this is true for all f in $X - Y$, it follows that $X - Y \subseteq Y^\lambda$, and consequently $X \subseteq Y^\lambda$. Finally, from Definition 13.1 and $Y \subseteq X \subseteq Y^\lambda$ we deduce that $X^\lambda = Y^\lambda$, that is $X = Y^\lambda$. □

We note the analogous result: if X is in the image of the μ operator, then

$$X = Y^\mu \Leftrightarrow X \subseteq Y \subseteq X^+.$$

Proposition 13.8 *With the notation for the diagram of operators introduced above, let T belong to $\mathscr{D}$, W belong to $\mathscr{C}$ and $W^\lambda = T$. Then $W^\epsilon = T^\epsilon$.*

Proof Suppose that the edge e is externally active with respect to T. We shall show that the whole of cir (T, e) belongs to W, whence it follows that e is externally active with respect to W.

If there were an edge $f \neq e$ in cir (T, e) which is not in W, then, since (by Proposition 13.7) we have $T^- \subseteq W \subseteq T$, f must be internally active with respect to T. Now $f \in$ cir (T, e) implies that $e \in$ cut (T, f), and the internally active property of f means that $f < e$. This contradicts the externally active property of e. Hence cir $(T, e) \subseteq W$, and e is externally active with respect to W.

Conversely, if e is externally active with respect to W it follows immediately that e is externally active with respect to T. □

We can now set up our main theorem, which concerns the portion $\mathscr{A} \xrightarrow{\mu} \mathscr{C} \xrightarrow{\lambda} \mathscr{D}$ of our operator diagram. We define

$$\tilde{\rho}_{ij} = \{X \in \mathscr{A} \mid r(X) = r_0 - i,\ s(X) = j\},$$

$$\pi_{ij} = \{W \in \mathscr{C} \mid r(W) = r_0 - i,\ |W^\epsilon| = j\},$$

$$t_{ij} = \{T \in \mathscr{D} \mid |T^\iota| = i,\ |T^\epsilon| = j\}.$$

The last definition occurs in Definition 13.6, and is repeated here for the sake of comparison. We have three corresponding two-variable polynomials

$$\tilde{R}(\Gamma; x, y) = \Sigma\tilde{\rho}_{ij}x^i y^j, \quad P(\Gamma, \leqslant; x, y) = \Sigma\pi_{ij}x^i y^j,$$
$$T(\Gamma, \leqslant; x, y) = \Sigma t_{ij}x^i y^j,$$

where the *modified rank polynomial* $\tilde{R}$ is related to the usual one defined in Definition 10.1 by $\tilde{R}(\Gamma; x, y) = x^{r_0}R(\Gamma; x^{-1}, y)$.

THEOREM 13.9 *Let Γ be a connected graph and $\leqslant$ an ordering of $E\Gamma$. Then the Tutte polynomial is related to the modified rank polynomial:*

$$T(\Gamma, \leqslant; x+1, y+1) = \tilde{R}(\Gamma; x, y).$$

Proof We shall use the intermediate polynomial P, and prove the equalities

$$T(\Gamma, \leqslant; x+1, y+1) = P(\Gamma, \leqslant; x, y+1) = \tilde{R}(\Gamma; x, y),$$

which are equivalent to the following relationships among the coefficients:

$$\pi_{ij} = \sum_k \binom{k}{i} t_{kj}, \quad \tilde{\rho}_{ij} = \sum_l \binom{l}{j} \pi_{il}.$$

For the first identity, we consider $\lambda: \mathscr{C} \to \mathscr{D}$. By Proposition 13.7, if T is in $\mathscr{D}$, then $T = W^\lambda$ if and only if $T^- \subseteq W \subseteq T$; also, by Proposition 13.8, the external activities of T and W are the same. Consequently, for each one of the t_{kj} spanning trees T with $|X^\iota| = k$ and $|X^\epsilon| = j$ there are $\binom{k}{i}$ edge-subgraphs W in $\mathscr{C}$ with $r(W) = r_0 - i$ and $|W^\epsilon| = j$. These subgraphs are obtained by removing from T any set of i edges contained in the k internally active edges of T. This proves the first identity.

For the second identity, we consider $\mu: \mathscr{A} \to \mathscr{C}$. By the analogue of Proposition 13.7 for μ, if X is in $\mathscr{C}$, then $X = Y^\mu$ if and only if $X \subseteq Y \subseteq X^+$. Consequently, for each one of the π_{il} edge-subgraphs X in $\mathscr{C}$ with $r(X) = r_0 - i$ and $|X^\epsilon| = l$, there are $\binom{l}{j}$ edge-subgraphs Y with $r(Y) = r_0 - i$ and $s(Y) = j$. These subgraphs are obtained by adding to X any set of

j edges contained in the l externally active edges of X. This proves the second identity. $\square$

COROLLARY 13.10 *The Tutte polynomial of a connected graph* Γ *is independent of the ordering* $\leqslant$ *used in its definition.*

Proof This statement follows from Theorem 13.9 and the fact that the modified rank polynomial is independent of the ordering. $\square$

We shall henceforth write $T(\Gamma; x, y)$ for the Tutte polynomial of Γ. It should be noted that, although each coefficient t_{ij} is independent of the ordering, the corresponding set of spanning trees (having internal activity i and external activity j) does depend on the ordering.

The original proof of Theorem 13.9 (Tutte 1954) was inductive; the first constructive proof was given by Crapo (1969), and our proof is a considerable simplification of that one.

13*A* *The commutative diagram* If $X \subseteq E\Gamma$, define

$$T = X^{\mu} \cup (X^{\lambda} - X) = X^{\lambda} - (X - X^{\mu}).$$

Then $X^{\lambda\mu} = T = X^{\mu\lambda}$ (Crapo 1969).

13*B* *Counting forests* Let

$$T(\Gamma; 1, 1+t) = \Sigma \phi_i t^i,$$

then ϕ_i is the number of forests in Γ which have $|V\Gamma| - i - 1$ edges. $T(\Gamma; 1, 2)$ is the total number of forests in Γ, and $T(\Gamma; 1, 1)$ is the complexity of Γ.

13*C* *Planar graphs* If Γ and Γ^* are dual planar graphs, then

$$T(\Gamma; x, y) = T(\Gamma^*; y, x).$$

13*D* *Recursive families* A family $\{\Gamma_l\}$ of graphs is said to be a *recursive family* if there is a linear recurrence of the form

$$T(\Gamma_{l+\rho}; x, y) + a_1 T(\Gamma_{l+\rho-1}; x, y) + \ldots + a_\rho T(\Gamma_l; x, y) = 0.$$

The coefficients $a_1, \ldots, a_\rho$ are polynomial functions of (x, y), and are independent of l. The family $\{M_h\}$ of Möbius ladders (§ 3C) is a recursive family with $\rho = 6$ and a recurrence whose auxiliary equation is

$$(t-1)(t-x)(t^2-(x+y+2)t+xy) \\ \times (t^2-(x^2+x+y+1)t+x^2y) = 0.$$

From this we can deduce the complexity (§6B) and the chromatic polynomial (§9D) of the graph M_h (Biggs, Damerell and Sands 1972).

14. The chromatic polynomial and spanning trees

In this chapter we shall apply the main result of Chapter 13 to the study of the chromatic polynomial of a connected graph.

THEOREM 14.1 *Let Γ be a connected graph with n vertices. Then*

$$C(\Gamma; u) = (-1)^{n-1} u \sum_{i=1}^{n-1} t_{i0}(1-u)^i,$$

where t_{i0} is the number of spanning trees of Γ which have internal activity i and external activity zero (with respect to any fixed ordering of $E\Gamma$).

Proof We have only to invoke some identities derived in earlier chapters. The chromatic polynomial is related to the rank polynomial by Corollary 10.5, and the modified rank polynomial is related to the Tutte polynomial by Theorem 13.9. Thus we have

$$\begin{aligned} C(\Gamma; u) &= u^n R(\Gamma; -u^{-1}, -1) \\ &= u^n(-u^{-1})^{n-1} \tilde{R}(\Gamma; -u, -1) \\ &= (-1)^{n-1} uT(\Gamma; 1-u, 0). \end{aligned}$$

The result now follows from the definition of the Tutte polynomial. □

This theorem indicates a purely algebraic way of calculating chromatic polynomials. For if we are given the incidence matrix of a graph Γ, then the basic circuits and cutsets associated with each spanning tree T of Γ can be found by matrix operations, as explained in Chapter 5. From this information we can compute the internal and external activities of T, using the results of Proposition 13.5. The method is impracticable for hand calculation, but it is well-adapted to automatic computation in view of the availability of sophisticated programs for carrying out matrix algebra.

Theorem 14.1 also has theoretical consequences in the study of chromatic polynomials, and the remainder of this chapter is devoted to some of these consequences.

PROPOSITION 14.2 *Let Γ be a connected graph, and let (t_{ij}) denote the matrix of coefficients of its Tutte polynomial. Then*

$$t_{10} = t_{01}.$$

Proof Suppose that the ordering of $E\Gamma = \{e_1, e_2, \ldots, e_m\}$ is chosen so that $e_i \leqslant e_j$ if and only if $i \leqslant j$. If T is a spanning tree with internal activity 1 and external activity 0, then e_1 must be an edge of T, otherwise it would be externally active. Further, e_2 is not an edge of T, otherwise both e_1 and e_2 would be internally active; also $e_1 \in \text{cir}\,(T, e_2)$, otherwise e_2 would be externally active. Consequently $T_* = (T - e_1) \cup e_2$ is a spanning tree, with internal activity 0 and external activity 1.

Reversing the argument shows that $T \mapsto T_*$ is a bijection, and hence t_{10} (the number of spanning trees T with $|T^\iota| = 1$ and $|T^\epsilon| = 0$) is equal to t_{10} (the number of spanning trees T_* with $|T_*^\iota| = 0$ and $|T_*^\epsilon| = 1$). □

The number t_{10} appears in different guises in the work of several authors, for example Crapo (1967) and Essam (1971). It is sufficiently important to warrant a name.

DEFINITION 14.3 The *chromatic invariant* $\theta(\Gamma)$ of a connected graph Γ, is the number of spanning trees of Γ which have internal activity 1 and external activity 0.

The number $\theta(\Gamma)$ is related to the chromatic polynomial of Γ, as a consequence of Theorem 14.1. Let C' denote the derivative of C; then a simple calculation shows that

$$C'(\Gamma; 1) = (-1)^n t_{10} = (-1)^n \theta(\Gamma).$$

Thus, for a graph with an even number of vertices, C increases through its zero at the point $u = 1$, whereas for a graph with an odd number of vertices it decreases.

This relationship can also be used to justify the use of the name 'invariant' for $\theta(\Gamma)$. Recall that two graphs are said to be *homeo-*

morphic [W, p. 60] if they can both be obtained from the same graph by inserting extra vertices of valency 2 into its edges.

PROPOSITION 14.4 *If* Γ_1 *and* Γ_2 *are homeomorphic, then*

$$\theta(\Gamma_1) = \theta(\Gamma_2).$$

Proof Let Γ be a graph which has a vertex of valency 2, and let e and f be the edges incident with this vertex. The deletion of e from Γ results in a graph $\Gamma^{(e)}$ in which the edge f is attached at a cut-vertex; hence $C(\Gamma^{(e)}; u)$ is of the form $(u-1)C(\Gamma_0; u)$. The contraction of e in Γ results in a graph homeomorphic with Γ. We have

$$\begin{aligned} C(\Gamma; u) &= C(\Gamma^{(e)}; u) - C(\Gamma_{(e)}; u) \\ &= (u-1)C(\Gamma_0; u) - C(\Gamma_{(e)}; u), \end{aligned}$$

and, on differentiating and putting $u = 1$, we find

$$C'(\Gamma; 1) = -C'(\Gamma_{(e)}; 1).$$

Since Γ has one more vertex than $\Gamma_{(e)}$, it follows that

$$\theta(\Gamma) = \theta(\Gamma_{(e)}).$$

Now, if two graphs are homeomorphic, then they are related to some graph by a sequence of operations like that by which $\Gamma_{(e)}$ was obtained from Γ; hence we have the result. □

We end this chapter with an application of Theorem 14.1 to a conjecture of Read (1968). This is the *unimodal conjecture* for chromatic polynomials, which postulates that if

$$C(\Gamma; u) = u^n - c_1 u^{n-1} + \ldots + (-1)^{n-1} c_{n-1} u,$$

then, for some number M in the range $1 \leqslant M \leqslant n-1$, we have

$$c_1 \leqslant c_2 \leqslant \ldots \leqslant c_M \geqslant c_{M+1} \geqslant \ldots \geqslant c_{n-1}.$$

There is strong numerical evidence to support this conjecture, but a proof seems surprisingly elusive. We do have the following partial result.

PROPOSITION 14.6 (Heron 1972) *Using the above notation for the chromatic polynomial of connected graph* Γ *with* n *vertices, we have*

$$c_{i-1} \leqslant c_i \quad \textit{for all} \quad i \leqslant \tfrac{1}{2}(n-1).$$

Proof The result of Theorem 14.1 leads to the following expression for the coefficients of the chromatic polynomial:

$$c_i = \sum_{l=0}^{i} t_{n-1-l,\,0} \binom{n-1-l}{n-1-i} = \sum_{l=0}^{i} t_{n-1-l,\,0} \binom{n-1-l}{i-l}.$$

Now if $i \leqslant \frac{1}{2}(n-1)$, then $i-l \leqslant \frac{1}{2}(n-1-l)$ for all $l \geqslant 0$. Hence, by the well-known unimodal property of the binomial coefficients, we have

$$\binom{n-1-l}{i-l} \geqslant \binom{n-1-l}{i-1-l} \quad \text{for} \quad i \leqslant \tfrac{1}{2}(n-1), \quad l \geqslant 0.$$

Thus, since each number $t_{n-1-l,\,0}$ is a non-negative integer, it follows that $c_i \geqslant c_{i-1}$ for $i \leqslant \frac{1}{2}(n-1)$, as required. □

14*A* *The chromatic invariants of dual graphs* Let Γ and Γ^* be dual planar connected graphs. Then

$$\theta(\Gamma) = \theta(\Gamma^*).$$

For instance,

$$\theta(Q_3) = \theta(K_{2,2,2}) = 11.$$

14*B* *Some explicit formulae* For the complete graphs K_n, and the graphs M_h, T_h (§9D), we have

$$\begin{aligned} \theta(K_n) &= (n-2)! && (n \geqslant 2), \\ \theta(M_h) &= 2^h - h && (h \geqslant 2), \\ \theta(T_h) &= 2^h - h - 1 && (h \geqslant 3). \end{aligned}$$

14*C* *The flow polynomial* Let $C^*(\Gamma; u)$ denote the number of flows mod u which are zero on no edge of a connected graph Γ (see §4E). Supposing that Γ has n vertices and m edges, we have:

(*a*) $C^*(\Gamma; u) = (-1)^m R(\Gamma; -1, -u)$

$$= (-1)^{m-n+1} T(\Gamma; 0, 1-u).$$

(*b*) If Γ is planar and Γ^* is its dual, then

$$C(\Gamma^*; u) = uC^*(\Gamma; u)$$

(Tutte 1954).

14*D* *The flow polynomials of* $K_{3,3}$ *and* O_3

$$C^*(K_{3,3}; u) = (u-1)(u-2)(u^2-6u+10);$$

$$C^*(O_3; u) = (u-1)(u-2)(u-3)(u-4)(u^2-5u+10).$$

PART THREE

Symmetry and regularity of graphs

15. *General properties of graph automorphisms*

An *automorphism* of a (simple) graph Γ is a permutation π of $V\Gamma$ which has the property that $\{u, v\}$ is an edge of Γ if and only if $\{\pi(u), \pi(v)\}$ is an edge of Γ. We shall take for granted the elementary facts about automorphisms [B, p. 82], including the fact that the set of all automorphisms of Γ, with the operation of composition, is a group. This group is called the *automorphism group* of Γ, and is written $G(\Gamma)$.

The study of automorphisms of graphs illuminates both graph theory and group theory. If a graph has a high degree of symmetry then we may hope to use this fact in the investigation of its graph-theoretical properties; conversely, these properties may be used to determine the structure of its group of automorphisms.

We say that Γ is *vertex-transitive* if $G(\Gamma)$ acts transitively on $V\Gamma$. The action of $G(\Gamma)$ on $V\Gamma$ induces an action on $E\Gamma$, by the rule $\pi\{u, v\} = \{\pi(u), \pi(v)\}$, and we say that Γ is *edge-transitive* if this action is transitive. Our first proposition is typical of the kind of result which follows directly from the definitions.

PROPOSITION 15.1 *If a connected graph* Γ *is edge-transitive but not vertex-transitive, then it is bipartite.*

Proof Let $\{u, v\}$ be an edge of Γ, and let U and V denote the orbits of u and v under the action of $G(\Gamma)$ on $V\Gamma$. Since Γ is edge-transitive and connected any vertex of Γ belongs either to U or to V. If there is some vertex in both U and V, then Γ would be vertex-transitive; consequently $U \cap V$ is empty and every edge of Γ has one end in U and one end in V. That is, Γ is bipartite. $\square$

The graph $K_{a,b}$ with $a \neq b$ is a simple example of a graph which is edge-transitive but not vertex-transitive. There are also regular graphs which are edge-transitive but not vertex-transitive, although examples of this type are by no means easy to find (Bouwer 1972).

One of the most important facts in the study of automorphisms is the link between the spectrum of a graph and its automorphism group; this is a consequence of Proposition 15.2. It enables us to draw very strong conclusions about the spectra of graphs with 'large' automorphism groups. These conclusions can then be applied, with the results of earlier chapters, to the solution of graph-theoretical problems.

We shall suppose that $V\Gamma = \{v_1, v_2, \ldots, v_n\}$ and that the rows and columns of the adjacency matrix of Γ are labelled in the usual way. A permutation π of $V\Gamma$ can be represented by a *permutation matrix* $\mathbf{P} = (p_{ij})$, where $p_{ij} = 1$ if $v_i = \pi(v_j)$, and $p_{ij} = 0$ otherwise.

PROPOSITION 15.2 *Let* $\mathbf{A}$ *be the adjacency matrix of a graph* Γ, *and* π *a permutation of* $V\Gamma$. *Then* π *is an automorphism of* Γ *if and only if* $\mathbf{PA} = \mathbf{AP}$, *where* $\mathbf{P}$ *is the permutation matrix representing* π.

Proof Let $v_h = \pi(v_i)$ and $v_k = \pi(v_j)$. Then we have

$$(\mathbf{PA})_{hj} = \Sigma\, p_{hl} a_{lj} = a_{ij};$$
$$(\mathbf{AP})_{hj} = \Sigma\, a_{hl} p_{lj} = a_{hk}.$$

Consequently, $\mathbf{AP} = \mathbf{PA}$ if and only if v_i and v_j are adjacent whenever v_h and v_k are adjacent; that is, if and only if π is an automorphism of Γ. $\square$

As a general rule, it appears that the size of the automorphism group of Γ is inversely related to the number of distinct eigenvalues of Γ. We shall formulate a result of this kind which is a consequence of the following lemma.

LEMMA 15.3 *Let* λ *be an eigenvalue of* Γ *whose multiplicity is one, and let* $\mathbf{x}$ *be a corresponding eigenvector. If the permutation matrix* $\mathbf{P}$ *represents an automorphism of* Γ *then* $\mathbf{Px} = \pm\mathbf{x}$.

Proof We can choose $\mathbf{x}$ to be a real vector. Since $\mathbf{Ax} = \lambda\mathbf{x}$ and $\mathbf{P}$ represents an automorphism we have

$$\mathbf{APx} = \mathbf{PAx} = \mathbf{P}\lambda\mathbf{x} = \lambda\mathbf{Px}.$$

Thus $\mathbf{Px}$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ. Now, λ has multiplicity one, so $\mathbf{x}$ and $\mathbf{Px}$ are linearly dependent; that is $\mathbf{P}\mathbf{x} = \mu\mathbf{x}$ for some complex number μ. Since $\mathbf{x}$ and $\mathbf{P}$ are real, μ is real; and, since $\mathbf{P}^s = \mathbf{I}$ for some natural number $s \geqslant 1$, it follows that $|\mu| = 1$. Consequently $\mu = \pm 1$ and the lemma is proved. □

THEOREM 15.4 (Mowshowitz 1969, Petersdorf and Sachs 1969) *If the graph* Γ *has an automorphism whose order is greater than two, then some eigenvalue of* Γ *has multiplicity greater than one.*

Proof Suppose that each eigenvalue of Γ has multiplicity one. Then, by Lemma 15.3, $\mathbf{P}^2\mathbf{x} = \mathbf{x}$ for each $\mathbf{P}$ representing an automorphism of Γ and eigenvector $\mathbf{x}$. But the space spanned by the eigenvectors is the whole space of column vectors, and so $\mathbf{P}^2 = \mathbf{I}$. Thus, if all the eigenvalues are simple, every non-identity automorphism has order two. □

In Chapter 2 we showed that, if Γ has n vertices and diameter d, then the number of distinct eigenvalues of Γ is bounded by $d+1$ and n. Theorem 15.4 describes the group of a graph for which the upper bound is attained: every element of the group is an involution, and so the group is an elementary Abelian 2-group. In Chapter 20 we shall study the remarkable properties of graphs for which the lower bound is attained.

In its most general form, the question of a relationship between the structure of a graph and the structure of its group of automorphisms has a rather disappointing answer: there is no relationship. Frucht (1938) was the first to prove that, for every abstract finite group G, there is some graph Γ whose automorphism group is abstractly isomorphic to G. His work has been extended by several authors to show that the conclusion remains true, even if we specify in advance that Γ must satisfy a number of graph-theoretical conditions.

However, if we strengthen the question by asking whether every permutation group is *equivalent* [B, p.15] to the automorphism group of some graph, then the answer is negative, as we shall see in the next chapter. The example we shall give is of a transitive permutation group which is not equivalent to the group of any (vertex-transitive) graph. This tends to confirm our intuitive impression that there must be some constraints upon the possible symmetry of graphs.

We now introduce some conditions whose implications will be investigated in subsequent chapters. If Γ is a connected graph, then $\partial(u, v)$ will denote the distance in Γ between the vertices u and v; for each automorphism g in $G(\Gamma)$ we have

$$\partial(u, v) = \partial(g(u), g(v)).$$

DEFINITION 15.5 Let Γ be a graph with automorphism group $G(\Gamma)$. Then:

(1) Γ is *symmetric* if, for all vertices u, v, x, y of Γ such that u and v are adjacent, and x and y are adjacent, there is some automorphism g in $G(\Gamma)$ satisfying $g(u) = x$ and $g(v) = y$.

(2) Γ is *distance-transitive* if, for all vertices u, v, x, y of Γ such that $\partial(u, v) = \partial(x, y)$, there is some automorphism g in $G(\Gamma)$ satisfying $g(u) = x$ and $g(v) = y$.

It should be remarked that the second condition applies only to connected graphs, and that in the case of a connected graph we have a hierarchy of conditions:

$$\text{distance-transitive} \Rightarrow \text{symmetric} \Rightarrow \text{vertex-transitive}.$$

We shall investigate these conditions in turn, beginning with the weakest one.

15*A* *Graphs with a given group* Let an abstract finite group G and natural numbers r and s satisfying $r \geqslant 3$, $2 \leqslant s \leqslant r$ be given. Then there are infinitely many graphs Γ with the properties:

(*a*) $G(\Gamma) \approx G$,

(*b*) Γ is regular of valency r,

(*c*) $\nu(\Gamma) = s$

(Izbicki 1960).

15*B* *The graphs* $P(h, t)$ The graph $P(h, t)$ is a trivalent graph with $2h$ vertices $x_0, x_1, \ldots, x_{h-1}, y_0, y_1, \ldots, y_{h-1}$, and edges $\{x_i, y_i\}$, $\{x_i, x_{i+1}\}$, $\{y_i, y_{i+t}\}$, for all $i \in \{0, 1, \ldots, h-1\}$, where the subscripts are reduced modulo h.

We have:

(*a*) $P(h,t)$ is vertex-transitive if and only if $t^2 \equiv \pm 1 (\bmod h)$, or $(h,t) = (10,2)$.

(*b*) $P(h,t)$ is symmetric if and only if (h,t) is one of (4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5).

(*c*) $P(h,t)$ is distance-transitive if and only if (h,t) is one of (4, 1), (5, 2), (10, 3) (Frucht, Graver and Watkins 1971).

15*C* *The automorphism groups of trees* Let $\mathscr{G}$ denote the class of permutation groups constructed according to the following rules:

(*a*) $\mathscr{G}$ contains all the symmetric groups.

(*b*) $\mathscr{G}$ is closed under the operation of taking direct products.

(*c*) $\mathscr{G}$ is closed under the operation of taking wreath products.

Then the automorphism group of any tree is equivalent to a member of $\mathscr{G}$ (Pólya 1937).

15*D* *Homogeneous graphs* A graph Γ is said to be *homogeneous* if, whenever two subsets U_1, U_2 of $V\Gamma$ are such that $\langle U_1 \rangle$ and $\langle U_2 \rangle$ are isomorphic, then there is an automorphism of Γ taking U_1 to U_2. A homogeneous graph is a distance-transitive graph with diameter at most 2; consequently its girth is at most 5. The only homogeneous graph of girth five is C_5, and the only homogeneous graphs of girth four are the graphs $K_{s,s}$ $(s \geqslant 2)$. The only known graphs of girth three which are homogeneous are:

(*a*) the complete graphs K_n $(n \geqslant 3)$;

(*b*) the complete multipartite graphs $K_{s,s,\dots,s}$ with t $(\geqslant 3)$ parts;

(*c*) the line graph $L(K_{3,3})$ (Sheehan).

16. *Vertex-transitive graphs*

In this chapter we study graphs Γ for which the automorphism group $G(\Gamma)$ acts transitively on $V\Gamma$. We recall the standard results [B, Chapter 1] on transitive permutation groups, among them the following.

Let $G = G(\Gamma)$ and let G_v denote the *stabilizer* subgroup of the vertex v; that is, the subgroup of G which fixes v. Then all stabilizer subgroups G_v $(v \in V\Gamma)$ are conjugate in G, and consequently isomorphic. For the index of G_v in G, we have

$$|G : G_v| = |G|/|G_v| = |V\Gamma|.$$

If each stabilizer G_v $(v \in V\Gamma)$ is the identity group, then no member of G fixes any vertex, and we say that G acts *regularly* on $V\Gamma$. In this case, the order of G is equal to the number of vertices.

There is a standard construction, due originally to Cayley (1878), which enables us to construct many, but not all, vertex-transitive graphs. We give a streamlined version which has proved to be well-adapted to the needs of algebraic graph theory. Let G be any abstract finite group, with identity 1; and suppose Ω is a set of generators for G, with the properties:

$$(*) \quad x \in \Omega \Rightarrow x^{-1} \in \Omega; \quad 1 \notin \Omega.$$

DEFINITION 16.1 The *Cayley graph* $\Gamma = \Gamma(G, \Omega)$ is the simple graph whose vertex-set and edge-set are

$$V\Gamma = G; \quad E\Gamma = \{\{g, h\} \,|\, g^{-1}h \in \Omega\}.$$

Simple verifications show that $E\Gamma$ is well-defined, and that $\Gamma(G, \Omega)$ is a connected graph.

PROPOSITION 16.2 (1) *The Cayley graph $\Gamma(G, \Omega)$ is vertex-transitive.*

(2) *Suppose that π is an automorphism of the group G such that $\pi(\Omega) = \Omega$. Then π, regarded as a permutation of the vertices of $\Gamma(G, \Omega)$, is a graph automorphism fixing the vertex* 1.

Proof (1) For each g in G we may define a permutation $\bar{g}$ of $V\Gamma = G$ by the rule $\bar{g}(h) = gh$ $(h \in G)$. This permutation is an automorphism of Γ, for

$$\begin{aligned} \{h,k\} \in E\Gamma &\Rightarrow h^{-1}k \in \Omega \\ &\Rightarrow (gh)^{-1}gk \in \Omega \\ &\Rightarrow \{\bar{g}(h), \bar{g}(k)\} \in E\Gamma. \end{aligned}$$

The set of all $\bar{g}$ $(g \in G)$ constitutes a group $\bar{G}$ (isomorphic with G), which is a subgroup of the full group of automorphisms of $\Gamma(G,\Omega)$, and acts transitively on the vertices.

(2) Since π is a group automorphism it must fix the vertex 1. Further, π is a graph automorphism, since

$$\begin{aligned} \{h,k\} \in E\Gamma &\Rightarrow h^{-1}k \in \Omega \\ &\Rightarrow \pi(h^{-1}k) \in \Omega \\ &\Rightarrow \pi(h)^{-1}\pi(k) \in \Omega \\ &\Rightarrow \{\pi(h), \pi(k)\} \in E\Gamma. \quad \square \end{aligned}$$

The second part of this proposition implies that the automorphism group of a Cayley graph $\Gamma(G,\Omega)$ will often be strictly larger than $\bar{G}$. For example, if G is the symmetric group S_3, and $\Omega = \{(12), (23), (13)\}$, then the Cayley graph $\Gamma(G,\Omega)$ is isomorphic to $K_{3,3}$ (Fig. 6).

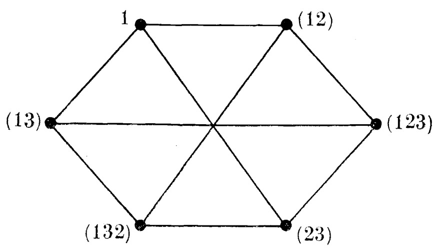


Fig. 6

Since every group automorphism of S_3 fixes Ω setwise, it follows that the stabilizer of the vertex 1 has order at least 6. (In fact, the order of the stabilizer is 12.)

Not every vertex-transitive graph is a Cayley graph; the simplest counter-example is Petersen's graph. (This statement can be checked directly by examining the small class of pairs (G, Ω) satisfying (*) with $|G| = 10$ and $|\Omega| = 3$.) However, a slightly more general construction will yield all vertex-transitive graphs. There is, in addition, a similar construction which uses, instead of an abstract group and a generating set, a permutation group and one of its orbits [B, p. 83].

We begin our study of the hierarchy of symmetry conditions with the minimal case: that is, when $G(\Gamma)$ acts regularly on $V\Gamma$.

LEMMA 16.3 *Let Γ be a connected graph. The automorphism group $G(\Gamma)$ has a subgroup H which acts regularly on $V\Gamma$ if and only if Γ is a Cayley graph $\Gamma(H, \Omega)$, for some set Ω generating H.*

Proof Suppose $V\Gamma = \{v_1, v_2, \ldots, v_n\}$, and H is a subgroup of $G(\Gamma)$ acting regularly on $V\Gamma$. Then, for $1 \leqslant i \leqslant n$, there is a unique $h_i \in H$ such that $h_i(v_1) = v_i$. Let

$$\Omega = \{h_i \in H \,|\, v_i \text{ is adjacent to } v_1 \text{ in } \Gamma\}.$$

Simple checks show that Ω satisfies the two conditions (*) required by Definition 16.1 and that the bijection $v_i \leftrightarrow h_i$ is a graph isomorphism of Γ with $\Gamma(H, \Omega)$.

Conversely if $\Gamma = \Gamma(H, \Omega)$ then the group $\bar{H}$ defined in the proof of Proposition 16.2 acts regularly on $V\Gamma$, and $\bar{H} \approx H$. □

Lemma 16.3 shows that if $G(\Gamma)$ itself acts regularly on $V\Gamma$, then Γ is a Cayley graph $\Gamma(G(\Gamma), \Omega)$. The next definition will facilitate discussion of this case.

DEFINITION 16.4 A finite abstract group G admits a *graphical regular representation* if there is a graph Γ with $G \approx G(\Gamma)$ and $G(\Gamma)$ acting regularly on $V\Gamma$.

We can easily show that the group S_3 admits no graphical regular representation. For, if there were an admissible graph Γ, then it would be a Cayley graph $\Gamma(S_3, \Omega)$. Now, for any generating set Ω, satisfying (*), there is some automorphism of S_3 fixing Ω setwise; thus, by part (2) of Proposition 16.2, the automorphism group of a Cayley graph $\Gamma(S_3, \Omega)$ is strictly larger than S_3. This example suffices to show that not every permutation group is equivalent to the group of a graph.

In the case of transitive Abelian groups, precise information is provided by the next proposition.

PROPOSITION 16.5 *Let Γ be a vertex-transitive graph whose automorphism group $G = G(\Gamma)$ is Abelian. Then G acts regularly on $V\Gamma$, and G is an elementary Abelian 2-group.*

Proof Any transitive Abelian group G is regular. For, if $g, h \in G$, and g fixes v, then $gh(v) = hg(v) = h(v)$ so that g fixes $h(v)$ also. Since G is transitive, g fixes everything; that is, $g = 1$.

Thus $G(\Gamma)$ acts regularly on $V\Gamma$ and so, by Lemma 16.3, Γ is a Cayley graph $\Gamma(G, \Omega)$. Now since G is Abelian, the function $g \mapsto g^{-1}$ is an automorphism of G, and it fixes Ω setwise. If this automorphism were non-trivial, then part (2) of Proposition 16.2 would imply that G is not regular. Thus $g = g^{-1}$ for all $g \in G$, and every element of G has order 2. $\square$

Proposition 16.5 shows that the only Abelian groups which can admit graphical regular representations are the groups $(\mathbb{Z}_2)^s$. In fact, it has been proved that all values of s can occur, with the exception of $s = 2, 3, 4$ (Imrich 1970).

We now turn to a discussion of some simple spectral properties of vertex-transitive graphs. A vertex-transitive graph Γ is necessarily a regular graph, and so its spectrum has the properties which are stated in Proposition 3.1. In particular, if Γ is connected and k-valent, then k is a simple eigenvalue of Γ. We can use the vertex-transitivity property to characterize all the simple eigenvalues of Γ.

PROPOSITION 16.6 (Petersdorf and Sachs 1969) *Let Γ be a vertex-transitive graph which has valency k, and let λ be a simple eigenvalue of Γ. If $|V\Gamma|$ is odd, then $\lambda = k$. If $|V\Gamma|$ is even, then λ is one of the integers $2\alpha - k$ $(0 \leqslant \alpha \leqslant k)$.*

Proof Let $\mathbf{x}$ be a real eigenvector corresponding to the simple eigenvalue λ, and let $\mathbf{P}$ be a permutation matrix representing an automorphism π of Γ. If $\pi(v_i) = v_j$, then, by Lemma 15.3,

$$x_i = (\mathbf{Px})_j = \pm x_j.$$

Since Γ is vertex-transitive, we deduce that all the entries of $\mathbf{x}$ have the same absolute value.

Now, since $\mathbf{u} = [1, 1, \ldots, 1]^t$ is an eigenvector corresponding to the eigenvalue k, if $\lambda \neq k$ we must have $\mathbf{u}^t\mathbf{x} = 0$, that is

$$\Sigma x_i = 0.$$

This is impossible for an odd number of summands of equal absolute value, and so our first statement is proved.

If Γ has an even number of vertices, choose a vertex v_i of Γ and suppose that, of the vertices v_j adjacent to v_i, a number α have $x_j = x_i$ while $k-\alpha$ have $x_j = -x_i$. Since $(\mathbf{Ax})_i = \lambda x_i$ it follows that

$$\Sigma' x_j = \alpha x_i - (k-\alpha) x_i = \lambda x_i,$$

whence $\lambda = 2\alpha - k$. □

For example, the only numbers which can be simple eigenvalues of a trivalent vertex-transitive graph are 3, 1, -1, -3. This statement is false if we assume merely that the graph is regular of valency 3.

If we strengthen our assumptions by postulating that Γ is symmetric, then the simple eigenvalues are restricted still further.

PROPOSITION 16.7 *Let Γ be a symmetric graph of valency k, and let λ be a simple eigenvalue of Γ. Then $\lambda = \pm k$.*

Proof We continue with the notation of the previous proof. Let v_j and v_l be any two vertices adjacent to v_i; then there is an automorphism π of Γ such that $\pi(v_i) = v_i$ and $\pi(v_j) = v_l$. If $\mathbf{P}$ is the permutation matrix representing π, then $\pi(v_i) = v_i$ implies that $\mathbf{Px} = \mathbf{x}$, and so $x_j = x_l$. Thus $\alpha = 0$ or k, and $\lambda = \pm k$. □

We remark that the eigenvalue $-k$ occurs, and is necessarily simple, if and only if Γ is bipartite.

16*A* *Circulant graphs* A circulant graph is vertex-transitive. A vertex-transitive graph with a prime number of vertices must be a circulant graph.

16*B* *Cayley graphs for the polyhedral groups* There are three special finite groups of rotations in $\mathbb{R}^3$, which are known as the tetrahedral, octahedral, and icosahedral groups; each can be represented by a planar Cayley graph. A Cayley graph for the tetrahedral group (the alternating group A_4) is shown in Fig. 7.

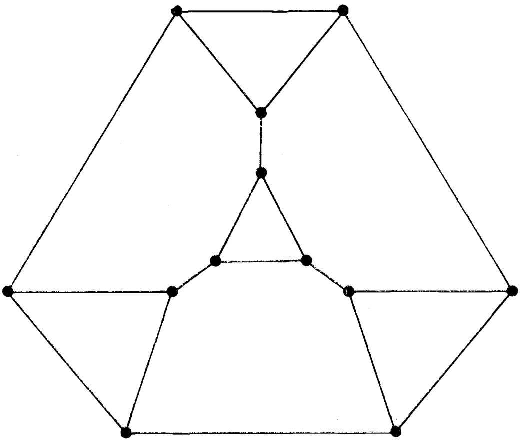


Fig. 7

16*C* *A generalization of Cayley's construction* Let G be an abstract finite group, H a subgroup of G, and Ω a subset of $G-H$ such that (*) is satisfied and $H \cup \Omega$ generates G. Then we may define a general graph $\Gamma(G, H, \Omega)$ whose vertices are the right cosets of H in G, with Hg_1 and Hg_2 being adjacent whenever $g_1 g_2^{-1}$ is in Ω. The graph $\Gamma(G, H, \Omega)$ is connected and vertex-transitive.

16*D* *Graphical regular representations* (*a*) The dihedral group of order $2r$ has a graphical regular representation if and only if $r \neq 3, 4, 5$ (Watkins 1971).

(*b*) If G is non-Abelian and $|G|$ is not divisible by 2 or 3, then G has a graphical regular representation (Nowitz and Watkins 1972).

17. *Symmetric graphs*

The condition of vertex-transitivity is not a very powerful one, as is demonstrated by the fact that we can construct at least one vertex-transitive graph from each finite group, by means of the Cayley graph construction. The condition that a graph be symmetric is apparently only slightly stronger, for a vertex-transitive graph is symmetric if and only if each vertex-stabilizer G_v is transitive on the set of vertices adjacent to v. For example, there are just two distinct connected trivalent graphs with 6 vertices; one is $K_{3,3}$ and the other is the prism T_3 (as defined in §9D). Both these graphs are vertex-transitive, but only $K_{3,3}$ is symmetric.

It is strange that symmetric graphs do have distinctive properties which are not shared by all vertex-transitive graphs. This was first demonstrated by Tutte (1947) in the case of trivalent graphs.

We begin by defining a *t-arc* in a graph Γ to be an ordered set $[\alpha] = (\alpha_0, \alpha_1, \ldots, \alpha_t)$ of $t+1$ vertices of Γ, with the properties that $\{\alpha_{i-1}, \alpha_i\}$ is in $E\Gamma$ for $1 \leqslant i \leqslant t$, and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leqslant i \leqslant t-1$. In other words, a t-arc is the ordered set of vertices underlying a path of length t. It is sometimes convenient to regard a single vertex v as a 0-arc $[v]$. If $\beta = (\beta_0, \beta_1, \ldots, \beta_s)$ is an s-arc in Γ, then we write $[\alpha . \beta]$ for the ordered set of vertices $(\alpha_0, \ldots, \alpha_t, \beta_0, \ldots, \beta_s)$, provided that this is a $(t+s+1)$-arc; that is, provided α_t is adjacent to β_0 and $\alpha_{t-1} \neq \beta_0$, $\alpha_t \neq \beta_1$.

The concept defined in the following definition will play a central role in our studies of symmetric graphs.

DEFINITION 17.1 A graph Γ is *t-transitive* ($t \geqslant 1$) if its automorphism group $G(\Gamma)$ is transitive on the set of t-arcs in Γ, but not transitive on the set of $(t+1)$-arcs in Γ.

(There is little risk of confusion with the concept of multiple transitivity used in the general theory of permutation groups, since the only graphs which are multiply transitive in that sense are the complete graphs.)

The automorphism group $G(\Gamma)$ is transitive on 1-arcs if and only if Γ is symmetric (since a 1-arc is just a pair of adjacent vertices). Consequently, a symmetric graph is t-transitive for some $t \geqslant 1$.

The only connected graph of valency one is K_2, and this graph is 1-transitive. The only connected graphs of valency two are the circuit graphs C_n ($n \geqslant 3$), and these are anomalous in that they are transitive on t-arcs for all $t \geqslant 1$. From now on, we shall usually assume that the graphs under consideration have valency not less than three. For such graphs we have the following elementary inequality.

PROPOSITION 17.2 *Let Γ be a t-transitive graph whose valency is at least three and whose girth is g. Then*

$$g \geqslant 2t-2.$$

Proof Γ contains a circuit of length g and (it is easy to see) a non-closed path of length g. In both cases there is a corresponding g-arc in Γ, and no automorphism of Γ can take a g-arc of the first kind to a g-arc of the second kind; thus $t < g$.

Consequently, if we select a circuit of length g in Γ, then there is a t-arc $[\alpha]$, without repeated vertices, contained in it. Let $[\beta]$ be the $(g-t)$-arc beginning at α_t and ending at α_0 which completes the circuit of length g. Also let v be a vertex adjacent to α_{t-1}, but which is not α_{t-2} or α_t; this situation is depicted in Fig. 8.

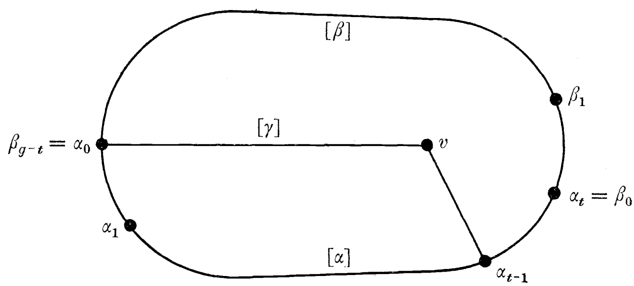


Fig. 8

Since Γ is t-transitive, there is an automorphism taking the t-arc $[\alpha]$ to the t-arc $(\alpha_0, \alpha_1, \ldots, \alpha_{t-1}, v)$. This automorphism must take the $(g-t+1)$-arc $[\alpha_{t-1}.\beta]$ to another $(g-t+1)$-arc $[\alpha_{t-1}.\gamma]$, where $\gamma_0 = v$ and $\gamma_{g-t} = \alpha_0$. The two arcs $[\alpha_{t-1}.\beta]$ and $[\alpha_{t-1}.\gamma]$ may overlap, but together they underly a circuit of length at most $2(g-t+1)$. Hence $g \leqslant 2(g-t+1)$, and $g \geqslant 2t-2$. □

DEFINITION 17.3 Let $[\alpha]$ and $[\beta]$ be any two s-arcs in a graph Γ. We say that $[\beta]$ is a *successor* of $[\alpha]$ if $\beta_i = \alpha_{i+1}$ $(0 \leqslant i \leqslant s-1)$.

It is helpful to think of the operation of taking a successor of $[\alpha]$ in terms of 'shunting' $[\alpha]$ through one step in Γ. Suppose we ask whether repeated shunting will transform one s-arc into any other. If there are vertices of valency one in Γ then our shunting might be halted in a 'siding', while if all vertices have valency two we cannot reverse the direction of our 'train'. However, if each vertex of Γ has valency not less than three, and Γ is connected, then our intuition is correct and the shunting procedure is universally adequate.

LEMMA 17.4 *Let Γ be a connected graph in which the valency of each vertex is at least three. If $s \geqslant 1$ and $[\alpha]$, $[\beta]$ are any two s-arcs in Γ, then there is a finite sequence $[\alpha^{(i)}]$ $(1 \leqslant i \leqslant l)$ of s-arcs in Γ such that $[\alpha^{(1)}] = [\alpha]$, $[\alpha^{(l)}] = [\beta]$, and $[\alpha^{(i+1)}]$ is a successor of $[\alpha^{(i)}]$ for $1 \leqslant i \leqslant s-1$.*

Proof The proof of this is not deep, but it does require careful examination of several different cases. We refer the reader to the book by Tutte (1966, pp. 56–58) for details. □

We can now state and prove a convenient test for t-transitivity. Let Γ be a connected graph in which the valency of each vertex is at least three, and let $[\alpha]$ be a t-arc in Γ. Suppose the vertices adjacent to α_t are α_{t-1} and $v^{(1)}, v^{(2)}, \ldots, v^{(l)}$ (Fig. 9), and let $[\beta^{(i)}]$ denote the t-arc $(\alpha_1, \alpha_2, \ldots, \alpha_t, v^{(i)})$ for $1 \leqslant i \leqslant l$, so that each $[\beta^{(i)}]$ is a successor of $[\alpha]$.

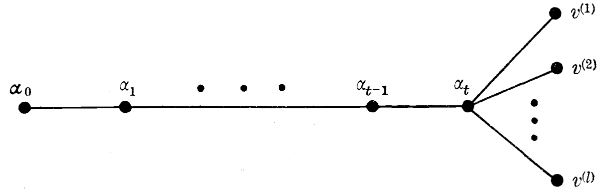


Fig. 9

THEOREM 17.5 *With the above notation, $G(\Gamma)$ is transitive on t-arcs if and only if there are automorphisms $g_1, g_2, \ldots, g_l$ in $G(\Gamma)$ such that $g_i[\alpha] = [\beta^{(i)}]$ $(1 \leqslant i \leqslant l)$.*

Proof The condition is clearly satisfied if $G(\Gamma)$ is transitive on t-arcs. Conversely, suppose the relevant automorphisms $g_1, g_2, \ldots, g_l$ can be found; then they generate a subgroup $H = \langle g_1, g_2, \ldots, g_l\rangle$ of $G(\Gamma)$, and we shall show that H is transitive on t-arcs.

Let $[\theta]$ be a t-arc in the orbit of $[\alpha]$ under H; thus $[\theta] = h[\alpha]$ for some $h \in H$. If $[\phi]$ is any successor of $[\theta]$, then $h^{-1}[\phi]$ is a successor of $[\alpha]$, and so $[\phi] = hg_i[\alpha]$ for some $i \in \{1, 2, \ldots, l\}$. That is, $[\phi]$ is also in the orbit of $[\alpha]$ under H. Now Lemma 17.4 tells us that all t-arcs can be obtained from $[\alpha]$ by repeatedly taking successors, and so all t-arcs are in the orbit of $[\alpha]$ under H. □

As an example, we consider Petersen's graph O_3, whose vertices are the unordered pairs from the set $\{1, 2, 3, 4, 5\}$ with disjoint pairs being adjacent. Any permutation of $\{1, 2, 3, 4, 5\}$ induces an automorphism of the graph. The girth of O_3 is five, so that Proposition 17.2 tells us that the graph is at most 3-transitive. Consider the 3-arc $[\alpha] = (12, 34, 15, 23)$, which has two successors: $[\beta^{(1)}] = (34, 15, 23, 14)$ and $[\beta^{(2)}] = (34, 15, 23, 45)$. The automorphism $(13)(245)$ takes $[\alpha]$ to $[\beta^{(1)}]$ and the automorphism (13524) takes $[\alpha]$ to $[\beta^{(2)}]$, hence O_3 is 3-transitive.

In addition to its usefulness as a test for t-transitivity, Theorem 17.5 is also the basis for theoretical investigations into the structure of t-transitive graphs. Suppose that Γ is a connected t-transitive graph $(t \geqslant 1)$; Γ is regular, of valency k, say. Let $[\alpha]$ be a given t-arc in Γ.

DEFINITION 17.6 The *stabilizer sequence* of $[\alpha]$ in Γ is the sequence

$$G(\Gamma) = G > F_t > F_{t-1} > \ldots > F_1 > F_0$$

of subgroups of $G(\Gamma)$, where F_i $(0 \leqslant i \leqslant t)$ is defined to be the pointwise stabilizer of the set $\{\alpha_0, \alpha_1, \ldots, \alpha_{t-i}\}$.

Since G is transitive on s-arcs $(1 \leqslant s \leqslant t)$ it follows that all stabilizer sequences of t-arcs are conjugate in G, and consequently we shall often omit explicit reference to $[\alpha]$.

The order of each group occurring in the stabilizer sequence is determined by the order of F_0, as follows. Since F_t is the stabilizer of the single vertex α_0 in the vertex-transitive group G, it follows that $|G: F_t| = n = |V\Gamma|$. Since G is transitive on 1-arcs, F_t acts transitively on the k vertices adjacent to α_0, and F_{t-1} is the stabilizer of the vertex α_1 in this action; consequently $|F_t: F_{t-1}| = k$. Since G is transitive on s-arcs ($2 \leqslant s \leqslant t$), the group F_{t-s+1} acts transitively on the $k-1$ vertices adjacent to α_{s-1} (other than α_{s-2}), and F_{t-s} is the stabilizer of the vertex α_s in this action; consequently $|F_{t-s+1}: F_{t-s}| = k-1$ for $2 \leqslant s \leqslant t$. Thus we have

$$\begin{aligned} |F_s| &= (k-1)^s\,|F_0| \quad (0 \leqslant s \leqslant t-1), \\ |F_t| &= k(k-1)^{t-1}\,|F_0|, \\ |G| &= nk(k-1)^{t-1}\,|F_0|. \end{aligned}$$

In the case of Petersen's graph, discussed above, we have $t = 3$, and it is easy to see that $|F_0| = 1$. Thus $|F_1| = 2$, $|F_2| = 6$, $|F_3| = 12$ and $|G| = 120$.

The properties of the stabilizer sequence can be conveniently discussed in terms of the set $\{g_1, g_2, \ldots, g_l\}$ of $l = k-1$ automorphisms whose existence is guaranteed by Theorem 17.5. We define an increasing sequence of subsets of G, denoted by $\{1\} = Y_0 \subseteq Y_1 \subseteq Y_2 \subseteq \ldots$, as follows:

$$Y_i = \{g_a^{-j} g_b^j \mid a, b \in \{1, 2, \ldots, l\} \text{ and } 1 \leqslant j \leqslant i\}.$$

PROPOSITION 17.7 (1) *If $1 \leqslant i \leqslant t$, then Y_i is a subset of F_i, but not of F_{i-1}.*

(2) *If $0 \leqslant i \leqslant t$, then F_i is the subgroup of G generated by Y_i and F_0.*

Proof (1) For $1 \leqslant a \leqslant l$, we have $g_a^r(\alpha_j) = \alpha_{j+r}$, provided that both j and $j+r$ lie between 0 and t. Also, $g_a^{t-j+1}(\alpha_j) = v^{(a)}$. It follows that $g_a^{-j} g_b^j$ fixes $\alpha_0, \alpha_1, \ldots, \alpha_{t-i}$ for all $j \leqslant i$, and so $Y_i \subseteq F_i$. If it were true that $Y_i \subseteq F_{i-1}$, then $g_a^{-i} g_b^i$ would fix α_{t-i+1}; but this means that $g_a^i(\alpha_{t-i+1}) = g_b^i(\alpha_{t-i+1})$, that is $v^{(a)} = v^{(b)}$. Since this is false for $a \neq b$, we have $Y_i \nsubseteq F_{i-1}$.

(2) Suppose $f \in F_i$, and $f[\alpha] = (\alpha_0, \alpha_1, \ldots, \alpha_{t-i}, \gamma_1, \ldots, \gamma_i)$. Pick

any g_b; since γ_1 is adjacent to α_{t-i}, $g_b^i(\gamma_1)$ is adjacent to $g_b^i(\alpha_{t-i}) = \alpha_t$, and so $g_b^i(\gamma_1) = v^{(a)}$ for some $a \in \{1, 2, \ldots, l\}$. Then

$$g_a^{-i} g_b^i f[\alpha] = (\alpha_0, \alpha_1, \ldots, \alpha_{t-i+1}, \delta_2, \ldots, \delta_i) \quad \text{say.}$$

By applying the same method with i replaced by $i-1$, we can find an automorphism $g_c^{-(i-1)} g_d^{i-1}$, which belongs to Y_{i-1} and Y_i, and takes δ_2 to α_{t-i+2} while fixing $\alpha_0, \alpha_1, \ldots, \alpha_{t-i+1}$. Continuing, we construct g in Y_i such that $gf[\alpha] = [\alpha]$, that is, gf is in F_0. Consequently f is in the group generated by Y_i and F_0. Conversely, both Y_i and F_0 are contained in F_i, so we have the result. □

All members of the sets $Y_0, Y_1, \ldots, Y_t$ fix the vertex α_0 and so belong to F_t, the stabilizer of α_0; further, we have shown that F_t is generated by Y_t and F_0. If we pass on to Y_{t+1}, then this set contains some automorphisms not fixing α_0, and we may ask whether Y_{t+1} and F_0 suffice to generate the entire automorphism group G. The following proposition shows that the answer is 'yes', with an important exception.

The exception is in the case of bipartite graphs. If Γ is a symmetric bipartite graph in which $V\Gamma$ is partitioned into two colour-classes V_1 and V_2, then the automorphisms which fix V_1 and V_2 setwise form a subgroup of index two in $G(\Gamma)$. We say that this subgroup *preserves the bipartition.*

PROPOSITION 17.8 *Let Γ be a t-transitive graph with $t \geqslant 2$ and girth greater than 3. Let G^* denote the subgroup of $G = G(\Gamma)$ generated by Y_{t+1} and F_0. Then either*

(1) $G^* = G$;

or (2) *Γ is bipartite, $|G\colon G^*| = 2$, and G^* is the subgroup of G preserving the bipartition.*

Proof Let u be any vertex of Γ such that $\partial(u, \alpha_0) = 2$; we show first that there is some g^* in G^* taking α_0 to u.

Since the girth of Γ is greater than 3, the vertices $v^{(a)} = g_a^{t+1}(\alpha_0)$ and $v^{(b)} = g_b^{t+1}(\alpha_0)$ satisfy $\partial(v^{(a)}, v^{(b)}) = 2$. Consequently, the distance between α_0 and $g_a^{-(t+1)} g_b^{t+1}(\alpha_0)$ is also 2. Now G^* contains F_t (since the latter is generated by Y_t, which is a subset of Y_{t+1}, and F_0), and F_t is transitive on the 2-arcs which begin at α_0 (since $t \geqslant 2$). Thus G^* contains an automorphism f fixing α_0 and taking $g_a^{-(t+1)} g_b^{t+1}(\alpha_0)$ to u, and $g^* = f g_a^{-(t+1)} g_b^{t+1}$ takes α_0 to u.

Let U denote the orbit of α_0 under the action of G^*. U contains all vertices whose distance from α_0 is two, and consequently all vertices whose distance from α_0 is even. If $U = V\Gamma$, then G^* is transitive on $V\Gamma$, and since it contains F_t, the vertex-stabilizer $(G^*)_{\alpha_0}$ is F_t. Thus $|G^*| = |V\Gamma|\,|F_t| = |G|$, and so $G^* = G$. If $U \neq V$, then U consists precisely of those vertices whose distance from α_0 is even, and Γ is bipartite, with colour-classes U and $V\Gamma - U$. Since G^* fixes these colour-classes setwise, G^* is the subgroup of G preserving the bipartition. □

We remark that a graph of girth three whose automorphism group is transitive on 2-arcs is necessarily a complete graph. Thus the girth constraint in Proposition 17.8 is not very restrictive.

In the next chapter we shall specialize the results of Propositions 17.7 and 17.8 to trivalent graphs, and give examples; our results will lead to very precise information about the stabilizer sequence. The case of valency $p+1$ (p prime) has recently been treated by Gardiner (1973), but the general case appears to present very great difficulties.

17*A* *The odd graphs again* The odd graphs O_k (defined in §8*E*) are 3-transitive, for all $k \geqslant 3$. The stabilizer sequence for O_k is

$$G = S_{2k-1},\quad F_3 = S_k \times S_{k-1},\quad F_2 = S_{k-1} \times S_{k-1},$$
$$F_1 = S_{k-1} \times S_{k-2},\quad F_0 = S_{k-2} \times S_{k-2}.$$

17*B* *A set of generators* Let Γ be a connected symmetric graph and let G_v denote the stabilizer of the vertex v. If h is an automorphism of Γ for which $\partial(v, h(v)) = 1$, then h and G_v generate $G = G(\Gamma)$.

17*C* *The double star stabilizer* Let u and v be adjacent vertices of a t-transitive graph Γ ($t \geqslant 2$), and let L denote the subgroup of $G(\Gamma)$ which fixes the vertices u, v, and all vertices adjacent to either of them. Then L is a p-group, for some prime p (Gardiner 1973).

18. *Trivalent symmetric graphs*

Suppose that Γ is a t-transitive graph; then, by definition, $G(\Gamma)$ is transitive on the t-arcs of Γ but not transitive on the $(t+1)$-arcs of Γ. We now show that, in the trivalent case, $G(\Gamma)$ acts regularly on the t-arcs.

PROPOSITION 18.1 *Let $[\alpha]$ be a t-arc in a trivalent t-transitive graph Γ. Then an automorphism of Γ which fixes $[\alpha]$ must be the identity.*

Proof Suppose f is an automorphism fixing $\alpha_0, \alpha_1, \ldots, \alpha_t$; if f is not the identity, then f does not fix all t-arcs in Γ. Thus, it follows from Lemma 17.4 that there is some t-arc $[\beta]$ such that f fixes $[\beta]$, but f does not fix both successors of $[\beta]$. In fact, if β_{t-1}, $u^{(1)}$, $u^{(2)}$ are the vertices adjacent to β_t, then f must interchange $u^{(1)}$ and $u^{(2)}$. Let $w \neq \beta_1$ be a vertex adjacent to β_0. Since Γ is t-transitive there is an automorphism $h \in G(\Gamma)$ taking the t-arc $(w, \beta_0, \ldots, \beta_{t-1})$ to $[\beta]$, and we may suppose the notation chosen so that $h(\beta_t) = u^{(1)}$. Then h and fh are automorphisms of Γ taking the $(t+1)$-arc $[w \,.\, \beta]$ to its two successors, and, by Theorem 17.5, $G(\Gamma)$ is transitive on $(t+1)$-arcs. This contradicts our hypothesis, and so we must have $f = 1$. $\square$

From now on we shall suppose that we are dealing with a trivalent t-transitive graph Γ, and that we have chosen a t-arc $[\alpha]$ in Γ. If the stabilizer sequence of this t-arc is $G > F_t > F_{t-1} > \ldots > F_0$, then Proposition 18.1 implies that $|F_0| = 1$. Consequently we know the orders of all the groups in the sequence:

$$\begin{aligned} |F_i| &= 2^i \qquad &(0 \leqslant i \leqslant t-1), \\ |F_t| &= 3 \times 2^{t-1}, & \\ |G| &= n \times 3 \times 2^{t-1} \qquad &(n = |V\Gamma|). \end{aligned}$$

The structure of these groups can be elucidated by investigating certain sets of generators for them. These generators are derived from the sets Y_i defined for the general case in Chapter 17.

Let α_{t-1}, $v^{(1)}$, $v^{(2)}$ be the vertices adjacent to α_t, and let g_r $(r = 1, 2)$ denote automorphisms taking $[\alpha]$ to $(\alpha_1, \alpha_2, \ldots, \alpha_t, v^{(r)})$. We note, as a consequence of Proposition 18.1, that these automorphisms are unique. We shall need the following notation:

$$g = g_1, \quad x_0 = g_1^{-1}g_2, \quad x_i = g^{-i}x_0g^i \quad (i = 0, 1, 2, \ldots).$$

For the rest of this chapter, $\langle X \rangle$ will denote the subgroup of $G(\Gamma)$ generated by the set X.

PROPOSITION 18.2 *With the above notation, and provided* $t \geqslant 2$, *we have:*

(1) $F_i = \langle x_0, x_1, \ldots, x_{i-1} \rangle$ *for* $i = 1, 2, \ldots, t$.
(2) *If* $G^* = \langle x_0, x_1, \ldots, x_t \rangle$ *then* $|G \colon G^*| \leqslant 2$.
(3) $G = \langle x_0, g \rangle$.

Proof We shall use the notation and results of Propositions 17.7 and 17.8. In the trivalent case we have $F_0 = 1$, and the set Y_i consists of the elements $g_1^{-j}g_2^j$ and their inverses $g_2^{-j}g_1^j$ for $1 \leqslant j \leqslant i$.

(1) From part (2) of Proposition 17.7, we deduce that $F_i = \langle Y_i \rangle$. Now

$$g_1^{-j}g_2^j = x_{j-1}g_1^{-(j-1)}g_2^{j-1} = x_{j-1}x_{j-2}\ldots x_0,$$

and so $F_i = \langle x_0, x_1, \ldots, x_{i-1} \rangle$.

(2) From Proposition 17.8 we deduce that

$$G^* = \langle Y_{t+1} \rangle = \langle x_0, x_1, \ldots, x_t \rangle$$

is a subgroup of index 1 or 2 in G, provided that the girth of Γ is greater than three. (If the girth is three, then $t = 2$, $\Gamma = K_4$, and we may verify the conclusion explicitly.)

(3) If $G^* = G$, then $\langle x_0, g \rangle$ contains $\langle x_0, x_1, \ldots, x_t \rangle = G^* = G$. If $|G \colon G^*| = 2$, then Γ is bipartite, and each element g^* of G^* moves vertices of Γ through an even distance in Γ. But the element $g = g_1$ moves some vertices to adjacent vertices, and so $g \notin G^*$. Thus, adjoining g to G^* must enlarge the group, and since there are no groups between G^* and G we have

$$\langle G^*, g \rangle = \langle x_0, g \rangle = G. \quad \square$$

We considered Petersen's graph in the previous chapter, and for the 3-arc $[\alpha] = (12, 34, 15, 23)$ we found the automorphisms $g_1 = (13)(245)$, $g_2 = (13524)$. Hence.

$$x_0 = (34), \quad x_1 = (12), \quad x_2 = (35), \quad x_3 = (14).$$

In this case $G^* = \langle x_0, x_1, x_2, x_3 \rangle = G \approx S_5$.

Another simple example is the 2-transitive graph Q_3 depicted in Fig. 10. Taking $[\alpha] = (1, 2, 5)$ we have the automorphisms listed.

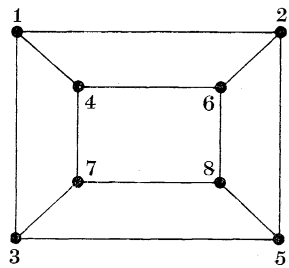


Fig. 10

$$g_1 = (125874)(36), \quad g_2 = (1253)(4687),$$
$$x_0 = (34)\,(56),$$
$$x_1 = (23)\,(67),$$
$$x_2 = (16)\,(38).$$

Here $G^* = \langle x_0, x_1, x_2 \rangle$ preserves the bipartition

$$V\Gamma = \{1, 5, 6, 7\} \cup \{2, 3, 4, 8\}$$

and $|G: G^*| = 2$.

Our aim now is to prove that there are no trivalent t-transitive graphs with $t > 5$. The proof of this very important result is due to Tutte (1947), with later improvements by Sims (1967) and Djoković (1972). The result is a purely algebraic consequence of the presentation of the stabilizer sequence given in Proposition 18.2.

We shall suppose that $t \geqslant 4$, as this assumption helps to avoid vacuous statements. We notice that each generator x_i $(i \geqslant 0)$ is an involution, and that each element of F_i $(1 \leqslant i \leqslant t-1)$ has a unique expression in the form

$$x_\rho x_\sigma \ldots x_\tau, \quad \text{where} \quad 0 \leqslant \rho < \sigma < \ldots < \tau \leqslant i-1.$$

(The empty set of subscripts corresponds to the identity element.) This is because there are 2^i such expressions, and $|F_i| = 2^i$ for $1 \leqslant i \leqslant t-1$.

The basic idea in the following arguments is the investigation of which stabilizers are Abelian and which are non-Abelian; since $|F_1| = 2$ and $|F_2| = 4$ we know that F_1 and F_2 are Abelian. Let λ denote the largest natural number such that F_λ is Abelian.

PROPOSITION 18.3 *If* $t \geqslant 4$, *then* $2 \leqslant \lambda < \frac{1}{2}(t+2)$.

Proof Suppose $F_\lambda = \langle x_0, \ldots, x_{\lambda-1}\rangle$ is Abelian; then its conjugate $g^{-t+\lambda-1}F_\lambda g^{t-\lambda+1} = \langle x_{t-\lambda+1}, \ldots, x_t\rangle$ is also Abelian. If

$$\lambda - 1 \geqslant t - \lambda + 1,$$

then both these groups contain $x_{\lambda-1}$, and together they generate G^*; hence $x_{\lambda-1}$ commutes with every element of G^*. Now $g^2 \in G^*$ (since $g \in G$ and $|G\colon G^*| \leqslant 2$) and so $x_{\lambda-1} = g^{-2}x_{\lambda-1}g^2 = x_{\lambda+1}$, whence $x_0 = x_2$. This is false, since $|F_3| > |F_2|$, and so

$$\lambda - 1 < t - \lambda + 1. \quad \square$$

Proposition 18.3 gives an upper bound for λ in terms of t. We shall find a lower bound of the same kind by means of arguments involving the commutators $(a,b) = a^{-1}b^{-1}ab$ of the canonical generators x_i. Since these generators are involutions,

$$(x_i, x_j) = (x_i x_j)^2.$$

LEMMA 18.4 (1) $(x_i, x_j) = 1$ if $|j-i| < \lambda$, *but* $(x_i, x_j) \neq 1$ *if* $|j-i| = \lambda$.

(2) *The centre of* $F_j = \langle x_0, \ldots, x_{j-1}\rangle$ *is the group* $\langle x_{j-\lambda}, \ldots, x_{\lambda-1}\rangle$ $(\lambda \leqslant j \leqslant 2\lambda)$.

(3) *The commutator subgroup of* F_{i+1} *is a subgroup of the group* $\langle x_1, \ldots, x_{i-1}\rangle = g^{-1}F_{i-1}g$ $(1 \leqslant i \leqslant t-2)$.

Proof (1) We may suppose without loss that $j > i$; then $(x_i, x_j) = g^{-i}(x_0, x_{j-i})\, g^i$ and so $(x_i, x_j) = 1$ if and only if x_0 and x_{j-i} commute. The result now follows from the fact that $F_\lambda = \langle x_0, \ldots, x_{\lambda-1}\rangle$ is the largest Abelian stabilizer.

(2) If the non-identity element x of F_j is written in the form $x_\rho x_\sigma \ldots x_\tau$ $(0 \leqslant \rho < \sigma < \ldots < \tau \leqslant j-1)$, then x does not commute with $x_{\rho+\lambda}$; further, if $\rho + \lambda < j$ then $x_{\rho+\lambda}$ belongs to F_j. Similarly,

x does not commute with $x_{\tau-\lambda}$, and if $\tau-\lambda > -1$, then $x_{\tau-\lambda}$ belongs to F_j. Thus, if x is in the centre of F_j, then $\rho \geqslant j-\lambda$ and $\tau \leqslant \lambda-1$, so that x is in $\langle x_{j-\lambda}, \ldots, x_{\lambda-1}\rangle$. Conversely, it follows from (1) that every element of this group is in the centre of F_j.

(3) Provided that $1 \leqslant i \leqslant t-2$, the groups $F_i = \langle x_0, \ldots, x_{i-1}\rangle$ and $g^{-1}F_i g = \langle x_1, \ldots, x_i\rangle$ are different, and they are both of index two in F_{i+1}, and consequently normal in F_{i+1}. Thus their intersection $\langle x_1, \ldots, x_{i-1}\rangle = g^{-1}F_{i-1}g$ is normal in F_{i+1}, and the quotient group $F_{i+1}/(g^{-1}F_{i-1}g)$ is Abelian, since it has order 4. Hence the commutator subgroup of F_{i+1} is contained in $g^{-1}F_{i-1}g$. □

The commutator (x_0, x_λ) belongs to the commutator subgroup of $F_{\lambda+1}$, and so it follows (from part (3) of Lemma 18.4 with $i = \lambda$) that (x_0, x_λ) belongs to the group $\langle x_1, \ldots, x_{\lambda-1}\rangle$. In other words, there is a unique expression

$$(x_0, x_\lambda) = x_\mu \ldots x_\nu \quad (1 \leqslant \mu \leqslant \nu \leqslant \lambda-1).$$

LEMMA 18.5 *With the above notation, we have:*

$$(1) \quad \mu+\lambda \geqslant t-1; \qquad (2) \quad 2\lambda-\nu \geqslant t-1.$$

Proof (1) Suppose that $\mu+\lambda \leqslant t-2$. Then (by part (3) of Lemma 18.4) the element $(x_0, x_{\mu+\lambda})$ of the commutator subgroup of $F_{\mu+\lambda+1}$ is contained in $\langle x_1, \ldots, x_{\mu+\lambda-1}\rangle$. The centre of $\langle x_1, \ldots, x_{\mu+\lambda-1}\rangle$ is $\langle x_\mu, \ldots, x_\lambda\rangle$, and since this contains both x_λ and (x_0, x_λ) it follows that $(x_0, x_{\mu+\lambda})$ commutes with x_λ and with (x_0, x_λ). Also x_λ commutes with $x_{\mu+\lambda}$, since $\mu \leqslant \lambda-1$. Hence we may argue as follows:

$$\begin{aligned}
x_{\mu+\lambda}^{-1}(x_0, x_\lambda)\,x_{\mu+\lambda} &= (x_{\mu+\lambda}^{-1}x_0 x_{\mu+\lambda}, x_\lambda)\\
&= (x_0(x_0, x_{\mu+\lambda}), x_\lambda)\\
&= (x_0, x_{\mu+\lambda})^{-1}\,(x_0, x_\lambda)\,(x_0, x_{\mu+\lambda})\,((x_0, x_{\mu+\lambda}), x_\lambda)\\
&= (x_0, x_\lambda).
\end{aligned}$$

This implies that $x_{\mu+\lambda}$ commutes with $(x_0, x_\lambda) = x_\mu \ldots x_\nu$. But this is false, since $x_{\mu+\lambda}$ does not commute with x_μ but does commute with any other term in the expression for (x_0, x_λ). Thus our hypothesis was wrong, and $\mu+\lambda \geqslant t-1$.

(2) If $2\lambda-\nu \leqslant t-2$, then using arguments parallel to those in (1), we may prove that $(x_{2\lambda-\nu}, x_0)$ commutes with $x_{\lambda-\nu}$ and with $(x_{\lambda-\nu}, x_{2\lambda-\nu})$; also $x_{\lambda-\nu}$ commutes with x_0, since $\nu \geqslant 1$. A calculation like that in (1) then implies that x_0 commutes with

$$(x_{\lambda-\nu}, x_{2\lambda-\nu}) = x_{\mu+\lambda-\nu} \ldots x_\lambda,$$

which is false. Hence $2\lambda-\nu \geqslant t-1$. □

THEOREM 18.6 (Tutte 1947) *There is no t-transitive trivalent graph with* $t > 5$.

Proof If t is at least four, then Proposition 18.3 tells us that $\lambda < \frac{1}{2}(t+2)$. However, the results of Lemma 18.5 show that $t-1-\lambda \leqslant \mu \leqslant \nu \leqslant 2\lambda-t+1$; that is, $\lambda \geqslant \frac{2}{3}(t-1)$. Now the inequality

$$\tfrac{2}{3}(t-1) < \tfrac{1}{2}(t+2)$$

is satisfied (for $t \geqslant 4$) only by $t = 4, 5, 7$. It remains to exclude the possibility $t = 7$, which is done by means of the following special argument.

If Γ is a 7-transitive trivalent graph, then the inequalities for λ, μ and ν show that $\lambda = 4$, $\mu = \nu = 2$; thus $(x_0, x_4) = x_2$. Also, by part (3) of Lemma 18.4, (x_0, x_5) belongs to the group $\langle x_1, x_2, x_3, x_4\rangle$. If the standard expression for (x_0, x_5) actually contains x_4, then we can write $(x_0, x_5) = hx_4$, where $h \in \langle x_1, x_2, x_3\rangle$ so that h commutes with x_0 and x_4. Hence

$$\begin{aligned} x_2 = (x_0, x_4) &= (x_0 x_4)^2 = (x_0 h x_4)^2 = (x_0(x_0 x_5)^2)^2 \\ &= (x_5 x_0 x_5)^2 = x_5 x_0^2 x_5 = 1. \end{aligned}$$

Since this is absurd, $(x_0, x_5) = (x_0 x_5)^2$ must belong to $\langle x_1, x_2, x_3\rangle$.

Now the original definitions show that x_1, x_2 and x_3 fix the vertex α_3 of the 7-arc $[\alpha]$, and so $x_0 x_5(\alpha_3) = x_5 x_0(\alpha_3) = x_5(\alpha_3)$. That is, x_0 fixes $x_5(\alpha_3)$.

Further, since x_5 fixes α_1 but not α_2 we have a 7-arc $[\theta] = (x_5(\alpha_3), x_5(\alpha_2), \alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6)$ in Γ. The three vertices adjacent to α_1 are α_0, α_2 and $x_5(\alpha_2)$, and since x_0 fixes α_0, α_1 and α_2 it must fix $x_5(\alpha_2)$ also. Consequently x_0 fixes the whole 7-arc $[\theta]$, and this contradicts Proposition 18.1. Hence $t = 7$ cannot occur in the trivalent case. □

The simplest example of a 5-transitive trivalent graph is constructed as follows. Represent the group S_6 on 6 symbols $\{a, b, c, d, e, f\}$, and take the vertices of a graph Ω to be the 15 permutations of shape (ab) and the 15 permutations of shape $(ab)(cd)(ef)$. Join two vertices by an edge if and only if the corresponding permutations have different shape and they commute. For instance, (ab) is joined to $(ab)(cd)(ef)$, $(ab)(ce)(df)$ and $(ab)(cf)(de)$, while $(ab)(cd)(ef)$ is joined to (ab), (cd) and (ef). Any automorphism of the group S_6 is an automorphism of Ω, and so

$$|G(\Omega)| = |\mathrm{Aut}\, S_6| = 1440 = 30 \times 3 \times 2^4,$$

as we expect for a 5-transitive graph with 30 vertices.

Let us define our generators in terms of the following 5-arc in Ω:

$$(ab), \quad (ab)(cd)(ef), \quad (cd), \quad (ae)(bf)(cd), \quad (ae), \quad (ae)(bd)(cf).$$

If π is an element of S_6, denote the corresponding inner automorphism of S_6 by $|\pi|$. Then the generators for the stabilizer sequence may be chosen as follows:

$$x_0 = |(cd)|, \quad x_1 = |(ab)(cd)(ef)|, \quad x_2 = |(ab)|,$$
$$x_3 = |(ab)(cf)(de)|, \quad x_4 = |(cf)|.$$

The groups which occur in the stabilizer sequence are

$$F_5 = S_4 \times \mathbb{Z}_2, \quad F_4 = D_8 \times \mathbb{Z}_2, \quad F_3 = (\mathbb{Z}_2)^3,$$
$$F_2 = (\mathbb{Z}_2)^2, \quad F_1 = \mathbb{Z}_2.$$

Finally, we may choose x_5 so that $G^* = \langle x_0, \ldots, x_5 \rangle$ is isomorphic to S_6, and so $|G: G^*| = 2$ in accordance with the fact that Ω is bipartite.

18*A* *A primitive 5-transitive trivalent graph* A 5-transitive trivalent graph, whose automorphism group acts primitively [B, p. 12] on the vertices, can be constructed as follows. The vertices correspond to the 234 triangles in $PG(2, 3)$ and two vertices are adjacent whenever the corresponding triangles have one common point and their remaining four points are distinct and collinear. The automorphism group is the group $\mathrm{Aut}\, PSL(3, 3)$.

18*B* *An infinite family of 4-transitive trivalent graphs* If p is a prime congruent to $\pm 1 \pmod{16}$, then $PSL(2, p)$ acts transitively on the 4-arcs of a trivalent graph with $p(p^2-1)/48$ vertices. In each case the group acts primitively on the vertices (Wong 1967).

18*C* *The structure of the stabilizer sequence* For a t-transitive trivalent graph, $1 \leqslant t \leqslant 5$, not only is the order of each group in the stabilizer sequence known, but also its abstract structure. The abstract groups are (Wong 1967):

t	F_1	F_2	F_3	F_4	F_5
1	$\mathbb{Z}_3$				
2	$\mathbb{Z}_2$	S_3			
3	$\mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	D_{12}		
4	$\mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	D_8	S_4	
5	$\mathbb{Z}_2$	$(\mathbb{Z}_2)^2$	$(\mathbb{Z}_2)^3$	$D_8 \times \mathbb{Z}_2$	$S_4 \times \mathbb{Z}_2$

18*D* *Generalization to* $(p+1)$*-valent graphs* Let Γ be a t-transitive graph (with $t \geqslant 4$), and let the valency of Γ be $p+1$, where p is a prime. Then:

(*a*) $t = 4$, 5, or 7;

(*b*) the order of a vertex-stabilizer is $(p+1)\,p^{t-1}m$, where m divides $(p-1)^2$ (Gardiner 1973).

19. The covering-graph construction

In this short chapter we shall study a technique which enables us to construct a new symmetric graph from a given one. A particular form of this construction is sufficient to imply the existence of infinitely many connected trivalent 5-transitive graphs.

We shall use the symbol $S\Gamma$ to denote the set of 1-arcs or *sides* of a graph Γ; each edge $\{u, v\}$ of Γ gives rise to two sides, (u, v) and (v, u). For any group K, a *K-chain* on Γ is a function $\phi : S\Gamma \to K$ such that $\phi(u, v) = (\phi(v, u))^{-1}$ for all sides (u, v) of Γ.

DEFINITION 19.1 The *covering graph* $\tilde{\Gamma} = \tilde{\Gamma}(K, \phi)$ of Γ, with respect to a given K-chain ϕ on Γ, is defined as follows. The vertex-set of $\tilde{\Gamma}$ is $K \times V\Gamma$, and two vertices (κ_1, v_1), (κ_2, v_2) are joined by an edge if and only if

$$(v_1, v_2) \in S\Gamma \quad \text{and} \quad \kappa_2 = \kappa_1 \phi(v_1, v_2).$$

It is easy to check that the definition of adjacency depends only on the unordered pair of vertices.

As an example let $\Gamma = K_4$, and let K be the group $\mathbb{Z}_2$ whose elements are 1 and z; the function ϕ which assigns z to each side of K_4 is a $\mathbb{Z}_2$-chain on K_4. The covering graph $\tilde{\Gamma}(\mathbb{Z}_2, \phi)$ is depicted in Fig. 11.

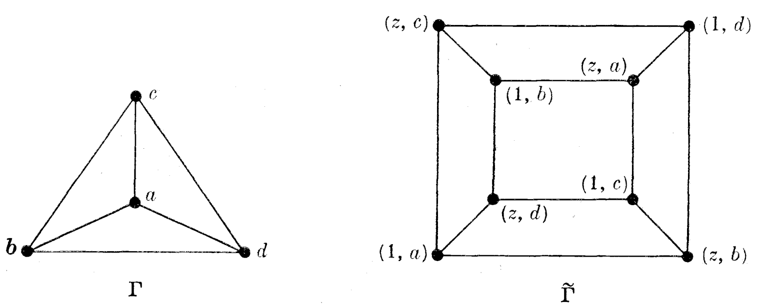


Fig. 11

Suppose that a group G acts as a group of automorphisms of a group K; that is, for each g in G we have an automorphism $\hat{g}$ of K such that the function $g \mapsto \hat{g}$ is a group homomorphism from G to $\operatorname{Aut} K$. In this situation we may define a *split extension* of K by G, denoted by $[K]G$. This is a group whose elements are the ordered pairs (κ, g) in $K \times G$, with the group operation given by

$$(\kappa_1, g_1)(\kappa_2, g_2) = (\hat{g}_2(\kappa_1)\kappa_2, g_1 g_2).$$

Let Γ be a graph, ϕ a K-chain on Γ, and let $G = G(\Gamma)$. Then G acts on the sides of Γ by the rule $g(u,v) = (g(u), g(v))$, and we may postulate a special relationship between the action of G on K and its action on $S\Gamma$.

DEFINITION 19.2 The K-chain ϕ is *compatible* with the given actions of G on K and $S\Gamma$, if the following diagram is commutative for each g in G:

$$\begin{array}{ccc} S\Gamma & \xrightarrow{\phi} & K \\ {\scriptstyle g}\downarrow & & \downarrow{\scriptstyle \hat{g}} \\ S\Gamma & \xrightarrow{\phi} & K \end{array}$$

PROPOSITION 19.3 *Suppose that Γ is a graph whose automorphism group $G = G(\Gamma)$ acts as a group of automorphisms of a group K. Suppose further that there is a K-chain ϕ on Γ which is compatible with the actions of G on K and $S\Gamma$. Then the split extension $[K]G$ is a group of automorphisms of the covering graph $\tilde{\Gamma} = \tilde{\Gamma}(K, \phi)$.*

Proof Define the effect of an element (κ, g) of $[K]G$ on a vertex (κ', v) of $\tilde{\Gamma}$ by the rule

$$(\kappa, g)(\kappa', v) = (\kappa\hat{g}(\kappa'), g(v)).$$

A straightforward calculation, using the definition of compatibility, shows that this permutation of $V\tilde{\Gamma}$ is an automorphism of $\tilde{\Gamma}$. □

The usefulness of the covering graph construction lies in the fact that the following result (which is considerably stronger than Proposition 19.3) is true.

PROPOSITION 19.4 *With the notation and hypotheses of Proposition* 19.3, *suppose also that G is transitive on the t-arcs of* Γ. *Then* $[K]G$ *is transitive on the t-arcs of* $\tilde{\Gamma}$.

Proof Let $((\kappa_0, v_0), \ldots, (\kappa_t, v_t))$ and $((\kappa'_0, v'_0), \ldots, (\kappa'_t, v'_t))$ be two t-arcs in $\tilde{\Gamma}$. Then $(v_0, \ldots, v_t)$ and $(v'_0, \ldots, v'_t)$ are t-arcs in Γ, and so there is some g in G such that $g(v_i) = v'_i$ $(0 \leqslant i \leqslant t)$.

Suppose we choose κ^* in K such that (κ^*, g) takes (κ_0, v_0) to (κ'_0, v'_0); that is, we choose $\kappa^* = \kappa'_0(\hat{g}(\kappa_0))^{-1}$. Then we claim that (κ^*, g) takes (κ_i, v_i) to (κ'_i, v'_i) for all $i \in \{0, 1, \ldots, t\}$.

Our claim is true when $i = 0$, and we make the inductive hypothesis that it is true when $i = j-1$, so that

$$(\kappa'_{j-1}, v'_{j-1}) = (\kappa^*, g)\,(\kappa_{j-1}, v_{j-1}) = (\kappa^*\hat{g}(\kappa_{j-1}), g(v_{j-1})).$$

Since (κ_j, v_j) is adjacent to (κ_{j-1}, v_{j-1}) we have $\kappa_j = \kappa_{j-1}\phi(v_{j-1}, v_j)$; also, the corresponding equation holds for the primed symbols. Thus:

$$\begin{aligned}\kappa'_j &= \kappa'_{j-1}\phi(v'_{j-1}, v'_j) = \kappa^*\hat{g}(\kappa_{j-1})\,\phi(g(v_{j-1}), g(v_j))\\ &= \kappa^*\hat{g}(\kappa_{j-1})\,\hat{g}(\phi(v_{j-1}, v_j)) = \kappa^*\hat{g}(\kappa_{j-1}\phi(v_{j-1}, v_j))\\ &= \kappa^*\hat{g}(\kappa_j).\end{aligned}$$

Consequently, (κ^*, g) takes (κ_j, v_j) to (κ'_j, v'_j), and the result follows by the principle of induction. □

The requirement that a compatible K-chain should exist is rather restrictive. In fact, for a given graph Γ and group K, it is very likely that the only covering graph is the trivial one consisting of $|K|$ components each isomorphic with Γ. However, it is possible to choose K (depending on Γ) in such a way that a non-trivial covering graph always exists.

Let us suppose that a t-transitive graph Γ is given. We define K to be the free $\mathbb{Z}_2$-module on the set $E\Gamma$; thus, K is the direct product of $|E\Gamma|$ copies of $\mathbb{Z}_2$, and its elements are the formal products $\Pi e_\alpha^{\sigma(\alpha)}$, where $\sigma(\alpha) = 0$ or 1 and the product is over all e_α in $E\Gamma$. The automorphism group $G = G(\Gamma)$ acts on K through its action on $E\Gamma$.

Also, there is a K-chain ϕ on Γ defined by the rule $\phi(u, v) = e_i$, where $e_i = \{u, v\}$. This K-chain is compatible with the actions of

G on K and $S\Gamma$, and so the covering graph $\tilde{\Gamma} = \tilde{\Gamma}(K, \phi)$ exists and (by Proposition 19.4) its automorphism group is transitive on t-arcs.

THEOREM 19.5 *Let Γ be a t-transitive graph whose rank and co-rank are $r(\Gamma)$ and $s(\Gamma)$. Then, with the special choices of K and ϕ given above, the covering graph $\tilde{\Gamma}$ consists of $2^{r(\Gamma)}$ connected components, each having $2^{s(\Gamma)}|V\Gamma|$ vertices.*

Proof Pick a vertex v of Γ, and let $\tilde{\Gamma}_0$ denote the component of $\tilde{\Gamma}$ which contains the vertex $(1, v)$. If $v = u_0, u_1, \ldots, u_l = v$ are the vertices of a circuit in Γ, with edges $e_i = \{u_{i-1}, u_i\}$, then we have the following path in $\tilde{\Gamma}_0$:

$$(1, v),\ (e_1, u_1),\ (e_1 e_2, u_2),\ \ldots,\ (e_1 e_2 \ldots e_l, v).$$

Conversely, the vertex (κ, v) is in $\tilde{\Gamma}_0$ only if κ represents the edges of a circuit in Γ.

Since there are $s(\Gamma)$ basic circuits in Γ, there are $2^{s(\Gamma)}$ elements κ in K such that (κ, v) is in $\tilde{\Gamma}_0$. It follows that $\tilde{\Gamma}_0$ has $2^{s(\Gamma)}|V\Gamma|$ vertices; further, $\tilde{\Gamma}$ is vertex-transitive and so each component has this number of vertices. Finally, since

$$|V\tilde{\Gamma}| = |K|\,|V\Gamma| = 2^{|E\Gamma|}|V\Gamma| \quad \text{and} \quad r(\Gamma)+s(\Gamma) = |E\Gamma|,$$

there must be $2^{r(\Gamma)}$ components. □

COROLLARY 19.6 (Conway, unpublished) *There are infinitely many connected trivalent 5-transitive graphs.*

Proof We know that there is at least one connected trivalent 5-transitive graph Ω (it was constructed at the end of the previous chapter). Applying the construction of Theorem 19.5 to Ω, we obtain a connected trivalent 5-transitive graph $\tilde{\Omega}_0$ with $2^{s(\Omega)}|V\Omega|$ vertices, and since $s(\Omega) > 0$ this graph is not isomorphic with Ω. We may repeat this process as often as we please, obtaining an infinite sequence of graphs with the required properties. □

We notice that the number of vertices required in the construction of Corollary 19.6 quickly becomes astronomical; for instance, the two graphs which follow Ω in the sequence have about 2^{21} and $2^{100\,000}$ vertices respectively. In fact, the only known connected trivalent 5-transitive graphs with a relatively small number of vertices are Ω and two graphs with 90 and 234 vertices. The last graph is described in § 18A.

19*A* *Double coverings* Let G be the automorphism group of a connected graph Γ, and let G act on the group $\mathbb{Z}_2$ by the rule that $\hat{g}$ is the identity automorphism of $\mathbb{Z}_2$, for each g in G. Then the $\mathbb{Z}_2$-chain ϕ on Γ which assigns the non-identity element of $\mathbb{Z}_2$ to each side of Γ is compatible with the actions of G on $S\Gamma$ and $\mathbb{Z}_2$. The covering graph $\tilde{\Gamma}(\mathbb{Z}_2, \phi)$ is connected if and only if Γ is not bipartite.

19*B* *The Desargues graph* The construction of § 19A applied to Petersen's graph results in a connected trivalent 3-transitive graph with 20 vertices. This graph represents the configuration of Desargues's theorem in projective geometry (Coxeter 1950).

20. *Distance-transitive graphs*

We have already explained that a connected graph Γ is said to be distance-transitive if, for any vertices u, v, x, y of Γ satisfying $\partial(u,v) = \partial(x,y)$, there is an automorphism g of Γ which takes u to x and v to y. This is the most restrictive of our hierarchy of symmetry conditions, and it has the most interesting consequences.

It is helpful to recast the definition. For any connected graph Γ, and each v in $V\Gamma$ we define

$$\Gamma_i(v) = \{u \in V\Gamma \mid \partial(u,v) = i\},$$

where $0 \leqslant i \leqslant d$, and d is the diameter of Γ. Of course, $\Gamma_0(v) = \{v\}$, and $V\Gamma$ is partitioned into the disjoint subsets $\Gamma_0(v), \ldots, \Gamma_d(v)$, for each v in $V\Gamma$. Small graphs may be depicted in a manner which emphasizes this partition by arranging their vertices in columns, according to distance from an arbitrary vertex v. For example, $K_{3,3}$ is displayed in this way in Fig. 12.

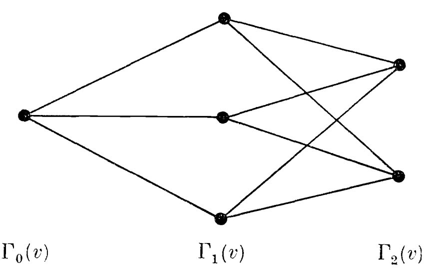


Fig. 12

LEMMA 20.1 *A connected graph Γ, with diameter d and automorphism group $G = G(\Gamma)$, is distance-transitive if and only if it is vertex-transitive and the vertex-stabilizer G_v is transitive on the set $\Gamma_i(v)$, for each $i \in \{0, 1, \ldots, d\}$, and each $v \in V\Gamma$.*

Proof Let us suppose that Γ is distance-transitive, and use the notation of the first sentence of this chapter. Taking $u = v$

and $x = y$, we see that Γ is vertex-transitive. Taking $y = v$, we see that G_v is transitive on $\Gamma_i(v)$ $(0 \leqslant i \leqslant d)$. The converse is proved by reversing the argument. $\square$

We shall study distance-transitive graphs by means involving their adjacency algebras (Chapter 2); we prepare for this by investigating some purely combinatorial consequences of the definition.

For any connected graph Γ, and any vertices u, v of Γ, let

$$s_{hi}(u, v) = |\{w \in V\Gamma \,|\, \partial(u, w) = h \text{ and } \partial(v, w) = i\}|;$$

that is, $s_{hi}(u, v)$ is the number of vertices of Γ whose distance from u is h and whose distance from v is i. In a distance-transitive graph the numbers $s_{hi}(u, v)$ depend, not on the individual pair (u, v) but only on the distance $\partial(u, v)$; and if $\partial(u, v) = j$ we write $s_{hij} = s_{hi}(u, v)$.

DEFINITION 20.2 The *intersection numbers* of a distance-transitive graph with diameter d are the $(d+1)^3$ numbers s_{hij}, where h, i and j belong to the set $\{0, 1, \ldots, d\}$.

Fortunately, there are many identities involving the intersection numbers, and we shall find that just $2d$ of them are sufficient to determine the rest.

Let us consider the intersection numbers with $h = 1$. For a fixed j, the number s_{1ij} counts the vertices w such that w is adjacent to u and $\partial(v, w) = i$, where $\partial(u, v) = j$. Now, if w is adjacent to u and $\partial(u, v) = j$, then $\partial(v, w)$ must be one of the numbers $j-1, j, j+1$; in other words

$$s_{1ij} = 0 \quad \text{if} \quad i \neq j-1, j, j+1.$$

We introduce the notation

$$c_j = s_{1, j-1, j}, \quad a_j = s_{1, j, j}, \quad b_j = s_{1, j+1, j},$$

where $0 \leqslant j \leqslant d$, except that c_0 and b_d are undefined. These numbers have the following simple interpretation in terms of the diagrammatic representation of Γ introduced at the beginning of this chapter. Let us pick an arbitrary vertex v and a vertex u in

$\Gamma_j(v)$. Then u is adjacent to c_j vertices in $\Gamma_{j-1}(v)$, a_j vertices in $\Gamma_j(v)$, and b_j vertices in $\Gamma_{j+1}(v)$. These numbers are independent of u and v, provided that $\partial(u, v) = j$.

DEFINITION 20.3 The *intersection array* of a distance-transitive graph is

$$\iota(\Gamma) = \begin{Bmatrix} * & c_1 & \cdots & c_j & \cdots & c_d \\ a_0 & a_1 & \cdots & a_j & \cdots & a_d \\ b_0 & b_1 & \cdots & b_j & \cdots & * \end{Bmatrix}.$$

This array can be read off from a diagram of the graph. For example, the cube Q_3 is a distance-transitive graph with diameter 3; from the representation in Fig. 13 we may write down its intersection array.

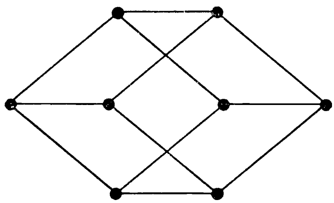


Fig. 13

$$\iota(Q_3) = \begin{Bmatrix} * & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 3 & 2 & 1 & * \end{Bmatrix}.$$

We remark that a distance-transitive graph is vertex-transitive, and consequently regular, of valency k say. We clearly have $a_0 = 0$, $b_0 = k$, $c_1 = 1$. Further, since each column of the intersection array sums to k, if we are given the first and third rows we can calculate the middle row. Thus it is both logically sufficient and typographically convenient to use the alternative notation

$$\iota(\Gamma) = \{k, b_1, \ldots, b_{d-1};\ 1, c_2, \ldots, c_d\}.$$

However, the original notation of Definition 20.3 is intuitively helpful, and we shall continue to use it occasionally.

We shall use the symbol k_i $(0 \leqslant i \leqslant d)$ to denote the number of vertices in $\Gamma_i(v)$ for any vertex v; clearly $k_0 = 1$ and $k_1 = k$.

PROPOSITION 20.4 *Let Γ be a distance-transitive graph whose intersection array is $\{k, b_1, \ldots, b_{d-1};\ 1, c_2, \ldots, c_d\}$. Then we have the following equations and inequalities:*

(1) $k_{i-1}b_{i-1} = k_i c_i \quad (1 \leqslant i \leqslant d)$.
(2) $1 \leqslant c_2 \leqslant c_3 \leqslant \ldots \leqslant c_d$.
(3) $k \geqslant b_1 \geqslant b_2 \geqslant \ldots \geqslant b_{d-1}$.

Proof (1) For any v in $V\Gamma$, there are k_{i-1} vertices in $\Gamma_{i-1}(v)$ and each is joined to b_{i-1} vertices in $\Gamma_i(v)$. Also there are k_i vertices in $\Gamma_i(v)$ and each is joined to c_i vertices in $\Gamma_{i-1}(v)$. Thus the number of edges with one end in $\Gamma_{i-1}(v)$ and one end in $\Gamma_i(v)$ is $k_{i-1}b_{i-1} = k_i c_i$.

(2) Suppose $u \in \Gamma_{i+1}(v)$ $(1 \leqslant i \leqslant d-1)$. Pick a path $v, x, \ldots, u$ of length $i+1$; then $\partial(x, u) = i$. If w is in $\Gamma_{i-1}(x) \cap \Gamma_1(u)$, then $\partial(v, w) = i$, and so w is in $\Gamma_i(v) \cap \Gamma_1(u)$. It follows that

$$c_i = |\Gamma_{i-1}(x) \cap \Gamma_1(u)| \leqslant |\Gamma_i(v) \cap \Gamma_1(u)| = c_{i+1}.$$

(3) This is proved by an argument analogous to that used in (2). $\square$

The results of Proposition 20.4 give some simple constraints which must be satisfied if an arbitrary array is to be the intersection array of some distance-transitive graph. We shall obtain much more restrictive conditions in the next chapter. However, in order to derive these conditions, we need not postulate that our graph is distance-transitive, but merely that it has the combinatorial regularity implied by the existence of an intersection array. In fact, this situation is encountered in important applications, and we are justified in making the following definition.

DEFINITION 20.5 A *distance-regular* graph, with diameter d, is a regular connected graph of valency k with the following property. There are natural numbers

$$b_0 = k, \quad b_1, \ldots, b_{d-1}, \quad c_1 = 1, \quad c_2, \ldots, c_d,$$

such that for each pair (u, v), of vertices satisfying $\partial(u, v) = j$, we have

(1) the number of vertices in $\Gamma_{j-1}(v)$ adjacent to u is c_j $(1 \leqslant j \leqslant d)$;

(2) the number of vertices in $\Gamma_{j+1}(v)$ adjacent to u is b_j $(0 \leqslant j \leqslant d-1)$.
The *intersection array* of Γ is $\iota(\Gamma) = \{k, b_1, \ldots, b_{d-1};\ 1, c_2, \ldots, c_d\}$.

A distance-transitive graph is distance-regular, but the converse is not necessarily true (see § 20D).

We shall now construct a simple basis for the adjacency algebra of a distance-regular graph. For any graph Γ, which has diameter d and vertex-set $\{v_1, \ldots, v_n\}$, let us define a set $\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_d\}$ of $n \times n$ matrices as follows:

$$(\mathbf{A}_h)_{rs} = \begin{cases} 1 & \text{if } \partial(v_r, v_s) = h; \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{A}_0 = \mathbf{I}$, $\mathbf{A}_1$ is the usual adjacency matrix $\mathbf{A}$ of Γ, and we notice that $\mathbf{A}_0 + \mathbf{A}_1 + \ldots + \mathbf{A}_d = \mathbf{J}$.

LEMMA 20.6 *Let Γ be a distance-regular graph with intersection array $\{k, b_1, \ldots, b_{d-1};\ 1, c_2, \ldots, c_d\}$. For $1 \leqslant i \leqslant d-1$, put $a_i = k - b_i - c_i$; then*

$$\mathbf{A}\mathbf{A}_i = b_{i-1}\mathbf{A}_{i-1} + a_i \mathbf{A}_i + c_{i+1}\mathbf{A}_{i+1} \quad (1 \leqslant i \leqslant d-1).$$

Proof From the definition of $\mathbf{A}$ and $\mathbf{A}_i$ it follows that $(\mathbf{A}\mathbf{A}_i)_{rs}$ is the number of vertices w of Γ such that $\partial(v_r, w) = 1$ and $\partial(v_s, w) = i$. If there are any such vertices w, then $\partial(v_r, v_s)$ must be one of the numbers $i-1$, i, $i+1$, and the number of vertices w in these three cases is b_{i-1}, a_i, c_{i+1}, respectively. Thus $(\mathbf{A}\mathbf{A}_i)_{rs}$ is equal to the (r, s)-entry of the matrix on the right-hand side. □

THEOREM 20.7 (Damerell 1973) *Let Γ be a distance-regular graph with diameter d. Then $\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_d\}$ is a basis for the adjacency algebra $\mathscr{A}(\Gamma)$, and consequently the dimension of $\mathscr{A}(\Gamma)$ is $d+1$.*

Proof By recursive application of the lemma we see that $\mathbf{A}_i$ is a polynomial $p_i(\mathbf{A})$, for $i = 2, \ldots, d$. The form of the recursion shows that the degree of p_i is at most i, and since $\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_d$ are linearly independent (exactly one of them has a non-zero entry in any given position) the degree of p_i is exactly i.

Since $\mathbf{A}_0 + \mathbf{A}_1 + \ldots + \mathbf{A}_d = \mathbf{J}$, and Γ is regular, with valency k say, we have $(\mathbf{A} - k\mathbf{I})(\mathbf{A}_0 + \mathbf{A}_1 + \ldots + \mathbf{A}_d) = \mathbf{0}$. The left-hand side is a polynomial in $\mathbf{A}$ of degree $d+1$, so the dimension of $\mathscr{A}(\Gamma)$ is at most $d+1$. However, Proposition 2.6 shows that the dimension of $\mathscr{A}(\Gamma)$ is at least $d+1$, and so we have equality. Since $\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_d\}$ is a set of $d+1$ linearly independent members of $\mathscr{A}(\Gamma)$, it is a basis for $\mathscr{A}(\Gamma)$. $\square$

It follows from Theorem 20.7 that a distance-regular graph has just $d+1$ distinct eigenvalues, the minimum number possible for a graph of diameter d. These eigenvalues, and the strange story of their multiplicities, form the subject of the next chapter.

We shall need to know that the full set of $(d+1)^3$ intersection numbers can be defined for a distance-regular graph; this is a trivial remark for a distance-transitive graph, but it requires proof in the distance-regular case. Also, we shall relate these intersection numbers to the basis $\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_d\}$ of $\mathscr{A}(\Gamma)$.

PROPOSITION 20.8 *Let Γ be a distance-regular graph with diameter d.*

(1) *The numbers $s_{hi}(u,v)$ depend only on $\partial(u,v)$, for*

$$h, i \in \{0, 1, \ldots, d\}.$$

(2) *If $s_{hi}(u,v) = s_{hij}$ when $\partial(u,v) = j$, then*

$$\mathbf{A}_h \mathbf{A}_i = \sum_{j=0}^{d} s_{hij} \mathbf{A}_j.$$

Proof We prove both parts in one argument. Since $\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_d\}$ is a basis for $\mathscr{A}(\Gamma)$, the product $\mathbf{A}_h\mathbf{A}_i$ is a linear combination $\Sigma t_{hij}\mathbf{A}_j$. Now $(\mathbf{A}_h\mathbf{A}_i)_{rs}$ is equal to $s_{hi}(v_r, v_s)$, and there is just one member of the basis whose (r,s)-entry is 1: it is that $\mathbf{A}_j$ for which $\partial(v_r, v_s) = j$. Thus $s_{hi}(v_r, v_s) = t_{hij}$, and so $s_{hi}(v_r, v_s)$ depends only on $\partial(v_r, v_s)$. Further, the coefficient t_{hij} is just the intersection number s_{hij}. $\square$

The theory which underlies our treatment of the adjacency algebra of a distance-regular graph was developed in two quite different contexts. First, the association schemes used by Bose in the statistical design of experiments led to an association algebra

(Bose and Mesner 1959), which corresponds to our adjacency algebra. Concurrently, the work of Schur (1933) and Wielandt (1964) on the commuting algebra, or centralizer ring, of a permutation group, culminated in the paper of Higman (1967) which employs graph-theoretic ideas very closely related to those of this chapter. The connection between the theory of the commuting algebra and distance-transitive graphs is also treated in [B, pp. 85–88].

We end this chapter by remarking that several well-known families of graphs have the distance-transitivity property. For instance, the complete graphs K_n and the complete bipartite graphs $K_{k,k}$ have this property. Their diameters are 1 and 2 respectively, and the intersection arrays are:

$$\iota(K_n) = \begin{Bmatrix} * & 1 \\ 0 & n-1 \\ n-1 & * \end{Bmatrix}, \quad \iota(K_{k,k}) = \begin{Bmatrix} * & 1 & k \\ 0 & 0 & 0 \\ k & k-1 & * \end{Bmatrix}.$$

The triangle graphs Δ_t (p. 19) are distance-transitive, with diameter 2, and for $t \geqslant 4$:

$$\iota(\Delta_t) = \begin{Bmatrix} * & 1 & 4 \\ 0 & t-2 & 2t-8 \\ 2t-4 & t-3 & * \end{Bmatrix}.$$

Other distance-transitive graphs are described in the following examples and in the next three chapters. A survey of distance-transitive graphs whose diameter is two, and their connection with sporadic simple groups, can be found in [B, pp. 96–112].

20*A* *The cube graphs* The *k-cube*, Q_k, is the graph defined as follows: the vertices of Q_k are the 2^k symbols $(\epsilon_1, \epsilon_2, \ldots, \epsilon_k)$, where $\epsilon_i = 0$ or 1 $(1 \leqslant i \leqslant k)$, and two vertices are adjacent when their symbols differ in exactly one coordinate. The graph Q_k $(k \geqslant 2)$ is distance-transitive, with valency k, diameter k, and intersection array

$$\{k, k-1, k-2, \ldots, 1;\ 1, 2, 3, \ldots, k\}.$$

20*B* *The odd graphs yet again* The odd graphs O_k $(k \geqslant 2)$

are distance-transitive, with diameter $k-1$. The intersection array, when k is odd and even respectively, is

$$\iota(O_{2l-1}) = \{2l-1, 2l-2, 2l-2, \ldots, l+1, l+1, l; \\ 1, 1, 2, 2, \ldots, l-1, l-1\},$$

$$\iota(O_{2l}) = \{2l, 2l-1, 2l-1, \ldots, l+1, l+1; \\ 1, 1, 2, 2, \ldots, l-1, l-1, l\}.$$

20*C* *More inequalities for intersection numbers* If Γ is a distance-regular graph with intersection array $\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$, then we have the following inequalities:

(*a*) If $1 \leqslant i \leqslant \frac{1}{2}d$, then $b_i \geqslant c_i$.

(*b*) If $1 \leqslant i \leqslant d-1$, then $b_1 \geqslant c_i$.

(*c*) $c_2 \geqslant k - 2b_1$.

20*D* *A distance-regular graph which is not distance-transitive* Let Ψ denote the graph whose vertices are the 26 symbols a_i, b_i $(i \in \mathbb{Z}_{13})$, and in which:

$$\begin{aligned} a_i \text{ and } a_j \text{ are adjacent} &\Leftrightarrow |i-j| = 1, 3, 4; \\ b_i \text{ and } b_j \text{ are adjacent} &\Leftrightarrow |i-j| = 2, 5, 6; \\ a_i \text{ and } b_j \text{ are adjacent} &\Leftrightarrow i-j = 0, 1, 3, 9. \end{aligned}$$

Then Ψ is distance-regular, with intersection array $\{10, 6; 1, 4\}$. But Ψ is not distance-transitive; in fact there is no automorphism taking a vertex a_i to a vertex b_j (Adel'son-Velskii 1969).

21. The feasibility of intersection arrays

In this chapter we shall study the question of when an arbitrary array is the intersection array of a distance-regular graph. The theoretical material is an improved formulation of that given in [B, pp. 86–95].

Suppose that the given array is $\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$. The results of Proposition 20.4 yield some conditions which are necessary for an affirmative answer to our question. For instance, the numbers $k_i = |\Gamma_i(v)|$ can be found from the recursion in part (1) of Proposition 20.4:

$$k_i = (kb_1 \ldots b_{i-1})/(c_2 c_3 \ldots c_i) \quad (2 \leqslant i \leqslant d),$$

and so the numbers on the right-hand side must be integers. In addition, the inequalities of parts (2) and (3) of Proposition 20.4 must be satisfied. There are also some elementary parity conditions: let $n = 1 + k_1 + \ldots + k_d$ be the number of vertices of the putative graph, then if k is odd, n must be even. That is, $nk \equiv 0 \pmod 2$. Similarly, considering the vertex-subgraph defined by the vertices in $\Gamma_i(v)$, we see that $k_i a_i \equiv 0 \pmod 2$ for $1 \leqslant i \leqslant d$, where $a_i = k - b_i - c_i$.

These conditions are not very restrictive, and they are satisfied by many arrays which are not realized by any graph. For example, $\{3, 2, 1; 1, 1, 3\}$ passes all these tests, and would represent a trivalent graph with diameter 3 and 12 vertices. In this case, simple (but special) arguments are sufficient to prove that there is no graph. Our aim now is to obtain a strong condition which rules out a multitude of examples of this kind.

Recall that the adjacency algebra $\mathscr{A}(\Gamma)$ of a distance-regular graph Γ has a basis $\{\mathbf{A}_0, \mathbf{A}_1, \ldots, \mathbf{A}_d\}$, and $\mathbf{A}_h \mathbf{A}_i = \Sigma s_{hij} \mathbf{A}_j$. This equation can be interpreted as saying that left-multiplication by $\mathbf{A}_h$ (regarded as a linear mapping of $\mathscr{A}(\Gamma)$ with respect to the given basis) is faithfully represented by the $(d+1) \times (d+1)$ matrix $\mathbf{B}_h$ with

$$(\mathbf{B}_h)_{ij} = s_{hij}.$$

(This representation is the transpose of the one most commonly employed, but since the algebra $\mathscr{A}(\Gamma)$ is commutative, that is immaterial.) Thus, we have proved:

PROPOSITION 21.1 *The adjacency algebra $\mathscr{A}(\Gamma)$ of a distance-regular graph Γ with diameter d can be faithfully represented by an algebra of matrices with $d+1$ rows and columns. A basis for this representation is the set $\{\mathbf{B}_0, \mathbf{B}_1, \ldots, \mathbf{B}_d\}$, where $(\mathbf{B}_h)_{ij}$ is the intersection number s_{hij} for $h, i, j \in \{0, 1, \ldots, d\}$.* $\square$

The members of $\mathscr{A}(\Gamma)$ can now be regarded as square matrices of size $d+1$ (instead of n), a considerable simplification. What is more, the matrix $\mathbf{B}_1$ alone is sufficient! To see this, we notice first that, since $(\mathbf{B}_1)_{ij} = s_{1ij}$, $\mathbf{B}_1$ is *tridiagonal*:

$$\mathbf{B}_1 = \begin{bmatrix} 0 & 1 & & & & & \mathbf{0} \\ k & a_1 & c_2 & & & & \\ & b_1 & a_2 & \cdot & & & \\ & & b_2 & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & c_d \\ \mathbf{0} & & & & & \cdot & a_d \end{bmatrix}.$$

We shall often write $\mathbf{B}_1 = \mathbf{B}$, and refer to $\mathbf{B}$ as the *intersection matrix* of Γ. Now, since the matrices $\mathbf{B}_i$ are images of the matrices $\mathbf{A}_i$ under a faithful representation, the equation of Lemma 20.6 carries over:

$$\mathbf{B}\mathbf{B}_i = b_{i-1}\mathbf{B}_{i-1} + a_i\mathbf{B}_i + c_{i+1}\mathbf{B}_{i+1} \quad (1 \leqslant i \leqslant d-1).$$

Consequently, each $\mathbf{B}_i$ $(i \geqslant 2)$ is a polynomial in $\mathbf{B}$ with coefficients which depend only on the entries of $\mathbf{B}$. It follows from this (in theory) that $\mathscr{A}(\Gamma)$ and the spectrum of Γ are determined by $\mathbf{B}$, which in turn is given by the intersection array $\iota(\Gamma)$. We shall now give a practical demonstration of this fact.

PROPOSITION 21.2 *Let Γ be a distance-regular graph with valency k and diameter d. Then Γ has $d+1$ distinct eigenvalues $k = \lambda_0, \lambda_1, \ldots, \lambda_d$, which are the eigenvalues of the intersection matrix $\mathbf{B}$.*

Proof We have already remarked, in Chapter 20, that Γ has exactly $d+1$ distinct eigenvalues. Since $\mathbf{B}$ is the image of the adjacency matrix $\mathbf{A}$ under a faithful representation, the minimum polynomials of $\mathbf{A}$ and $\mathbf{B}$ coincide, and so the eigenvalues of $\mathbf{A}$ are those of $\mathbf{B}$. $\square$

Each eigenvalue λ, common to $\mathbf{A}$ and $\mathbf{B}$, is a simple eigenvalue of $\mathbf{B}$ (since $\mathbf{B}$ is a $(d+1)\times(d+1)$ matrix). However, the multiplicity $m(\lambda)$ of λ as an eigenvalue of $\mathbf{A}$ will usually be greater than one; we claim that $m(\lambda)$ can be calculated from $\mathbf{B}$ alone.

Let us regard λ as an indeterminate, and define a sequence of polynomials in λ, with rational coefficients, by the recursion:

$$v_0(\lambda) = 1, \quad v_1(\lambda) = \lambda,$$

$$c_{i+1}v_{i+1}(\lambda)+(a_i-\lambda)v_i(\lambda)+b_{i-1}v_{i-1}(\lambda) = 0 \qquad (i = 1, 2, \ldots, d-1).$$

The polynomial $v_i(\lambda)$ has degree i in λ, and comparing the definition with Lemma 20.6 we see that

$$\mathbf{A}_i = v_i(\mathbf{A}) \quad (i = 0, 1, \ldots, d).$$

Another interpretation of the sequence $\{v_i(\lambda)\}$ is as follows. If we introduce the column vector $\mathbf{v}(\lambda) = [v_0(\lambda), v_1(\lambda), \ldots, v_d(\lambda)]^t$, then the defining equations are those which arise when we put $v_0(\lambda) = 1$ and solve the system $\mathbf{Bv}(\lambda) = \lambda\mathbf{v}(\lambda)$, using one row of $\mathbf{B}$ at a time. The last row of $\mathbf{B}$ is not used; it gives rise to an equation representing the condition that $\mathbf{v}(\lambda)$ is an eigenvector of $\mathbf{B}$ corresponding to the eigenvalue λ. The roots of this equation in λ are the eigenvalues $\lambda_0, \lambda_1, \ldots, \lambda_d$ of $\mathbf{B}$, and so a *right* eigenvector $\mathbf{v}_i$ corresponding to λ_i has components $(\mathbf{v}_i)_j = v_j(\lambda_i)$.

It is convenient to introduce the *left* eigenvector $\mathbf{u}_i$ corresponding to λ_i; this is a row vector satisfying $\mathbf{u}_i\mathbf{B} = \lambda_i\mathbf{u}_i$. We shall say that a vector $\mathbf{x}$ is *standard* when $x_0 = 1$.

LEMMA 21.3 *Suppose that* $\mathbf{u}_i$ *and* $\mathbf{v}_i$ *are standard left and right eigenvectors corresponding to the eigenvalue* λ_i *of* $\mathbf{B}$. *Then* $(\mathbf{v}_i)_j = k_j(\mathbf{u}_i)_j$, *for all* $i, j \in \{0, 1, \ldots, d\}$.

Proof Each eigenvalue of $\mathbf{B}$ is simple, and so there is a one-dimensional space of corresponding eigenvectors. It follows that there are unique standard eigenvectors $\mathbf{u}_i$ and $\mathbf{v}_i$. (If $(\mathbf{u}_i)_0$ or $(\mathbf{v}_i)_0$ were zero, then the special form of $\mathbf{B}$ would imply that $\mathbf{u}_i = \mathbf{0}$, $\mathbf{v}_i = \mathbf{0}$.)

Let $\mathbf{K}$ denote the diagonal matrix with diagonal entries $k_0, k_1, \ldots, k_d$. Using the equations $b_{i-1}k_{i-1} = c_i k_i$ $(2 \leqslant i \leqslant d)$ we may check that $\mathbf{BK}$ is a symmetric matrix; that is,

$$\mathbf{BK} = (\mathbf{BK})^t = \mathbf{KB}^t.$$

Thus, if $\mathbf{u}_i \mathbf{B} = \lambda_i \mathbf{u}_i$ $(0 \leqslant i \leqslant d)$, we have

$$\mathbf{BKu}_i^t = \mathbf{KB}^t\mathbf{u}_i^t = \mathbf{K}(\mathbf{u}_i\mathbf{B})^t = \mathbf{K}(\lambda_i\mathbf{u}_i)^t = \lambda_i\mathbf{Ku}_i^t.$$

In other words, $\mathbf{Ku}_i^t$ is a right eigenvector of $\mathbf{B}$ corresponding to λ_i. Also $(\mathbf{Ku}_i^t)_0 = 1$, and so, by the uniqueness of $\mathbf{v}_i$, $\mathbf{Ku}_i^t = \mathbf{v}_i$. □

We notice that, when $i \neq l$, the inner product $(\mathbf{u}_i, \mathbf{v}_l)$ is zero, since

$$\lambda_i(\mathbf{u}_i, \mathbf{v}_l) = \mathbf{u}_i\mathbf{B}\mathbf{v}_l = \lambda_l(\mathbf{u}_i, \mathbf{v}_l).$$

The inner product with $i = l$ gives the multiplicity $m(\lambda_i)$.

THEOREM 21.4 *With the notation above, the multiplicity of the eigenvalue λ_i of a distance-regular graph with n vertices is*

$$m(\lambda_i) = \frac{n}{(\mathbf{u}_i, \mathbf{v}_i)} \qquad (0 \leqslant i \leqslant d).$$

Proof For $i = 0, 1, \ldots, d$, define

$$\mathbf{L}_i = \sum_{j=0}^{d} (\mathbf{u}_i)_j \mathbf{A}_j.$$

We calculate the trace of $\mathbf{L}_i$ in two ways. First, the trace of $\mathbf{A}_j$ is zero $(j \neq 0)$, and $\mathbf{A}_0 = \mathbf{I}$, so that

$$\operatorname{tr}\mathbf{L}_i = (\mathbf{u}_i)_0 \,.\, \operatorname{tr}\mathbf{I} = n.$$

On the other hand, since $\mathbf{A}_j = v_j(\mathbf{A})$, the eigenvalues of $\mathbf{A}_j$ are $v_j(\lambda_0), \ldots, v_j(\lambda_d)$ with multiplicities $m(\lambda_0), \ldots, m(\lambda_d)$; consequently the trace of $\mathbf{A}_j$ is $\Sigma m(\lambda_l)\, v_j(\lambda_l)$. Thus

$$\begin{aligned} \operatorname{tr}\mathbf{L}_i &= \sum_j (\mathbf{u}_i)_j \sum_l m(\lambda_l)\,(\mathbf{v}_l)_j \\ &= \sum_l m(\lambda_l)\,(\mathbf{u}_i, \mathbf{v}_l) \\ &= m(\lambda_i)\,(\mathbf{u}_i, \mathbf{v}_i). \quad \square \end{aligned}$$

In the context of our question about the realizability of a given array, we shall view Theorem 21.4 in the following way. The numbers $n/(\mathbf{u}_i, \mathbf{v}_i)$, which are completely determined by the array, represent multiplicities, and consequently they must be positive integers. This turns out to be a very powerful condition.

DEFINITION 21.5 The array $\{k, b_1, \ldots, b_{d-1}; 1, c_2, \ldots, c_d\}$ is *feasible* if the following conditions are satisfied.

(1) The numbers $k_i = (kb_1 \ldots b_{i-1})/(c_2 c_3 \ldots c_i)$ are integers $(2 \leqslant i \leqslant d)$.

(2) $k \geqslant b_1 \geqslant \ldots \geqslant b_{d-1}$; $1 \leqslant c_2 \leqslant \ldots \leqslant c_d$.

(3) If $n = 1 + k + k_2 + \ldots + k_d$ and $a_i = k - b_i - c_i$ $(1 \leqslant i \leqslant d-1)$, $a_d = k - c_d$, then $nk \equiv 0 \pmod 2$ and $k_i a_i \equiv 0 \pmod 2$.

(4) The numbers $n/(\mathbf{u}_i, \mathbf{v}_i)$ $(0 \leqslant i \leqslant d)$ are positive integers.

It is strange that, although these conditions are not sufficient for the existence of a graph with the given array, they are nevertheless so restrictive that most known feasible arrays are in fact realizable.

We shall now give an example of the practical application of the feasibility conditions in a case where the conditions are satisfied. (An important set of examples where the conditions are not satisfied will be encountered in Chapter 23.) The calculation of $n/(\mathbf{u}_i, \mathbf{v}_i)$ is facilitated by the formula of Lemma 21.3, which implies that

$$(\mathbf{u}_i, \mathbf{v}_i) = \Sigma k_j (\mathbf{u}_i)_j^2 = \Sigma \frac{(\mathbf{v}_i)_j^2}{k_j}.$$

Let us consider the array $\{2r, r-1; 1, 4\}$ $(r \geqslant 2)$, and the corresponding matrix

$$\begin{bmatrix} 0 & 1 & 0 \\ 2r & r & 4 \\ 0 & r-1 & 2r-4 \end{bmatrix}.$$

It is easy to verify that conditions (1), (2) and (3) of Definition 21.5 are fulfilled; we have $k = 2r$, $k_2 = \frac{1}{2}r(r-1)$, $n = \frac{1}{2}(r+1)(r+2)$.

The eigenvalues are $\lambda_0 = 2r$, $\lambda_1 = r-2$, $\lambda_2 = -2$, and the calculation of the multiplicities goes as follows:

$$\mathbf{v}_0 = \begin{bmatrix} 1 \\ 2r \\ \frac{1}{2}r(r-1) \end{bmatrix}, \quad \mathbf{v}_1 = \begin{bmatrix} 1 \\ r-2 \\ 1-r \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

$$m(\lambda_1) = \frac{n}{(\mathbf{u}_1, \mathbf{v}_1)} = \frac{\frac{1}{2}(r+1)(r+2)}{1+(r-2)^2/2r+(1-r)^2/\frac{1}{2}r(r-1)} = r+1;$$

$$m(\lambda_2) = \frac{n}{(\mathbf{u}_2, \mathbf{v}_2)} = \frac{\frac{1}{2}(r+1)(r+2)}{1+4/2r+1/\frac{1}{2}r(r-1)} = \tfrac{1}{2}(r-1)(r-2).$$

Thus condition (4) is satisfied and our array is feasible. In fact the array is realized by the triangle graph Δ_{r+2}, as we remarked in the previous chapter. (The eigenvalues and multiplicities of this graph were found in a different way in Chapter 3.)

Another example is the graph Σ representing the 27 lines on a cubic surface (§ 3E, and Chapter 8, p. 55). The information given by Henderson (1912) suffices to show that Σ is a distance-transitive graph with diameter 2 and intersection array $\{16, 5; 1, 8\}$. From this we may calculate that

$$\operatorname{Spec}\Sigma = \begin{pmatrix} 16 & 4 & -2 \\ 1 & 6 & 20 \end{pmatrix},$$

which justifies our earlier statements about Σ.

21*A* *The spectra of Q_k and O_k* (*a*) The eigenvalues of the k-cube Q_k are $\lambda_i = k-2i$ $(0 \leqslant i \leqslant k)$ and $m(\lambda_i) = \binom{k}{i}$. Consequently, the complexity of Q_k is

$$\kappa(Q_k) = 2^{2^k-\frac{1}{2}(k^2-3k-2)} 3^{\binom{k}{3}} 4^{\binom{k}{4}} \dots k^{\binom{k}{k}}$$

(*b*) The eigenvalues of the odd graph O_k are $\lambda_i = (-1)^i(k-i)$ $(0 \leqslant i \leqslant k-1)$, and

$$m(\lambda_i) = \binom{2k-1}{i} - \binom{2k-1}{i-1}.$$

21*B* *Trivalent distance-transitive graphs* There are exactly 12 trivalent distance-transitive graphs. Their diameters (d) and numbers of vertices (n) are (Biggs and Smith 1971):

d:	1	2	2	3	3	4	4	4	5	5	7	8
n:	4	6	10	8	14	18	30	28	20	20	102	90.

21*C* *A feasible array which is not realizable* The array $\{9, 8; 1, 4\}$ is feasible, but there is no graph with this intersection array [B, p. 106].

21*D* *The friendship theorem* If, in a finite set of people, each pair of people has precisely one common friend, then someone is everyone's friend. (Friendship is interpreted as a symmetric, irreflexive relation.) This result may be proved as follows: let Γ denote the graph whose vertices represent people and whose edges join friends. Then Γ is either a graph consisting of a number of triangles all with a common vertex or a distance-regular graph of diameter 2 with intersection array $\{k, k-2; 1, 1\}$. This array is not feasible, so the first possibility must hold (G. Higman, unpublished).

21*E* *The permutation character* If Γ is a distance-transitive graph with diameter d, then the permutation character χ corresponding to the representation of $G(\Gamma)$ on $V\Gamma$ is the sum of $d+1$ irreducible complex characters:

$$\chi = 1 + \chi_1 + \ldots + \chi_d$$

and the labelling can be chosen so that the degree of χ_i is $m(\lambda_i)$ $(0 \leqslant i \leqslant d)$. This can be deduced from the results of Wielandt (1964).

21*F* *Janko's little group* There is a distance-transitive graph with valency 11, diameter 4, 266 vertices, and intersection array $\{11, 10, 6, 1; 1, 1, 5, 11\}$. The automorphism group of this graph is Janko's simple group of order 175 560 (Conway 1971).

22. *Primitivity and imprimitivity*

In this chapter we shall investigate the permutation-group-theoretic notion of primitivity in the context of symmetric and distance-transitive graphs. In the latter case we shall prove that the automorphism group is imprimitive if and only if one of two simple graph-theoretic conditions is satisfied.

We begin by summarizing some terminology which is explained at greater length in [B, Section 1.5]. A *block* B, in the action of a group G on a set X, is a subset of X such that B and $g(B)$ are either disjoint or identical, for each g in G. If G is transitive on X, then we say that the permutation group (X, G) is *primitive* if the only blocks are the *trivial* blocks, that is, those with cardinality 0, 1 or $|X|$. In the case of an imprimitive permutation group (X, G), the set X is partitioned into a disjoint union of non-trivial blocks, which are permuted by G. We shall refer to this partition as a *block system*.

A graph Γ is said to be primitive or imprimitive according as the group $G(\Gamma)$ acting on $V\Gamma$ has the corresponding property.

PROPOSITION 22.1 *Let Γ be an imprimitive symmetric connected graph. Then a block system for the action of $G(\Gamma)$ on $V\Gamma$ is a colour-partition of Γ.*

Proof Suppose that $V\Gamma$ is partitioned by the block system $B^{(1)}, B^{(2)}, \ldots, B^{(l)}$; then we may select a block C, and elements $g^{(i)}$ in $G = G(\Gamma)$, such that $B^{(i)} = g^{(i)}C$ $(1 \leqslant i \leqslant l)$. Suppose C contains two adjacent vertices u and v. Since Γ is symmetric, for each vertex w adjacent to u there is an automorphism g such that $g(u) = u$ and $g(v) = w$. Now u belongs to $C \cap g(C)$ and C is a block, so $C = g(C)$ and w belongs to C. Thus the set $\Gamma_1(u)$ is contained in C, and by repeating the argument we may prove successively that $\Gamma_2(u), \Gamma_3(u), \ldots$ are contained in C. Since Γ is connected, we have $C = V\Gamma$. This contradicts the hypothesis of imprimitivity, and so our assumption that C contains a pair of adjacent vertices is false. Since each block $B^{(i)}$ is the image of C under an automorphism, the block system is a colour-partition. □

This result is false under the weaker hypothesis of vertex-transitivity. For example, the prism T_3 (see §9D) is vertex-transitive and has a block system consisting of two blocks, the vertices of the two triangles in T_3. Clearly, this block system is not a colour-partition of T_3.

The rest of this chapter is devoted to an investigation of the consequences of the stronger hypothesis of distance-transitivity. We shall show that, in the imprimitive case, the vertex-colouring induced by a block system is either a 2-colouring, or a colouring of another quite specific kind.

LEMMA 22.2 *Let Γ be a distance-transitive graph with diameter d, and suppose B is a block in the action of $G(\Gamma)$ on $V\Gamma$. If B contains two vertices u and v such that $\partial(u,v) = j$ $(1 \leqslant j \leqslant d)$, then B contains each set $\Gamma_{rj}(u)$, where r is an integer satisfying $0 \leqslant rj \leqslant d$.*

Proof Let w be any vertex in $\Gamma_j(u)$. Since Γ is distance-transitive there is some automorphism g such that $g(u) = u$ and $g(v) = w$. Thus $u \in B \cap g(B)$ and since B is a block, $B = g(B)$ and $w \in B$. So $\Gamma_j(u) \subseteq B$.

If $z \in \Gamma_{2j}(u)$, there is some vertex $y \in \Gamma_j(u)$ for which $\partial(y,z) = j$. Since $\partial(z,y) = \partial(u,y)$, and both u and y are in B, it follows by a repetition of the argument in the previous paragraph that $z \in B$, and so $\Gamma_{2j}(u) \subseteq B$.

Further repetitions of the argument show that $\Gamma_{rj}(u) \subseteq B$ for each r such that $rj \leqslant d$. □

PROPOSITION 22.3 *Let Γ be a distance-transitive graph with diameter d and valency $k \geqslant 3$, and suppose d' is the largest even integer such that $d' \leqslant d$. Then a non-trivial block (in the action of $G(\Gamma)$ on $V\Gamma$) which contains the vertex u is one of the two sets*

(1) $\{u\} \cup \Gamma_2(u) \cup \Gamma_4(u) \cup \ldots \cup \Gamma_{d'}(u)$;

(2) $\{u\} \cup \Gamma_d(u)$.

Proof Suppose B is a non-trivial block, containing u, and not equal to the set (2); then B contains a vertex v such that $\partial(u, v) = j < d$, and consequently $\Gamma_j(u) \subseteq B$.

Consider the numbers c_j, a_j, b_j in the intersection array of Γ. If a_j were non-zero, then B would contain two adjacent vertices, which is false, by Proposition 22.1; thus $a_j = 0$. Now

$$c_j + a_j + b_j = k \geqslant 3,$$

and so one of c_j, b_j is at least 2. From parts (2) and (3) of Proposition 20.4 it follows that one of c_{j+1}, b_{j-1} is at least 2, and consequently $\Gamma_j(u)$ contains a pair of vertices whose distance is two. Thus B contains the set (1) above, and if it contained any other vertices, it would contain two adjacent vertices and would be trivial. We deduce that B is the set (1). □

The cube Q_3 is an example of an imprimitive distance-transitive graph with diameter 3. One block system consists of four sets $\{u\} \cup \Gamma_3(u)$ of cardinality two, while another block system consists of two sets $\{u\} \cup \Gamma_2(u)$ of cardinality four. It is quite common for an imprimitive group to be imprimitive in more ways than one, and our example demonstrates this behaviour in the two ways allowed by Proposition 22.3.

We now turn to an investigation of the graph-theoretical consequences of the two kinds of imprimitivity in distance-transitive graphs.

LEMMA 22.4 *Let Γ be a distance-transitive graph with girth* 3 *and diameter $d \geqslant 2$, in which the set $X = \{u\} \cup \Gamma_2(u) \cup \ldots \cup \Gamma_{d'}(u)$ is a block. Then $d = 2$ (and consequently $X = \{u\} \cup \Gamma_2(u)$).*

Proof Since Γ contains triangles and is distance-transitive, every ordered pair of adjacent vertices belongs to a triangle. Choose adjacent vertices $v_1 \in \Gamma_1(u)$, $v_2 \in \Gamma_2(u)$; then there is some vertex z such that $v_1 v_2 z$ is a triangle. If $z \in \Gamma_2(u)$, then X contains two adjacent vertices, contrary to Proposition 22.1. Thus $z \in \Gamma_1(u)$.

If $d \geqslant 3$, we can find a vertex $v_3 \in \Gamma_3(u)$ adjacent to v_2 (Fig. 14).

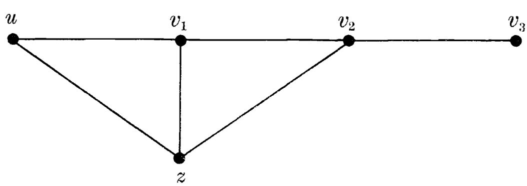


Fig. 14

Now $\Gamma_2(v_3)$ contains the adjacent vertices v_1 and z, and if h is an automorphism of Γ taking u to v_3, $h(X)$ is a block containing adjacent vertices, again contradicting Proposition 22.1. Thus $d = 2$. □

PROPOSITION 22.5 *Let Γ be a distance-transitive graph with diameter $d \geqslant 3$, and valency $k \geqslant 3$. Then*

$$X = \{u\} \cup \Gamma_2(u) \cup \ldots \cup \Gamma_{d'}(u)$$

is a block if and only if Γ is bipartite.

Proof Suppose Γ is bipartite. If X were not a block, then we should have an automorphism g of Γ such that X and $g(X)$ intersect but are not identical. This would imply that there are vertices x and y in X, for which $g(x) \in X$ but $g(y) \notin X$, so that $\partial(x, y)$ is even and $\partial(g(x), g(y))$ is odd. From this contradiction we conclude that X is a block.

Conversely, suppose X is a block. A minimal odd circuit in Γ has length $2j+1 > 3$ (by Lemma 22.4), and we may suppose this circuit to be $uu_1 \ldots w_1 v_1 v_2 w_2 \ldots u_2 u$, where

$$u_1, u_2 \in \Gamma_1(u), \quad w_1, w_2 \in \Gamma_{j-1}(u), \quad v_1, v_2 \in \Gamma_j(u),$$

(and if $j = 2$, then $u_1 = w_1$ and $u_2 = w_2$). If j is even, then X contains the adjacent vertices v_1 and v_2, and so $X = V\Gamma$, a contradiction. If j is odd we have, for $i = 1, 2$, $\partial(u, w_i) = \partial(u_i, v_i)$, and so there is no automorphism h_i taking u to u_i and w_i to v_i. Thus $Y_i = h_i(X)$ is a block containing u_i and v_i. But, since Γ contains no triangles, $\partial(u_1, u_2) = 2$ and so $u_2 \in Y_1$. Consequently $Y_1 = Y_2$ and we have adjacent vertices v_1, v_2 in Y_1, so that $Y_1 = V\Gamma$, $X = V\Gamma$. From this contradiction we deduce that Γ has no odd circuits and its bipartite. □

In summary, Lemma 22.4 and Proposition 22.5 lead to the conclusion that, if a block of the type (1) described in Proposition 22.3 exists in a distance-transitive graph Γ, then either $d = 2$, in which case the block is simultaneously of type (2), or $d \geqslant 3$ and Γ is bipartite. The complete tripartite graphs $K_{r,r,r}$ exemplify the case dealt with in Lemma 22.4 and are clearly not bipartite.

The graphs which have blocks of type (2) can also be given a simple graph-theoretical characterization, which now follows.

DEFINITION 22.6 A graph of diameter d is said to be *antipodal* if, when we are given vertices u, v, w such that $\partial(u,v) = \partial(u,w) = d$, then it follows that $\partial(v,w) = d$ or $v = w$.

The cubes Q_k are trivially antipodal, since every vertex has a unique vertex at maximum distance from it; these graphs are at the same time bipartite. The dodecahedron is also trivially antipodal and not bipartite. Examples of graphs which are non-trivially antipodal and not bipartite are the complete tripartite graphs $K_{r,r,r}$ which have diameter 2, and the line graph of Petersen's graph, which has diameter 3.

PROPOSITION 22.7 *A distance-transitive graph* Γ *of diameter* d *has a block* $X = \{u\} \cup \Gamma_d(u)$ *if and only if* Γ *is antipodal.*

Proof Suppose Γ is antipodal. Then if $X = \{u\} \cup \Gamma_d(u)$ and x is in X, it follows that $X = \{x\} \cup \Gamma_d(x)$. Consequently, if g is any automorphism of Γ, and z is in $X \cap g(X)$ then

$$X = \{z\} \cup \Gamma_d(z) = g(X);$$

thus X is a block.

Conversely, suppose X is a block, and v, w belong to $\Gamma_d(u)$ $(v \neq w)$. Let $\partial(v,w) = j$ $(1 \leqslant j \leqslant d)$, and let h be any automorphism of Γ such that $h(v) = u$. Then $h(w)$ is in $\Gamma_j(u)$; also $h(w)$ belongs to $h(X) = X$, since $h(X)$ intersects X (in u) and X is a block. This is impossible for $1 \leqslant j < d$, so that $\partial(v,w) = d$, and Γ is antipodal. $\square$

THEOREM 22.8 (Smith 1971) *An imprimitive distance-transitive graph is either bipartite or antipodal.* (*Both possibilities can occur in the same graph.*)

Proof A non-trivial block is of the type (1) or (2) described in Proposition 22.3. In the case of a block of type (1), Proposition 22.5 tells us that the graph is either bipartite, or its diameter is less than 3. If the diameter is 1, then the graph is complete, and consequently primitive. If the diameter is 2, a block of type (1) is simultaneously of type (2). Consequently, if the graph is not bipartite, it must be antipodal. $\square$

DEFINITION 22.9 An *automorphic* graph is a distance-transitive graph which is primitive (and not a complete graph or a line graph).

Automorphic graphs are apparently very rare. For instance, there are exactly three automorphic graphs whose valency is three (Biggs and Smith 1971) and just one whose valency is four. The three trivalent graphs can be neatly described by the following diagrams (Fig. 15).

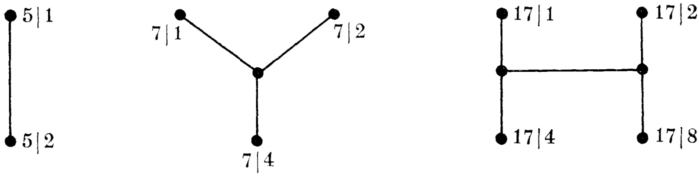


Fig. 15

By way of explanation we should say that the second diagram represents a graph with 28 vertices, consisting of 7 copies of the given four-vertex configuration, with the free ends joined up by star polygons as indicated. In the same way, the first diagram represents Petersen's graph, and the last diagram represents a graph with 102 vertices.

22*A* *The derived graph of an antipodal graph* Let Γ be a distance-transitive antipodal graph, with valency k and diameter $d > 2$. Define the *derived graph* Γ' by taking the vertices of Γ' to be the blocks $\{u\} \cup \Gamma_d(u)$ in Γ, two blocks being joined in Γ' whenever they contain adjacent vertices of Γ. Then Γ' is a distance-transitive graph with valency k and diameter equal to the integer part of $\frac{1}{2}d$ (Smith 1971).

22*B* *The icosahedron and the dodecahedron* These graphs were described in §§10C and 10D. The graph I is distance-transitive with $\iota(I) = \{5, 2, 1;\ 1, 2, 5\}$; the graph D is distance-transitive with $\iota(D) = \{3, 2, 1, 1, 1;\ 1, 1, 1, 2, 3\}$. Both graphs are antipodal, and their derived graphs are K_6 and O_3 respectively. From the intersection arrays we may calculate the spectra, and hence the complexity: $\kappa(I) = \kappa(D) = 5\,184\,000$.

22*C* *Antipodal graphs covering* $K_{k,k}$ Let Γ be a distance-transitive antipodal graph, whose derived graph is $K_{k,k}$, and suppose each block in Γ contains r vertices. Then r must divide k, and if this is so the intersection array for Γ is

$$\{k, k-1, k-t, 1;\ 1, t, k-1, k\},$$

where $rt = k$. This array is feasible (provided that t divides k) and the spectrum of Γ is

$$\operatorname{Spec}\Gamma = \begin{pmatrix} k & \sqrt{k} & 0 & -\sqrt{k} & -k \\ 1 & k(r-1) & 2(k-1) & k(r-1) & 1 \end{pmatrix}.$$

In the case $r = k$, the existence of Γ implies the existence of a projective plane of order k (Gardiner).

22*D* *A 5-valent automorphic graph* Let $L = \{a, b, c, d, e, f\}$, and $N = \{1, 2, 3, 4, 5, 6\}$. Define a graph Γ whose vertex-set is $L \times N$, and in which (l_1, n_1) is adjacent to (l_2, n_2) if and only if the transposition $(n_1 n_2)$ occurs in position (l_1, l_2) of the following 'syntheme-duad' table (Coxeter 1958):

	b	c	d	e	f
a	(15) (23) (46)	(14) (26) (35)	(13) (24) (56)	(12) (36) (45)	(16) (25) (34)
b		(12) (34) (56)	(14) (25) (36)	(16) (24) (35)	(13) (26) (45)
c			(16) (23) (45)	(13) (25) (46)	(15) (24) (36)
d				(15) (26) (34)	(12) (35) (46)
e					(14) (23) (56)

Then Γ is an automorphic graph of valency 5 and diameter 3. Its intersection array is $\{5, 4, 2;\ 1, 1, 4\}$ and its automorphism group is $\operatorname{Aut} S_6$.

23. *Minimal regular graphs with given girth*

The results of Chapter 21, on the feasibility of intersection arrays, can be applied to a wide range of combinatorial problems. We shall conclude this tract by applying those results to a graph-theoretical problem which has attracted considerable attention, and which has equivalent formulations in other fields (for example, in terms of the generalized polygons defined in § 23A).

We shall be dealing with regular graphs whose valency ($k \geqslant 3$) and girth ($g \geqslant 3$) are given; such graphs exist for all these values of k and g (Tutte 1966, p. 81).

PROPOSITION 23.1 (1) *The number of vertices in a graph with valency k and odd girth g $(= 2d+1)$ is at least*

$$n_0(k, g) = 1 + k + k(k-1) + \ldots + k(k-1)^{\frac{1}{2}(g-3)}.$$

If there is such a graph, having exactly $n_0(k, g)$ vertices, then it is distance-regular with diameter d, and its intersection array is

$$\{k, k-1, k-1, \ldots, k-1;\ 1, 1, 1, \ldots, 1\}.$$

(2) *The number of vertices in a graph with valency k and even girth g $(= 2d)$ is at least*

$$n_0(k, g) = 1 + k + k(k-1) + \ldots + k(k-1)^{\frac{1}{2}g-2} + (k-1)^{\frac{1}{2}g-1}.$$

If there is such a graph, having exactly $n_0(k, g)$ vertices, then it is bipartite, distance-regular with diameter d, and its intersection array is

$$\{k, k-1, k-1, \ldots, k-1;\ 1, 1, 1, \ldots, 1, k\}.$$

Proof (1) Suppose that Γ is a graph with valency k and girth $g = 2d+1$, and let (u, v) be any pair of vertices such that $\partial(u, v) = j$ $(1 \leqslant j \leqslant d)$. The number of vertices in $\Gamma_{j-1}(v)$ adjacent to u is precisely one, otherwise we should have a circuit of length at most $2j < 2d+1$ in Γ. Using the notation of Definition 20.5, we have shown the existence of the numbers $c_1 = 1, \ldots, c_d = 1$.

Similarly, if $1 \leqslant j \leqslant d$, then there are no vertices in $\Gamma_j(v)$ adjacent to u, otherwise we should have a circuit of length at most $2j+1 < 2d+1$. In the usual notation, this means that $a_j = 0$ and consequently $b_j = k - a_j - c_j = k-1$, for $1 \leqslant j < d$.

We have now proved that the diameter of Γ is at least d, and that Γ has at least $n_0(k,g)$ vertices. If Γ has just $n_0(k,g)$ vertices, its diameter must be precisely d, which implies that $a_d = 0$, and Γ has the stated intersection array.

(2) In this case the argument proceeds as in (1), except that c_d may be greater than one. Now the recurrence for the numbers $k_i = |\Gamma_i(v)|$ shows that k_d is smallest when $c_d = k$; if this is so, then Γ has at least $n_0(k,g)$ vertices. If Γ has exactly $n_0(k,g)$ vertices, then its diameter is d, and it has the stated intersection array. The form of this array shows that Γ has no odd circuits, and so it is bipartite. □

DEFINITION 23.2 A (k,g)-*graph* is a graph with valency k, girth g, and $n_0(k,g)$ vertices.

It is important to remark that, for a given k and g, there will always be some graph such that there are no smaller graphs with the same valency and girth. However, this graph need not be a (k,g)-graph, for it will not necessarily attain the lower bound, $n_0(k,g)$, for the number of vertices. For example, there is a unique smallest graph with valency 3 and girth 7; it has 24 vertices, whereas $n_0(3,7) = 22$. Consequently, there is no $(3,7)$-graph.

Proposition 23.1 allows us to use the powerful methods of Chapter 21 to study the existence of (k,g)-graphs. In the cases $g = 3$ and $g = 4$ the intersection arrays in question are

$$\{k;\ 1\} \quad \text{and} \quad \{k, k-1;\ 1, k\}$$

and these are feasible for all $k \geqslant 3$. It is very easy to see that each array has a unique realization – the complete graph K_{k+1} and the complete bipartite graph $K_{k,k}$, respectively. Thus, for $g = 3$ and $g = 4$, the question of the existence of (k,g)-graphs is easily settled in the affirmative.

For $g \geqslant 5$ the problem is much more subtle, both in the technical details and in the nature of the final solution. The remarkable results (which are not yet quite complete) are due to several mathematicians: among those who study the case when g is even are Feit and Higman (1964), Singleton (1966), Benson (1966); the case when g is odd has been investigated by Hoffman and Singleton (1960), Vijayan (1972), Damerell (1973) and Bannai and Ito (1973). The methods of Damerell are most closely related to those of our tract, and we shall apply the techniques of his paper to both odd and even values of g.

We shall study the feasibility of the following intersection matrix:

$$\mathbf{B} = \begin{bmatrix} 0 & 1 & & & & & \\ k & 0 & 1 & & & & \\ & k-1 & 0 & \cdot & & & \\ & & k-1 & \cdot & \cdot & & \\ & & & \cdot & \cdot & 1 & \\ & & & & \cdot & 0 & c \\ & & & & & k-1 & k-c \end{bmatrix},$$

which subsumes, when $c = 1$ and $c = k$, the intersection arrays relevant to the existence of (k, g)-graphs. Suppose that λ is an eigenvalue of $\mathbf{B}$ and that the corresponding standard left eigenvector is $\mathbf{u}(\lambda) = [u_0(\lambda), u_1(\lambda), \ldots, u_d(\lambda)]$. Then, from the equations $\mathbf{u}(\lambda)\mathbf{B} = \lambda\mathbf{u}(\lambda)$ and $u_0(\lambda) = 1$, we deduce that $u_1(\lambda) = \lambda/k$ and

$$(*) \quad (k-1)u_i(\lambda) - \lambda u_{i-1}(\lambda) + u_{i-2}(\lambda) = 0 \quad (i = 2, 3, \ldots, d),$$

$$(**) \quad cu_{d-1}(\lambda) + (k-c-\lambda)u_d(\lambda) = 0.$$

It is helpful to view these equations in the following way. The equations (*) give a recursion which enables us to express $u_i(\lambda)$ as a polynomial of degree i in λ for $0 \leqslant i \leqslant d$; the equation (**) then becomes a polynomial equation of degree $d+1$ in λ. In fact (**) represents the condition that λ is an eigenvalue; it is the characteristic equation of $\mathbf{B}$.

We now put $q = \sqrt{(k-1)}$ and suppose that $|\lambda| < 2q$, so that we may write $\lambda = 2q\cos\alpha$ with $0 < \alpha < \pi$ (this assumption will be justified in due course). The solution to the recursion (*) can be found explicitly:

$$u_i = \frac{q^2 \sin(i+1)\alpha - \sin(i-1)\alpha}{kq^i \sin\alpha} \quad (1 \leqslant i \leqslant d.)$$

LEMMA 23.3 *With the above notation, the number* $2q\cos\alpha$ *is an eigenvalue of* $\mathbf{B}$ *if and only if*

$$q\sin(d+1)\alpha + c\sin d\alpha + \left(\frac{c-1}{q}\right)\sin(d-1)\alpha = 0.$$

Proof The stated equation results from substituting the explicit forms of u_{d-1} and u_d in the equation (**), which is the characteristic equation of $\mathbf{B}$. □

PROPOSITION 23.4 (1) *Let* $g = 2d$ *and suppose* Γ *is a* (k, g)*-graph. Then* Γ *has* $d+1$ *distinct eigenvalues:*

$$k, \ -k, \quad 2q\cos\pi j/d \quad (j = 1, 2, \ldots, d-1).$$

(2) *Let* $g = 2d+1$ *and suppose* Γ *is a* (k, g)*-graph. Then* Γ *has* $d+1$ *distinct eigenvalues:*

$$k, \quad 2q\cos\alpha_j \quad (j = 1, 2, \ldots, d),$$

where the numbers $\alpha_1, \ldots, \alpha_d$ *are the distinct solutions in the interval* $0 < \alpha < \pi$ *of the equation* $q\sin(d+1)\alpha + \sin d\alpha = 0$.

Proof (1) The existence of the eigenvalues k and $-k$ follows from the fact that Γ is k-valent and bipartite. Now the eigenvalues of Γ are (by Proposition 21.2) the $d+1$ eigenvalues of its intersection matrix, which is the matrix given above, with $c = k$. In this case, $\lambda = 2q\cos\alpha$ is an eigenvalue of $\mathbf{B}$ if and only if

$$q\sin(d+1)\alpha + k\sin d\alpha + q\sin(d-1)\alpha = 0.$$

This reduces to $(2q\cos\alpha + k)\sin d\alpha = 0$, and since $|k/2q| > 1$ when $k \geqslant 3$, the only possibility is that $\sin d\alpha = 0$. Thus there are $d-1$ solutions $\alpha = \pi j/d\ (j = 1, \ldots, d)$ in the range $0 < \alpha < \pi$, and we have the required total of $d+1$ eigenvalues in all.

(2) Since Γ is k-valent, k is an eigenvalue. As in (1), we now seek eigenvalues $\lambda = 2q\cos\alpha$ of $\mathbf{B}$, this time with $c = 1$. The equation of Lemma 23.3 reduces to

$$\Delta(\alpha) \equiv q\sin(d+1)\alpha + \sin d\alpha = 0.$$

For $1 \leqslant j \leqslant d$, Δ is strictly positive at $\theta_j = (j-\frac{1}{2})\pi/(d+1)$ and strictly negative at $\phi_j = (j+\frac{1}{2})\pi/(d+1)$. Hence there is a zero α_j of Δ in each one of the d intervals (θ_j, ϕ_j). Thus we have the required total of $d+1$ eigenvalues in all. □

We now have sufficient information to calculate the multiplicities of the eigenvalues of a (k, g)-graph, and consequently, to test the feasibility of the corresponding intersection array.

Suppose that λ is an eigenvalue of our matrix $\mathbf{B}$. The multiplicity of λ as an eigenvalue of the putative graph is given by the formula $m(\lambda) = n/(\mathbf{u}(\lambda), \mathbf{v}(\lambda))$ of Theorem 21.4, which we shall utilize in the form $m(\lambda) = n/\Sigma k_i u_i(\lambda)^2$. We have, for our particular matrix $\mathbf{B}$, $k_0 = 1$, $k_i = k(k-1)^{i-1}$ $(1 \leqslant i < d)$, and $k_d = c^{-1}k(k-1)^{d-1}$. Further, for any eigenvalue $\lambda = 2q\cos\alpha$ we have

$$k_i u_i(\lambda)^2 = kq^{2i-2}\left[\frac{q^2\sin(i+1)\alpha - \sin(i-1)\alpha}{kq^i\sin\alpha}\right]^2$$
$$= (2hk\sin^2\alpha)^{-1}(E + F\cos 2i\alpha + G\sin 2i\alpha)$$
$$(1 \leqslant i < d),$$

where we have written $h = q^2 = k-1$, and $E = (h^2+1) - 2h\cos\alpha$, $F = 2h - (h^2+1)\cos 2\alpha$, $G = (h^2-1)\sin 2\alpha$. Allowing for the anomalous form of k_d by means of a compensating term, we can sum the trigonometric series involved in $\Sigma k_i u_i(\lambda)^2$ to obtain:

$$1 + (2hk\sin^2\alpha)^{-1}\left[dE + \{F\cos(d+1)\alpha + G\sin(d+1)\alpha\}\frac{\sin d\alpha}{\sin\alpha}\right.$$
$$\left. + (c^{-1}-1)(E + F\cos 2d\alpha + G\sin 2d\alpha)\right].$$

Fortunately this expression can be simplified considerably in the two cases $(c = 1, k)$ which are of special interest.

PROPOSITION 23.5 *Let λ be an eigenvalue of a (k, g)-graph, with $|\lambda| \neq k$. If g is even, the multiplicity of λ is given by*

$$m(\lambda) = \frac{nk}{g}\left[\frac{4h-\lambda^2}{k^2-\lambda^2}\right] \quad (h = k-1).$$

If g is odd, the multiplicity of λ is given by

$$m(\lambda) = \frac{nk}{g}\left[\frac{4h-\lambda^2}{(k-\lambda)(f+\lambda)}\right] \quad (h = k-1,\ f = k + (k-2)/g).$$

Proof In the case of even girth ($c = k$), we know that $\lambda = 2q\cos\alpha$ is an eigenvalue if and only if $\sin d\alpha = 0$. In this case our expression for $\Sigma k_i u_i(\lambda)^2$ becomes

$$1 + (2hk\sin^2\alpha)^{-1}[dE + hk^{-1}(E+F)] = (2hk\sin^2\alpha)\, dE,$$

which leads, on putting $2d = g$, $\lambda = 2q\cos\alpha$ to the stated formula for $m(\lambda)$.

In the case of odd girth ($c = 1$), we know that $\lambda = 2q\cos\alpha$ is an eigenvalue if and only if $q\sin(d+1)\alpha + \sin d\alpha = 0$. From this equation we have

$$\tan d\alpha = \frac{-q\sin\alpha}{1+q\cos\alpha}; \quad \sin d\alpha = \frac{-q\cos\alpha}{\sqrt{(k+\lambda)}};$$

$$\sin(d+1)\alpha = \frac{\sin\alpha}{\sqrt{(k+\lambda)}}; \quad \cos(d+1)\alpha = \frac{q+\cos\alpha}{\sqrt{(k+\lambda)}}.$$

Substituting for the relevant quantities in our general expression, and putting $g = 2d+1$, we obtain, after much tedious algebra, the stated formula for $m(\lambda)$. $\square$

We now come to a major theorem, which is the result of the combined efforts of the several mathematicians mentioned earlier in this chapter.

THEOREM 23.6 *The intersection array for a $(k,\ g)$-graph with $k \geqslant 3$, $g \geqslant 4$ is feasible if and only if either*

$$(1)\ \ g \in \{4, 6, 8, 12\} \quad \textit{or} \quad (2)\ \ g = 5 \textit{ and } k \in \{3, 7, 57\}.$$

Proof Suppose g is even, $g = 2d$. Then a (k, g)-graph has $d-1$ eigenvalues $\lambda_j = 2\sqrt{(k-1)}\cos(\pi j/d)$ with multiplicities

$$m(\lambda_j) = \frac{nk}{g}\left[\frac{4h - \lambda_j^2}{k^2 - \lambda_j^2}\right].$$

If $m(\lambda_1)$ is a natural number, λ_1^2 is rational, which means that $\cos 2\pi/d$ is rational. But it is well known [I. Niven, *Irrational numbers* (Wiley, 1956), p. 37] that this is so if and only if $d \in \{2, 3, 4, 6\}$. Thus we have the case (1).

The case when g is odd presents more problems. We deal with $g = 5$ and $g = 7$ separately, and then dispose of $g \geqslant 9$.

Suppose $g = 5$. Then the characteristic equation

$$q \sin 3\alpha + \sin 2\alpha = 0$$

reduces in terms of $\lambda = 2q \cos\alpha$ to $\lambda^2 + \lambda - (k-1) = 0$. Thus there are two eigenvalues $\lambda_1 = \frac{1}{2}(-1+\sqrt{D})$ and $\lambda_2 = \frac{1}{2}(-1-\sqrt{D})$, where $D = 4k-3$. In our formula for $m(\lambda)$ we have $n = 1+k^2$, so

$$m(\lambda) = \frac{(k+k^3)(4k-4-\lambda^2)}{(k-\lambda)(6k-2+5\lambda)}.$$

If $\sqrt{D}$ is irrational, we multiply out the expression above, substitute $\lambda = \frac{1}{2}(-1 \pm \sqrt{D})$ and equate the coefficients of $\sqrt{D}$. This gives $5m+k-2 = k+k^3$, where $m = m(\lambda_1) = m(\lambda_2)$. But there are three eigenvalues in all: k, λ_1, λ_2 with multiplicities $1, m, m$; hence $1+2m = n = 1+k^2$. Thus $5k^2-4 = 2k^3$, which has no solution for $k \geqslant 3$.

Consequently $\sqrt{D}$ must be rational, $s = \sqrt{D}$, say. Then $k = \frac{1}{4}(s^2+3)$ and substituting for λ_1 and k in terms of s in our expression for $m_1 = m(\lambda_1)$ we obtain the following polynomial equation in s:

$$s^5 - s^4 + 6s^3 - 2s^2 + (9-m_1)s - 15 = 0.$$

Thus s divides 15, and the possibilities are $s = 1, 3, 5, 15$, giving $k = 1, 3, 7, 57$. Discounting the first possibility, we can check explicitly that the three others do lead to feasible intersection arrays.

Suppose $g = 7$. Then the characteristic equation

$$q \sin 4\alpha + \sin 3\alpha = 0$$

reduces in terms of $\lambda = 2q\cos\alpha$ to $\lambda^3 + \lambda^2 - 2(k-1)\lambda - (k-1) = 0$. This equation has no rational (and consequently integral) roots, since we may write it in the form $k-1 = \lambda^2(\lambda+1)/(2\lambda+1)$, and if any prime divisor of $2\lambda+1$ divides $x = \lambda$ or $\lambda+1$ it must divide $2\lambda+1-x = \lambda+1$ or λ, which is impossible. If the roots λ_1, λ_2 λ_3 are all irrational, then their multiplicities are all equal, to m say, and $1+3m = n = 1+k-k^2+k^3$, whereas $k+m(\lambda_1+\lambda_2+\lambda_3) = 0$ (the trace of $\mathbf{A}$). But $\lambda_1+\lambda_2+\lambda_3 = -1$, hence

$$m = k = \tfrac{1}{3}(k^3-k^2+k),$$

which is impossible for $k \geqslant 3$. Thus there are no (k, g)-graphs when $g = 7$.

Suppose $g \geqslant 9$. We obtain a contradiction here by first proving that $-1 < \lambda_1+\lambda_d < 0$, and then showing that all eigenvalues must in fact be integers. (Our argument just fails in the case $k = 3$, $g = 9$, but this can be discarded by an explicit calculation of eigenvalues and 'multiplicities'.)

Let α_i $(1 \leqslant i \leqslant d)$ be the roots of

$$\Delta(\alpha) \equiv q \sin(d+1)\alpha + \sin d\alpha = 0,$$

and set $\omega = \pi/(d+1)$. We saw in the proof of Proposition 23.4 that α_1 lies between $\omega/2$ and $3\omega/2$, and these bounds can be improved by noting that

$$\Delta(\alpha) \equiv (q+\cos\alpha)\sin(d+1)\alpha - \sin\alpha\cos(d+1)\alpha$$

is positive at ω and negative at $\omega(1+1/2q)$. Thus

$$\omega < \alpha_1 < \omega(1+1/2q)$$

and
$$\begin{aligned} 0 < 2q\cos\omega - 2q\cos\alpha_1 &< 2q\cos\omega - 2q\cos\omega(1+1/2q) \\ &= 2q\cos\omega(1-\cos\omega/2q) \\ &\qquad + 2q\sin\omega\sin\omega/2q \\ &< 2q\times\tfrac{1}{2}(\omega/2q)^2 + 2q\omega(\omega/2q) \\ &= (1/4q+1)\,\omega^2 \\ &< 5\omega^2/4. \end{aligned}$$

In a similar fashion we can prove that $d\omega < \alpha_d < \omega(d+1/2q)$ and deduce

$$0 < 2q\cos d\omega - 2q\cos\alpha_d < \omega^2.$$

Adding our two inequalities, noting that

$$\lambda_1 = 2q\cos\alpha_1, \quad \lambda_d = 2q\cos\alpha_d, \quad \text{and} \quad \cos d\omega = -\cos\omega,$$

we have
$$-9\omega^2/4 < \lambda_1+\lambda_d < 0.$$

Now $\omega^2 = \pi^2/(d+1)^2 \leqslant \pi^2/5^2 < 4/9$, so $-1 < \lambda_1+\lambda_d < 0$, as promised.

To show that the eigenvalues must be integers we note first that since the characteristic equation is monic with integer coefficients, the eigenvalues are algebraic integers. Now the formula for $m(\lambda)$ is the quotient of two quadratic expressions in λ, and so $m(\lambda)$ is integral only if λ is at worst a quadratic irrational. Suppose λ is a quadratic irrational. Then

$$R(\lambda) = gm(\lambda)/nk = (4h-\lambda^2)/(k-\lambda)(f+\lambda)$$

is rational, and we may write this in the form

$$(R(\lambda)-1)\lambda^2 + R(\lambda)(f-k)\lambda - (R(\lambda)fk-4h) = 0.$$

But this must be a multiple of the minimal equation for λ, which is monic with integer coefficients. In particular

$$S(\lambda) = \frac{(f-k)R(\lambda)}{R(\lambda)-1} = \frac{4h-\lambda^2}{\lambda-t} \quad \left(t = \frac{fk-4h}{f-k}\right),$$

must be an integer.

However, $f = k+(k-2)/g > k$, so $t > (k^2-4h)/(f-k) = g(k-2)$, and consequently $|\lambda-t| > g(k-2)-k$ since $|\lambda| < k$. Thus

$$|S(\lambda)| \leqslant \frac{4h}{|\lambda-t|} \leqslant \frac{4k-4}{g(k-2)-k} < 1,$$

for all $k \geqslant 3$, $g \geqslant 9$ (except $k = 3$, $g = 9$, which we have noted). Since $S(\lambda)$ is to be an integer, we must have $S(\lambda) = 0$, which leads to the absurdity $R(\lambda) = m(\lambda) = 0$.

Thus all eigenvalues λ must be integers, which is incompatible with our inequality $-1 < \lambda_1 + \lambda_d < 0$, and consequently disposes of all cases with $g \geqslant 9$. □

The existence of the graphs allowed by Theorem 23.6 is not completely settled. In the case of even girth, we have already remarked that a $(k,4)$-graph (the graph $K_{k,k}$) exists for all $k \geqslant 3$, but (k,g)-graphs with $g \in \{6, 8, 12\}$ are known to exist only when $k-1$ is a prime power. For instance, a $(k,6)$-graph exists if and only if there is a projective plane with k points on each line (Singleton 1966). Also, the only methods we have for constructing $(k,8)$-graphs and $(k,12)$-graphs are based on properties of finite projective geometries in four and six dimensions (Benson 1966).

(The $(3, 12)$-graph is an exception to this statement, and it can be constructed by the simpler means described in § 23D.)

In the case of odd girth $g > 3$, the only possibilities allowed by Theorem 23.6 are those with $g = 5$ and $k \in \{3, 7, 57\}$. The $(3, 5)$-graph is Petersen's graph, and the $(7, 5)$-graph was constructed and proved unique by Hoffman and Singleton (1960); we give an equivalent construction in § 23C. The $(57, 5)$-graph is rather enigmatic; many claims of its non-existence have been made, but none published. However, the results of Aschbacher (1971) show that there is no distance-transitive $(57, 5)$-graph, and so the construction of the graph (if there is one) is certain to be highly complicated.

23*A* *Generalized polygons* Suppose (P, L, I) is an incidence system consisting of two disjoint finite sets, P (points) and L (lines), and an incidence relation I between points and lines. A sequence whose terms are alternately points and lines, each term being incident with its successor, is called a chain; it is a proper chain if there are no repeated terms, except possibly when the first and last terms are identical (when we speak of a closed chain). A non-degenerate *generalized m-gon* is an incidence system with the properties:

(*a*) each pair of elements of $P \cup L$ is joined by a chain of length at most m;

(*b*) for some pairs of elements of $P \cup L$ there is no proper chain of length less than m joining them;

(*c*) there are no closed chains of length less than $2m$. Suppose there are x points on each line and y lines through each point $(x > 2, y > 2)$; then if $x = y = k$ we have a $(k, 2m)$-graph, and so $m = 2, 3, 4, 6$. More generally, if $x \neq y$ we must have $m = 2, 3, 4, 6, 8, 12$ (Feit and Higman 1964).

23*B* *A minimal graph which is not a (k, g)-graph* There is a unique minimal graph of valency 4 and girth 5. It has 19 vertices, whereas $n_0(4, 5) = 17$ (Robertson 1964).

23*C* *The Hoffman–Singleton graph* The unique $(7, 5)$-graph may be constructed by extending the graph of § 22D as follows. Add 14 new vertices, called L, N, a, b, c, d, e, f, 1, 2, 3, 4, 5, 6; join L to a, b, c, d, e, f and N; join N to 1, 2, 3, 4, 5, 6 and L. Also, join the vertex (l, n) of § 22D to l and n. The automorphism group of this graph is the group of order 252 000 obtained from $PSU(3, 5^2)$ by adjoining the field automorphism of $GF(5^2)$ (Hoffman and Singleton 1960).

23*D The* (3, 12)-*graph* Suppose we are given a unitary polarity of the plane $PG(2, 3^2)$. There are 63 points of the plane which do not lie on their polar lines, and they form 63 self-polar triangles (Edge 1963). The (3, 12)-graph is the graph whose 126 vertices are these 63 points and 63 triangles, with adjacent vertices corresponding to an incident {point, triangle} pair. The graph is not vertex-transitive, since there is no automorphism taking a 'point' vertex to a 'triangle' vertex.

23*E A* 7-*transitive graph* The (4, 12)-graph, constructed by Benson (1966), is the smallest 7-transitive graph. It has 728 vertices, and its automorphism group has order 8 491 392.

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Index