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Original Article

Complexity of graphs generated by wheel graph and their asymptotic limits

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ABSTRACT

The literature is very rich with works deal with the enumerating the spanning trees in any graph G since the pioneer Kirchhoff (1847). Generally, the number of spanning trees in a graph can be acquired by directly calculating an associated determinant corresponding to the graph. However, for a large graph, evaluating the pertinent determinant is ungovernable. In this paper, we introduce a new technique for calculating the number of spanning trees which avoids the strenuous computation of the determinant for calculating the number of spanning trees. Using this technique, we can obtain the number of spanning trees of any graph generated by the wheel graph. Finally, we give the numerical result of asymptotic growth constant of the spanning trees of studied graphs.

1. Introduction

The research of the complexity of a graph has a comparatively long history. The importance of this research line is in fact due to:

1- Investigating the possible particle transitions of masers using energy analysis,
2- Estimating the accuracy of a network,
3- Recounting specific chemical isomers,
4- Electrical circuits layout,
5- Enumerating the number of Eulerian tours in a graph [1–10].

The complexity (the number of spanning trees) $\tau(G)$ of a finite connected undirected graph G is defined as the total number of distinct connected acyclic spanning subgraphs.

There are many techniques to compute this number. Kirchhoff [11] gave the famous matrix tree theorem. In which $\tau(G) =$ any cofactor of $L(G)$, where $L(G)$ is equal to the degree matrix $D(G)$ of G minus the adjacency matrix $A(G)$ of G.

Another method to count the complexity of a graph is using Laplacian eigenvalues. Let G be a connected graph with n vertices. Kelmans and Chelnoknov [12] derived the following formula:

$$\tau(G) = \frac{1}{n}\prod_{k=1}^{n-1} \mu_k. \tag{1}$$

Where $n = \mu_1 \geq \mu_2 \geq \ldots \geq \mu_n = 0$ are the eigenvalues of the Laplacian matrix $L(G)$.

Degenerating the graph through successive elimination of contraction of its edges represent the core of another way to compute the complexity of a graph [13]. In this way, the summation of complexities in small well known graphs yields directly the complexity of an unknown graph G. Let e be an edge with endpoints u and v in the graph G, the deletion $G - e$ of e from G is the graph gained by removing e and the contraction $G \cdot e$ of e from G is the graph obtained by removing e and identifying u and v. The formula for computing the complexity of a graph G is given by

$$\tau(G) = \tau(G - e) + \tau(G \cdot e). \tag{2}$$

Recently, Daoud [14] introduced some new theorems which generalized this method. We will make use of these theorems in this work.

2. Main results

Theorem 1. For $n \geq 3$, the number of spanning trees of the wheel graph W_n is given by $\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2.$

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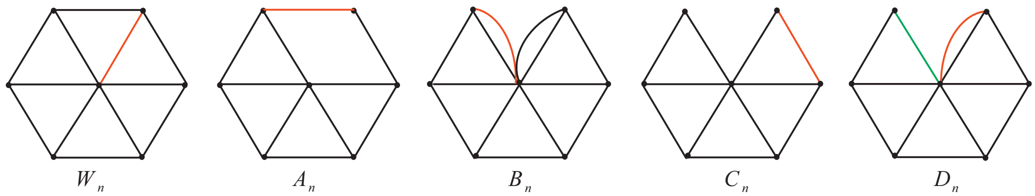

Fig. 1. The five families of graphs which we use to find an explicit formula for the complexity in the wheel graph W_n.

Proof: Consider the following five different families of graphs denoted by W_n, A_n, B_n, C_n and D_n as shown in Fig. 1, where n denote the number of vertices.

We use Eq. (2) on the indicated edges to find a system of recurrence relations:

$\tau(W_n) = \tau(A_n) + \tau(B_{n-1})$
$\tau(A_n) = \tau(C_{n-1}) + \tau(W_{n-1})$
$\tau(B_n) = \tau(D_n) + \tau(B_{n-1})$
$\tau(C_n) = \tau(C_{n-1}) + \tau(D_{n-1})$
$\tau(D_n) = \tau(C_n) + \tau(D_{n-1}) = \tau(D_{n-1}) + \tau(B_{n-1})$.

Consider the last two relations for $\tau(C_n)$ and $\tau(D_n)$, we have: $\tau(C_{n+1}) = \tau(C_n) + \tau(D_n) = 2\tau(C_n) + \tau(D_{n-1}) = 3\tau(C_n) - \tau(C_{n-1})$, or $\tau(C_{n+1}) - 3\tau(C_n) + \tau(C_{n-1}) = 0$, thus $\tau(C_n) - 3\tau(C_{n-1}) + \tau(C_{n-2}) = 0$ and $\tau(C_{n-1}) - 3\tau(C_{n-2}) + \tau(C_{n-3}) = 0$.

Subtracting these two equations, we get $\tau(C_n) - 4\tau(C_{n-1}) + 4\tau(C_{n-2}) - \tau(C_{n-3}) = 0$, which is the final recurrence relation for $\tau(C_n)$.

Consider the first two relations for $\tau(W_n)$ and $\tau(A_n)$.

Since $\tau(B_{n-1}) = \tau(C_n)$, we have $\tau(W_n) = \tau(A_n) + \tau(C_n)$ and hence $\tau(W_{n-1}) = \tau(A_{n-1}) + \tau(C_{n-1})$.

Substituting into the second relation, we obtain $\tau(A_n) = \tau(A_{n-1}) + 2\tau(C_{n-1})$, therefore $\tau(A_n) - \tau(A_{n-1}) = 2\tau(C_{n-1})$, since $\tau(C_{n-1}) - 3\tau(C_{n-2}) + \tau(C_{n-3}) = 0$, we have $2\tau(C_{n-1}) - 2(3)\tau(C_{n-2}) + 2\tau(C_{n-3}) = 0$, $[\tau(A_n) - \tau(A_{n-1})] - 3[\tau(A_{n-1}) - \tau(A_{n-2})] + [\tau(A_{n-2}) - \tau(A_{n-3})] = 0$, thus $\tau(A_n) - 4\tau(A_{n-1}) + 4\tau(A_{n-2}) - \tau(A_{n-3}) = 0$, which is the final recurrence relation for $\tau(A_n)$.

Now both $\tau(A_n)$ and $\tau(C_n)$ have the third order homogeneous recurrence relation:

$$x_n - 4x_{n-1} + 4x_{n-2} - x_{n-3} = 0. \tag{3}$$

Thus $\tau(W_n) = \tau(A_n) + \tau(C_n)$ must have the same relation. Therefore the characteristic equation corresponding to this recurrence relation is $r^3 - 4r^2 + 4r - 1 = 0$, which has characteristic roots $r = \frac{3\pm\sqrt{5}}{2}$ and $r = 1$. Therefore, the general solution of $\tau(W_n)$ is $\tau(W_n) = \alpha(\frac{3+\sqrt{5}}{2})^n + \beta(\frac{3-\sqrt{5}}{2})^n + \gamma$.

Solution of the recurrence relation (3) now reduces to find the values of the constants α, β and γ such that the general solution conforms with the given initial conditions $\tau(W_3) = 16$, $\tau(W_4) = 45$ and $\tau(W_5) = 121$. Substituting the initial conditions in the general solution we obtain

$$\tau(W_n) = \alpha\left(\tfrac{3+\sqrt{5}}{2}\right)^3 + \beta\left(\tfrac{3-\sqrt{5}}{2}\right)^3 + \gamma = 16$$
$$\tau(W_n) = \alpha\left(\tfrac{3+\sqrt{5}}{2}\right)^4 + \beta\left(\tfrac{3-\sqrt{5}}{2}\right)^4 + \gamma = 45$$
$$\tau(W_n) = \alpha\left(\tfrac{3+\sqrt{5}}{2}\right)^5 + \beta\left(\tfrac{3-\sqrt{5}}{2}\right)^5 + \gamma = 121.$$

This system of equations have a unique solution $\alpha = \beta = 1$ and $\gamma = -2$, and hence the result follows. □

The gear graph G_n, is the graph obtained from W_n by inserting a vertex between any two adjacent vertices in its cycle C_n. See Fig. 2.

Theorem 2. For $n \geq 3$, the number of spanning trees of the gear graph G_n is given by $(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2$.

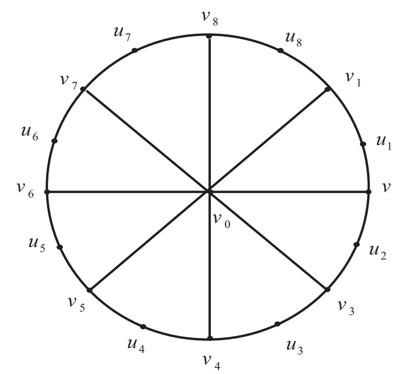

Fig. 2. The gear graph G_8.

Proof: Consider the following five different families of graphs denoted by G_n, A_n, B_n, C_n and D_n as shown in Fig. 3, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.2 in [14] on the indicated edges and paths to find a system of recurrence relations:

$\tau(G_n) = \tau(A_n) + \tau(B_{n-1})$
$\tau(A_n) = 2\tau(C_{n-1}) + \tau(G_{n-1})$
$\tau(B_n) = 2\tau(D_n) + \tau(B_{n-1})$
$\tau(C_n) = \tau(C_{n-1}) + \tau(D_{n-1})$
$\tau(D_n) = 2\tau(C_n) + \tau(D_{n-1}) = \tau(D_{n-1}) + \tau(B_{n-1})$.

Consider the last two relations for $\tau(C_n)$ and $\tau(D_n)$, we have $\tau(C_{n+1}) = \tau(C_n) + \tau(D_n) = 3\,\tau(C_n) + \tau(D_{n-1}) = 4\tau(C_n) - \tau(C_{n-1})$, or $\tau(C_{n+1}) - 4\tau(C_n) + \tau(C_{n-1}) = 0$, Thus $\tau(C_n) - 4\tau(C_{n-1}) + \tau(C_{n-2}) = 0$ and $\tau(C_{n-1}) - 4\tau(C_{n-2}) + \tau(C_{n-3}) = 0$.

Subtracting these two equations, we get $\tau(C_n) - 5\tau(C_{n-1}) + 5\tau(C_{n-2}) - \tau(C_{n-3}) = 0$, which is the final recurrence relation for $\tau(C_n)$. Consider the first two relations for $\tau(G_n)$ and $\tau(A_n)$.

Since $\tau(B_{n-1}) = 2\tau(C_n)$, we have $\tau(G_n) = \tau(A_n) + 2\tau(C_n)$ and hence $\tau(G_{n-1}) = \tau(A_{n-1}) + 2\tau(C_{n-1})$.

Substituting into the second relation, we obtain $\tau(A_n) = \tau(A_{n-1}) + 4\tau(C_{n-1})$, therefore $\tau(A_n) - \tau(A_{n-1}) = 4\tau(C_{n-1})$, since $\tau(C_{n-1}) - 4\tau(C_{n-2}) + \tau(C_{n-3}) = 0$, we have $4\tau(C_{n-1}) - 4(4)\tau(C_{n-2}) + 4\tau(C_{n-3}) = 0$, $[\tau(A_n) - \tau(A_{n-1})] - 4[\tau(A_{n-1}) - \tau(A_{n-2})] + [\tau(A_{n-2}) - \tau(A_{n-3})] = 0$, thus $\tau(A_n) - 5\tau(A_{n-1}) + 5\tau(A_{n-2}) - \tau(A_{n-3}) = 0$, which is the final recurrence relation for $\tau(A_n)$.

Now both $\tau(A_n)$ and $\tau(C_n)$ have the third order homogeneous recurrence relation:

$$x_n - 5x_{n-1} + 5x_{n-2} - x_{n-3} = 0. \tag{4}$$

Thus the characteristic equation corresponding to this recurrence relation is $r^3 - 5r^2 + 5r - 1 = 0$, which has characteristic roots $r = 2 \pm \sqrt{3}$ and $r = 1$. Thus the general solution of $\tau(G_n)$ is $\tau(G_n) = \alpha(2+\sqrt{3})^n + \beta(2-\sqrt{3})^n + \gamma$.

Solution of the recurrence relation (4) now reduces to find the values of the constants α, β and γ such that the general solution conforms with the given initial conditions $\tau(G_3) = 50$, $\tau(G_4) = 192$ and $\tau(G_5) = 722$. Substituting the initial conditions in the gen-

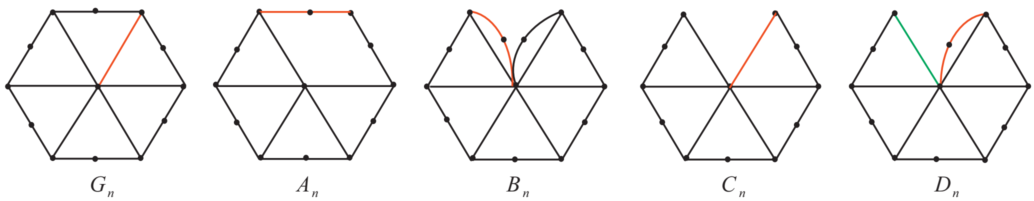

Fig. 3. The five families of graphs which we use to find an explicit formula for the complexity in the gear graph G_n.

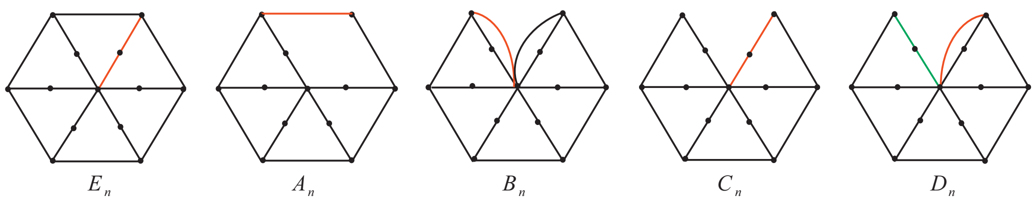

Fig. 4. The five families of graphs which we use to find an explicit formula for the complexity in the graph E_n.

eral solution we obtain

$$\tau(G_3) = \alpha\,(2+\sqrt{3})^3 + \beta\,(2-\sqrt{3})^3 + \gamma = 50$$
$$\tau(G_4) = \alpha\,(2+\sqrt{3})^4 + \beta\,(2-\sqrt{3})^4 + \gamma = 192$$
$$\tau(G_5) = \alpha\,(2+\sqrt{3})^5 + \beta\,(2-\sqrt{3})^5 + \gamma = 722.$$

This system of equations have a unique solution $\alpha = \beta = 1$ and $\gamma = -2$, and hence the result follows. □

Theorem 3. Let G_n^* be the graph obtained from the wheel graph W_n by replacing each edge on the rim by a path consisting k edges, then $\tau(G_n^*) = (\frac{k+2}{2} + \sqrt{(\frac{k+2}{2})^2 - 1}\,)^n + (\frac{k+2}{2} - \sqrt{(\frac{k+2}{2})^2 - 1}\,)^n - 2$.

Proof: Consider the following five different families of graphs denoted by G_n^*, A_n, B_n, C_n and D_n, where n denote the number of vertices.

We use Eq. (2) and Theorem 2.2 in [14] on the indicated edges and paths to find a system of recurrence relations:

$$\tau(G_n^*) = \tau(A_n) + \tau(B_{n-1})$$
$$\tau(A_n) = k\,\tau(C_{n-1}) + \tau(G_{n-1}^*)$$
$$\tau(B_n) = k\,\tau(D_n) + \tau(B_{n-1})$$
$$\tau(C_n) = \tau(C_{n-1}) + \tau(D_{n-1})$$
$$\tau(D_n) = k\,\tau(C_n) + \tau(D_{n-1}) = \tau(D_{n-1}) + \tau(B_{n-1}).$$

The proof can be completed via the same technique used in Theorem 2. □

Theorem 4. Let E_n be the graph obtained from the wheel graph W_n by inserting a vertex between the central vertex and each vertex in its cycle, then for $n \geq 3$, $\tau(E_n) = 2^{2n} - 2^{n+1} + 1$.

Proof: Consider the following five different families of graphs denoted by E_n, A_n, B_n, C_n and D_n as shown in Fig. 4, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.2 in [14] on the indicated edges and paths to find a system of recurrence relations:

$$\tau(E_n) = 2\,\tau(A_n) + \tau(B_{n-1})$$
$$\tau(A_n) = \tau(C_{n-1}) + \tau(E_{n-1})$$
$$\tau(B_n) = \tau(D_n) + 2\tau(B_{n-1})$$
$$\tau(C_n) = 2\,\tau(C_{n-1}) + \tau(D_{n-1})$$
$$\tau(D_n) = \tau(C_n) + 2\,\tau(D_{n-1}) = 2\,\tau(D_{n-1}) + \tau(B_{n-1}).$$

Consider the last two relations for $\tau(C_n)$ and $\tau(D_n)$, we have $\tau(C_{n+1}) = 2\,\tau(C_n) + \tau(D_n) = 4\,\tau(C_n) + 2\,\tau(D_{n-1}) = 5\,\tau(C_n) - 4\,\tau(C_{n-1})$ or $\tau(C_{n+1}) - 5\,\tau(C_n) + 4\,\tau(C_{n-1}) = 0$, Thus $\tau(C_n) - 5\,\tau(C_{n-1}) + 4\,\tau(C_{n-2}) = 0$ and $\tau(C_{n-1}) - 5\,\tau(C_{n-2}) + 4\,\tau(C_{n-3}) = 0$.

Subtracting these two equations, we get $\tau(C_n) - 6\,\tau(C_{n-1}) + 9\,\tau(C_{n-2}) - 4\,\tau(C_{n-3}) = 0$, which is the final recurrence relation for $\tau(C_n)$. Consider the first two relations for $\tau(G_n)$ and $\tau(A_n)$.

Since $\tau(B_{n-1}) = \tau(C_n)$, we have $\tau(E_n) = 2\,\tau(A_n) + \tau(C_n)$ and hence $\tau(E_{n-1}) = 2\,\tau(A_{n-1}) + \tau(C_{n-1})$.

Substituting into the second relation, we obtain $\tau(A_n) = 2\,\tau(A_{n-1}) + 2\,\tau(C_{n-1})$, therefore $\tau(A_n) - 2\,\tau(A_{n-1}) = 2\,\tau(C_{n-1})$, since $\tau(C_{n-1}) - 5\,\tau(C_{n-2}) + 4\,\tau(C_{n-3}) = 0$, we have $2\,\tau(C_{n-1}) - 5\,(2)\,\tau(C_{n-2}) + 4\,(2)\,\tau(C_{n-3}) = 0$, $[\tau(A_n) - 2\,\tau(A_{n-1})] - 5\,[\tau(A_{n-1}) - 2\,\tau(A_{n-2})] + 4\,[\tau(A_{n-2}) - 2\,\tau(A_{n-3})] = 0$, thus $\tau(A_n) - 7\,\tau(A_{n-1}) + 14\,\tau(A_{n-2}) - 8\,\tau(A_{n-3}) = 0$, which is the final recurrence relation for $\tau(A_n)$.

Now $\tau(A_n)$ has the third order homogeneous recurrence relation:

$$x_n - 7\,x_{n-1} + 14\,x_{n-2} - 8\,x_{n-3} = 0 \tag{5}$$

Thus the characteristic equation corresponding to this recurrence relation is $r^3 - 7\,r^2 + 14\,r - 8 = 0$, which has characteristic roots $r = 1$, $r = 1$ and $r = 4$.

Also $\tau(C_n)$ has the third order homogeneous recurrence relation:

$$y_n - 6\,y_{n-1} + 9\,y_{n-2} - 4\,y_{n-3} = 0 \tag{6}$$

Thus the characteristic equation corresponding to this recurrence is $s^3 - 6\,s^2 + 9\,s - 4 = 0$ which has characteristic roots $r = 1$, $r = 2$ and $r = 4$.

Therefore $\tau(E_n) = \tau(A_n) + 2\,\tau(C_n)$ has the general solution $\tau(E_n) = \alpha\,(2)^{2n} + \beta\,(2)^n + \gamma$.

Using the initial conditions $\tau(E_3) = 49$, $\tau(E_4) = 225$ and $\tau(E_5) = 961$, we have $\alpha = 1$, $\beta = -2$ and $\gamma = 1$ and hence the result follows. □

Theorem 5. Let E_n^* be the graph obtained from the wheel graph W_n by replacing each internal edge by a path consisting k edges, then $\tau(E_n^*) = \left(\frac{2k+1+\sqrt{4k+1}}{2}\right)^n + \left(\frac{2k+1-\sqrt{4k+1}}{2}\right)^n - 2k^n$.

Proof: Consider the five different families of graphs denoted by E_n^*, A_n, B_n, C_n and D_n, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.2 in [14] to find a system of recurrence relations

$$\tau(E_n^*) = k\,\tau(A_n) + \tau(B_{n-1})$$
$$\tau(A_n) = \tau(C_{n-1}) + \tau(E_{n-1}^*)$$
$$\tau(B_n) = \tau(D_n) + k\,\tau(B_{n-1})$$
$$\tau(C_n) = k\,\tau(C_{n-1}) + \tau(D_{n-1})$$
$$\tau(D_n) = \tau(C_n) + k\,\tau(D_{n-1}) = k\,\tau(D_{n-1}) + \tau(B_{n-1}).$$

The proof can be completed via the same technique used in Theorem 4. □

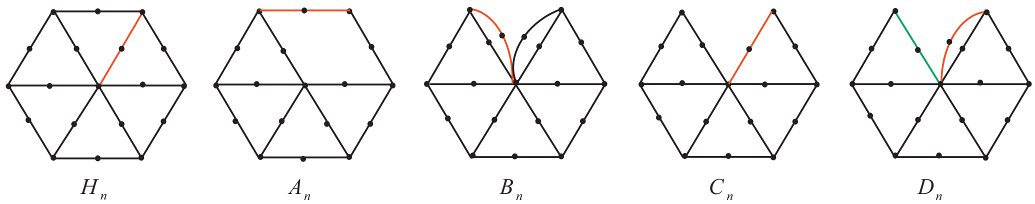

Fig. 5. The five families of graphs which we use to find an explicit formula for the complexity in the graph H_n.

Theorem 6. Let H_n be a graph resulting from a wheel graph W_n by inserting into each edge of W_n a new vertex of degree two. Then for $n \geq 3$, $\tau(H_n) = 2^n\left[\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2\right] = 2^n\,\tau(W_n)$.

Proof: Consider the following five different families of graphs denoted by H_n, A_n, B_n, C_n and D_n as shown in Fig. 5, where n denote the number of vertices of W_n.

We use Theorem 2.2 in [14] on the indicated paths to find a system of recurrence relations

$$\begin{aligned}
&\tau(H_n) = 2\,\tau(A_n) + \tau(B_{n-1})\\
&\tau(A_n) = 2\,\tau(C_{n-1}) + \tau(H_{n-1})\\
&\tau(B_n) = 2\,\tau(D_n) + 2\tau(B_{n-1})\\
&\tau(C_n) = 2\,\tau(C_{n-1}) + \tau(D_{n-1})\\
&\tau(D_n) = 2\,\tau(C_n) + 2\,\tau(D_{n-1}) = 2\,\tau(D_{n-1}) + \tau(B_{n-1}).
\end{aligned}$$

Consider the last two relations for $\tau(C_n)$ and $\tau(D_n)$, we have: $\tau(C_{n+1}) = 2\,\tau(C_n) + \tau(D_n) = 4\,\tau(C_n) + 2\,\tau(D_{n-1}) = 6\,\tau(C_n) - 4\,\tau(C_{n-1})$ or $\tau(C_{n+1}) - 6\,\tau(C_n) + 4\,\tau(C_{n-1}) = 0$, Thus $\tau(C_n) - 6\,\tau(C_{n-1}) + 4\,\tau(C_{n-2}) = 0$ and $\tau(C_{n-1}) - 6\,\tau(C_{n-2}) + 4\,\tau(C_{n-3}) = 0$.

Subtracting these two equations, we get $\tau(C_n) - 7\,\tau(C_{n-1}) + 10\,\tau(C_{n-2}) - 4\,\tau(C_{n-3}) = 0$, which is the final recurrence relation for $\tau(C_n)$. Consider the first two relations for $\tau(G_n)$ and $\tau(A_n)$.

Since $\tau(B_{n-1}) = 2\,\tau(C_n)$, we have $\tau(H_n) = 2\,\tau(A_n) + 2\,\tau(C_n)$ and hence $\tau(H_{n-1}) = 2\,\tau(A_{n-1}) + 2\,\tau(C_{n-1})$.

Substituting into the second relation, we obtain $\tau(A_n) = 2\,\tau(A_{n-1}) + 4\,\tau(C_{n-1})$, therefore $\tau(A_n) - 2\,\tau(A_{n-1}) = 4\,\tau(C_{n-1})$, since $\tau(C_{n-1}) - 6\,\tau(C_{n-2}) + 4\,\tau(C_{n-3}) = 0$, we have $4\,\tau(C_{n-1}) - 6\;(4)\;\tau(C_{n-2}) + 4\;(4)\;\tau(C_{n-3}) = 0$, $[\tau(A_n) - 2\,\tau(A_{n-1})] - 6\,[\tau(A_{n-1}) - 2\,\tau(A_{n-2})] + 4\,[\tau(A_{n-2}) - 2\,\tau(A_{n-3})] = 0$, thus $\tau(A_n) - 8\;\tau(A_{n-1}) + 16\;\tau(A_{n-2}) - 8\;\tau(A_{n-3}) = 0$, which is the final recurrence relation for $\tau(A_n)$.

Now $\tau(A_n)$ has the third order homogeneous recurrence relation:

$$x_n - 8\,x_{n-1} + 16\,x_{n-2} - 8\,x_{n-3} = 0 \tag{7}$$

Thus the characteristic equation corresponding to this recurrence relation is $r^3 - 8r^2 + 16r - 8 = 0$ which has characteristic roots $r = 2$ and $r = 3 \pm \sqrt{5}$.

Also $\tau(C_n)$ has the third order homogeneous recurrence relation:

$$y_n - 7\,y_{n-1} + 10\,y_{n-2} - 4\,y_{n-3} = 0. \tag{8}$$

Thus the characteristic equation corresponding to this recurrence relation is $s^3 - 7\,s^2 + 10\,s - 4 = 0$ which has characteristic roots $s = 1$ and $s = 3 \pm \sqrt{5}$.

Thus $\tau(H_n) = 2\,\tau(A_n) + 2\,\tau(C_n)$ must have the general solution.

$$\tau(H_n) = \alpha\;(3+\sqrt{5})^n + \beta\;(3-\sqrt{5})^n + \gamma + \sigma\;(2)^n.$$

Substituting the initial conditions $\tau(H_3) = 128$, $\tau(H_4) = 720$, $\tau(H_5) = 3872$ and $\tau(H_6) = 20480$ in the general solution, we obtain

$$\begin{aligned}
&\tau(H_3) = \alpha\;(3+\sqrt{5})^3 + \beta\;(3-\sqrt{5})^3 + \gamma + \sigma\;(2)^3 = 128\\
&\tau(H_4) = \alpha\;(3+\sqrt{5})^4 + \beta\;(3-\sqrt{5})^4 + \gamma + \sigma\;(2)^4 = 720\\
&\tau(H_5) = \alpha\;(3+\sqrt{5})^5 + \beta\;(3-\sqrt{5})^5 + \gamma + \sigma\;(2)^5 = 3872\\
&\tau(H_6) = \alpha\;(3+\sqrt{5})^6 + \beta\;(3-\sqrt{5})^6 + \gamma + \sigma\;(2)^6 = 20480.
\end{aligned}$$

This system of equations have a unique solution $\alpha = \beta = 1$, $\gamma = 0$ and $\sigma = -2$, and hence the result follows. □

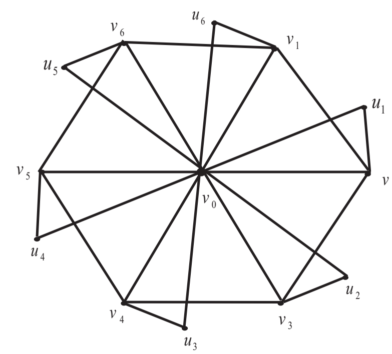

Fig. 6. The flower graph $Fl_6^{(3)}$.

Theorem 7. Let H_n^* be a graph resulting from a wheel graph W_n by inserting k vertices of degree two into each edge of W_n. Then for $n \geq 3$, $\tau(H_n^*) = k^n\,\tau(W_n)$.

Proof: Consider the five different families of graphs denoted by H_n^*, A_n, B_n, C_n and D_n, where n denote the number of vertices of W_n.

We use Theorem 2.2 in [14] to find a system of recurrence relations

$$\begin{aligned}
&\tau(H_n^*) = k\,\tau(A_n) + \tau(B_{n-1})\\
&\tau(A_n) = k\,\tau(C_{n-1}) + \tau(H_{n-1}^*)\\
&\tau(B_n) = k\,\tau(D_n) + k\tau(B_{n-1})\\
&\tau(C_n) = k\,\tau(C_{n-1}) + \tau(D_{n-1})\\
&\tau(D_n) = k\,\tau(C_n) + k\,\tau(D_{n-1}) = k\,\tau(D_{n-1}) + \tau(B_{n-1}).
\end{aligned}$$

The proof can be completed via the same technique used in Theorem 6. □

The Flower $Fl_n^{(3)}$, is the graph obtained from the wheel graph W_n with a pendent edge at each vertex of its cycle after joining each pendent vertex to its center. See Fig. 6.

Theorem 8. For $n \geq 3$, the number of spanning trees of the flower graph $Fl_n^{(3)}$ is given by $\left(\frac{7+\sqrt{33}}{2}\right)^n + \left(\frac{7-\sqrt{33}}{2}\right)^n - 2^{n+1}$.

Proof: Consider the following five different families of graphs denoted by $Fl_n^{(3)}$, A_n, B_n, C_n and D_n as shown in Fig. 7, where n denote the number of vertices of W_n.

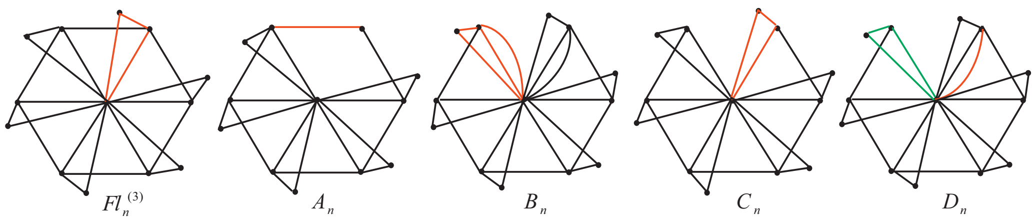

Fig. 7. The five families of graphs which we use to find an explicit formula for the complexity in the flower graph $Fl_n^{(3)}$.

We use Eq. (2) together with Theorem 2.5 in [14] on the indicated edges and triangles to find a system of recurrence relations:

$$\tau(Fl_n^{(3)}) = 2\,\tau(A_n) + 3\,\tau(B_{n-1})$$
$$\tau(A_n) = \tau(C_{n-1}) + \tau(Fl_{n-1}^{(3)})$$
$$\tau(B_n) = 2\tau(D_{n-1}) + 5\tau(B_{n-1})$$
$$\tau(C_n) = 2\,\tau(C_{n-1}) + 3\,\tau(D_{n-1})$$
$$\tau(D_n) = \tau(C_n) + 2\,\tau(D_{n-1}) = 2\,\tau(D_{n-1}) + 3\,\tau(B_{n-1}).$$

Consider the last two relations for $\tau(C_n)$ and $\tau(D_n)$, we have $\tau(C_{n+1}) = 2\,\tau(C_n) + 3\,\tau(D_n) = 5\,\tau(C_n) + 6\,\tau(D_{n-1}) = 7\,\tau(C_n) - 4\,\tau(C_{n-1})$ or $\tau(C_{n+1}) - 7\,\tau(C_n) + 4\,\tau(C_{n-1}) = 0$, thus $\tau(C_n) - 7\,\tau(C_{n-1}) + 4\,\tau(C_{n-2}) = 0$ and $\tau(C_{n-1}) - 7\,\tau(C_{n-2}) + 4\,\tau(C_{n-3}) = 0$.

Subtracting these two equations, we get $\tau(C_n) - 8\,\tau(C_{n-1}) + 11\,\tau(C_{n-2}) - 4\,\tau(C_{n-3}) = 0$, which is the final recurrence relation for $\tau(C_n)$. Consider the first two relations for $\tau(Fl_n^{(3)})$ and $\tau(A_n)$.

Since $3\,\tau(B_{n-1}) = \tau(C_n)$, we have $\tau(Fl_n^{(3)}) = 2\,\tau(A_n) + \tau(C_n)$ and hence $\tau(Fl_{n-1}^{(3)}) = 2\,\tau(A_{n-1}) + \tau(C_{n-1})$.

Substituting into the second relation, we obtain $\tau(A_n) = 2\,\tau(A_{n-1}) + 2\,\tau(C_{n-1})$, therefore $\tau(A_n) - 2\,\tau(A_{n-1}) = 2\,\tau(C_{n-1})$. Since $\tau(C_{n-1}) - 7\,\tau(C_{n-2}) + 4\,\tau(C_{n-3}) = 0$, we have $2\,\tau(C_{n-1}) - 2\,(7)\,\tau(C_{n-2}) + 2\,(4)\,\tau(C_{n-3}) = 0$, $[\tau(A_n) - 2\,\tau(A_{n-1})] - 7\,[\tau(A_{n-1}) - 2\,\tau(A_{n-2})] + 4\,[\tau(A_{n-2}) - 2\,\tau(A_{n-3})] = 0$, thus $\tau(A_n) - 9\,\tau(A_{n-1}) + 18\,\tau(A_{n-2}) - 8\,\tau(A_{n-3}) = 0$, which is the final recurrence relation for $\tau(A_n)$.

Now $\tau(A_n)$ has the third order homogeneous recurrence relation:

$$x_n - 9\,x_{n-1} + 18\,x_{n-2} - 8\,x_{n-3} = 0 \tag{9}$$

Thus the characteristic equation corresponding to this recurrence relation is $r^3 - 9\,r^2 + 18\,r - 8 = 0$ which has characteristic roots $r = 2$ and $r = \frac{7\pm\sqrt{33}}{2}$.

Also $\tau(C_n)$ have the third order homogeneous recurrence relation:

$$y_n - 8\,y_{n-1} + 11\,y_{n-2} - 4\,y_{n-3} = 0. \tag{10}$$

Thus the characteristic equation corresponding to this recurrence relation is $s^3 - 8\,s^2 + 11\,s - 4 = 0$, which has characteristic roots $s = 1$ and $s = \frac{7\pm\sqrt{33}}{2}$.

Therefore $\tau(Fl_n^{(3)}) = 2\,\tau(A_n) + \tau(C_n)$ has the general solution $\tau(Fl_n^{(3)}) = \alpha(\frac{7+\sqrt{33}}{2})^n + \beta(\frac{7-\sqrt{33}}{2})^n + \gamma + \sigma\ (2)^n$.

Substituting the initial conditions $\tau(Fl_3^{(3)}) = 243$, $\tau(Fl_4^{(3)}) = 1617$, $\tau(Fl_5^{(3)}) = 10443$ and $\tau(Fl_6^{(3)}) = 66825$ in the general solution, we obtain

$$\tau(Fl_3^{(3)}) = \alpha\left(\frac{7+\sqrt{33}}{2}\right)^3 + \beta\left(\frac{7-\sqrt{33}}{2}\right)^3 + \gamma + \sigma\ (2)^3 = 234$$
$$\tau(Fl_4^{(3)}) = \alpha\left(\frac{7+\sqrt{33}}{2}\right)^4 + \beta\left(\frac{7-\sqrt{33}}{2}\right)^4 + \gamma + \sigma\ (2)^4 = 1617$$
$$\tau(Fl_5^{(3)}) = \alpha\left(\frac{7+\sqrt{33}}{2}\right)^5 + \beta\left(\frac{7-\sqrt{33}}{2}\right)^5 + \gamma + \sigma\ (2)^5 = 10443$$
$$\tau(Fl_6^{(3)}) = \alpha\left(\frac{7+\sqrt{33}}{2}\right)^6 + \beta\left(\frac{7-\sqrt{33}}{2}\right)^6 + \gamma + \sigma\ (2)^6 = 66825.$$

This system of equations have a unique solution $\alpha = \beta = 1$, $\gamma = 0$ and $\sigma = -2$, and hence the result follows. □

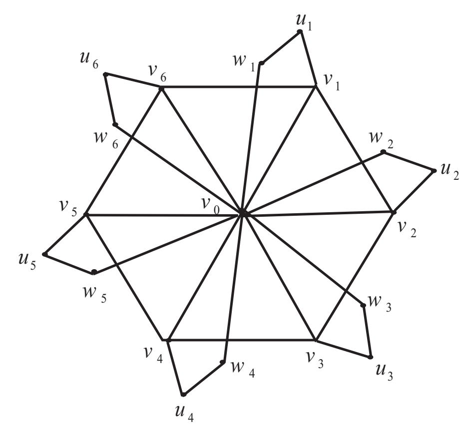

Fig. 8. The flower graph $Fl_6^{(4)}$.

Theorem 9. Let $H_n^{(3)}$, be the graph obtained from the wheel graph W_n with k pendent edges at each vertex of its cycle after joining each pendent vertex to its center. Then

$$\tau\ (H_n^{(3)}) = 2^{kn}\left[\left(\frac{k+6+\sqrt{(k+6)^2-16}}{4}\right)^n + \left(\frac{k+6-\sqrt{(k+6)^2-16}}{4}\right)^n - 2\right].$$

Proof: Consider the five different families of graphs denoted by $H_n^{(3)}$, A_n, B_n, C_n and D_n, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.5 and Lemma 2.10 in [14] to find a system of recurrence relations:

$$\tau(H_n^{(3)}) = 2^k\,\tau(A_n) + 2^{k-1}\,(k+2)\,\tau(B_{n-1})$$
$$\tau(A_n) = \tau(C_{n-1}) + \tau(H_{n-1}^{(3)})$$
$$\tau(B_n) = 2^k\,\tau(D_{n-1}) + 2^{k-1}\,(k+4)\,\tau(B_{n-1})$$
$$\tau(C_n) = 2^k\,\tau(C_{n-1}) + 2^{m-1}\,(m+2)\,\tau(D_{n-1})$$
$$\tau(D_n) = \tau(C_n) + 2^k\,\tau(D_{n-1}) = 2^k\,\tau(D_{n-1}) + 2^{k-1}\,(k+2)\,\tau(B_{n-1}).$$

The proof can be completed via the same technique used in Theorem 8. □

The Flower $Fl_n^{(4)}$, is the graph obtained from the wheel graph W_n with a pendent edge at each vertex of its cycle after joining each pendent vertex to its center by a path of length 2. See Fig. 8.

Theorem 10. For $n \geq 3$, the number of spanning trees of the flower graph $Fl_n^{(4)}$ is given by $3^{2n} - 2 \times 3^n + 1$.

Proof: Consider the following five different families of graphs denoted by $Fl_n^{(4)}$, A_n, B_n, C_n and D_n as shown in Fig. 9, where n denote the number of vertices of W_n.

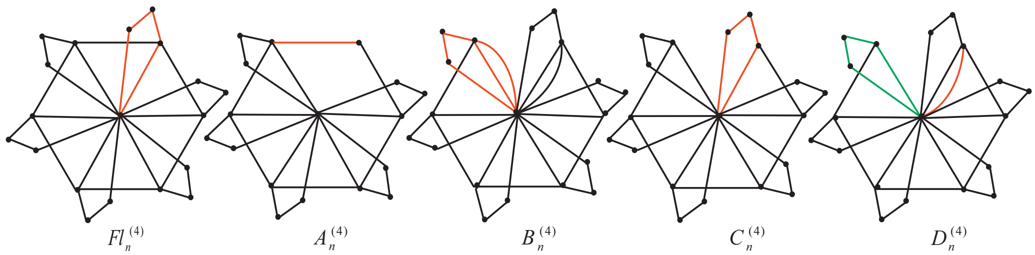

Fig. 9. The five families of graphs which we use to find an explicit formula for the complexity in the flower graph $Fl_n^{(4)}$.

We use Eq. (2) together with Theorem 2.5 in [14] on the indicated edges and squares to find a system of recurrence relations:

$$\begin{aligned}
&\tau(Fl_n^{(4)}) = 3\,\tau(A_n) + 4\,\tau(B_{n-1})\\
&\tau(A_n) = \tau(C_{n-1}) + \tau(Fl_{n-1}^{(4)})\\
&\tau(B_n) = 3\,\tau(D_n) + 7\,\tau(B_{n-1})\\
&\tau(C_n) = 3\,\tau(C_{n-1}) + 4\,\tau(D_{n-1})\\
&\tau(D_n) = \tau(C_n) + 3\,\tau(D_{n-1}) = 3\,\tau(D_{n-1}) + 4\,\tau(B_{n-1}).
\end{aligned}$$

Consider the last two relations for $\tau(C_n)$ and $\tau(D_n)$, we have $\tau(C_{n+1}) = 3\,\tau(C_n) + 4\,\tau(D_n) = 7\,\tau(C_n) + 12\,\tau(D_{n-1}) = 10\,\tau(C_n) - 9\,\tau(C_{n-1})$ or $\tau(C_{n+1}) - 10\,\tau(C_n) + 9\,\tau(C_{n-1}) = 0$. Thus $\tau(C_n) - 10\,\tau(C_{n-1}) + 9\,\tau(C_{n-2}) = 0$ and $\tau(C_{n-1}) - 10\,\tau(C_{n-2}) + 9\,\tau(C_{n-3}) = 0$.

Subtracting these two equations, we get $\tau(C_n) - 11\,\tau(C_{n-1}) + 19\,\tau(C_{n-2}) - 9\,\tau(C_{n-3}) = 0$, which is the final recurrence relation for $\tau(C_n)$. Consider the first two relations for $\tau(Fl_n^{(4)})$ and $\tau(A_n)$.

Since $4\,\tau(B_{n-1}) = \tau(C_n)$, we have $\tau(Fl_n^{(4)}) = 3\,\tau(A_n) + 2\,\tau(C_n)$ and hence $\tau(Fl_{n-1}^{(4)}) = 3\,\tau(A_{n-1}) + \tau(C_{n-1})$.

Substituting into the second relation, we obtain $\tau(A_n) = 3\,\tau(A_{n-1}) + 2\,\tau(C_{n-1})$, therefore $\tau(A_n) - 3\,\tau(A_{n-1}) = 2\,\tau(C_{n-1})$, since $\tau(C_{n-1}) - 10\,\tau(C_{n-2}) + 9\,\tau(C_{n-3}) = 0$, we have $2\,\tau(C_{n-1}) - 2\,(10)\,\tau(C_{n-2}) + 2\,(9)\,\tau(C_{n-3}) = 0$, $[\tau(A_n) - 3\,\tau(A_{n-1})] - 10\,[\tau(A_{n-1}) - 3\,\tau(A_{n-2})] + 9\,[\tau(A_{n-2}) - 3\,\tau(A_{n-3})] = 0$, thus $\tau(A_n) - 13\,\tau(A_{n-1}) + 39\,\tau(A_{n-2}) - 27\,\tau(A_{n-3}) = 0$, which is the final recurrence relation for $\tau(A_n)$.

Now $\tau(A_n)$ has the third order homogeneous recurrence relation:

$$x_n - 13\,x_{n-1} + 39\,x_{n-2} - 27\,x_{n-3} = 0. \tag{11}$$

Thus the characteristic equation corresponding to this recurrence relation is $r^3 - 13\,r^2 + 39\,r - 27 = 0$, which has characteristic roots $r = 1,\ r = 3$ and $r = 9$.

Also $\tau(C_n)$ have the third order homogeneous recurrence relation:

$$y_n - 11\,y_{n-1} + 19\,y_{n-2} - 9\,y_{n-3} = 0. \tag{12}$$

Thus the characteristic equation corresponding to this recurrence relation is $s^3 - 11\,s^2 + 19\,s - 9 = 0$, which has characteristic roots $s = 1,\ s = 1$ and $s = 9$.

Therefore, $\tau(Fl_n^{(4)}) = 3\,\tau(A_n) + \tau(C_n)$ has the general solution $\tau(Fl_n^{(4)}) = \alpha\ (9)^n + \beta\ (3)^n + \gamma$.

Substituting the initial conditions $\tau(Fl_3^{(4)}) = 676$, $\tau(Fl_4^{(4)}) = 6400$ and $\tau(Fl_5^{(4)}) = 58564$ in the general solution, we obtain

$$\begin{aligned}
&\tau(Fl_3^{(4)}) = \alpha\ (9)^3 + \beta\ (3)^3 + \gamma = 676,\\
&\tau(Fl_4^{(4)}) = \alpha\ (9)^4 + \beta\ (3)^4 + \gamma = 6400,\\
&\tau(Fl_5^{(4)}) = \alpha\ (9)^5 + \beta\ (3)^5 + \gamma = 58564.
\end{aligned}$$

This system of equations have a unique solution $\alpha = \gamma = 1$ and $\beta = -2$ and hence the result follows. □

Theorem 11. Let $H_n^{(4)}$, be the graph obtained from the wheel graph W_n with k pendent edges at each vertex of its cycle after joining each pendent vertex to its center by a path P_3. Then

$$\tau(H_n^{(4)}) = 3^{kn}\left[\left(\frac{k+9+\sqrt{(k+9)^2-36}}{6}\right)^n + \left(\frac{k+9+\sqrt{(k+9)^2-36}}{6}\right)^n - 2\right].$$

Proof: Consider the five different families of graphs denoted by $H_n^{(4)}$, A_n, B_n, C_n and D_n, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.5 and Lemma 2.10 in [14] to find a system of recurrence relations:

$$\begin{aligned}
&\tau(H_n^{(4)}) = 2^k\,\tau(A_n) + 2^{k-1}\,(k+2)\,\tau(B_{n-1})\\
&\tau(A_n) = \tau(C_{n-1}) + \tau(H_{n-1}^{(4)})\\
&\tau(B_n) = 2^k\,\tau(D_{n-1}) + 2^{k-1}\,(k+4)\,\tau(B_{n-1})\\
&\tau(C_n) = 2^k\,\tau(C_{n-1}) + 2^{m-1}\,(m+2)\,\tau(D_{n-1})\\
&\tau(D_n) = \tau(C_n) + 2^k\,\tau(D_{n-1}) = 2^k\,\tau(D_{n-1}) + 2^{k-1}\,(k+2)\,\tau(B_{n-1}).
\end{aligned}$$

The proof can be completed via the same technique used in Theorem 10 □

The Flower $Fl_n^{(m)}$, is the graph obtained from the wheel graph W_n with a pendent edge at each vertex of its cycle after joining each pendent vertex to its center by a path of length $m-2$.

Theorem 12. For $n \ge 3$, the number of spanning trees of the flower graph $Fl_n^{(m)}$ is given by $\tau\left(Fl_n^{(m)}\right) = \left(\dfrac{3m-2+\sqrt{m\,(5m-4)}}{2}\right)^n + \left(\dfrac{3m-2-\sqrt{m\,(5m-4)}}{2}\right)^n - 2(m-1)^n$.

Proof: Consider the five different families of graphs denoted by $Fl_n^{(m)}$, A_n, B_n, C_n and D_n, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.5 and Lemma 2.10 in [14] to find a system of recurrence relations:

$$\begin{aligned}
&\tau(Fl_n^{(m)}) = (m-1)\,\tau(A_n) + m\,\tau(B_{n-1})\\
&\tau(A_n) = \tau(C_{n-1}) + \tau(Fl_{n-1}^{(m)})\\
&\tau(B_n) = (m-1)\,\tau(D_{n-1}) + (2m-1)\,\tau(B_{n-1})\\
&\tau(C_n) = (m-1)\,\tau(C_{n-1}) + m\,\tau(D_{n-1})\\
&\tau(D_n) = \tau(C_n) + (m-1)\,\tau(D_{n-1}) = (m-1)\,\tau(D_{n-1}) + m\,\tau(D_{n-1}).
\end{aligned}$$

The proof can be completed via the same technique used in Theorem 10. □

Theorem 13. Let $H_n^{(m)}$, be the graph obtained from the wheel graph W_n with k pendent edges at each vertex of its cycle after joining each pendent vertex to its center by a path length $m-2$. Then

$$\tau(H_n^{(m)}) = (m-1)^{kn}\left[\left(\frac{k+3\,(m-1)+\sqrt{(k+3\,(m-1))^2-4\,(m-1)^2}}{2\,(m-1)}\right)^n + \left(\frac{k+3\,(m-1)+\sqrt{(k+3\,(m-1))^2-4\,(m-1)^2}}{2\,(m-1)}\right)^n - 2\right].$$

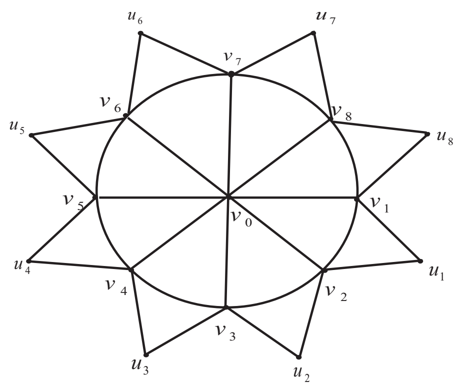

Fig. 10. The sun flower graph Sf_8.

Proof: Consider the five different families of graphs denoted by $H_n^{(m)}$, A_n, B_n, C_n and D_n, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.5 and Lemma 2.10 in [14] to find a system of recurrence relations:

$$\tau(H_n^{(m)}) = (m-1)^k\,\tau(A_n) + [k\,m^{k-1} + (m-1)^k]\,\tau(B_{n-1})$$
$$\tau(A_n) = \tau(C_{n-1}) + \tau(H_{n-1}^{(m)})$$
$$\tau(B_n) = (m-1)^k\,\tau(D_{n-1}) + [(k\,m^{k-1} + 2\,(m-1)^k]\,\tau(B_{n-1})$$
$$\tau(C_n) = (m-1)^k\,\tau(C_{n-1}) + [k\,m^{k-1} + (m-1)^k]\,\tau(D_{n-1})$$
$$\tau(D_n) = \tau(C_n) + (m-1)^k\,\tau(D_{n-1}) = (m-1)^k\tau(D_{n-1}) + [k\,m^{k-1} + (m-1)^k]\,\tau(B_{n-1}).$$

The proof can be completed via the same technique used in Theorem 10. □

The sunflower graph Sf_n, is the graph obtained from W_n by joining a path of length 2 between any two adjacent vertices in its cycle C_n. See Fig. 10.

Theorem 14. For $n \geq 3$, the number of spanning trees of the sun flower graph Sf_n, is given by $(4+\sqrt{7})^n + (4-\sqrt{7})^n - 2\times 3^n$.

Proof: Consider the following five different families of graphs denoted by Sf_n, A_n, B_n, C_n and D_n as shown in Fig. 11, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.5 in [14] on the indicated edges and triangles to find a system of recurrence relations:

$$\tau(Sf_n) = \tau(A_n) + \tau(B_{n-1})$$
$$\tau(A_n) = 6\,\tau(C_{n-1}) + 3\,\tau(Sf_{n-1})$$
$$\tau(B_n) = 2\,\tau(D_n) + 3\,\tau(B_{n-1})$$
$$\tau(C_n) = 3\,\tau(C_{n-1}) + \tau(D_{n-1})$$
$$\tau(D_n) = 2\,\tau(C_n) + 3\,\tau(D_{n-1}) = 3\,\tau(D_{n-1}) + \tau(B_{n-1}).$$

Consider the last two relations for $\tau(C_n)$ and $\tau(D_n)$, we have $\tau(C_{n+1}) = 3\,\tau(C_n) + \tau(D_n) = 5\,\tau(C_n) + 3\,\tau(D_{n-1}) = 8\,\tau(C_n) - 9\,\tau(C_{n-1})$ or $\tau(C_{n+1}) - 8\,\tau(C_n) + 9\,(C_{n-1}) = 0$, Thus $\tau(C_n) - 8\,\tau(C_{n-1}) + 9\,(C_{n-2}) = 0$ and $\tau(C_{n-1}) - 8\,\tau(C_{n-2}) + 9\,\tau(C_{n-3}) = 0$.

Subtracting these two equations, we get $\tau(C_n) - 9\,\tau(C_{n-1}) + 17\,\tau(C_{n-2}) - 9\,\tau(C_{n-3}) = 0$, which is the final recurrence relation for $\tau(C_n)$. Consider the first two relations for $\tau(Sf_n)$ and $\tau(A_n)$.

Since $\tau(B_{n-1}) = 2\,\tau(C_n)$, we have $\tau(Sf_n) = \tau(A_n) + 2\,\tau(C_n)$ and hence $\tau(Sf_{n-1}) = \tau(A_{n-1}) + 2\,\tau(C_{n-1})$.

Substituting into the second relation, we obtain $\tau(A_n) = 3\,\tau(A_{n-1}) + 12\,\tau(C_{n-1})$, therefore $\tau(A_n) - 3\,\tau(A_{n-1}) = 12\,\tau(C_{n-1})$, since $\tau(C_{n-1}) - 8\,\tau(C_{n-2}) + 9\,\tau(C_{n-3}) = 0$, we have $12\,\tau(C_{n-1}) - 12\,(8)\,\tau(C_{n-2}) + 12\,(9)\,\tau(C_{n-3}) = 0$, $[\tau(A_n) - 3\,\tau(A_{n-1})] - 8\,[\tau(A_{n-1}) - 3\,\tau(A_{n-2})] + 9\,[\tau(A_{n-2}) - 3\,\tau(A_{n-3})] = 0$, thus $\tau(A_n) - 11\,\tau(A_{n-1}) + 33\,\tau(A_{n-2}) - 27\,\tau(A_{n-3}) = 0$, which is the final recurrence relation for $\tau(A_n)$.

Now $\tau(A_n)$ has the third order homogeneous recurrence relation:

$$x_n - 11\,x_{n-1} + 33\,x_{n-2} - 27\,x_{n-3} = 0 \tag{13}$$

Thus the characteristic equation corresponding to this recurrence relation is $r^3 - 11\,r^2 + 33\,r - 27 = 0$, which has characteristic roots $r = 3$ and $r = 4 \pm \sqrt{7}$.

Also $\tau(C_n)$ have the third order homogeneous recurrence relation:

$$y_n - 9\,y_{n-1} + 17\,y_{n-2} - 9\,y_{n-3} = 0. \tag{14}$$

Thus the characteristic equation corresponding to this recurrence relation is $s^3 - 9\,s^2 + 17\,s - 9 = 0$, which has characteristic roots $s = 1$ and $s = 4 \pm \sqrt{7}$.

Therefore $\tau(Sf_n) = \tau(A_n) + 2\tau(C_n)$ has the general solution $\tau(Sf_n) = \alpha\,(4+\sqrt{7})^n + \beta\,(4-\sqrt{7})^n + \gamma + \sigma\,(3)^n$.

Substituting the initial conditions $\tau(Sf_3) = 242$, $\tau(Sf_4) = 1792$, $\tau(Sf_5) = 12482$ and $\tau(Sf_6) = 84700$ in the general solution, we obtain

$$\tau(Sf_3) = \alpha\,(4+\sqrt{7})^3 + \beta\,(4+\sqrt{7})^3 + \gamma + \sigma\,(3)^3 = 242$$
$$\tau(Sf_4) = \alpha\,(4+\sqrt{7})^4 + \beta\,(4+\sqrt{7})^4 + \gamma + \sigma\,(3)^4 = 1792$$
$$\tau(Sf_5) = \alpha\,(4+\sqrt{7})^5 + \beta\,(4+\sqrt{7})^5 + \gamma + \sigma\,(3)^5 = 12482$$
$$\tau(Sf_6) = \alpha\,(4+\sqrt{7})^6 + \beta\,(4+\sqrt{7})^6 + \gamma + \sigma\,(3)^6 = 84700.$$

This system of equations have a unique solution $\alpha = \beta = 1$, $\gamma = 0$ and $\sigma = -2$ and hence the result follows. □

Theorem 15. Let Q_n, be the graph obtained from W_n by joining a path of length $k-1$ between any two adjacent vertices in its cycle C_n. Then

$$\tau\ (Q_n) = \left(\frac{3k-1+\sqrt{5k^2-6k+1}}{2}\right)^n + \left(\frac{3k-1-\sqrt{5k^2-6k+1}}{2}\right)^n - 2\,k^n.$$

Proof: Consider the five different families of graphs denoted by G_n, A_n, B_n, C_n and D_n, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.5 and Lemma 2.10 in [14] to find a system of recurrence relations:

$$\tau(Q_n) = \tau(A_n) + \tau(B_{n-1})$$
$$\tau(A_n) = k(k-1)\,\tau(C_{n-1}) + k\,\tau(Q_{n-1})$$
$$\tau(B_n) = (k-1)\,\tau(D_n) + k\,\tau(B_{n-1})$$
$$\tau(C_n) = k\,\tau(C_{n-1}) + \tau(D_{n-1})$$
$$\tau(D_n) = (k-1)\,\tau(C_n) + k\,\tau(D_{n-1}) = k\,\tau(D_{n-1}) + \tau(B_{n-1}).$$

The proof can be completed via the same technique used in Theorem 14. □

Theorem 16. Let Q_n^*, be the graph obtained from W_n by joining 2−uniform k−skein between any two adjacent vertices in its cycle C_n. Then

$$\tau\ (Q_n^*) = 2^{(k-1)n}\,[(k+3+\sqrt{2k+5})^n + (k+3+\sqrt{2k+5}))^n - 2\,(k+2)^n].$$

Proof: Consider the five different families of graphs denoted by Q_n, A_n, B_n, C_n and D_n, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.5 and Lemma 2.10 in [14] to find a system of recurrence relations:

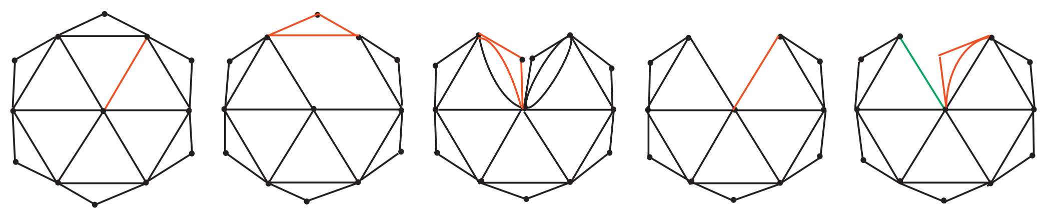

Fig. 11. The five families of graphs which we use to find an explicit formula for the complexity in the sun flower graph Sf_n.

$\tau(Q_n^*) = \tau(A_n) + \tau(B_{n-1})$
$\tau(A_n) = 2^{2k-1}(k+2)\,\tau(C_{n-1}) + 2^{k-1}(k+2)\,\tau(Q_{n-1}^*)$
$\tau(B_n) = 2^k\,\tau(D_n) + 2^{k-1}(k+2)\,\tau(B_{n-1})$
$\tau(C_n) = 2^{k-1}(k+2)\,\tau(C_{n-1}) + \tau(D_{n-1})$
$\tau(D_n) = 2^k\,\tau(C_n) + 2^{k-1}(k+2)\,\tau(D_{n-1}) = 2^{k-1}(k+2)\,\tau(D_{n-1}) + \tau(B_{n-1}).$

The proof can be completed via the same technique used in Theorem 14. □

Theorem 17. Let Q_n^{**}, be the graph obtained from W_n by joining $m-$uniform $k-$skein between any two adjacent vertices in its cycle C_n. Then

$$\tau(Q_n^{**}) = k^{(m-1)n}\left[\left(\frac{3k+2m+\sqrt{(3k+2m)^2-4(k+m)^2}}{2}\right)^n + \left(\frac{3k+2m-\sqrt{(3k+2m)^2-4(k+m)^2}}{2}\right)^n - 2(k+m)^n\right].$$

Proof: Consider the five different families of graphs denoted by Q_n^*, A_n, B_n, C_n and D_n, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.5 and Lemma 2.10 in [14] to find a system of recurrence relations:

$\tau(Q_n^{**}) = \tau(A_n) + \tau(B_{n-1})$
$\tau(A_n) = k^{2m-1}(m+k)\,\tau(C_{n-1}) + k^{m-1}(m+k)\,\tau(Q_{n-1}^{**})$
$\tau(B_n) = k^m\,\tau(D_n) + k^{m-1}(m+k)\,\tau(B_{n-1})$
$\tau(C_n) = k^{m-1}(m+k)\,\tau(C_{n-1}) + \tau(D_{n-1})$
$\tau(D_n) = k^m\,\tau(C_n) + k^{m-1}(m+k)\,\tau(D_{n-1}) = k^{m-1}(m+k)\,\tau(D_{n-1}) + \tau(B_{n-1}).$

The proof can be completed via the same technique used in Theorem 14. □

Theorem 18. Let M_n, be the graph obtained from W_n by joining $K_m - e$ between any two adjacent vertices in its cycle C_n. Then

$$\tau(M_n) = m^{(m-3)n}[(m+1+\sqrt{2m+1})^n + (m+1+\sqrt{2m+1}))^n - 2m^n].$$

Proof: Consider the five different families of graphs denoted by M_n, A_n, B_n, C_n and D_n, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.5 and Lemma 2.3 in [14] to find a system of recurrence relations:

$\tau(M_n) = \tau(A_n) + \tau(B_{n-1})$
$\tau(A_n) = 2m^{2m-5}\,\tau(C_{n-1}) + m^{m-2}\,\tau(M_{n-1})$
$\tau(B_n) = 2m^{m-3}\,\tau(D_n) + m^{m-2}\,\tau(B_{n-1})$
$\tau(C_n) = m^{m-2}\,\tau(C_{n-1}) + \tau(D_{n-1})$
$\tau(D_n) = 2m^{m-3}\,\tau(C_n) + m^{m-2}\,\tau(D_{n-1}) = m^{m-2}\,\tau(D_{n-1}) + \tau(B_{n-1}).$

The proof can be completed via the same technique used in Theorem 14. □

The very remarkable result is that when we compute the number of spanning trees of a graph and we obtain a system of recurrence relations, we obtain the same number of spanning trees of other graph with different system of recurrence relations as shown in the following Theorems:

Theorem 19. Let X_n be the graph obtained from the wheel graph W_n by adding parallel edge to each edge in the internal edges of W_n. Then for $n \geq 3$, $\tau(X_n) = \tau(G_n) = (2+\sqrt{3})^n + (2-\sqrt{3})^n - 2.$

Proof: Consider the following five different families of graphs denoted by X_n, A_n, B_n, C_n and D_n as shown in Fig. 12, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.2 in [14] on the indicated edges to find a system of recurrence relations:

$\tau(X_n) = \tau(A_n) + 2\,\tau(B_{n-1})$
$\tau(A_n) = \tau(C_{n-1}) + \tau(X_{n-1})$
$\tau(B_n) = \tau(D_n) + 3\,\tau(B_{n-1})$
$\tau(C_n) = \tau(C_{n-1}) + \tau(D_{n-1})$
$\tau(D_n) = \tau(C_n) + \tau(D_{n-1}) = \tau(D_{n-1}) + 2\,\tau(B_{n-1}).$

Consider the last two relations for $\tau(C_n)$ and $\tau(D_n)$, we have $\tau(C_{n+1}) = \tau(C_n) + 2\,\tau(D_n) = 3\,\tau(C_n) + 2\,\tau(D_{n-1}) = 4\,\tau(C_n) - \tau(C_{n-1})$ or $\tau(C_{n+1}) - 4\,\tau(C_n) + \tau(C_{n-1}) = 0$. Thus $\tau(C_n) - 4\,\tau(C_{n-1}) + \tau(C_{n-2}) = 0$ and $\tau(C_{n-1}) - 4\,\tau(C_{n-2}) + \tau(C_{n-3}) = 0$.

Subtracting these two equations, we get $\tau(C_n) - 5\,\tau(C_{n-1}) + 5\,\tau(C_{n-2}) - \tau(C_{n-3}) = 0$, which is the final recurrence relation for $\tau(C_n)$. Consider the first two relations for $\tau(X_n)$ and $\tau(A_n)$.

Since $2\tau(B_{n-1}) = \tau(C_n)$, we have $\tau(X_n) = \tau(A_n) + \tau(C_n)$ and hence $\tau(X_{n-1}) = \tau(A_{n-1}) + \tau(C_{n-1})$.

Substituting into the second relation, we obtain $\tau(A_n) = \tau(A_{n-1}) + 2\,\tau(C_{n-1})$, therefore $\tau(A_n) - \tau(A_{n-1}) = 2\,\tau(C_{n-1})$, since $\tau(C_{n-1}) - 4\,\tau(C_{n-2}) + \tau(C_{n-3}) = 0$, we have $2\,\tau(C_{n-1}) - 2\,(4)\,\tau(C_{n-2}) + 2\,\tau(C_{n-3}) = 0$, $[\tau(A_n) - \tau(A_{n-1})] - 4[\tau(A_{n-1}) - \tau(A_{n-2})] + [\tau(A_{n-2}) - \tau(A_{n-3})] = 0$, thus $\tau(A_n) - 5\,\tau(A_{n-1}) + 5\,\tau(A_{n-2}) - \tau(A_{n-3}) = 0$, which is the final recurrence relation for $\tau(A_n)$.

Now both $\tau(A_n)$ and $\tau(C_n)$ have the third order homogeneous recurrence relation:

$$x_n - 5\,x_{n-1} + 5\,x_{n-2} - x_{n-3} = 0 \tag{15}$$

Thus the characteristic equation is $r^3 - 5r^2 + 5r - 1 = 0$ which has characteristic roots $r = 2 \pm \sqrt{3}$ and $r = 1$. Thus the general solution of $\tau(X_n)$ is $\tau(X_n) = \alpha\,(2+\sqrt{3})^n + \beta\,(2-\sqrt{3})^n + \gamma$.

Which the same general solution as $\tau(G_n)$ in Theorem 2 with the same initial conditions $\tau(X_3) = 50$, $\tau(X_4) = 192$ and $\tau(X_5) = 722$. The proof is complete. □

Theorem 20. Let X_n^* be the graph obtained from the wheel graph W_n by adding k parallel edges to each edge in the internal edges of W_n. Then for $n \geq 3$,

$$\tau(X_n^*) = \tau(G_n^*) = \left(\frac{k+2+\sqrt{k(k+4)}}{2}\right)^n + \left(\frac{k+2-\sqrt{k(k+4)}}{2}\right)^n - 2.$$

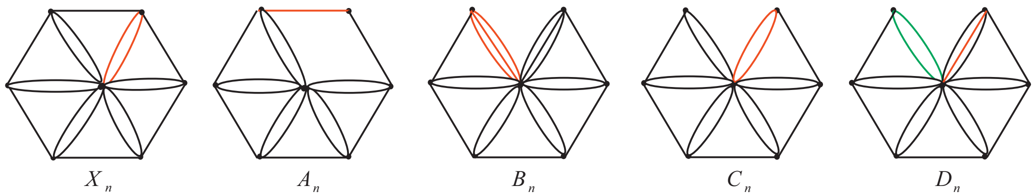

Fig. 12. The five families of graphs which we use to find an explicit formula for the complexity in the graph X_n.

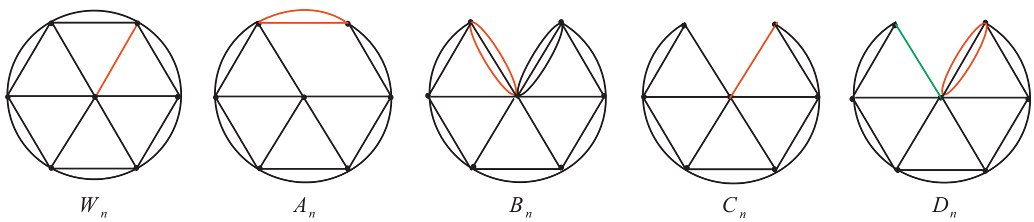

Fig. 13. The five families of graphs which we use to find an explicit formula for the complexity in the graph Y_n.

Proof: Consider the five different families of graphs denoted by $X_n^*,\ A_n,\ B_n,\ C_n$ and D_n, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.2 in [14] to find a system of recurrence relations:

$$\begin{aligned}
&\tau(X_n^*) = \tau(A_n) + k\tau(B_{n-1})\\
&\tau(A_n) = \tau(C_{n-1}) + \tau(X_{n-1}^*)\\
&\tau(B_n) = \tau(D_n) + (k+1)\,\tau(B_{n-1})\\
&\tau(C_n) = \tau(C_{n-1}) + k\,\tau(D_{n-1})\\
&\tau(D_n) = \tau(C_n) + \tau(D_{n-1}) = \tau(D_{n-1}) + k\,\tau(B_{n-1}).
\end{aligned}$$

The proof can be completed via the same technique used in Theorem 19. □

Theorem 21. Let Y_n be the graph obtained from the wheel graph W_n by adding parallel edge to each edge in the cycle of W_n. Then for $n \geq 3$, $\tau(Y_n) = \tau(E_n) = 2^{2n} - 2^{n+1} + 1$.

Proof: Consider the following five different families of graphs denoted by $Y_n,\ A_n,\ B_n,\ C_n$ and D_n as shown in Fig. 13, where n denote the number of vertices of W_n.

We use Eq. (2) together with Theorem 2.2 in [14] on the indicated edges to find a system of recurrence relations:

$$\begin{aligned}
&\tau(Y_n) = \tau(A_n) + \tau(B_{n-1})\\
&\tau(A_n) = 2\,\tau(C_{n-1}) + 2\tau(Y_{n-1})\\
&\tau(B_n) = \tau(D_n) + 2\,\tau(B_{n-1})\\
&\tau(C_n) = 2\,\tau(C_{n-1}) + \tau(D_{n-1})\\
&\tau(D_n) = \tau(C_n) + 2\,\tau(D_{n-1}) = 2\,\tau(D_{n-1}) + \tau(B_{n-1}).
\end{aligned}$$

Consider the last two relations for $\tau(C_n)$ and $\tau(D_n)$, we have $\tau(C_{n+1}) = 2\,\tau(C_n) + \tau(D_n) = 3\,\tau(C_n) + 2\,\tau(D_{n-1}) = 5\,\tau(C_n) - 4\,\tau(C_{n-1})$ or $\tau(C_{n+1}) - 5\,\tau(C_n) + 4\,(C_{n-1}) = 0$, thus $\tau(C_n) - 5\,\tau(C_{n-1}) + 4\,\tau(C_{n-2}) = 0$, $\tau(C_{n-1}) - 5\,\tau(C_{n-2}) + 4\,\tau(C_{n-3}) = 0$.

Subtracting these two equations, we get $\tau(C_n) - 6\,\tau(C_{n-1}) + 9\,\tau(C_{n-2}) - 4\,\tau(C_{n-3}) = 0$, which is the final recurrence relation for $\tau(C_n)$. Consider the first two relations for $\tau(Y_n)$ and $\tau(A_n)$.

Since $\tau(B_{n-1}) = \tau(C_n)$, we have $\tau(Y_n) = \tau(A_n) + \tau(C_n)$ and hence $\tau(Y_{n-1}) = \tau(A_{n-1}) + \tau(C_{n-1})$.

Substituting into the second relation, we obtain $\tau(A_n) = 2\,\tau(A_{n-1}) + 4\,\tau(C_{n-1})$, therefore $\tau(A_n) - \tau(A_{n-1}) = 4\,\tau(C_{n-1})$, since $\tau(C_{n-1}) - 5\,\tau(C_{n-2}) + 4\,\tau(C_{n-3}) = 0$, we have $4\,\tau(C_{n-1}) - 4\,(5)\ \tau(C_{n-2}) + 4\,(4)\ \tau(C_{n-3}) = 0$, $[\tau(A_n) - 2\,\tau(A_{n-1})] - 5\,[\tau(A_{n-1}) - 2\tau(A_{n-2})] + 4\,[\tau(A_{n-2}) - 2\,\tau(A_{n-3})] = 0$, thus $\tau(A_n) - 7\,\tau(A_{n-1}) + 14\,\tau(A_{n-2}) - 8\,\tau(A_{n-3}) = 0$, which is the final recurrence relation for $\tau(A_n)$.

We obtain $\tau(A_n)$ and $\tau(C_n)$ have the same recurrence relations (5) and (6) respectively and so $\tau(Y_n) = \tau(A_n) + \tau(B_{n-1})$ has the same general solution of $\tau(E_n)$ in Theorem 4.

Since the initial conditions $\tau(Y_3) = 49$, $\tau(Y_4) = 225$ and $\tau(Y_5) = 961$ are also the same as $\tau(E_n)$.

The proof is complete. □

Theorem 22. Let Y_n^* be the graph obtained from the wheel graph W_n by adding k parallel edges to each edge in the cycle of W_n. Then for $n \geq 3$,

$$\tau(Y_n^*) = \tau(E_n^*) = \left(\frac{2k+1+\sqrt{4k+1}}{2}\right)^n + \left(\frac{2k+1-\sqrt{4k+1}}{2}\right)^n - 2\,k^n.$$

Proof: Consider the five different families of graphs denoted by $Y_n^*,\ A_n,\ B_n,\ C_n$ and D_n, where n denote the number of vertices of W_n.

We use Eq. (2) together Theorem 2.2 in [14] to find a system of recurrence relations:

$$\begin{aligned}
&\tau(Y_n^*) = \tau(A_n) + \tau(B_{n-1})\\
&\tau(A_n) = k\tau(C_{n-1}) + k\,\tau(Y_{n-1}^*)\\
&\tau(B_n) = \tau(D_n) + k\,\tau(B_{n-1})\\
&\tau(C_n) = k\,\tau(C_{n-1}) + \tau(D_{n-1})\\
&\tau(D_n) = \tau(C_n) + k\,\tau(D_{n-1}) = k\,\tau(D_{n-1}) + \tau(B_{n-1}).
\end{aligned}$$

The proof can be completed via the same technique used in Theorem 21. □

3. Spanning tree entropy

After having explicit Formulas for the number of spanning trees of wheel graph W_n and the graphs generated by W_n, we can calculate its spanning tree entropy which is a finite number and a very interesting quantity characterizing the network structure, defined as in [15,16] as:

$$Z(G) = \lim_{n\to\infty} \frac{\ln \tau(G)}{|V(G)|}. \tag{16}$$

This limit is known as the asymptotic tree number entropy, asymptotic growth constant, or thermodynamical limit.

$$Z(W_n) = \lim_{n\to\infty} \frac{\ln\left[\left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2\right]}{n+1} = \ln\left(\frac{3+\sqrt{5}}{2}\right) \approx 0.9624.$$

$$Z(G_n) = Z(X_n) = \lim_{n\to\infty} \frac{\ln\left[(2+\sqrt{3})^n + (2-\sqrt{3})^n - 2\right]}{2n+1} = \ln\left(\sqrt{2+\sqrt{3}}\right) \approx 0.6585.$$

$$Z(E_n) = Z(Y_n) = \lim_{n\to\infty} \frac{\ln\left[4^n - 2\times 2^n + 1\right]}{2n+1} = \ln(2) \approx 0.6931.$$

$$Z(H_n) = \lim_{n\to\infty} \frac{\ln\left[(3+\sqrt{5})^n + (3-\sqrt{5})^n - 2\times 2^n\right]}{3n+1} = \ln\left(\sqrt[3]{3+\sqrt{5}}\right) \approx 0.5519.$$

$$Z(Fl_n^{(3)}) = \lim_{n\to\infty} \frac{\ln\left[\left(\frac{7+\sqrt{33}}{2}\right)^n + \left(\frac{7-\sqrt{33}}{2}\right)^n - 2^{n+1}\right]}{2n+1} = \ln\left(\sqrt{\frac{7+\sqrt{33}}{2}}\right) \approx 0.9260.$$

$$Z(Fl_n^{(4)}) = \lim_{n\to\infty} \frac{\ln\left[9^n - 2\times 3^n + 1\right]}{3n+1} = \ln(\sqrt[3]{9}) \approx 0.7324.$$

$$Z(Sf_n) = \lim_{n\to\infty} \frac{\ln\left[(4+\sqrt{7})^n + (4-\sqrt{7})^n - 2\times 3^n\right]}{2n+1} = \ln(\sqrt{4+\sqrt{7}}) \approx 0.9470.$$

4. Conclusions

In this work, we have proposed a new method for counting the number of spanning trees of a wheel graph. Using this method, one can obtain the number of spanning trees of any graph generated by the wheel graph. Also we investigate the asymptotic limit of these graphs. An advantage of our technique lies in the avoidance of laborious computation of Laplacian spectra that is needed for a generic method for determining spanning trees.

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