

# Algebraic Statistics in A State of the Union



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BAYLOR  
UNIVERSITY

DEPARTMENT OF STATISTICAL SCIENCE

- Historical Perspective
- The **algstat** Ecosystem
  - Installing the packages
  - **mpoly**
  - **latter**
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# Historical Perspective

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- a variety of ways to create new objects

- flexible mechanisms for implementing new methods

  - (including infix operators and overloading)

- access to the operating system (rw, sockets, ...)

- a simple way to incorporate C++ routines (Rcpp)

2011 : `mpoly` – data structures and methods for multivariate polynomials

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`mpoly`

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2014 : **algstat** – algebraic statistical data analysis

**mpoly**

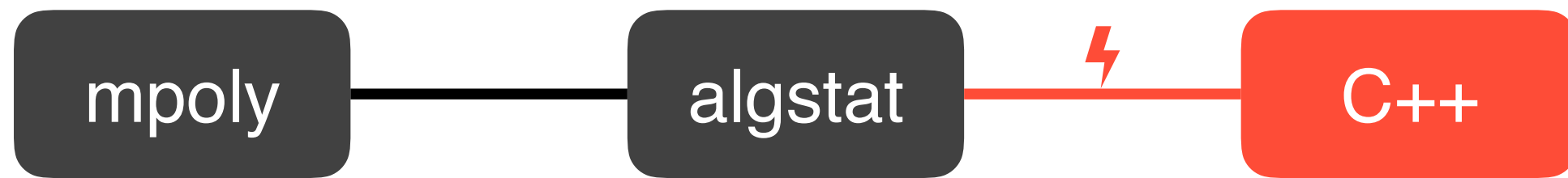
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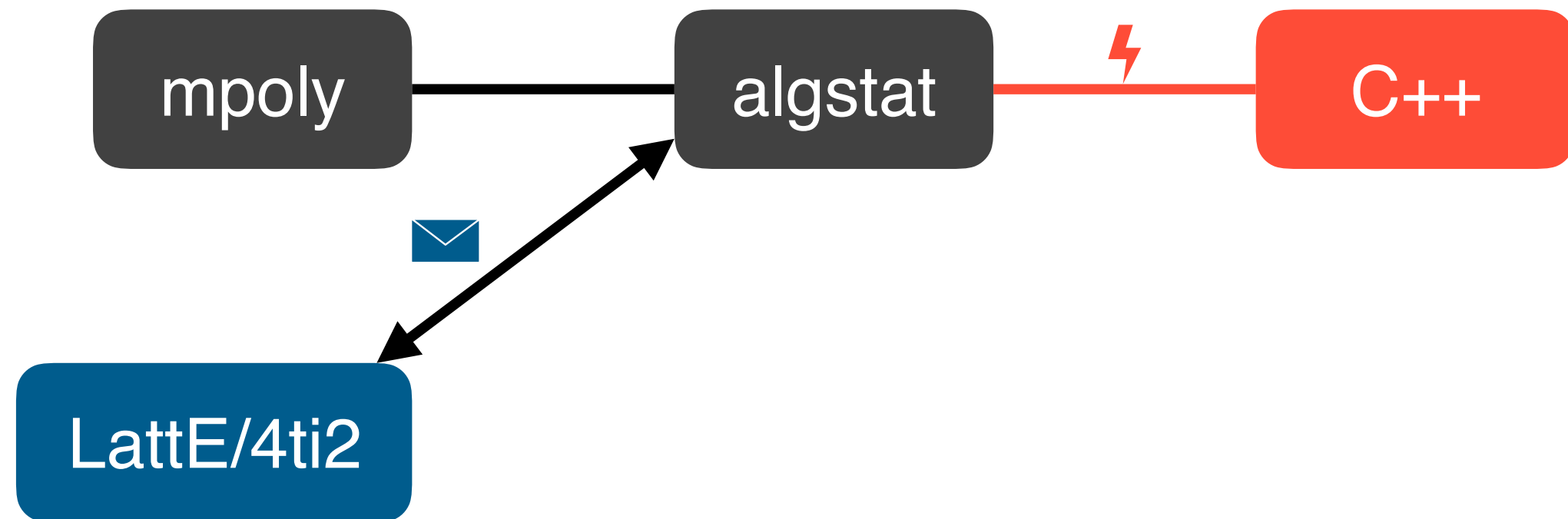
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 Objects computed on in place

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Objects computed on in place

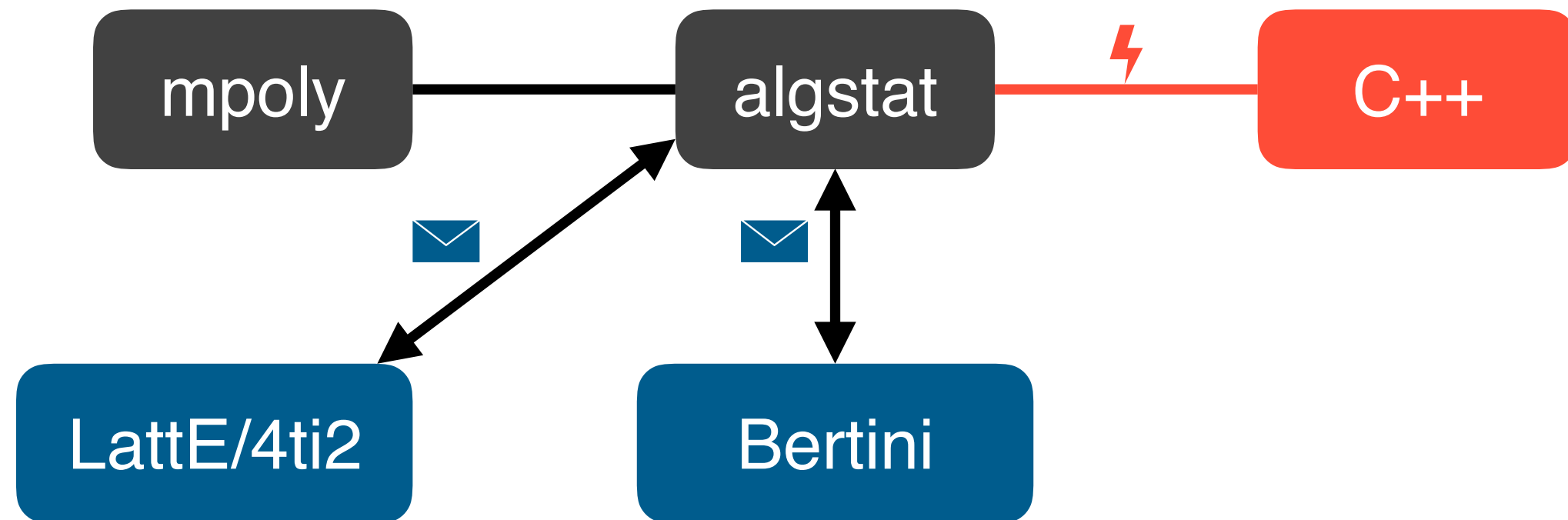


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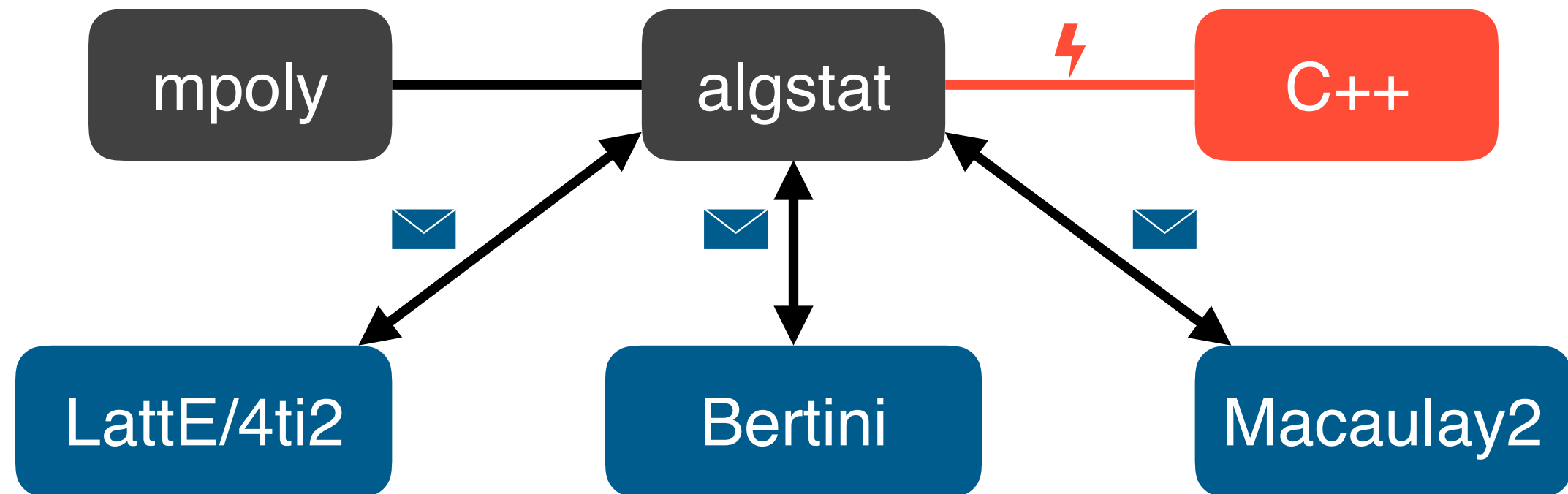
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2016 : **m2r** – Macaulay2

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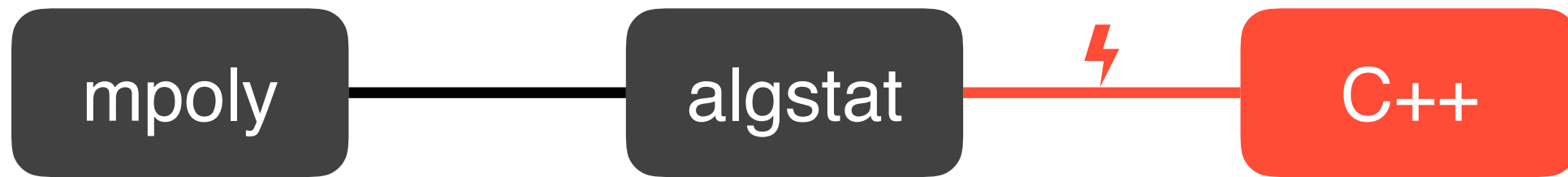
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2017 : **tropical** – Tropical geometry

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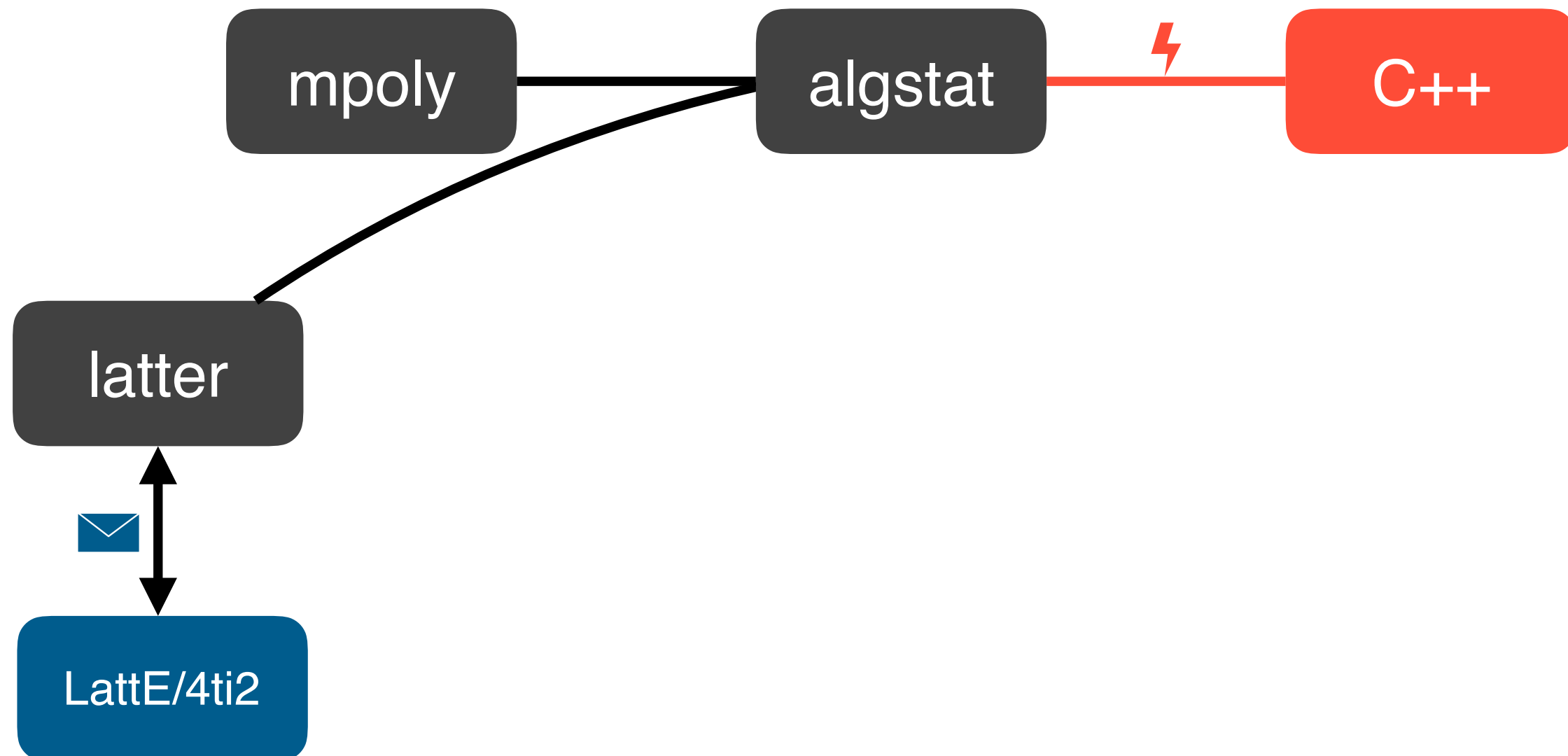
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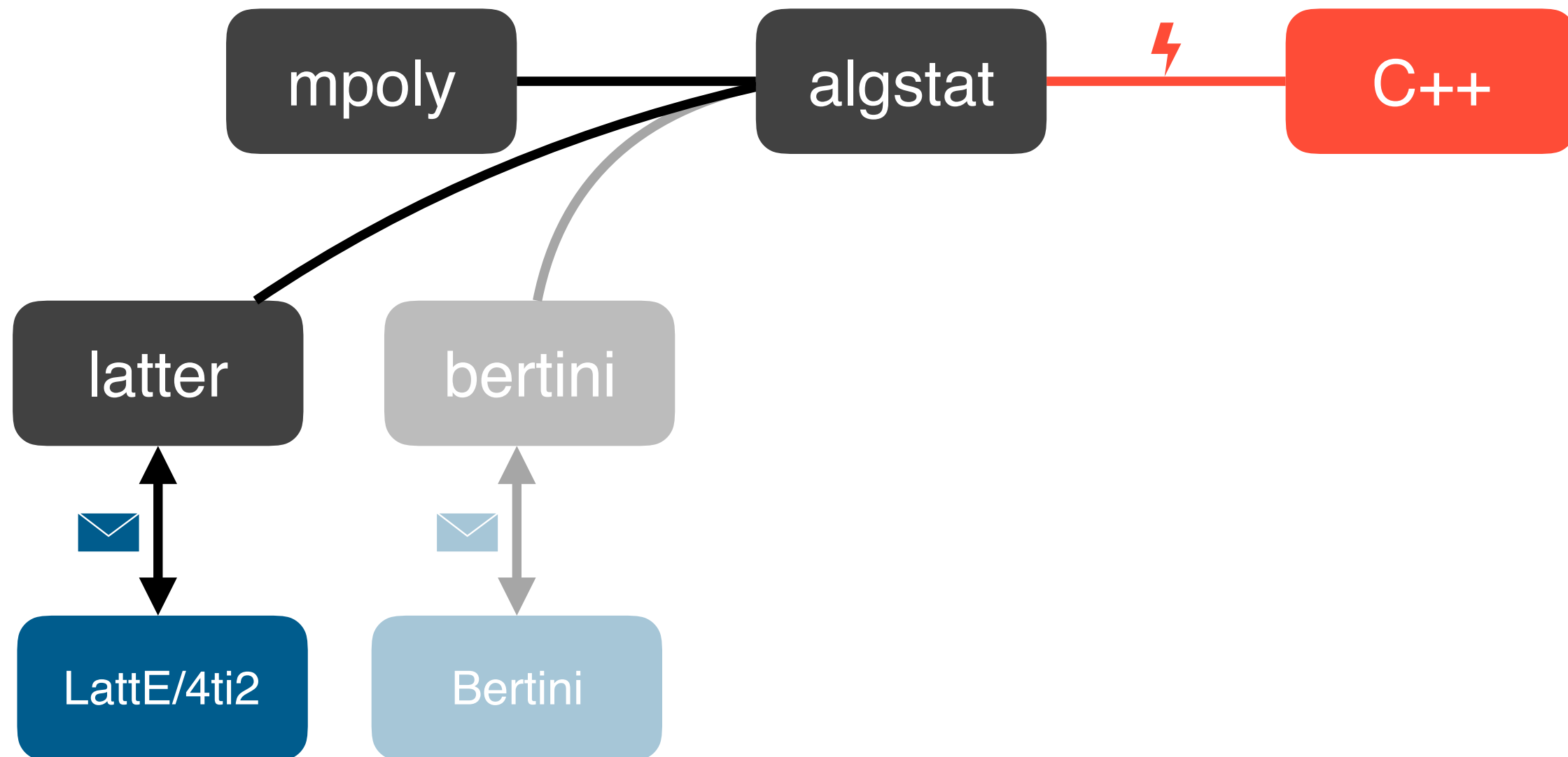
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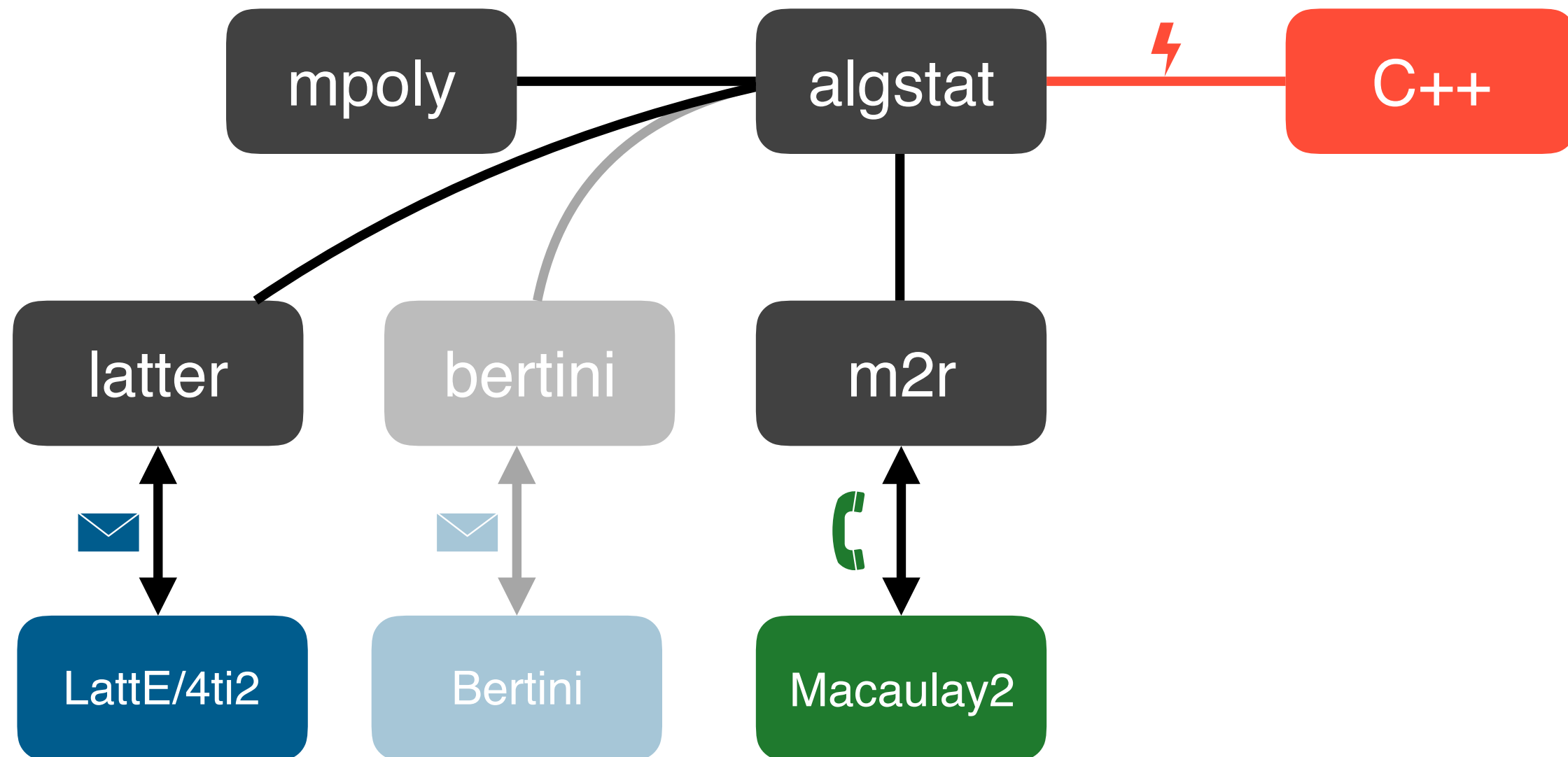
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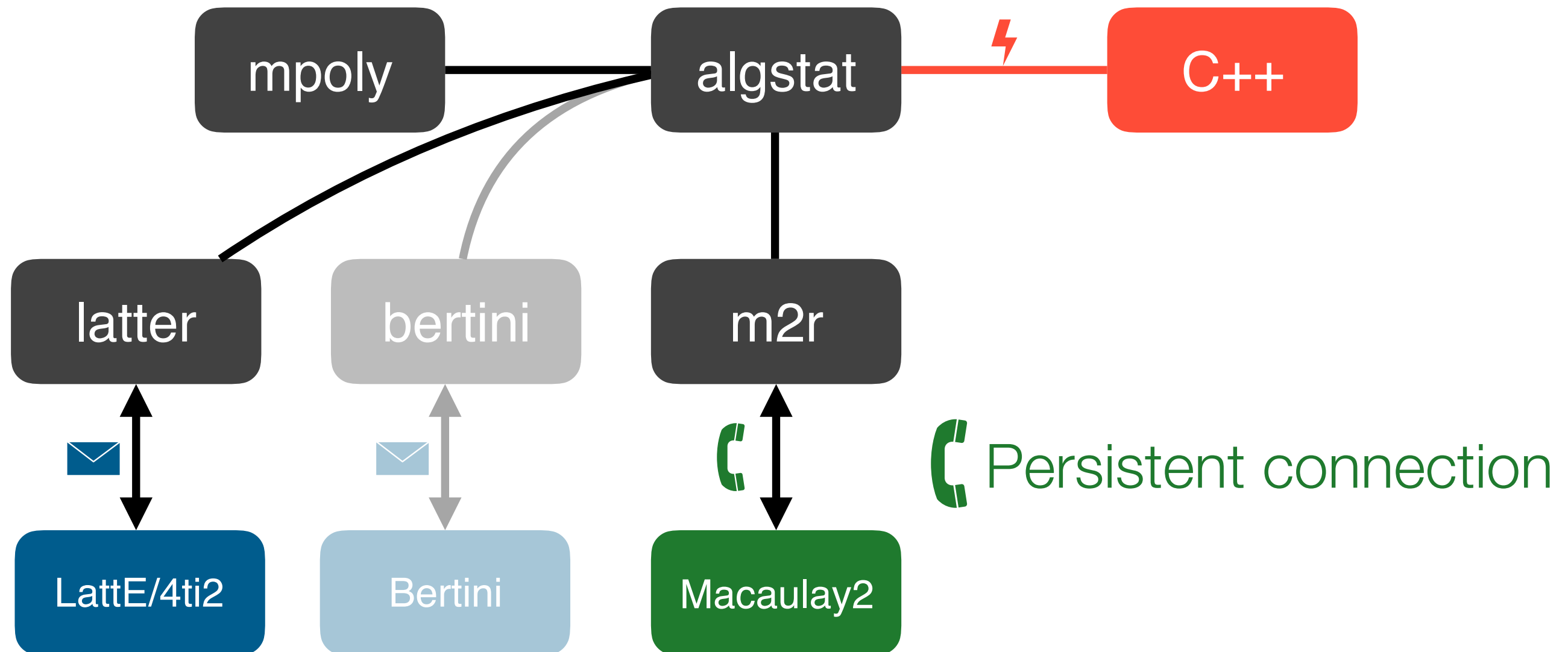
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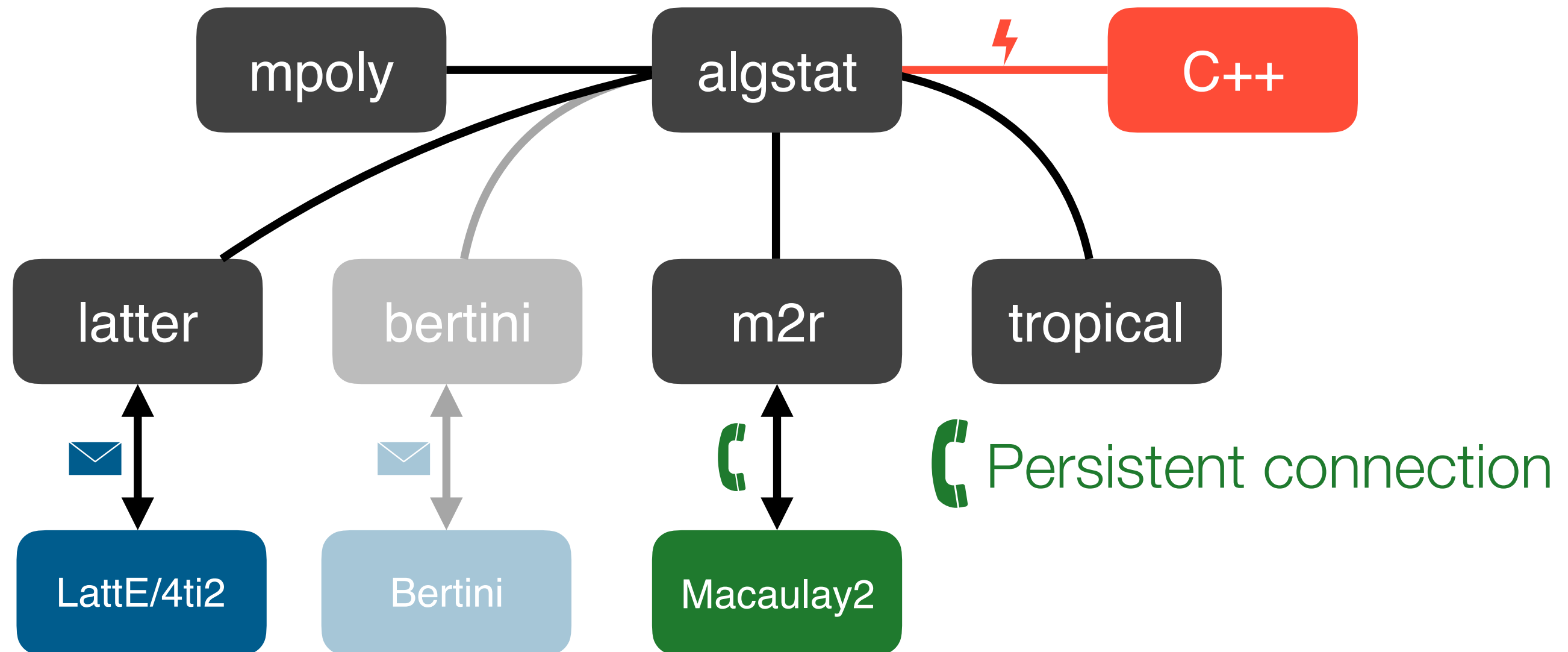
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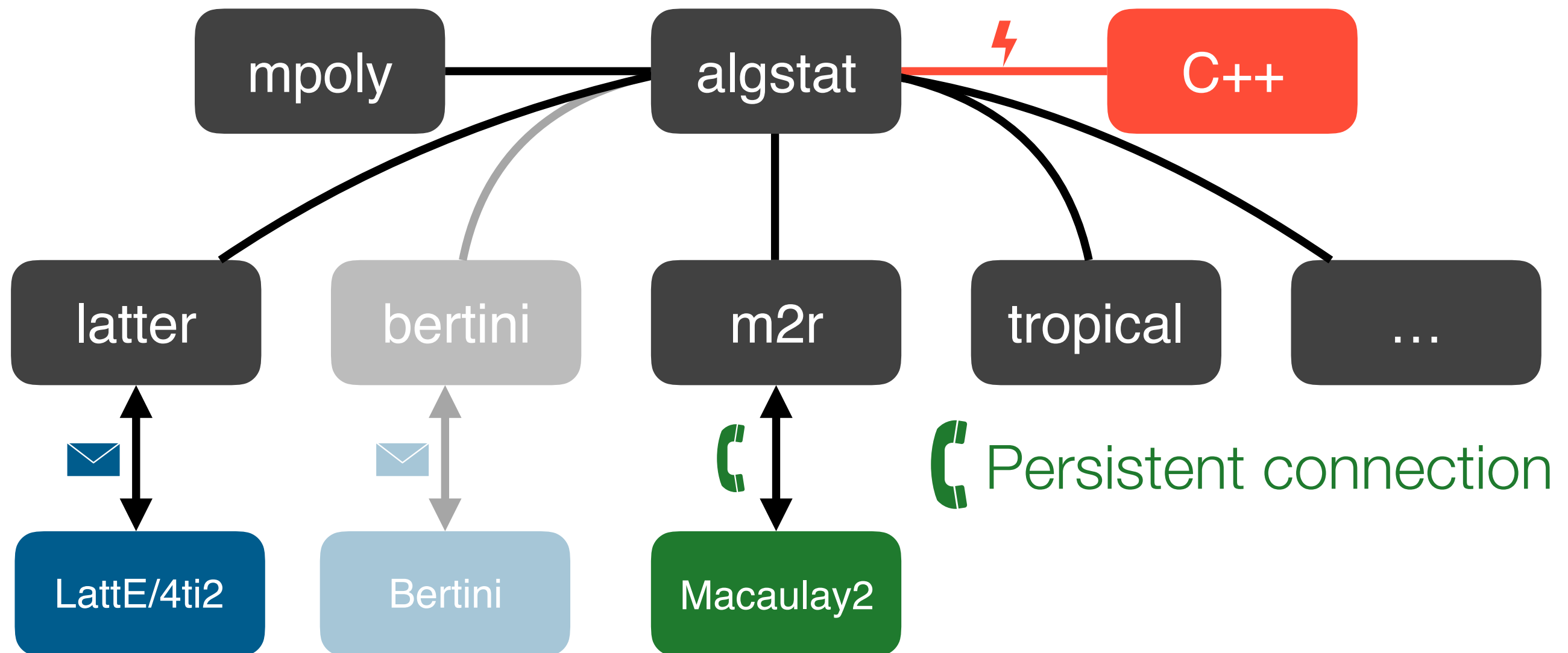
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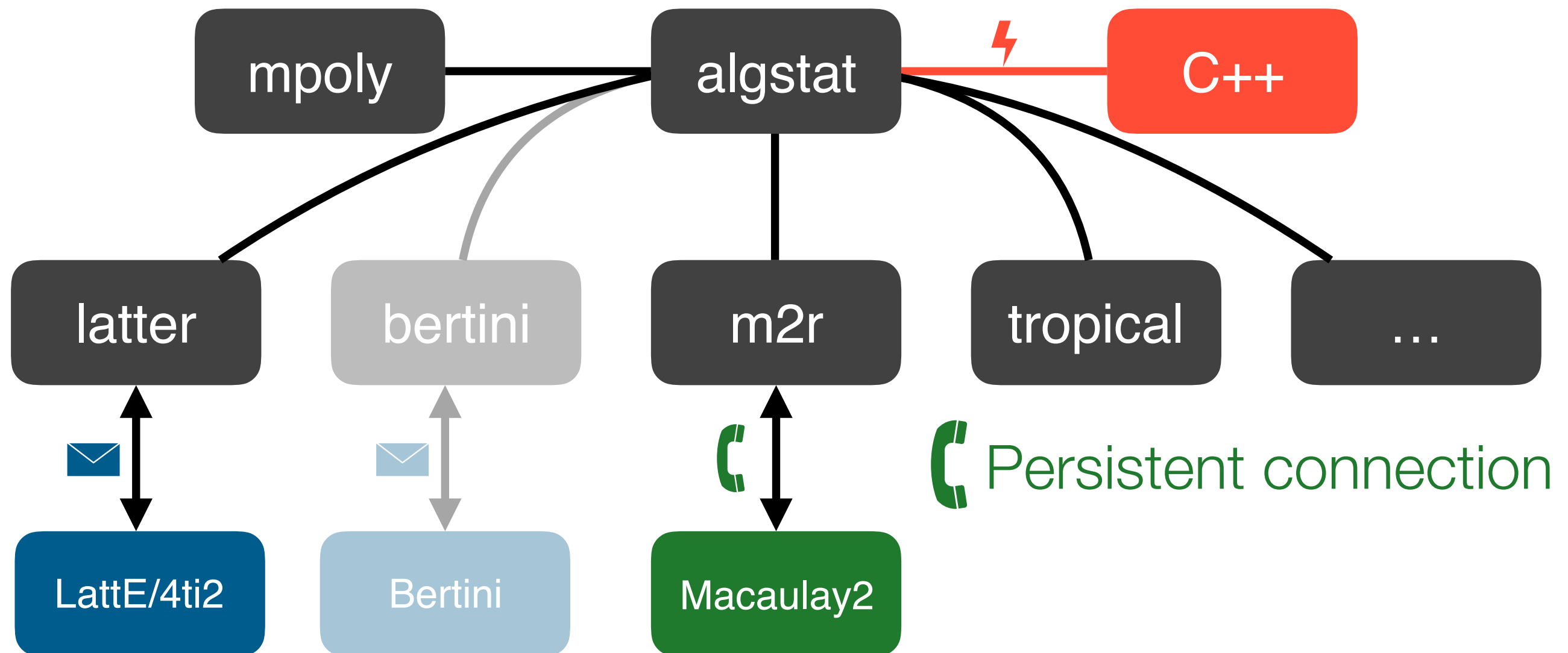
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\* Many other interconnections not shown

# The algstat Ecosystem



To install the packages I'm using in this talk, copy/paste the following code into R

```
if(!requireNamespace("devtools")) install.packages("devtools")
devtools::install_github("dkahle/mpoly", ref = "670645f")
devtools::install_github("dkahle/latter", ref = "9d2fd27")
devtools::install_github("dkahle/tropical", ref = "5599df2")
devtools::install_github("coneill-math/m2r", ref = "572aadd")
devtools::install_github("dkahle/algstat", ref = "1f063ab")
```

For **latter**, you'll need LattE/4ti2, which you can get [here](#)

For **algstat**, you'll want Bertini, which you can get [here](#)

mpoly contains data structures and methods for polynomials

`mpoly` contains data structures and methods for polynomials

Understanding `mpoly` requires understanding a bit about the R language

homogeneous

must hold the same kinds of objects

heterogeneous

can hold different kinds of objects

1d

atomic vector

int, num, char, logi, ...

list

2d

matrix

data frame

list of equal-length atomic vectors

nd

array

# R's main data structures



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```
(x <- c(1, 2, 3)) # a vector  
# [1] 1 2 3
```

```
(x <- c(1, 2, 3)) # a vector  
# [1] 1 2 3
```

```
(y <- list(1:3, c("a", "b"), TRUE)) # a list  
# [[1]]  
# [1] 1 2 3  
#  
# [[2]]  
# [1] "a" "b"  
#  
# [[3]]  
# [1] TRUE  
#
```

R objects can also have a list of metadata attached to them called **attributes**



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Two basic types of attributes are **names** and **class**

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**Classes** are R's most basic object oriented framework

R objects can also have a list of metadata attached to them called **attributes**

Two basic types of attributes are **names** and **class**

**Names** are used for easy, non-index based referencing

**Classes** are R's most basic object oriented framework

**Classes** are also used for method dispatch

(Methods are not contained in objects in R)

```
attr(x, "names") <- c("x", "y", "z")
```

```
x
```

```
# x y z
```

```
# 1 2 3
```

```
attr(x, "names") <- c("x", "y", "z")
```

```
x
```

```
# x y z
```

```
# 1 2 3
```

```
attr(x, "class") <- "foo"
```

```
x
```

```
# x y z
```

```
# 1 2 3
```

```
# attr(,"class")
```

```
# [1] "foo"
```

```
attr(x, "names") <- c("x", "y", "z")
```

```
x
```

```
# x y z
```

```
# 1 2 3
```

```
attr(x, "class") <- "foo"
```

```
x
```

```
# x y z
```

```
# 1 2 3
```

```
# attr("class")
```

```
# [1] "foo"
```

```
print(x)
```

```
# x y z
```

```
# 1 2 3
```

```
# attr("class")
```

```
# [1] "foo"
```

```
attr(x, "names") <- c("x", "y", "z")
```

```
x
```

```
# x y z
```

```
# 1 2 3
```

```
attr(x, "class") <- "foo"
```

```
x
```

```
# x y z
```

```
# 1 2 3
```

```
# attr("class")
```

```
# [1] "foo"
```

```
print(x)
```

```
# x y z
```

```
# 1 2 3
```

```
# attr("class")
```

```
# [1] "foo"
```

Typing x at the command line implicitly calls print(x)



```
print(x)
```

```
# x y z
```

```
# 1 2 3
```

```
# attr("class")
```

```
# [1] "foo"
```

```
print.foo <- function(.) {  
  cat( paste0(names(.), "^", .) )  
}
```

```
x
```

```
# x^1 y^2 z^3
```

```
print(x)
```

```
# x y z
```

```
# 1 2 3
```

```
# attr("class")
```

```
# [1] "foo"
```

```
print.foo <- function(.) {  
  cat( paste0(names(.), "^", .) )  
}
```

```
x
```

```
# x^1 y^2 z^3
```

```
str(x)
```

```
# Class 'foo' Named num [1:3] 1 2 3
```

```
# ..- attr(*, "names")= chr [1:3] "x" "y" "z"
```

mpoly's main basic data structure is the mpoly,  
a list of numeric vectors representing terms

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```
library(mpoly)
```

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```
library(mpoly)
mp("x^2 + 2 x y - 1")
# x^2 + 2 x y - 1
```

mpoly's main basic data structure is the mpoly,  
a list of numeric vectors representing terms

```
library(mpoly)
mp("x^2 + 2 x y - 1")
# x^2 + 2 x y - 1

str( mp("x^2 + 2 x y - 1") )
# List of 3
# $ : Named num [1:2] 2 1
# ..- attr(*, "names")= chr [1:2] "x" "coef"
# $ : Named num [1:3] 1 1 2
# ..- attr(*, "names")= chr [1:3] "x" "y" "coef"
# $ : Named num -1
# ..- attr(*, "names")= chr "coef"
# - attr(*, "class")= chr "mpoly"
```

`mpoly` provides all basic arithmetic operations as well as well as related tools

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```
p <- mp("x + y"); q <- mp("x - y")
```



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```
p <- mp("x + y"); q <- mp("x - y")
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```
p + q  
# 2 x
```

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```
p <- mp("x + y"); q <- mp("x - y")
```

```
p + q
```

```
# 2 x
```

```
p * q
```

```
# x^2 - y^2
```

`mpoly` provides all basic arithmetic operations as well as well as related tools

```
p <- mp("x + y"); q <- mp("x - y")
```

```
p + q
```

```
# 2 x
```

```
p * q
```

```
# x^2 - y^2
```

```
p^2
```

```
# x^2 + 2 x y + y^2
```

`mpoly` provides all basic arithmetic operations as well as well as related tools

```
p <- mp("x + y"); q <- mp("x - y")
```

```
p + q
```

```
# 2 x
```

```
p * q
```

```
# x^2 - y^2
```

```
p^2
```

```
# x^2 + 2 x y + y^2
```

```
f <- as.function(p, vector = FALSE)
```

```
# f(x, y)
```

```
f(1, 2)
```

```
# [1] 3
```

Vectors of mpolys are also defined with similar methods

Vectors of mpolys are also defined with similar methods

```
(ps <- mp(c("x + y", "x - y^2")))
```

```
# x + y
```

```
# x - y^2
```

Vectors of mpolys are also defined with similar methods

```
(ps <- mp(c("x + y", "x - y^2")))
```

```
# x + y
```

```
# x - y^2
```

```
g <- as.function(ps, vector = FALSE)
```

```
g(1, 2)
```

```
# [1] 3 -3
```

Vectors of mpolys are also defined with similar methods

```
(ps <- mp(c("x + y", "x - y^2")))
```

```
# x + y
# x - y^2
```

```
g <- as.function(ps, vector = FALSE)
```

```
g(1, 2)
```

```
# [1] 3 -3
```

These kinds of lists are used by most other packages in the **algstat** ecosystem



`latter` implements back-end connections to LattE/4ti2

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It is used mostly for computing Markov and related bases

```
library(latter)
```

```
# LattE found in /Applications/latte/dest/bin
```

```
# 4ti2 found in /Applications/latte/dest/bin
```

```
library(latter)
# LattE found in /Applications/latte/dest/bin
# 4ti2 found in /Applications/latte/dest/bin

(A <- genmodel(c(2, 2), 1:2)) # 2x2 independence model
#      [,1] [,2] [,3] [,4]
# [1,]    1    0    1    0
# [2,]    0    1    0    1
# [3,]    1    1    0    0
# [4,]    0    0    1    1
```

```
library(latter)
```

```
# LattE found in /Applications/latte/dest/bin
```

```
# 4ti2 found in /Applications/latte/dest/bin
```

```
(A <- genmodel(c(2, 2), 1:2)) # 2x2 independence model
```

```
#      [,1] [,2] [,3] [,4]
```

```
# [1,]    1    0    1    0
```

```
# [2,]    0    1    0    1
```

```
# [3,]    1    1    0    0
```

```
# [4,]    0    0    1    1
```

```
markov(A)
```

```
#      [,1]
```

```
# [1,]    1
```

```
# [2,]   -1
```

```
# [3,]   -1
```

```
# [4,]    1
```

```
library(latter)
```

```
# LattE found in /Applications/latte/dest/bin
```

```
# 4ti2 found in /Applications/latte/dest/bin
```

```
(A <- genmodel(c(2, 2), 1:2)) # 2x2 independence model
```

```
#      [,1] [,2] [,3] [,4]
```

```
# [1,]     1     0     1     0
```

```
# [2,]     0     1     0     1
```

```
# [3,]     1     1     0     0
```

```
# [4,]     0     0     1     1
```

```
markov(A)
```

```
#      [,1]
```

```
# [1,]     1
```

```
# [2,]    -1
```

```
# [3,]    -1
```

```
# [4,]     1
```

```
graver(A)
```

```
#      [,1]
```

```
# [1,]     1
```

```
# [2,]    -1
```

```
# [3,]    -1
```

```
# [4,]     1
```



LattE functions are also available



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```
spec <- c("x + y <= 10", "x >= 1", "y >= 1")
```

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```
spec <- c("x + y <= 10", "x >= 1", "y >= 1")  
count(spec)  
# [1] 45
```

LattE functions are also available

```
spec <- c("x + y <= 10", "x >= 1", "y >= 1")
```

```
count(spec)
```

```
# [1] 45
```

```
count(spec, dilation = 10)
```

```
# [1] 3321
```

LattE functions are also available

```
spec <- c("x + y <= 10", "x >= 1", "y >= 1")
```

```
count(spec)
```

```
# [1] 45
```

```
count(spec, dilation = 10)
```

```
# [1] 3321
```

```
latte_max(
```

```
  "-2 x + 3 y",
```

```
  c("x + y <= 10", "x >= 0", "y >= 0")
```

```
)
```

LattE functions are also available

```
spec <- c("x + y <= 10", "x >= 1", "y >= 1")
```

```
count(spec)
```

```
# [1] 45
```

```
count(spec, dilation = 10)
```

```
# [1] 3321
```

```
latte_max(
```

```
  "-2 x + 3 y",
```

```
  c("x + y <= 10", "x >= 0", "y >= 0")
```

```
)
```

```
# $par
```

```
#   x   y
```

```
#   0 10
```

```
#
```

```
# $value
```

```
# [1] 30
```

# m2r connections, data structures, and methods for Macaulay2

m2r connections, data structures, and methods for Macaulay2

Since Macaulay2 is a full computer algebra system, a persistent connection is needed

m2r connections, data structures, and methods for Macaulay2

Since Macaulay2 is a full computer algebra system, a persistent connection is needed

So... stay tuned for Chris O'Neill's talk next!



Bertini was one of the original algstat connections, but it hasn't been implemented in its "Version 2" form yet

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```
library(algstat)
```

Bertini was one of the original algstat connections, but it hasn't been implemented in its "Version 2" form yet

```
library(algstat)
polySolve(
  c("y == x^2", "y == 2 - x^2"),
  varOrder = c("x", "y")
)
# 2 solutions (x,y) found.
# (2 real, 0 complex; 2 nonsingular, 0 singular.)
#      (-1,1) (R)
#      ( 1,1) (R)
```

A new direction of **algstat** applications is phylogenetics

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**tropical** is a (very!) new package intended to supply the necessary tropical geometry computations

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```
library(tropical)  
# Using min-plus algebra.
```

A new direction of **algstat** applications is phylogenetics

**tropical** is a (very!) new package intended to supply the necessary tropical geometry computations

```
library(tropical)  
# Using min-plus algebra.
```

$$x \oplus y = \min(x, y)$$

$$x \otimes y = x + y$$

A new direction of **algstat** applications is phylogenetics

**tropical** is a (very!) new package intended to supply the necessary tropical geometry computations

```
library(tropical)
```

```
# Using min-plus algebra.
```

$$x \oplus y = \min(x, y)$$

$$x \otimes y = x + y$$

Max-plus is available with `set_plus_max()`



```
# basic tropical arithmetic
```

```
# basic tropical arithmetic  
1 %+% 5  
# [1] 1
```

```
# basic tropical arithmetic
1 +% 5
# [1] 1
1 %.% 5
# [1] 6
```

```
# basic tropical arithmetic
1 +% 5
# [1] 1
1 %.% 5
# [1] 6
5 %^% 3
# [1] 15
```

```
# basic tropical arithmetic
```

```
1 +% 5
```

```
# [1] 1
```

```
1 %.% 5
```

```
# [1] 6
```

```
5 %^% 3
```

```
# [1] 15
```

```
# vectorized for R-users
```

```
1:3 +% 3:1
```

```
# [1] 1 2 1
```

```
1:3 %.% 3:1
```

```
# [1] 4 4 4
```

```
# basic tropical arithmetic
```

```
1 +% 5
```

```
# [1] 1
```

```
1 %.% 5
```

```
# [1] 6
```

```
5 %^% 3
```

```
# [1] 15
```

```
# tropical mat. mult
```

```
# vectorized for R-users
```

```
1:3 +% 3:1
```

```
# [1] 1 2 1
```

```
1:3 %.% 3:1
```

```
# [1] 4 4 4
```

```
# basic tropical arithmetic
```

```
1 %+% 5
```

```
# [1] 1
```

```
1 %.% 5
```

```
# [1] 6
```

```
5 %^% 3
```

```
# [1] 15
```

```
# tropical mat. mult
```

```
1:3 %..% 4:6
```

```
# [1] 5
```

```
# vectorized for R-users
```

```
1:3 %+% 3:1
```

```
# [1] 1 2 1
```

```
1:3 %.% 3:1
```

```
# [1] 4 4 4
```

```
# basic tropical arithmetic
```

```
1 %+% 5
```

```
# [1] 1
```

```
1 %.% 5
```

```
# [1] 6
```

```
5 %^% 3
```

```
# [1] 15
```

```
# vectorized for R-users
```

```
1:3 %+% 3:1
```

```
# [1] 1 2 1
```

```
1:3 %.% 3:1
```

```
# [1] 4 4 4
```

```
# tropical mat. mult
```

```
1:3 %..% 4:6
```

```
# [1] 5
```

```
(m1 <- matrix(1:6, 2, 3))
```

```
#      [,1] [,2] [,3]
```

```
# [1,]    1    3    5
```

```
# [2,]    2    4    6
```

```
(m2 <- matrix(6:1, 3, 2))
```

```
#      [,1] [,2]
```

```
# [1,]    6    3
```

```
# [2,]    5    2
```

```
# [3,]    4    1
```



```
# basic tropical arithmetic
```

```
1 %+% 5
```

```
# [1] 1
```

```
1 %.% 5
```

```
# [1] 6
```

```
5 %^% 3
```

```
# [1] 15
```

```
# vectorized for R-users
```

```
1:3 %+% 3:1
```

```
# [1] 1 2 1
```

```
1:3 %.% 3:1
```

```
# [1] 4 4 4
```

```
# tropical mat. mult
```

```
1:3 %..% 4:6
```

```
# [1] 5
```

```
(m1 <- matrix(1:6, 2, 3))
```

```
#      [,1] [,2] [,3]
```

```
# [1,]      1      3      5
```

```
# [2,]      2      4      6
```

```
(m2 <- matrix(6:1, 3, 2))
```

```
#      [,1] [,2]
```

```
# [1,]      6      3
```

```
# [2,]      5      2
```

```
# [3,]      4      1
```

```
m1 %..% m2
```

```
#      [,1] [,2]
```

```
# [1,]      7      4
```

```
# [2,]      8      5
```

```
data(politics)
```

```
politics
```

```
#           Party
# Personality Democrat Republican
#   Introvert         3         7
#   Extrovert         6         4
```

```
data(politics)
```

```
politics
```

```
#           Party
# Personality Democrat Republican
#  Introvert           3           7
#  Extrovert           6           4
```

```
(A <- hmat(c(2, 2), list(1, 2))) # alternative to genmodel
```

```
#      11 12 21 22
# 1+   1  1  0  0
# 2+   0  0  1  1
# +1   1  0  1  0
# +2   0  1  0  1
```

```
data(politics)
```

```
politics
```

```
#           Party
# Personality Democrat Republican
#  Introvert           3           7
#  Extrovert           6           4
```

```
(A <- hmat(c(2, 2), list(1, 2))) # alternative to genmodel
```

```
#      11 12 21 22
# 1+   1  1  0  0
# 2+   0  0  1  1
# +1   1  0  1  0
# +2   0  1  0  1
```

```
countTables(politics, A)
```

```
# [1] 10
```

```
loglinear(~ Personality + Party, data = politics)
```

```
loglinear(~ Personality + Party, data = politics)  
# Computing Markov moves (4ti2)... done.  
# Running chain (C++)... done.
```

```
loglinear(~ Personality + Party, data = politics)
# Computing Markov moves (4ti2)... done.
# Running chain (C++)... done.
# Call:
# loglinear(model = ~Personality + Party, data = politics)
#
# Fitting method:
# Iterative proportional fitting (with stats::loglin)
#
# MCMC details:
# N = 10000 samples (after thinning), burn in = 1000, thinning = 10
#
#           Distance      Stat      SE p.value      SE mid.p.value
#           P(samp)                0.3677 0.0048      0.2201
#   Pearson X^2 1.8182 0.0146 0.3677 0.0048      0.2201
# Likelihood G^2 1.848 0.0155 0.3677 0.0048      0.2201
# Freeman-Tukey 1.8749 0.0167 0.3677 0.0048      0.2201
# Cressie-Read 1.8247 0.0148 0.3677 0.0048      0.2201
```

# GitHub and Contributing



To submit a feature request or report a bug:

- Go to <https://github.com/> and create a free account
- Go to <https://github.com/dkahle/algstat>
- Click *Issues*
- Click *New Issue*

To join the fray, submit a pull request (PR)!

- Go to <https://github.com/> and create a free account
- Go to <https://github.com/dkahle/algstat>
- Click *Fork* to make your own copy of the repository
- In RStudio...
  - File > New Project... > Version Control > Git
  - Enter the URL of the repo, <https://github.com/dkahle/algstat.git>
  - Make changes to the code and commit them, see [tutorial here](#)
  - Push changes to GitHub
- On GitHub, click *Submit a Pull Request*

# Upcoming Projects

tropical with Grant Innerst, Rudy Yoshida, Leon Zhang  
and Xu Zhang

m2r with Chris O'Neill and Jeff Sommars

algstat with Luis Garcia, Rudy Yoshida, ...

bertini with Grant Innerst

...?

# Thank you!!

[www.kahle.io](http://www.kahle.io)

<https://github.com/dkahle/2017-SIAM-Talk>

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