Lecture 3: Identification using time-series variation

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Motivation

- Last time, we introduced the impulse-propagation paradigm
$$y_t = \sum_{l=0}^{\infty} \Theta_l \varepsilon_{t-l}$$
From the Θ's we get our objects of interest: IRFs, FEVDs, HDs
- How can we use data to learn about the Θ's? Three broad approaches:
  1. **Structural:** specify full-blown model, then *estimate* model using micro/macro data
     This will be the focus of Matt's course
  2. **Semi-structural, time-series**: use time-series properties of y_t + identifying assumptions that hold for *families* of models
     This will be our focus for the next few lectures
  3. **Semi-structural, cross-sectional**: what can microeconometric causal designs teach us?
     We will discuss this towards the end of the course

Outline

Identifying SVMA models

- We assume that our observed macro aggregates y_t follow a structural SVMA model

$$y_t = \sum_{l=0}^{\infty} \Theta_l \varepsilon_{t-l}, \quad \varepsilon_t \sim WN(0, I)$$

- **Identification challenge**
  - We can estimate the second-moment properties of y_t (i.e. autocovariances/spectrum)
  - However, the structural SVMA model is not identified by these moments – infinitely many VMA models are consistent with those second-moment properties
- This should not come as a surprise: why should simple covariances tell us something about the origins of business cycles or the effects of policy interventions?

The Wold decomposition

- From the Wold decomposition we know that we can always find a VMA representation

$$y_t = \tilde{\Psi}(L)\tilde{\varepsilon}_t$$

where $\tilde{\varepsilon}_t \equiv \Sigma^{-1/2} u_t \sim WN(0, I)$ and $u_t = y_t - \text{Proj}(y_t | \{y_\tau\}_{-\infty<\tau<t})$

  - Clearly there is nothing "structural" about the $\tilde{\varepsilon}_t$'s – they are just scaled versions of the Wold innovations

- In fact, there exist many such **observationally equivalent** representations, none of which necessarily have any economic meaning

- This *indeterminacy* can take different forms

Indeterminacy: rotations

- The first indeterminacy is **static indeterminacy**
  - Start with the true SVMA model and assume that $n_\varepsilon = n_y = n$. Let $Q \in O(n)$ denote an $n \times n$ orthogonal matrix. It is easy to verify that the process

$$y_t = \sum_{l=0}^{\infty} \tilde{\Theta}_l \tilde{\varepsilon}_{t-l}, \quad \tilde{\varepsilon}_t \sim WN(0, I)$$

    with $\tilde{\Theta}_l = \Theta_l Q'$ and $\tilde{\varepsilon}_t = Q\varepsilon_t$ has the same second-moment properties:

$$\text{Cov}(y_t, y_{t-h}) = \sum_{l=0}^{\infty} \tilde{\Theta}_l Q' Q \tilde{\Theta}'_{l+h} = \sum_{l=0}^{\infty} \Theta_l \Theta'_{l+h}, \quad \text{as } Q'Q = I$$

- **In words**: orthogonal rotations of today's shocks leave covariances unchanged
  - This is a problem: we could just relabel the shocks or take linear combinations – the data wouldn't be able to reject it

Indeterminacy: root flipping

- The second indeterminacy is **dynamic indeterminacy**
  - To illustrate this consider a simple univariate MA(2), i.e. $n_y = n_\varepsilon = 1$:

$$y_t = \theta_0 \varepsilon_t + \theta_1 \varepsilon_{t-1}$$

  - Now compare this with the following alternative process

$$y_t = \theta_1 \varepsilon_t + \theta_0 \varepsilon_{t-1}$$

    Note how they share all second-moment properties! But of course the implied shock IRFs look very different as long as $\theta_0 \neq \theta_1$ [show what they are]
  - What did we do? We took the polynomial $\Theta(L) = \theta_0 + \theta_1 L$ and "flipped the root"
- **Lesson**: **flipping roots** of MA polynomials leaves second moments unchanged

  For a general multivariate treatment of this see Lippi and Reichlin (1994)

Indeterminacy: system size

- The third indeterminacy concerns the **size** of the system
  - So far we have assumed that $n_y = n_\varepsilon$. Our options for possible SVMA processes expand even more if we allow $n_\varepsilon > n_y$
  - To illustrate suppose again that the true model is the same one-shock MA(2) as before, but now we entertain the two-shock process

$$y_t = \theta_{01}\varepsilon_{1,t} + \theta_{02}\varepsilon_{2,t} + \theta_{11}\varepsilon_{1,t-1} + \theta_{12}\varepsilon_{2,t-1}$$

  - The process is consistent with the data as long as

$$\begin{aligned} \text{Var}(y_t) =& \theta_{01}^2 + \theta_{02}^2 + \theta_{11}^2 + \theta_{12}^2 = \theta_0^2 + \theta_1^2 \\ \text{Cov}(y_t, y_{t-1}) =& \theta_{01}\theta_{11} + \theta_{02}\theta_{12} = \theta_0\theta_1 \end{aligned}$$

- **Lesson**: we can always split any given n_y-dimensional covariance structure into many $(n_\varepsilon > n_y)$ **distinct shocks**

Summary: identification challenge

- From second moments alone, our SVMA model is severely **under-identified**

  We will discuss the feasibility and desirability of identification from higher-order moments later

- This should not be surprising: so far we have not made any **economic identifying assumptions** that would allow claims about *causality*

- What assumptions are necessary to make progress, imposing as *little structure* as possible? Two main approaches:
  - **Identification under invertibility**: e.g. using zero, sign restrictions, ...

    This is the traditional VAR literature
  - **Identification without invertibility**: e.g. using IVs, ...

    The macro IV approach is very popular these days and is more in spirit of the applied micro tradition

Invertibility

- Recall our definition of **invertibility**:

Definition (Invertibility)

A VMA process $\{y_t\}$ is said to be **invertible** with respect to $\{\varepsilon_t\}$ if

$$\varepsilon_t \in \text{span}(y_\tau, -\infty < \tau \leq t)$$

In this case, ε_t are said to be y_t-fundamental

- This is an economically **substantive assumption**:
  - Current and past y_t span the same space as current and past ε_t
    $\Rightarrow$ implies that an econometrician observing $\{y_\tau\}_{-\infty<\tau<t}$ would learn nothing from observing the actual structural shocks $\{\varepsilon_\tau\}_{-\infty<\tau<t}$
  - Need $n_\varepsilon \leq n_y$. The more (relevant) data we include, the more plausible this assumption [more on this later]
  - Not always satisfied in macro models, see e.g. Fernandez-Villaverde et al (2007)

Invertibility: discussion

- Invertibility may fail for **many reasons**
  - A necessary – but quite strong – condition is that we have as many observables as shocks
  - But even that is not sufficient. **Key culprits** for non-invertibility when $n_\varepsilon = n_y$ are:
    1. There are news shocks (e.g. forward guidance, fiscal spending plans)
       See Leeper et al. (2013) for a detailed discussion
    2. There are noise shocks (e.g. noise misperceived as news about future fundamentals)
       See Chahrour-Jurado (2018) for a detailed discussion

    Intuition: with those types of shocks we would need to be able to look into the future. $n_\varepsilon = n_y$ in general only ensures that $\varepsilon_t \in \text{span}(y_\tau, -\infty < \tau < \infty)$, not that just looking to the past is enough
- So what do we gain from this strong assumption? As it turns out, a lot ...

Invertibility & Wold innovations

- Invertibility has important implications for **Wold innovations**. Recall

$$u_t \equiv y_t - \mathsf{Proj}(y_t | \{y_\tau\}_{-\infty<\tau<t})$$

- Since y_t and u_t share the same span, it follows that under invertibility

$$\varepsilon_t \in \mathsf{span}(u_\tau, -\infty < \tau \leq t)$$

  But ε_t is uncorrelated with all u_τ for $\tau < t$ and u_t is white noise, so $\varepsilon_t \in \mathsf{span}(u_t)$ or

$$u_t = S\varepsilon_t$$

  for some matrix S, where $SS' = \Sigma = \mathsf{Var}(u_t)$. We call S the structural impact matrix

- Also recall that the Wold innovations u_t and their coefficients $\Psi(L)$ are identifiable. We have thus made a lot of progress ...

Invertibility & Wold innovations

- By imposing invertibility, we have identified the structural shocks ε_t and their dynamic causal effects up to a **static rotation problem**:
  - The structural shocks are given by

$$\varepsilon_t = Q\Sigma^{-1/2}u_t$$

    We define here the rotations Q relative to the basis $\Sigma^{-1/2}$. Intuitively, we just start from the standardized Wold innovations and rotate them

  - Their dynamic causal effects are given as

$$\Theta(L) = \Psi(L)\Sigma^{-1/2}Q'$$

- Invertibility simplified the identification problem by eliminating the dynamic & size indeterminacy. Remains to find identifying assumptions to pin down Q ...

Connection to VARs

- This approach to identification can be operationalized using VARs
  - In finite samples, can't estimate the Wold VMA(∞)
  - We have the SVMA model $y_t = \Theta(L)\varepsilon_t$. By invertibility, we know that the one-sided inverse $\tilde{A}(L) = \Theta(L)^{-1}$ exists
  - Letting $A(L) = S\tilde{A}(L)$, we thus have

$$A(L)y_t = u_t = S\varepsilon_t$$

  That is VAR residuals = Wold residuals, which we now just need to rotate

- The derivations above are why the literature often speaks of "VAR identification"
  - May mask the fact that we also have to retain invertibility. Important to remember that identifying assumptions are **invertibility + something** to find the right rotation

VAR approach

- Obtain an estimate of the reduced-form representation $A(L)$
- Problem: can't estimate VAR with infinite lags in finite sample
- Approach: approximate $A(L)$ by truncating VAR to p lags
  - How good is this approximation?
  - If the Wold representation has absolute summable coefficients, $\sum_{k=0}^{\infty} |\Psi_k| < \infty$, then it admits a VAR representation with coefficient matrices that decay to zero rapidly
  - Always the case for causal VARMA models
  - In this case, can be well approximated by a finite-order VAR
  - However, may still be in issue in small samples where we can't estimate many lags, see discussion in Nakamura and Steinsson (2018)
- Identify impact matrix $S = \Theta_0$ and compute structural impulse responses

Outline

Exact exclusion restrictions

- **Traditional approach**: point-identify the system through exact zero restrictions on $\Theta(L)$
  - Need to impose enough restrictions such that only one $Q \in O(n)$ is consistent with them
- We will review the most well-known examples of such restrictions
  1. Restrictions on short-run impulse responses Θ_0
     We will review Christiano, Eichenbaum, and Evans (1999)
  2. Restrictions on long-run impulse responses $\Theta(1) = \sum_{l=0}^{\infty} \Theta_l$
     We will review Blanchard-Quah (1991)

Short-run zero restrictions

- Recall that $\Theta_0\Theta_0' = \Sigma$. As Σ is symmetric, this gives us $n(n+1)/2$ distinct equations
- We want to identify Θ_0, which has n^2 unknown parameters. Thus, we need an additional $n(n-1)/2$ restrictions
- One approach: impose zero restrictions on Θ_0
  - Has the interpretation of timing assumptions: shock i does not contemporaneously affect variable j
- Popular assumption: Θ_0 is **lower-triangular**:
  - In economic terms: variable 1 only responds contemporaneously to shock 1, variable 2 responds contemporaneously to shocks 1&2, and so on
  - Mathematically $\Theta_0 = \text{chol}(\Sigma)$

Christiano, Eichenbaum, and Evans (1999)

- Christiano, Eichenbaum, and Evans (1999) operationalize this assumption to identify **monetary policy shocks**
- They impose a block recursive structure:
  - Observables y_t: real GDP, implicit GDP deflator, commodity price index, fed funds rate, total reserves, non-borrowed reserves, M1
  - Quarterly data, 4 lags
  - Identifying assumptions: invertibility $+$ recursive shock ordering
    - Here this means that all variables listed before the fed funds rate don't respond within the period to monetary shocks, while monetary policy doesn't respond to changes in the variables listed after
    - Can be equivalently thought of in terms of the information set of the monetary authority: monetary policy only looks at current prices and output when setting the policy rate

Christiano, Eichenbaum, and Evans (1999)

- Formally, they assume a block recursive structure for $A_0 = \Theta_0^{-1}$:

$$A_0 = \begin{pmatrix} \underset{(n_1 \times n_1)}{a_{11}} & \underset{(n_1 \times 1)}{0} & \underset{(n_1 \times n_2)}{0} \\ \underset{(1 \times n_1)}{a_{21}} & \underset{(1 \times 1)}{a_{22}} & \underset{(1 \times n_2)}{0} \\ \underset{(n_2 \times n_1)}{a_{31}} & \underset{(n_2 \times 1)}{a_{32}} & \underset{(n_2 \times n_2)}{a_{33}} \end{pmatrix}. \tag{1}$$

  where $y_t = (y_{1,t}, ffr_t, y_{2,t})'$ with $y_{1,t} = (gdp_t, p_t, pcom_t)$ and $y_{2,t} = (tr_t, nbr_t, m_t)$

- Assumes:
  - monetary authority does not monitor $y_{2,t}$ contemporaneously when setting the policy rate
  - monetary policy shocks are orthogonal to the central bank information set
    1. have no direct effect on $y_{1,t}$
    2. do not affect $y_{1,t}$ indirectly via $y_{2,t}$

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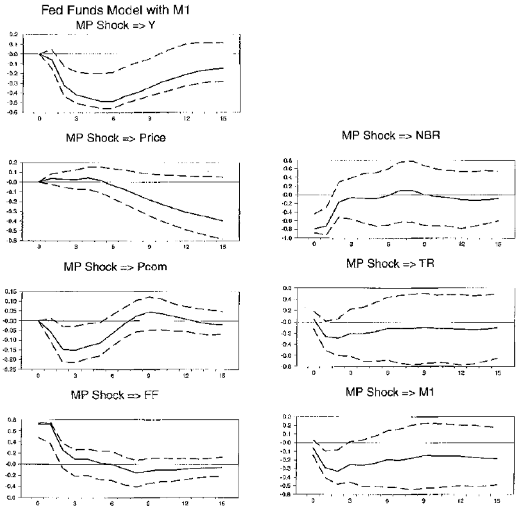
Christiano, Eichenbaum, and Evans (1999)

- Assumption can be implemented by setting $A_0^{-1} \equiv \Theta_0 = \text{chol}(\Sigma)$
- By definition, this satisfies $\Theta_0 \Theta_0' = \Sigma$
- Since the inverse of a block triangular matrix is also block triangular, it will also satisfy (1)
- Note that **recursiveness** assumption is not sufficient to identify all elements of A_0
    - easy to see that the first n_1 and last n_2 equations are indistinguishable
- But sufficient to identify the object of interest: the dynamic response of y_t to a monetary policy shock

Christiano, Eichenbaum, and Evans (1999)



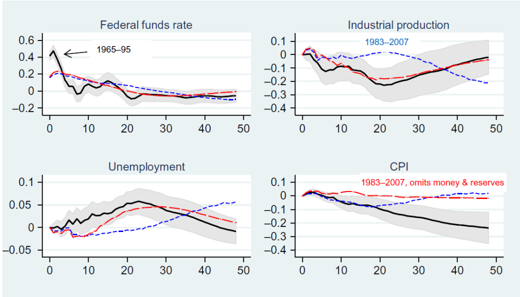
- persistent rise in the ffr and drop in nbr $\rightarrow$ liquidity effect
- sluggish fall in tr and M1
- sustained decline in real GDP after 2 quarters
  - hump-shaped response with trough after 4-6 quarters
- very sluggish fall in GDP deflator, persistent decline in the index of commodity prices

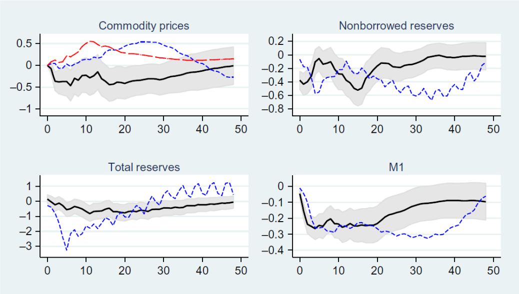
Christiano, Eichenbaum, and Evans (1999)

- Response confirm well with conventional views on the effect of monetary policy shocks
- Christiano, Eichenbaum, and Evans (1999) also show that results are robust along a number of dimensions
- But Ramey (2016) shows that approach does not work as well when sample period is updated
- What potential problems of short-run zero restrictions can you think of?

Christiano, Eichenbaum, and Evans (1999) revisited

From Ramey (2016):





Problems with short-run zero restrictions

- In general, it is unclear why the economy should possess a recursive structure
  - In fact, most DSGE models do not possess such a structure (Canova and Pina, 2005)
- Timing-based restrictions **particularly problematic**
  1. with lower-frequency data (e.g. quarterly or annual): questionable why output should not respond within a quarter – more plausible at monthly frequency
  2. in the presence of financial data: asset prices react near-instantaneously to shocks and news. At the same time, policymakers take financial conditions into account
     - Depending on ordering, recursiveness assumption implies that financial variables don't respond contemporaneously to monetary policy shocks or that monetary policy does not take financial variables into account
     - Both assumptions are clearly counterfactual

Agnostic identification

- Often times, we are only interested in one (or a subset of shocks): $\varepsilon_{j,t}$
- This simplifies the identification problem as we need fewer restrictions to identify a subset of shocks
- In this case, it is also possible to relax the invertibility assumption
- In particular, we only have to require that the shock of interest is invertible

$$\varepsilon_{j,t} \in \text{span}(y_\tau, -\infty < \tau \leq t) \Leftrightarrow \varepsilon_{j,t} = s' u_t \quad \text{for some } s$$

  - This condition is weaker than invertibility of all shocks. It's commonly referred to as partial invertibility, see Forni, Gambetti, and Sala (2019)

Outline

Long-run zero restrictions

- Economic theory has sometime stark predictions about the long-run impact of different shocks (e.g. long-run monetary neutrality)
- **Idea**: restrict the **long-run response** of certain variables i to certain shocks j to zero. Implies

$$0 = \sum_{l=0}^{\infty} \Theta_{i,j,l} = \sum_{l=0}^{\infty} \Psi_{1,\cdot,l} S_{\cdot,j} = \sum_{l=0}^{\infty} \Psi_{1,\cdot,l} \Sigma^{1/2} Q'_{\cdot,j}$$

  - this imposes additional restrictions on the orthogonal matrix Q, aiding with identification of the system
- Can be operationalized as follows:
  - The long-run impacts of shocks in the VAR is given by $A(1)^{-1}\Theta_0$
  - The long-run variance is $J \equiv A(1)^{-1}\Sigma(A(1)')^{-1} = A(1)^{-1}\Theta_0\Theta_0'(A(1)')^{-1} = \Xi\Xi'$
  - Popular choice: $\Xi = \text{chol}(J)$. Gives $\Theta_0 = A(1)\Xi$
  - Only properly defined in stationary VAR

Blanchard and Quah (1989)

- Blanchard and Quah (1989) study the dynamic effects of aggregate demand and supply disturbances
- Observables y_t: real output growth, unemployment rate
  - unemployment postulated to be stationary, log output assumed to be $I(1)$
- Identifying assumption: invertibility + long-run zero restrictions:
  - only first shock has a permanent effect on output
  - first shock interpreted as supply shock, second shock as demand shock

Blanchard and Quah (1989)

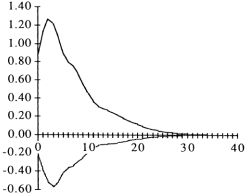


FIGURE 1. RESPONSE TO DEMAND, — = OUTPUT, ~~ = UNEMPLOYMENT

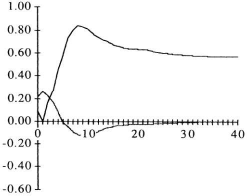


FIGURE 2. RESPONSE TO SUPPLY, — = OUTPUT, ~~ = UNEMPLOYMENT

- Demand disturbances have a hump-shaped mirror-image effect on output and unemployment that is temporary
- Positive supply shock temporarily increases unemployment and is only shock to permanently impact upon output: response increases steadily over time, peaking after two years and reaching a plateau after five years

Blanchard and Quah (1989)

TABLE 2A—VARIANCE DECOMPOSITION OF OUTPUT AND UNEMPLOYMENT
(NO DUMMY BREAK, TIME TREND IN UNEMPLOYMENT)

Percentage of Variance Due to Demand:

Horizon (Quarters)	Output	Unemployment
1	83.8	79.7
	(59.4, 93.9)	(55,3, 92.0)
2	87.5	88.2
	(62.8, 95.4)	(58.9, 95.2)
3	83.4	93.5
	(58.8, 93.3)	(61.3, 97.5)
4	78.9	95.7
	(53.5, 90.0)	(63.9, 98.2)
8	52.5	88.9
	(31.4, 68.6)	(63.5, 94.5)
12	37.8	79.7
	(21.3, 51.4)	(58.8, 90.3)
40	18.7	75.9
	(7.4, 23.5)	(56.9, 88.6)

- Demand disturbances make a substantial contribution to output fluctuations at short and medium-term horizons
- But sensitive to treatment of breaks and trends in data

Blanchard and Quah (1989)

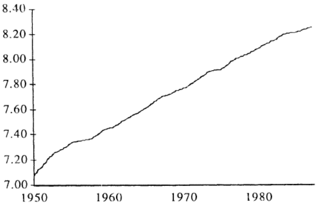


FIGURE 7. OUTPUT FLUCTUATIONS ABSENT DEMAND

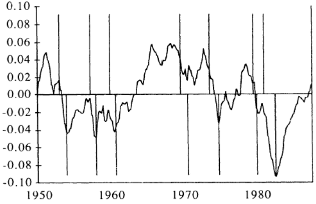


FIGURE 8. OUTPUT FLUCTUATIONS DUE TO DEMAND

- Using the identified shocks, we can also decompose output fluctuations into demand and supply factors
  - These are computed as counterfactuals, setting transitory or permanent shocks to zero respectively
- We see a large negative value in transitory output component during the Volker disinflation

Challenges with long-run zero restrictions

- Can you think of any problems with Blanchard and Quah (1989) and long-run restrictions more generally?

Challenges with long-run zero restrictions

- Can you think of any problems with Blanchard and Quah (1989) and long-run restrictions more generally?
- Very difficult to estimate **long-run variance** in short samples (Uhlig, 2004)
- Only two shocks may be too simplistic: conditions for **meaningful aggregation** of underlying structural shocks quite stringent (Faust and Leeper, 1997)
  - In reality, many shocks that satisfy properties of demand and supply shocks
  - Demand: monetary, fiscal policy, . . . ; Supply: Technology, labor force, . . .
  - The condition for aggregation involves that the responses for the underlying shocks are scaled versions of each other ought to be a ‘scaled-up’ or ‘scaled down’ version of one another $\Rightarrow$ unlikely to be satisfied in practice

Challenges with long-run zero restrictions

- Problems with **high-frequency feedbacks and time aggregation** (Faust and Leeper, 1997)
    - Even if true shocks are orthogonal in the DGP, they may become correlated when looking at infrequently sampled data
    - *"For example, suppose we view she stock-market drop in October of 1987 as a supply shock. The Federal Reserve Board reacted within hours, injecting reserves into the banking system. In quarterly data, these two changes will be contemporaneous; sorting out whether this is a nominal or a real shock will be impossible"*

Practical implications

- Apart from concerns of non-invertibility [more on this later], small scale models may be problematic because of **aggregation problems**
- Work with sufficient information and try to identify meaningful economic shocks
  - in this way, each shock is allowed to have 'its own properties' and to impact upon economy in its own way
- Because of **time-aggregation** issues, working with low-frequency data when using time-variation for identification can be problematic
  - Work with data sampled as frequently as possible
  - This is one of the motivations behind **high-frequency** identification [which we'll cover in detail later]

Medium-run restrictions

- Back to the problem of estimating the **long-run variance**
  - Even with 40-50 years of data, it is difficult to capture the infinite long run
- Uhlig (2004) proposes different focus:
  - Instead of infinite long-run, look at sufficiently long horizon (e.g. 10 years out)
  - At this horizon, transitory shocks should have essentially died out and permanent shocks explain the dominant **fraction of the forecast error variance** of relevant series
  - In context of Blanchard and Quah (1989): permanent shocks explain the **largest fraction** of the forecast error variance of GDP **at sufficiently long horizon**

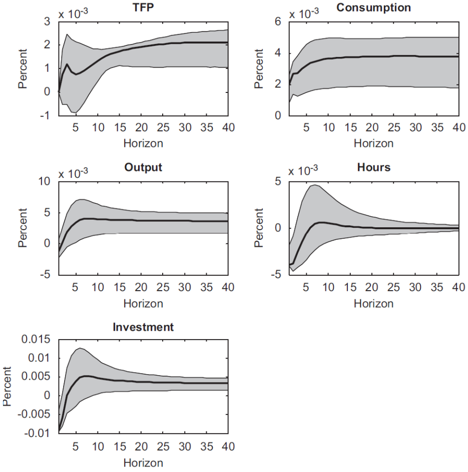
Medium-run restrictions

- Can be implemented as follows:
  - Start with an estimate of S, e.g. $\tilde{S} = \text{chol}(\Sigma)$
  - The contribution to the forecast error forecast variance of shock j up to horizon k is $M = \sum_{h=0}^{k} \tilde{S}' \Psi_h' E_{jj} \Psi_h \tilde{S}$, where E_{jj} is a zero matrix with 1 on jth diagonal entry
  - Compute the ordered eigendecomposition of M. The shock associated with the first eigenvector will explain the maximum of the FEV of $y_{j,t}$ at horizon k
  - Then obtain $S = \tilde{S}V$, where V are the corresponding eigenvectors
  - This approach nests short-run restrictions for $h = 0$ and long-run restrictions for $h = \infty$
- See Uhlig (2003, 2004) for more information

Barsky and Sims (2011)

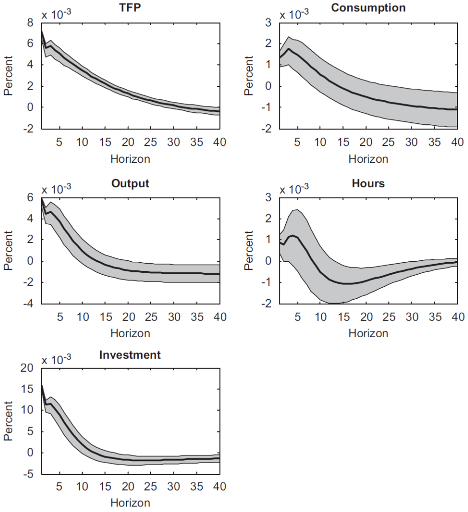
- Barsky and Sims (2011) study the role of technology news shocks in the business cycle
- Observables y_t: TFP, consumption, output, hours
- Identifying assumptions: invertibility + short-run zero and medium-run restrictions
  - Aggregate TFP driven by two shocks: surprise technology shock and technology news shock
  - News shocks have no contemporaneous effect on TFP
  - News shocks explain maximum possible fraction of residual variance of TFP at pre-specified long horizon

Barsky and Sims (2011)



- News shock does not affect TFP on impact but affects households' permanent income
- Consumption starts to increase right away, hours jump down
- Output and investment fall on impact but then start to increase
- Shock has permanent effects
- What about the surprise shock?

Barsky and Sims (2011)



- Interestingly, estimated surprise shock is a transitory one
- TFP jumps up but then converges back to zero
- Other variables essentially follow this pattern
- Importantly, this comes naturally out of the data, Barsky and Sims (2011) impose nothing that constrains the surprise shock to be transitory

Outline

Motivation

- Exact zero restrictions are widely regarded as implausible

  E.g.: effects of monetary policy on output may be delayed, not exactly 0 on impact

- One popular alternative: impose restrictions on **signs** of impulse responses
  - Example: a contractionary monetary shock should lead to an increase in interest rates and a decrease in prices (eventually)
  - This is promising: if central banks lean against inflation, then all non-policy shocks lead to a positive co-movement of output and inflation

- How can we operationalize this intuition? What can invertibility + sign restrictions tell us about the SVMA model?

Invertibility + sign restrictions

- Recall that IRFs for a given rotation Q are equal to

$$\Theta(L) = \Psi(L)S = \Psi(L)\Sigma^{1/2}Q'$$

- Suppose we want to impose that $\Theta_{i,j,l} \geq 0$. This rules out some Q's! We need

$$\Psi_{i,\cdot,l}\Sigma^{1/2}Q'_{\cdot,j} \geq 0 \tag{2}$$

  $\Rightarrow$ Sign restrictions give **identified sets**: keep all Q's such that (i) $QQ' = I$ and (ii) all imposed sign restrictions of the form (2) hold

- In this sense, sign restrictions are **weak** information: they generally do not point- but only set-identify the objects of interest

Example: sign restrictions for monetary policy

- Where do sign restrictions come from?
  - Simple economic frameworks (e.g. AS-AD etc), intuitive reasoning
  - More recently: derive signs consistent with a vast class of DSGE models
- Take the simple New Keynesian model as an example

$$y_t = \mathbb{E}_t[y_{t+1}] - (i_t - \mathbb{E}_t[\pi_{t+1}]) + \sigma_d \varepsilon_t^d \qquad \text{(IS)}$$

$$\pi_t = \kappa y_t + \beta\, \mathbb{E}_t[\pi_{t+1}] - \sigma_s \varepsilon_t^s$$

$$i_t = \phi_\pi \pi_t + \sigma_m \varepsilon_t^m$$

with $\varepsilon_t = (\varepsilon_t^d, \varepsilon_t^s, \varepsilon_t^m) \sim WN(0, I)$

Example: sign restrictions for monetary policy

- Solving the model (e.g. using method of undetermined coefficients) gives **static mapping** from shocks to observables [i.e. VMA(0)]

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \underbrace{\begin{pmatrix} + & + & - \\ + & - & - \\ + & - & + \end{pmatrix}}_{\Theta} \begin{pmatrix} \varepsilon_t^d \\ \varepsilon_t^s \\ \varepsilon_t^m \end{pmatrix}$$

  Denote the response of variable j to shock k as $\Theta_{j,k}$

- The signs are robust for reasonable parameter values

Example: sign restrictions for monetary policy

- Easy to see from the model solution:

$$\begin{pmatrix} y_t \\ \pi_t \\ i_t \end{pmatrix} = \underbrace{\frac{1}{1+\phi_\pi \kappa} \begin{pmatrix} \sigma_d & \phi_\pi \sigma_s & -\sigma_m \\ \kappa \sigma_d & -\sigma_s & -\kappa \sigma_m \\ \phi_\pi \kappa \sigma_d & -\phi_\pi \sigma_s & \sigma_m \end{pmatrix}}_{\Theta} \begin{pmatrix} \varepsilon_t^d \\ \varepsilon_t^s \\ \varepsilon_t^m \end{pmatrix}$$

- Can also show that as long as $\sigma_k > 0$ for $k = d, s, m$, $\kappa > 0$ and $\phi_\pi > 1$, Θ is invertible

Example: sign restrictions for monetary policy

- Interpreting the Q's
  - Let $x_t = (y_t, \pi_t, i_t)'$ and $\Sigma_x = \text{Var}(x_t)$. Note that

$$\Sigma_x = \Theta\Theta'$$

  - Before we defined the Q's relative to $\Sigma^{1/2}$. More generally we can define them wrt any matrix A such that $AA' = \Sigma_x$, including Θ. The Q's are just rotations wrt a basis
  - Choosing Θ as the basis has an advantage: if $\tilde{\Theta} = \Theta Q'$, then the matrix Q corresponds to 'mis-identified' structural shocks

$$\tilde{\varepsilon}_t \equiv Q\varepsilon_t$$

- We'd like to find identifying assumptions that are consistent with $Q = I$ (so $\tilde{\varepsilon}_t = \varepsilon_t$) but rule out (almost) everything else. Will **sign restrictions** on Θ do the trick?

Example: sign restrictions for monetary policy

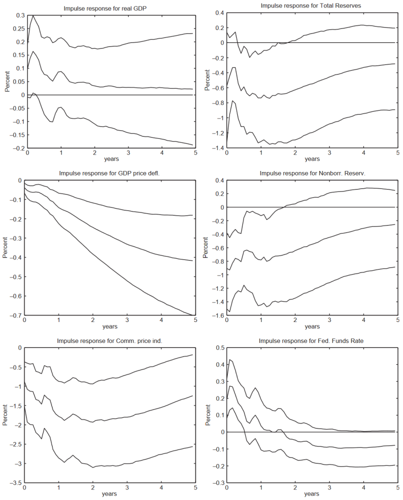
- Let's impose the following **minimalist sign restrictions:**

$$\Theta = \begin{pmatrix} ? & ? & ? \\ ? & ? & - \\ ? & ? & + \end{pmatrix} \tag{3}$$

- Is this enough to learn anything about the **real effects of monetary policy**, $\Theta_{y,m}$?
  - Promising: monetary shocks are the only shocks to move π_t and i_t in opposite directions
  - This implies that $\tilde{\varepsilon}_t^m = \varepsilon_t^m$ is consistent with (3) while $\tilde{\varepsilon}_t^m = \varepsilon_t^d$ or $\tilde{\varepsilon}_t^m = \varepsilon_t^s$ are not

Uhlig (2005)

- This is essentially what Uhlig (2005) does.
  - Agnostically identifies a monetary shock by imposing signs on the responses of the FFR, non-borrowed reserves and prices
  - But importantly leaves impact on output unrestricted
- Controversial result: output increases after contractionary monetary policy shock
- Can you think of any problems with this approach?



What do (minimalist) sign restrictions identify?

- Let's focus on the object in question: the response of output to monetary policy shocks $\Theta_{y,m}$
- Formally (3) gives us an *identified* set $[\underline{\Theta}_{y,m}, \bar{\Theta}_{y,m}]$ for $\Theta_{y,m}$, defined via the programs

$$\inf_q / \sup_q \Theta_{1,\cdot} q$$

  s.t. $||q|| = 1$ and $\Theta_{2,\cdot} q < 0$ and $\Theta_{3,\cdot} q > 0$

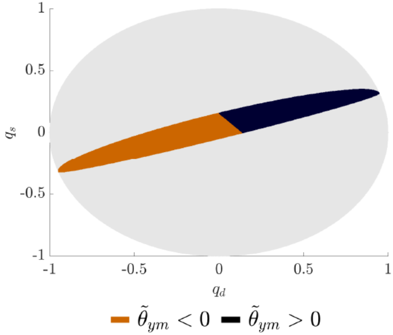
- What does the identified set $[\underline{\Theta}_{y,m}, \bar{\Theta}_{y,m}]$ look like?

What do (minimalist) sign restrictions identify?

- We can show that if Θ is invertible, we can't sign $\Theta_{y,m}$, i.e.

$$\underline{\Theta}_{y,m} < 0 < \bar{\Theta}_{y,m}$$

- Interpretation: "**masquerading shocks**", see Wolf (2020)
    - The truth $q = (0,0,1)'$ is in the identified set, while $q = (1,0,0)'$ and $q = (0,1,0)'$ are not
    - **But:** linear combos of expansionary supply and demand shocks can also move $i \uparrow$ & $\pi \downarrow$, but $y \uparrow$
- In general, the identified set for Q may be empty, contain a singleton or contain multiple elements, see Wolf (2022) for more info



Practical implications

- Minimal sign restrictions typically **not enough** to identify shock of interest
- Try to identify as many shocks as possible
- Shock for which some impacts are left unrestricted should explain a sufficiently large fraction of the variance of the data
    - typically not the case for monetary policy shocks
- Complement sign restrictions with **other restrictions**
    - zero and sign restrictions (Arias, Rubio-Ramírez, and Waggoner, 2018)
    - narrative sign restrictions (Antolín-Díaz and Rubio-Ramírez, 2018)
    - explicitly defended probabilistic priors on Q (Baumeister and Hamilton, 2015)
    - we will come back to this later ...

Digression: inference in sign identified models

- Much of the sign restrictions literature does not report identified sets
- Instead: additionally impose a prior on orthogonal rotation matrices, then characterize posterior
  - Common choice: uniform Haar prior
  - While uniform on the rotation matrices, prior can be informative for posterior objects of interest (Baumeister and Hamilton, 2015)
- May be **not desirable**: prior can be economically counterfactual and influence posterior tightness in actual applications
- Ongoing debate in the literature: Rubio-Ramirez, Waggoner, and Zha (2010); Arias, Rubio-Ramirez, and Waggoner (2020); Baumeister and Hamilton (2015, 2022); Watson (2019); Wolf (2020); Giacomini and Kitagawa (2021)

Outline

Statistical identification

- Identification through **higher-order moments** has recently gained in popularity
- Basic idea:
  - Lack of identification so far because we only looked at second moments. If shocks are not normal, higher-order moments can help
    1. Identification through heteroskedasticity point-identifies the model [discuss next]
    2. Alternative: if true shocks are independent and non-normal, then the model is also point-identified
       Intuition: only true rotation of the u_t's is independent, see Gouriéroux, Monfort, and Renne (2017)
  - **But**: even with statistical id, you still need economic reasoning to label the shocks
    E.g. which of the "statistically" identified shocks is the monetary policy shock?

Identification through heteroskedasticity in one slide

- Suppose that the structural shocks ε_t have different volatilities on two subsamples
  - Write the variances as $\{(\sigma_j^a)^2, (\sigma_j^b)^2\}_{j=1}^n$ for our two subsamples a and b. Note: now we're not normalizing shock variances, so we instead normalize the diagonal entries of S to 1
  - For each sub-sample $k = a, b$, the Wold residuals u_t^k satisfy

$$\text{Var}(u_t^k) = S \begin{pmatrix} (\sigma_1^k)^2 & \dots & 0 \\ & \ddots & \\ 0 & \dots & (\sigma_n^k)^2 \end{pmatrix} S' = \Sigma_k$$

- We thus have now more information that we can exploit

$$\Sigma_b \Sigma_a^{-1} = S \begin{pmatrix} (\sigma_1^b/\sigma_1^a)^2 & \dots & 0 \\ & \ddots & \\ 0 & \dots & (\sigma_n^b/\sigma_n^a)^2 \end{pmatrix} S^{-1}$$

- Columns of S are eigenvectors of $\Sigma_b \Sigma_a^{-1}$, which is identified. Given our normalization of the diagonal of S to 1, we see that S is identified if the volatility ratios differ across shocks

Brunnermeier et al. (2021)

- Brunnermeier et al. (2021) study the dynamic interactions between credit conditions, monetary policy, and real activity using a structural VAR identified by heteroskedasticity
- Observables y_t: IP, PCEPI, real estate and consumer loans, commercial and industrial loans, M1, FFR, PCOM, term spread, bond spread, TED spread
- Regimes:

TABLE 2—DATES FOR VARIANCE REGIMES

	Start	End	Description
1	Jan 1973	Sep 1979	Oil crisis and stagflation
2	Oct 1979	Dec 1982	Volcker disinflation
3	Jan 1983	Dec 1989	Major S&L crisis defaults
4	Jan 1990	Dec 2007	Great moderation
5	Jan 2008	Dec 2010	Financial crisis and Great Recession
6	Jan 2011	Jun 2015	Zero lower bound, recovery from Great Recession

- Give identified shocks a label, usually by the variable that they are most strongly associated with

Brunnermeier et al. (2021)

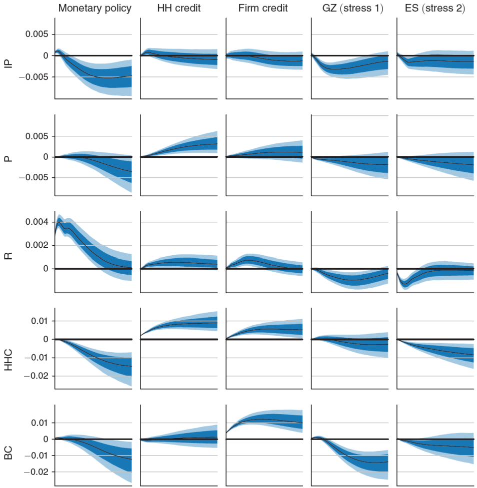


Figure 1. Impulse Response Functions (1/4)

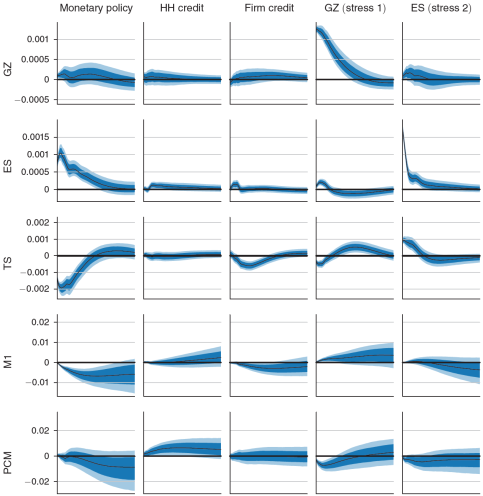


Figure 2. Impulse Response Functions (2/4)

Brunnermeier et al. (2021)

- Identified monetary policy shocks have somewhat larger effects than typically found in the literature $\Rightarrow$ shocks are amplified through interbank credit spreads
- Two other shocks look like: **"stress" shocks**, which originate in the financial sector and propagate to the real economy with some lag
    - shocks to corporate bond spreads
    - shocks to interbank rate spreads
- As the two shocks have different economic implications, may be important to separate them
    - This is an example where aggregation in the spirit of Faust and Leeper (1997) may fail $\Rightarrow$ one-dimensional metrics of financial conditions may be insufficient for capturing risks for the real economy

Brunnermeier et al. (2021)

- While credit spread shocks do have strong real effects, they do not provide more than a few months of "advance warning" of an output contraction
- The weak predictive value of credit aggregates for output growth contrasts with some results in the literature
- Can you think of any issues with the heteroskedasticity-based approach?

Thoughts on statistical identification

- Comparison with our **standard identification arguments**
  - Classical invertibility-based analysis assumes white noise shocks. We may allow for stochastic volatility, but do not exploit higher-order moments for identification
  - Statistical approaches achieve identification by strengthening assumptions on the shock process: independence/conditional orthogonality rather than just white noise
- Some words of caution: [see Montiel Olea, Plagborg-Møller, and Qian (2022) for details]
  - For heteroskedasticity: must assume that **only volatility ratios** changed, but not shock propagation. Also can't handle common volatility changes (Great Moderation?)
  - Higher-order moments are **hard to estimate**, in particular in time series. Analysis is necessarily fragile wrt statistical properties of shocks, e.g. weak identification issues arise if the shocks are nearly iid Gaussian
  - It can be very **hard to label/interpret** the identified shocks ex-post
- To me, reliance on economic identifying assumptions is a **virtue**, not a bug!
  Even with statistical identification need economic framework to interpret shocks

Outline

Identification without invertibility

- Recently there's been a push towards **invertibility-robust** identification
- Two main approaches:
  1. **IV** approach:
     - Postulates that the researcher can find a valid IV for a macro shock:
       $$z_t = \alpha\varepsilon_{j,t} + \sigma_\nu \nu_t, \quad \nu_t \sim WN(0,1) \tag{4}$$
     - This brings applied macro closer to microeconomic practice: try to find credible natural experiments that satisfy (4)
     - Main appeal: requires **no** further assumptions on the SVMA model, at least for IRFs
  2. Large VAR/FAVAR approach
     - Idea: increase information set in VAR to make invertibility assumption more plausible
       In the context of monetary shocks: make the information set of the econometrician align with the information set of the central banker
     - Main appeal: can retain assumption of invertibility, easy to compute FEVD, historical decompositions

What do we get if invertibility is violated?

- What does invertibility-based identification recover **without** invertibility?
- Can show that in this case

$$u_t = \sum_{l=0}^{\infty} M_l \varepsilon_{t-l}$$

  - Without invertibility, the identified shocks $\tilde{\varepsilon}_t = S^{-1} u_t$ are a linear combination of current and **past true shocks** ▸ Derivations
  - How bad is the bias? It depends ...
- Invertibility shouldn't be an either-or proposition. Intuitively, we should be able to do well if $\varepsilon_{j,t}$ is **"close to invertible"**. How to formalize that?

$$\mathcal{R}^2_{j,0} \equiv 1 - \frac{\text{Var}^*(\varepsilon_{j,t} | \{y_\tau\}_{-\infty<\tau\leq t})}{\text{Var}(\varepsilon_{j,t})} = 1 - \text{Var}^*(\varepsilon_{j,t} | \{y_\tau\}_{-\infty<\tau\leq t})$$

  - Can show: the asymptotic bias of many VAR-type procedures is a function of $\mathcal{R}^2_{j,0}$, vanishing as $\mathcal{R}^2_{j,0} \to 1$. Forni, Gambetti, and Sala (2019); Plagborg-Møller and Wolf (2020)

Outline

Setting & identification problem

- Let's suppose we have access to a **macro IV/proxy**

$$z_t = \alpha\varepsilon_{j,t} + \sigma_\nu \nu_t, \quad \nu_t \sim WN(0,1) \tag{5}$$

- How may we be able to find such z's? Examples:
    - Institutional knowledge of policy decisions (e.g. fiscal spending $\perp$ to macro conditions)
    - High-frequency financial market responses (e.g. idiosyncratic global oil supply changes)
    - We will cover different approaches to come up with macro IVs in detail . . .
- Even if found, such z's are unlikely to capture all of the hidden shocks $\varepsilon_{j,t} \Rightarrow$ Motivation for IV

Setting & identification problem

- How can we use such IVs to identify out objects of interest? Our econometric model is now the general SVMA + our IV equation

$$y_t = \Theta(L)\varepsilon_t$$

$$z_t = \alpha\varepsilon_{1,t} + \sigma_\nu \nu_t, \quad (\varepsilon_t', \nu_t')' \sim WN(0, I)$$

- It is trivial to identify relative **dynamic causal effects**
  - Note that we have

$$\text{Cov}(y_{i,t+l}, z_t) = \Theta_{i,1,l}\text{Cov}(\varepsilon_{1,t}, \alpha\varepsilon_{1,t} + \sigma_\nu \nu_t) = \alpha\Theta_{i,1,l}$$

  - Thus we can point identify the IRF ratio

$$\frac{\Theta_{i,1,l}}{\Theta_{1,1,0}} = \frac{\text{Cov}(y_{i,t+l}, z_t)}{\text{Cov}(y_{1,t}, z_t)}$$

    Often this is all we want: how does output (y_i) respond to a monetary shock that increases nominal rates by 100 bp on impact (y_1)?

Identifying assumptions

- Note that none of these derivations assumed anything about **invertibility** (or more generally, our three sources of indeterminacy). Simply not needed for IV.
- **Identifying assumptions** are
  (1) $\mathbb{E}[z_t \varepsilon_{1,t}] = \alpha \neq 0$ (Relevance)
  (2) $\mathbb{E}[z_t \varepsilon_{-1,t}] = 0$ (Contemporaneous exogeneity)
  (3) $\mathbb{E}[z_t \varepsilon_{t+j}] = 0$ for $j \neq 0$ (Lead-lag exogeneity)
- (1) and (2) are the standard IV relevance and exogeneity conditions (w/o controls)
- (3) arises because of dynamics: $y_{i,t+l}$ depends on entire history of shocks, for z_t to identify effect of $\varepsilon_{1,t}$ alone, it must be uncorrelated with shocks at all leads and lags
  - uncorrelated with future ε's not restrictive (follows from definition of shocks as unanticipated structural disturbances)
  - uncorrelated with past ε's is **restrictive and strong** assumption

Lag(-lead) exogeneity

- Lag exogeneity implies that z_t is unpredictable given past ε's
  - Note that z_t could be serially correlated yet satisfy this condition. Example:

$$z_t = \alpha \varepsilon_{j,t} + \sigma_\nu \nu_t,$$

    where ν_t is a serially correlated error that is independent of $\{\varepsilon_t\}$

- In contrast to contemporaneous exogeneity, lag exogeneity is testable
  - z_t should be unforecastable in a regression of z_t on lags of y_t
  - If lag exogeneity fails, then the IV approach is not valid
  - Problem can potentially be addressed by adding control variables

Adding controls

- If the instrument is truly exogenous, we do not need to add any controls
- Two main reasons for adding controls
  - Instrument may only be valid after including suitable controls
  - Including controls could help reduce the sampling variance of the IV estimator by reducing the variance of the error term
- With controls, the conditions for **instrument validity** are

(1) $\mathbb{E}[z_t^\perp \varepsilon_{1,t}^\perp] = \alpha \neq 0$ (Relevance)

(2) $\mathbb{E}[z_t^\perp \varepsilon_{-1,t}^\perp] = 0$ (Contemporaneous exogeneity)

(3) $\mathbb{E}[z_t^\perp \varepsilon_{t+j}^\perp] = 0$ for $j \neq 0$ (Lead-lag exogeneity)

where $x_t^\perp = x_t - \text{Proj}(x_t|w_t)$ for some variable x and controls w

VAR implementation: SVAR-IV

- We can also implement the IV approach in the VAR framework, retaining invertibility
  - This is the original external instruments/proxy SVAR approach developed by Stock and Watson (2012); Mertens and Ravn (2013)
- **Identifying assumptions:**
  (1) $\mathbb{E}[z_t \varepsilon_{1,t}] = \alpha \neq 0$ (Relevance)
  (2) $\mathbb{E}[z_t \varepsilon_{-1,t}] = 0$ (Exogeneity)
  (3) $u_t = S\varepsilon_t$ (Invertibility)

VAR implementation: SVAR-IV

- Identifies structural impact vector $s_1 = S_{\cdot,1}$ up to sign and scale:

$$s_1 \propto \frac{\mathbb{E}[z_t u_t]}{\mathbb{E}[z_t u_{i,t}]}$$

- **Proof**:

$$\begin{aligned} \mathbb{E}[z_t u_t] &= S\,\mathbb{E}[z_t \varepsilon_t] = s_1 \alpha \\ \mathbb{E}[z_t u_{1,t}] &= s_{1,1} \alpha \\ \Rightarrow \frac{\mathbb{E}[z_t u_t]}{\mathbb{E}[z_t u_{1,t}]} &= \frac{s_1}{s_{1,1}} \end{aligned}$$

provided that $\mathbb{E}[z_t u_{1,t}] \neq 0$.

VAR implementation: SVAR-IV

- If invertibility holds, we can recover the shock $\varepsilon_{1,t}$ as the projection of z_t on u_t (which is $\propto \varepsilon_{1,t}$), rescaled to have unit variance

$$\varepsilon_{1,t} = [\text{Cov}(z_t, u_t)\text{Var}(u_t)^{-1}\text{Cov}(z_t, u_t)']^{-1/2}\text{Cov}(z_t, u_t)\text{Var}(u_t)^{-1}u_t$$

- Can then recover variance decompositions, historical decompositions in the usual invertibility-based way

Partial invertibility

- Note that full invertibility is not necessary for agnostic identification in SVAR-IV
- Instead of (3) we can assume

(3a) $\varepsilon_{1,t} = s' u_t$ (Partial invertibility)

(3b) $\mathbb{E}[z_t \varepsilon'_{-1,t+j}] = 0 \;\; \forall j \neq 0$ for which $\mathbb{E}[u_t \varepsilon'_{-1,t+j}] \neq 0$ (Limited lead-lag exogeneity)

  - **In words**: we only have to assume that the shock of interest is invertible in combination with a lead-lag exogeneity condition of the instrument with the non-invertible shocks, see Miranda-Agrippino and Ricco (2018)

No free lunch

- We have seen that there is no free lunch dropping invertibility
- We can relax the assumption of (partial) invertibility but have to assume a **lead-lag exogeneity** condition for the instrument instead
- This condition is potentially also *quite restrictive*
- Which approach is more plausible has to be argued for in the particular empirical application
- Also note that if the controls are the same, the IV and SVAR-IV approach produce the same response at $h = 0$ and but may differ after

Testing invertibility

- If we have an IV available, this allows us to **test invertibility**
- The key insight is that under invertibility we must have

$$\text{Proj}(y_{i,t} | \{z_\tau, y_\tau\}_{-\infty<\tau<t}) = \text{Proj}(y_{i,t} | \{y_\tau\}_{-\infty<\tau<t})$$

  - In words: z does not help to predict future y's above and beyond the information already contained in y
  - Why? $z_t = \alpha\varepsilon_{1,t} + \sigma_\nu \nu_t$, where the second term is useless for forecasting future y, and the first term is captured by past $y's$
  - Suggests a simple test: does z Granger-cause y?
- If SVAR-IV is used even though invertibility fails, then we are subject to the issues reviewed in the slide appendix

Outline

Recoverability

- Alternative approach: **recoverablility-based identification**

Definition (Recoverability)

A VARMA process $\{y_t\}$ is said to be **recoverable** with respect to $\{\varepsilon_t\}$ if

$$\varepsilon_t \in \text{span}(y_\tau, -\infty < \tau < \infty)$$

- Thus the identification problem can be solved by looking into the past & future:

$$\varepsilon_t = \sum_{l=-\infty}^{\infty} \Psi_l y_{t-l} = \sum_{l=-\infty}^{\infty} Q_l u_{t+l}$$

  - Here $\Psi(L)$ is some two-sided lag polynomial, and $Q(L)$ satisfies $I = \sum_{l=0}^{\infty} Q_l \Sigma Q_l'$

Recoverability

- The identification problem is now even harder: we call some n_y-dimensional VMA "structural" (assume away size indeterminacy), but still need to contend with static & now **dynamic** indeterminacy
- See Lippi and Reichlin (1994); Mertens and Ravn (2010); Chahrour and Jurado (2022) for more info

Shock importance

- Without assuming invertibility, we were not able to characterize shock importance
- However, we can also say something about shock importance under the weaker recoverability assumption
- Intuition on the **identification challenge**
  - The IV consists of signal $(\alpha\varepsilon_{1,t})$ and noise $(\sigma_\nu \nu_t)$. We don't know the signal-to-noise ratio
  - This matters: e.g. can't know whether $\text{Cov}(y_{i,t}, z_t)$ is low because $\varepsilon_{1,t}$ is unimportant of because there's just a lot of noise
  - Formally our challenge is to say something about the forecast variance ratio

$$FVR_{i,1,h} \equiv 1 - \frac{\text{Var}(y_{i,t+h} | \{y_{t-l}\}_{l=0}^{\infty}, \{\varepsilon_{1,t+l}\}_{l=1}^{\infty})}{\text{Var}(y_{i,t+h} | \{y_{t-l}\}_{l=0}^{\infty})}$$
$$= \frac{\sum_{m=0}^{h-1} \frac{1}{\alpha^2} \text{Cov}(y_{i,t+h}, z_t)^2}{\text{Var}(y_{i,t+h} | \{y_{t-l}\}_{l=0}^{\infty})}$$

  - Plagborg-Møller and Wolf (2020) prove that α and so the FVR is **interval-identified**

Outline

FAVAR/Large VAR

- A different approach from relaxing invertibility is to make the assumption more plausible
- Idea: if the VAR spans a wide set of information, invertibility is more likely to be satisfied
- Problem: difficult to estimate large VARs on short samples
- Two main approaches
  1. Estimate large Bayesian VAR using shrinkage prior (Bańbura, Giannone, and Reichlin, 2010)
  2. Impose a factor structure and estimate dynamic factor model/FAVAR

FAVAR

- We focus here on FAVAR approach but intuition for large BVARs similar
- Let y_t be a vector of observable economic variables assumed to have pervasive effects throughout the economy
- We know that additional information not fully captured in y_t may be relevant
- Suppose that this additional information can be summarized by an $n_f \times 1$ vector of unobserved factors f_t, where n_f is “small”
    - We may think of the unobserved factors as diffuse concepts such as economic activity, credit conditions, etc, that cannot be easily represented by one or two series but rather are reflected in a wide range of economic variables

FAVAR

- The joint dynamics of the system are given by

$$\begin{pmatrix} f_t \\ y_t \end{pmatrix} = \Psi(L) \begin{pmatrix} f_{t-1} \\ y_{t-1} \end{pmatrix} + u_t \tag{6}$$

  - We call this a **factor-augmented** VAR (FAVAR)
  - If the terms that relate y_t to f_{t-1} are all zero, this collapses to a standard VAR in y_t
- We cant directly estimate (6) because the factors are unobservable
- Idea: if factors represent forces that affect many economic variables, we can hope to infer something about the factors from observations on a variety of economic time series

FAVAR

- Consider a wide range of macro and financial variables, collected in the $n_x \times 1$ vector x_t, where n_x is "large"
  - May be greater than T and much greater than the number of factors $n_x \gg n_f + n_y$
- Assume that the informational time series x_t are related to the unobservable factors f_t and observable factors y_t by

$$x_t = \Lambda_f f_t + \Lambda_y y_t + e_t$$

  - Captures the idea that both y_t and f_t represent pervasive forces that drive x_t
  - The implication that x_t does only depend on current and not lagged factors is not restrictive, as f_t can be interpreted to include arbitrary lags of fundamental factors

How to estimate the factors?

- Two approaches to estimate the factors:
  1. Classical two-step approach: estimate factors f_t as first n_f principal components of x_t (Stock and Watson, 2002)
  2. Bayesian single-step approach: estimate factors and FAVAR jointly using Bayesian maximum likelihood (Bernanke, Boivin, and Eliasz, 2005)

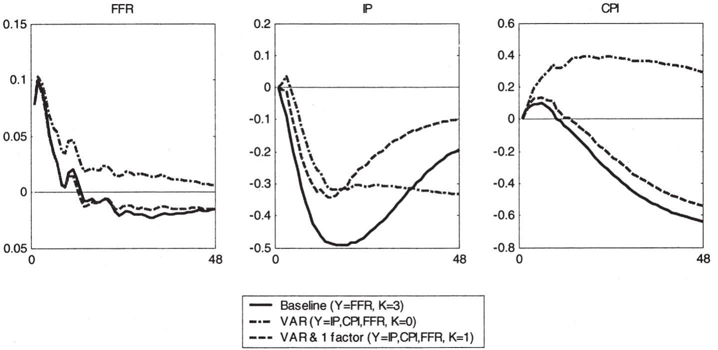
Test for invertibility

- Using the FAVAR we can do another test for invertibility
- Idea: As FAVAR nests VAR, potential differences may hint at non-invertibility issue
- Forni and Gambetti propose the following formal approach
    1. Extract first k principal components from large panel of macro and financial time series
    2. Test whether the principal components Granger-cause the series in the VAR. If they do, series in VAR are not informationally sufficient, i.e. VAR is non-invertible
- But: can only be used to reject sufficiency but not to validate it

Bernanke, Boivin, and Eliasz (2005)

- Bernanke, Boivin, and Eliasz (2005) estimate a FAVAR to address three potential issues with small-scale monetary SVARs:
  - Non-invertibility: to the extent that central banks and private sector have info not reflected in VAR, measurement of policy shocks likely contaminated
  - Variable selection: choice of a specific data series to represent a general economic concept such as “real activity” is often arbitrary to some degree
  - Wider effects: IRFs can be observed only for included variables, which is generally only a small subset of variables of interest
- Variables y_t: FFR (treated as observed), 3 unobservable factors estimated from 120 monthly macro time series
- Identifying assumptions: invertibility + short-run zero restrictions
  - monetary policy shock cannot impact upon factors within the month
  - but can impact variables within the panel. In particular, fast-moving variables (think of asset prices) are allowed to respond contemporaneously to policy shocks.

Bernanke, Boivin, and Eliasz (2005)



Bernanke, Boivin, and Eliasz (2005)

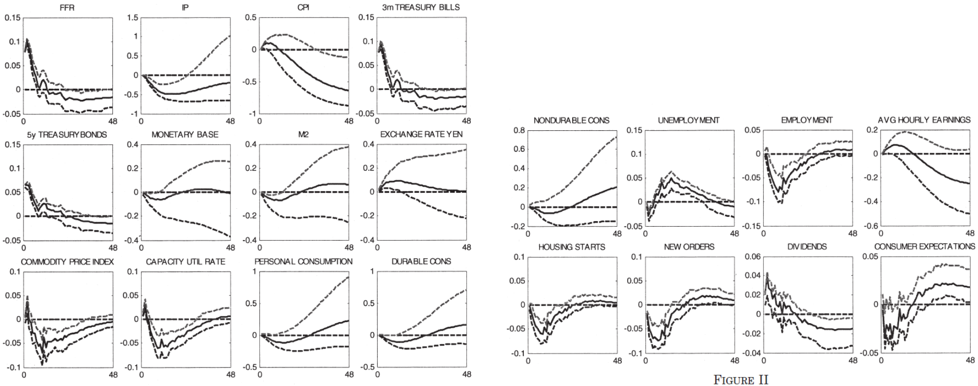


FIGURE II

- Price puzzle goes (almost) away when VAR is augmented with factors
- Responses paint a coherent picture of the effect of MP

Bernanke, Boivin, and Eliasz (2005)

TABLE I
CONTRIBUTION OF THE POLICY SHOCK TO VARIANCE OF THE COMMON COMPONENT

Variables	Variance decomposition	R^2
Federal funds rate	0.454	*1.000
Industrial production	0.054	0.707
Consumer price index	0.038	0.870
3-month treasury bill	0.433	0.975
5-year bond	0.403	0.925
Monetary base	0.005	0.104
M2	0.005	0.052
Exchange rate (Yen/$)	0.007	0.025
Commodity price index	0.049	0.652
Capacity utilization	0.100	0.753
Personal consumption	0.006	0.108
Durable consumption	0.005	0.062
Nondurable cons.	0.002	0.062
Unemployment	0.103	0.817
Employment	0.066	0.707
Aver. hourly earnings	0.007	0.072
Housing starts	0.032	0.387
New orders	0.081	0.624
S&P dividend yield	0.062	0.549
Consumer expectations	0.036	0.700

The column titled Variance decomposition reports the fraction of the variance of the forecast error, at the 60-month horizon, explained by the policy shock. R^2 refers to the fraction of the variance of the variable explained by the common factors, $(\hat{F}_t', Y_t')$. See text for details.
* This is by construction.

- For all real activity and prices variables, and some monetary variables, fraction of variance explained by monetary policy shocks is very low
- for a few financial variables, fraction of variance is significantly higher
- For some maybe implausibly high, e.g. 45.5% of Fed funds rate

Summary

- We have seen different time series methods for identifying the SVMA Θ's
  - We began by characterizing the **identification problem**: indeterminacy from three sources
  - We then reviewed potential solutions: invertibility + X, macro IVs, recoverability + X, invertibility in information-rich environments + X
- Next: how should we in finite samples implement these identification approaches & estimate the Θ's?

Outline

What do we get if invertibility is violated?

- What does invertibility-based identification recover **without** invertibility?
- We will derive everything from a **state-space** representation:

$$s_t = As_{t-1} + B\varepsilon_t$$
$$y_t = Cs_{t-1} + D\varepsilon_t$$

  - Invertibility is about the informativeness of y_t about s_t (and so ε_t)
  - We will compute this using the Kalman filter. Will rely on linear projections. Notation:

$$\hat{s}_{t|t} \equiv \mathsf{Proj}(s_t|\{y_\tau\}_{-\infty<\tau\leq t})$$
$$\Sigma^s_{t|t} \equiv \mathsf{Var}^*(s_t|\{y_\tau\}_{-\infty<\tau\leq t})$$

    and similarly for y_t
  - Algorithm: use standard linear projection formulas to update $\hat{s}_{t-1|t-1}$ and $\Sigma^s_{t-1|t-1}$ to $\hat{s}_{t|t}$ and $\Sigma^s_{t|t}$. Fixed point will give population limits (i.e. $t \to \infty$)

What do we get if invertibility is violated?

- Start with $\hat{s}_{t-1|t-1}$ and $\Sigma^s_{t-1|t-1}$. Predicting one period ahead:

$$\begin{aligned}
\hat{s}_{t|t-1} &= A\hat{s}_{t-1|t-1} \\
\Sigma^s_{t|t-1} &= A\Sigma^s_{t-1|t-1}A' + BB' \\
\hat{y}_{t|t-1} &= C\hat{s}_{t-1|t-1} \\
\Sigma^y_{t|t-1} &= C\Sigma^s_{t-1|t-1}C' + DD'
\end{aligned}$$

- Next we use y_t:

$$\hat{s}_{t|t} = \hat{s}_{t|t-1} + \mathbb{E}[(s_t - \hat{s}_{t|t-1})(y_t - \hat{y}_{t|t-1})']\,\mathbb{E}[(y_t - \hat{y}_{t|t-1})(y_t - \hat{y}_{t|t-1})']^{-1}(y_t - \hat{y}_{t|t-1})$$

Updating

- Plugging in we get

$$\hat{s}_{t|t} = \hat{s}_{t|t-1} + \underbrace{[A\Sigma^s_{t-1|t-1}C' + BD'][C\Sigma^s_{t-1|t-1}C' + DD']^{-1}}_{\text{Kalman gain } K_t}(y_t - \hat{y}_{t|t-1})$$

  Similarly

$$\Sigma^s_{t|t} = \Sigma^s_{t|t-1} - [A\Sigma^s_{t-1|t-1}C' + BD'][C\Sigma^s_{t-1|t-1}C' + DD']^{-1}[A\Sigma^s_{t-1|t-1}C' + BD']'$$

- Thus, given any history of the observables $\{y_t\}_{t=0}^T$, we can construct a sequence of estimates of the hidden states s_t

Innovations representation

- Letting $t \to \infty$, we obtain the steady-state Kalman gain and state uncertainty:

$$\Sigma^s = (A - KC)\Sigma^s(A - KC)' + BB' + KDD'K' - BD'K' - KDB'$$
$$K = (A\Sigma^s C' + BD')(C\Sigma^s C' + DD')^{-1}$$

- This allows us to re-write the state-space system in **innovations form**:

$$\hat{s}_t = A\hat{s}_{t-1} + K \underbrace{\left(y_t - \hat{y}_{t|t-1}\right)}_{u_t}$$
$$y_t = C\hat{s}_{t-1} + u_t$$

- Now we finally get the payoff: write **Wold innovations** in terms of the ε_t's ...

Innovations representation

- The innovations representation gives

$$\begin{pmatrix} s_t \\ \hat{s}_t \end{pmatrix} = \begin{pmatrix} A & 0 \\ KC & A - KC \end{pmatrix} \begin{pmatrix} s_{t-1} \\ \hat{s}_{t-1} \end{pmatrix} + \begin{pmatrix} B \\ KD \end{pmatrix} \varepsilon_t$$

$$u_t = (C \quad -C) \begin{pmatrix} s_{t-1} \\ \hat{s}_{t-1} \end{pmatrix} + D\varepsilon_t$$

to arrive at

$$u_t = \left\{ D + (C \quad -C) \left(I - \begin{pmatrix} A & 0 \\ KC & A - KC \end{pmatrix} L \right)^{-1} \begin{pmatrix} B \\ KD \end{pmatrix} L \right\} \varepsilon_t = \sum_{l=0}^{\infty} M_l \varepsilon_{t-l}$$

  - Under invertibility $s_t = \hat{s}_t$, so $u_t = D\varepsilon_t$, exactly as we have seen
  - Without invertibility, the identified shocks $\tilde{\varepsilon}_t = S^{-1} u_t$ are a linear combination of current and **past true shocks**

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