Problem 1: ①

$$p(x=0) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

$$p(x=1) = \frac{1}{3} = \frac{1}{3}.$$

$$p(y=0) = \frac{1}{3} + 0 = \frac{1}{3}$$

$$p(y=1) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

$p(x|y)$

$$p(x=0|y=0) = \frac{p(x=0, y=0)}{p(y=0)} = \frac{1/3}{1/3} = 1.$$

$$p(x=0|y=1) = \frac{p(x=0, y=1)}{p(y=1)} = \frac{1/3}{2/3} = \frac{1}{2}.$$

$$p(x=1|y=0) = \frac{p(x=1, y=0)}{p(y=0)} = \frac{0}{1/3} = 0.$$

$$p(x=1|y=1) = \frac{p(x=1, y=1)}{p(y=1)} = \frac{1/3}{2/3} = \frac{1}{2}.$$

$p(y|x)$ *

$$p(y=0|x=0) = \frac{p(y=0,x=0)}{p(x=0)} = \frac{1/3}{2/3} = \frac{1}{2}.$$

$$p(y=0|x=1) = \frac{p(y=0,x=1)}{p(x=1)} = \frac{0}{1/3} = 0.$$

$$p(y=1|x=0) = \frac{p(y=1,x=0)}{p(x=0)} = \frac{1/3}{2/3} = \frac{1}{2}.$$

$$p(y=1|x=1) = \frac{p(y=1,x=1)}{p(x=1)} = \frac{1/3}{1/3} = 1.$$

$$p(x,y) = \sum_i \sum_j p(x=x_i, y=y_i)$$

$$= \frac{1}{3} + \frac{1}{3} + 0 + \frac{1}{3}$$

$$= 1.$$

$$H(x) = -p(x=0)\log(p(x=0)) - p(x=1)\log(p(x=1)).$$

$$= -\frac{2}{3}\log\left(\frac{2}{3}\right) - \frac{1}{3}\log\left(\frac{1}{3}\right)$$

$$= 0.9183.$$

$$H(y) = -p(y=0)\log(p(y=0)) - p(y=1)\log(p(y=1)).$$

$$= -\frac{1}{3}\log\frac{1}{3} - \frac{2}{3}\log\frac{2}{3}$$

$$= 0.9183$$

$$H(y|x) = p(x=0)\,H(y|x=0) + p(x=1)\,H(y|x=1)$$

$$= p(x=0)\left[-p(y=0|x=0)\log_2 p(y=0|x=0) - p(y=1|x=0)\log_2 p(y=1|x=0)\right] +$$

$$p(x=1)\left[-p(y=1|x=1)\log_2 p(y=1|x=1) - p(y=0|x=1)\log_2 p(y=0|x=1)\right].$$

$$= \frac{2}{3}\left[-\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2}\right] + \frac{1}{3}\left[-1\log_2 1 - 0\right]$$

$$= \frac{2}{3}\left[-\log_2\frac{1}{2}\right] = \frac{2}{3}.$$

$$H(x|y) = p(y=0)\,H(x|y=0) + p(y=1)\,H(x|y=1).$$

$$= p(y=0)\left[-p((x=0)|y=0)\log p((x=0)|y=0)) + (-p(x=1|y=0)\log p(x=1|y=0))\right].$$

$$+ p(y=1)\left[-p((x=1)|y=0)\log p((x=1)|y=0)) + (-p(x=0|y=1)\log p(x=0|y=1)\right]$$

$$= \frac{1}{3}\left[-1\log 1 + 0\right] + \frac{2}{3}\left[-\frac{1}{2}\log\frac{1}{2} + \left(-\frac{1}{2}\right)\log\frac{1}{2}\right].$$

$$= \frac{2}{3}\left[\frac{1}{2} + \frac{1}{2}\right] = \frac{2}{3}.$$

$$IG(x|y) = H(x) - H(x|y)$$

$$= 0.9183 - \frac{2}{3}$$

$$IG(x|y) = 0.2516.$$